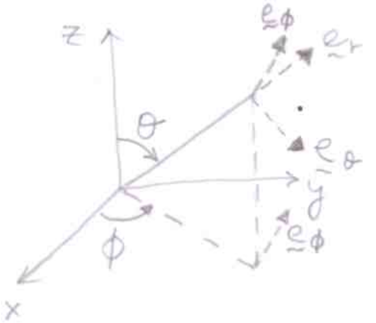
CE623 — HW2 — Solutions

T2, P4, long way

1) At boundary, $\sigma_{xx} = \dfrac{x^2 z}{a^4}$, $\sigma_{yy} = \dfrac{y^2 z}{a^4}$, $\sigma_{zz} = \dfrac{z^3}{a^4}$

$\sigma_{xy} = \dfrac{xyz}{a^4}$, $\sigma_{yz} = \dfrac{yz^2}{a^4}$, $\sigma_{xz} = \dfrac{xz^2}{a^4}$.

$x = r s\theta c\phi$, $y = r s\theta s\phi$, $z = r c\theta$



$$\begin{Bmatrix} \underset{\sim}{e}_\theta \\ e_\phi \\ \underset{\sim}{e}_r \end{Bmatrix} = \underbrace{\begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -s\phi & c\phi & 0 \\ s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix}}_{= \underline{\underline{a}}} \begin{Bmatrix} \underset{\sim}{i} \\ \underset{\sim}{j} \\ \underset{\sim}{k} \end{Bmatrix}$$

$$(\underline{\underline{\sigma}})_{r,\theta,\phi} = \underline{\underline{a}} (\underline{\underline{\sigma}})_{x,y,z}\, a^T$$

$$\begin{bmatrix} \sigma_{\theta\theta} & \sigma_{\theta\phi} & \sigma_{r\theta} \\ \sigma_{\theta\phi} & \sigma_{\phi\phi} & \sigma_{r\phi} \\ \sigma_{r\theta} & \sigma_{r\phi} & \sigma_{rr} \end{bmatrix} = \frac{1}{a^4} \begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -s\phi & c\phi & 0 \\ s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix} \begin{bmatrix} x^2 z & xyz & xz^2 \\ xyz & y^2 z & yz^2 \\ xz^2 & yz^2 & z^3 \end{bmatrix} \begin{bmatrix} c\theta c\phi & -s\phi & s\theta c\phi \\ c\theta s\phi & c\phi & s\theta s\phi \\ -s\theta & 0 & c\theta \end{bmatrix}$$

<u>Loading on surface</u>: only components involving r participate in b.c. Hence we need only σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\phi}$.

$$\sigma_{rr} = \Big[s\theta c\phi \underbrace{\left(s\theta c\phi . s^2\theta c^2\phi c\theta + s\theta s\phi . s\theta c\phi s\theta s\phi c\theta + c\theta . s\theta c\phi c^2\theta \right)}_{\text{I}}$$
$$+ s\theta s\phi \underbrace{\left(s\theta c\phi . s\theta c\phi s\theta s\phi c\theta + s\theta s\phi . s^2\theta s^2\phi c\theta + c\theta . s\theta s\phi c^2\theta \right)}_{\text{II}}$$
$$+ c\theta \underbrace{\left(s\theta c\phi . s\theta c\phi c^2\theta + s\theta s\phi . s\theta s\phi c^2\theta + c\theta . c^3\theta \right)}_{\text{III}} \Big] * \frac{1}{a}$$

Here, and in what follows σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\phi}$ represent stress components on surface, hence they represent the loading

$$\sigma_{r\phi} = \frac{1}{a}\left[-s\phi\,(\text{I}) + c\phi\,(\text{II}) \right]$$

$$\sigma_{r\theta} = \frac{1}{a}\left[c\theta c\phi\,(\text{I}) + c\theta s\phi\,(\text{II}) - s\theta\,(\text{III}) \right]$$

$$\text{I} = s^3\theta c\theta c^3\phi + s^3\theta c\theta s^2\phi c\phi + s\theta c^3\theta c\phi = s\theta c\theta c\phi$$

$$\text{II} = s^3\theta c\theta s\phi c^2\phi + s^3\theta c\theta s^3\phi + s\theta c^3\theta s\phi = s\theta c\theta s\phi$$

$$\text{III} = s^2\theta c^2\theta c^2\phi + s^2\theta c^2\theta s^2\phi + c^4\theta = c^2\theta$$

$$\sigma_{rr} = \left(s^2\theta c\theta c^2\phi + s^2\theta c\theta s^2\phi + c^3\theta \right) * \frac{1}{a} = c\theta / a$$

$$\sigma_{r\phi} = \left(-s\theta c\theta s\phi c\phi + s\theta c\theta s\phi c\phi \right)/a = 0$$

$$\sigma_{r\theta} = \left(s\theta c^2\theta c^2\phi + s\theta c^2\theta s^2\phi - s\theta c^2\theta \right)/a = 0$$

$\because \sigma_{rr}$ is function of θ alone, the stress distribution σ_{rr} along every latitudinal band cancels out $\Rightarrow F_x = F_y = 0$

$$F_z = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{\sigma_{rr}}_{c\theta/a} \underbrace{(a\,d\theta\; a s\theta\, d\phi)}_{dA} c\theta = 2\pi a \int_0^{\pi/2} s\theta c^2\theta\, d\theta = -\frac{2\pi a}{3} c^3\theta \Big|_0^{\pi/2}$$

$$F_z = \frac{2\pi a}{3}$$

Extra - I

In general,

$$F_z = \iint (\sigma_{rr}\, dA\, c\theta - \sigma_{r\theta}\, dA\, s\theta) = \int_0^{2\pi}\int_0^{\pi/2} (\sigma_{rr} c\theta - \sigma_{r\theta} s\theta)(a^2 s\theta\, d\theta\, d\phi)$$

$$F_x = \iint (\sigma_{rr}\, dA\, s\theta c\phi + \sigma_{r\theta}\, dA\, c\theta c\phi - \sigma_{r\phi}\, dA\, s\phi)$$
$$= \int_0^{2\pi}\int_0^{\pi/2} (\sigma_{rr} s\theta c\phi + \sigma_{r\theta} c\theta c\phi - \sigma_{r\phi} s\phi)(a^2 s\theta\, d\theta\, d\phi)$$

$$F_y = \iint (\sigma_{rr}\, dA\, s\theta s\phi + \sigma_{r\theta}\, dA\, c\theta s\phi + \sigma_{r\phi}\, dA\, c\phi)$$
$$= \int_0^{2\pi}\int_0^{\pi/2} (\sigma_{rr} s\theta s\phi + \sigma_{r\theta} c\theta s\phi + \sigma_{r\phi} c\phi)(a^2 s\theta\, d\theta\, d\phi)$$

T2, P4, short way

Extra - 2. Alternative & much shorter method.

Find stress vector on surface, i.e., $\boxed{\underline{\sigma} = \underline{\underline{\sigma}}\, \underline{n}}$

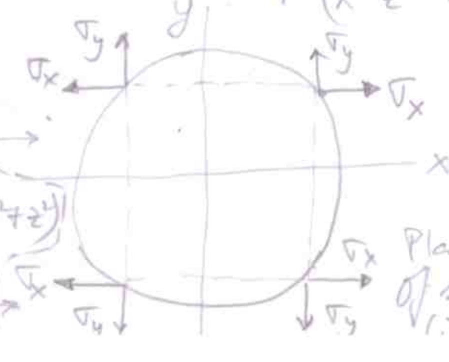
Surface $\equiv f(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$.

$$\underline{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\underline{i} + 2y\underline{j} + 2z\underline{k}}{2\underbrace{\sqrt{x^2+y^2+z^2}}_{=a}} = \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}$$

$$\Rightarrow \underline{\sigma} = \frac{1}{a^4}\begin{bmatrix} x^2 z & xyz & xz^2 \\ xyz & y^2 z & yz^2 \\ xz^2 & yz^2 & z^3 \end{bmatrix} \frac{1}{a}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \frac{1}{a^5}\Big[(x^3 z + xy^2 z + xz^3)\underline{i} + (x^2 yz + y^3 z + yz^3)\underline{j} + (x^2 z^2 + y^2 z^2 + z^4)\underline{k}\Big]$$

$$\Rightarrow \sigma_x = \frac{1}{a^5}(x^3 z + xy^2 z + xz^3) = \frac{xz}{a^5}\underbrace{(x^2+y^2+z^2)}_{a^2}$$
$$= \frac{xz}{a^3} = \text{odd in } x.$$

$$\sigma_y = \frac{1}{a^5}(x^2 yz + y^3 z + yz^3) = \frac{yz}{a^5}(x^2+y^2+z^2)$$
$$= \frac{yz}{a^3} - \text{odd in } y.$$



Plan of a frustrum of sphere at latitude (i.e. θ, z are const)

(1c)

So we see that in every strip of annular area $2\pi a \sin\theta \, a\, d\theta$, the contributions from opposite sides cancel when summing, i.e.,

$$F_x = \int_A \sigma_x \, dA = 0, \qquad F_y = \int_A \sigma_y \, dA = 0$$

$$F_z = \int \sigma_z \, dA = \frac{1}{a^5}\iint \underbrace{(x^2z^2 + y^2z^2 + z^4)}_{= z^2a^2} dA = \frac{1}{a^5}\iint a^2 z^2 \, dA$$

$$= \iint \frac{\cancel{a^2}\, a^2 c^2\theta}{\cancel{a^5}} (\cancel{a}\, d\theta \, a s\theta \, d\phi) = 2\pi a \int_0^{\pi/2} c^2\theta \, s\theta = \frac{2\pi a}{3}$$

same as by longer method involving stress transf.

T2, P5

(2). It can be intuitively shown that

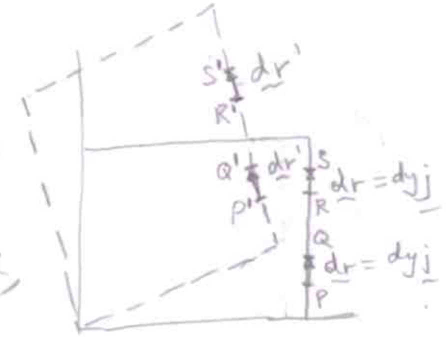
(a) $u_x = ax + by + cxy$, $u_y = dx + ey + gxy$ $\Big\}$ → Result.

from the fact that straight edges remain straight. However, below is a rigorous proof yielding this result.

$$\underline{r}' = \underline{r} + \underline{u} = x\underline{i} + y\underline{j} + u_x\underline{i} + u_y\underline{j}$$

$$d\underline{r} = (dx + du_x)\underline{i} + (dy + du_y)\underline{j}$$

Assume $u_x = ax + by + cxy + p(x,y)$, $u_y = dx + ey + gxy + q(x,y)$ $\Big\}$ → (1)



Corresponding to an initially vertical edge, we have,

$$d\underline{r}\Big|_{x=k_1} = \left(b + ck_1 + \frac{\partial p}{\partial y}\Big|_{x=k_1}\right) dy\,\underline{i} + \left(1 + e + gk_1 + \frac{\partial q}{\partial y}\Big|_{x=k_1}\right) dy\,\underline{j}$$

Now PQ and RS are same vector, i.e. $d\underline{r} = dy\,\underline{j}$, but corresponding to points at different locations y, along the $x = k_1$ edge. These deform to $\underline{P'Q'} = (d\underline{r})_{PQ}$ and $\underline{R'S'} = (d\underline{r})_{RS}$, which need not be same vector, since

stretching can take place. However the orientations (1d)
of $(\underline{dr}')_{PQ}$ and $(\underline{dr}')_{RS}$ must be same if deformed edge remains straight line. This means,

$$\frac{1+e+gk_1+\left.\frac{\partial q}{\partial y}\right|_{x=k_1}}{b+ck_1+\left.\frac{\partial p}{\partial y}\right|_{x=k_1}} = \text{function of } x \text{ only.}$$

$\Rightarrow$ q and p are linear in y.
Considering initially horizontal edge, we have (using similar procedure),

$$\left.\underline{dr}\right|_{y=k_2} = \left(1+a+ck_2+\left.\frac{\partial p}{\partial x}\right|_{y=k_2}\right)dx\,\underline{i} + \left(d+gk_2+\left.\frac{\partial q}{\partial x}\right|_{y=k_2}\right)dx\,\underline{j}$$

For line $y=k_2$ to remain straight after deformation, we require,

$$\frac{\left(1+a+ck_2+\left.\frac{\partial p}{\partial x}\right|_{y=k_2}\right)}{\left(d+gk_2+\left.\frac{\partial q}{\partial x}\right|_{y=k_2}\right)} = \text{function of } y \text{ only}$$

$\Rightarrow$ q and p are linear in x

$\Rightarrow$ q, p have bi-linear form (xy) which is already included in u_x, u_y. Hence $p(x,y)$, $q(x,y)$ are discarded form (1) $\rightarrow$ QED $\rightarrow$ contd on pg (1e)

T2, P6

P.3. You must use finite (large) strain theory, with given assumptions.

So, $\gamma_{xz} = 2E_{xz} = (1+\varepsilon_x)(1+\varepsilon_z)\cos\theta$
where θ = angle between A^*C^* & A^*B^* (ie between two line elements originally $\perp$ar) and ε_x, ε_z are engg ext. strains of elements originally along x & z directions (ie along AB & AC directions).
$\Rightarrow \gamma_{xz} = (1+0)(1+\{AC[1/\cos\alpha - 1]\}/AC)\cos\theta = \frac{1}{\cos\alpha}\cos\theta = \frac{1}{\cos\alpha}\sin\alpha = \tan\alpha$ ◀

$$u_x[1,0] = -0.002 = a$$
$$u_x[1,1] = -0.005 = a+b+c$$
$$u_x[0,1] = -0.003 = b$$

$\Rightarrow a = -0.002,\ b = -0.003,\ c = 0$

$$u_x = -0.002x - 0.003y$$

$$u_y[1,0] = 0.001 = d$$
$$u_y[1,1] = d+e+g = 0.0035$$
$$u_y[0,1] = e = 0.0025$$

$\Rightarrow d = 0.001,\ e = 0.0025,\ g = 0$

$$u_y = 0.001x + 0.0025y$$

Note: The above expressions give u_x, u_y in metres if x, y are in metres

(b) $\because (a, b, c, d) \ll 1$, you can use infinitesimal displ. gradient theory (ie linear theory) However we'll use nonlinear theory to start with.

$$E_{xx} = u_{x,x} + \frac{1}{2}u_{x,x}^2 + \frac{1}{2}u_{y,x}^2 = -0.002 + \frac{0.002^2 + 0.001^2}{2} = -0.0019975$$

$$E_{yy} = u_{y,y} + \frac{1}{2}u_{x,y}^2 + \frac{1}{2}u_{y,y}^2 = 0.0025 + \frac{0.003^2 + 0.0025^2}{2} = 0.002507625$$

$$E_{xy} = \frac{1}{2}\left(u_{x,y} + u_{y,x} + u_{x,x}u_{x,y} + u_{y,x}u_{y,y}\right)$$
$$= \frac{1}{2}\left(-0.003 + 0.001 + [-0.002][-0.003] + [0.001][0.0025]\right)$$
$$= -0.00099575$$

$$(\underline{\underline{E}})_{x,y} = \begin{pmatrix} -0.0019975 & -0.00099575 \\ -0.00099575 & 0.002507625 \end{pmatrix} \approx \begin{pmatrix} -0.002 & -0.001 \\ -0.001 & 0.0025 \end{pmatrix} = (\underline{\underline{e}})_{xy}$$

linear strain tensor

SO FROM HERE ON I USE LINEAR THEORY ($\underline{\underline{e}}$) FOR CONVENIENCE

(c) $$(\underline{\underline{e}})_{X,Y} = \underline{\underline{a}}\,(\underline{\underline{e}})_{x,y}\,\underline{\underline{a}}^T = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}\begin{pmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{pmatrix}\begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix}$$

$$= \begin{pmatrix} (e_{xx}c^2\theta + e_{yy}s^2\theta + 2e_{xy}\cos\theta) & (\cos\theta[-e_{xx}+e_{yy}] + e_{xy}[c^2\theta - s^2\theta]) \\ \text{symmetric} & (e_{xx}s^2\theta + e_{yy}c^2\theta - 2e_{xy}\cos\theta) \end{pmatrix}$$

$\theta = 30^\circ$

$$(\underline{\underline{e}})_{XY} = \begin{pmatrix} -0.001741 & 0.001449 \\ 0.001449 & 0.002241 \end{pmatrix}$$

(d) $|\underline{\underline{e}} - \lambda \underline{\underline{I}}| = 0$, for convenience, scale up $\underline{\underline{e}}$ by 10^3

$$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2.5-\lambda \end{pmatrix} = 0$$, so actual p-strains will be $10^{-3}\lambda$.

$$\lambda^2 - 4.5\lambda + 4 = 0 \Rightarrow \lambda = \frac{4.5 \pm \sqrt{4.5^2 - 16}}{2}$$

$$\lambda(1) = 3.2808, \; 1.2192 = \lambda(2)$$

p-strains are $e(1) = 3.2808 * 10^{-3}$, $e(2) = 1.2192 * 10^{-3}$

p-axes : $(2-\lambda(1))n_1(1) - n_2(1) = 0 \rightarrow$ (i)

$$n_1^2(1) + n_2^2(1) = 1 \rightarrow \text{(ii)}$$

$$\Rightarrow n_1^2(1)\left[1 + (2-\lambda(1))^2\right] = 1$$

$$\Rightarrow n_1(1) = 0.6154, \quad n_2(1) = -0.7882$$

$$\underline{n}(1) = (0.6154, -0.7882)^T$$

↳ p-axis corresponding to $e(1)$

$$(2-\lambda(2))n_1(2) - n_2(2) = 0$$

$$n_1^2(2) + n_2^2(2) = 1$$

$$\Rightarrow n_1^2(2)\left[1 + (2-\lambda(2))^2\right] = 1$$

$$\Rightarrow n_1(2) = 0.7882, \quad n_2(2) = 0.6154$$

$$\underline{n}(2) = (0.7882, 0.6154)^T$$

Observe that $\underline{n}(1) \cdot \underline{n}(2) = 0$; ie p-axes are orthogonal.

T3, P1

Problem (5)

$\because$ strains are linear in a_1, a_2, a_3 and compat eqns involve double differentiation of strain components w.r.t. a_i's, compat eqns are satisfied identically. Hence it is a possible strain distribution ◄

$$l_{11} = 2a_1 = \frac{\partial u_1}{\partial a_1} \Rightarrow u_1 = a_1^2 + f[a_2, a_3]$$

$$l_{22} = 2a_1 = \frac{\partial u_2}{\partial a_2} \Rightarrow u_2 = 2a_1a_2 + g[a_1, a_3]$$

$$l_{12} = a_1 + 2a_2 = \frac{1}{2}\left[\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right] = \frac{1}{2}\left[\frac{\partial f}{\partial a_2} + 2a_2 + \frac{\partial g}{\partial a_1}\right]$$

$$\Rightarrow \frac{\partial f[a_2,a_3]}{\partial a_2} + \frac{\partial g[a_1,a_3]}{\partial a_1} = 2a_1 + 2a_2 \longrightarrow (1)$$

$$l_{33} = 2a_3 = \frac{\partial u_3}{\partial a_3} \Rightarrow u_3 = a_3^2 + h[a_1, a_2]$$

$$l_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_3} + \frac{\partial u_3}{\partial a_1}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial a_3} + \frac{\partial h}{\partial a_1}\right) = 0 \Rightarrow \frac{\partial f[a_2,a_3]}{\partial a_3} + \frac{\partial h[a_1,a_2]}{\partial a_1} = 0 \rightarrow (2)$$

$$l_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_3} + \frac{\partial u_3}{\partial a_2}\right) = \frac{1}{2}\left(\frac{\partial g}{\partial a_3} + \frac{\partial h}{\partial a_2}\right) = 0 \Rightarrow \frac{\partial g[a_1,a_3]}{\partial a_3} + \frac{\partial h[a_1,a_2]}{\partial a_2} = 0 \rightarrow (3)$$

Now $\dfrac{\partial (1)}{\partial a_3} = \dfrac{\partial^2 f}{\partial a_2 \partial a_3} + \dfrac{\partial^2 g}{\partial a_1 \partial a_3} = 0$

From $\dfrac{\partial (2)}{\partial a_2} + \dfrac{\partial (3)}{\partial a_1} = 0$ and the above we get $\dfrac{\partial^2 h}{\partial a_1 \partial a_2} = 0 \Rightarrow h = A + p[a_1] + q[a_2] \rightarrow (4)$

From (2) & (4), $\dfrac{\partial f[a_2,a_3]}{\partial a_3} + p'[a_1] = 0 \Rightarrow \because f = f[a_2,a_3]$ that $p[a_1] = Ba_1$

From (3) & (4), $\dfrac{\partial g[a_1,a_3]}{\partial a_3} + q'[a_2] = 0 \Rightarrow \because g = g[a_1,a_3]$ that $q[a_2] = Ca_2$

constants omitted $\because$ 'A' already present

Thus, $h = A + Ba_1 + Ca_2 \rightarrow (5)$

From (2) & (5), $f = -Ba_3 + r[a_2] \rightarrow (6)$

From (3) & (5), $g = -Ca_3 + s[a_1] \rightarrow (7)$

$\rightarrow$ from (1), $r' + s' = 2a_1 + 2a_2$

$\Rightarrow r = a_2^2 + D + Ka_2$, $s = a_1^2 + E - Ka_1$

So $f = -Ba_3 + a_2^2 + D + Ka_2$, $g = -Ca_3 + a_1^2 + E - Ka_1$

Thus
$$u_1 = a_1^2 + a_2^2 - Ba_3 + D + Ka_2$$
$$u_2 = 2a_1a_2 + a_1^2 - Ca_3 + E - Ka_1$$
$$u_3 = a_3^2 + Ba_1 + Ca_2 + A$$

If origin has zero displ ($u_1 = u_2 = u_3 = 0$ @ origin) $\Rightarrow A = D = E = 0$

Further, if an (arbitrary) infinitesimal line element @ origin has zero rotation, $\Rightarrow \tilde{w}_i$ and hence $\tilde{w}_{ij}$ vanish @ origin.

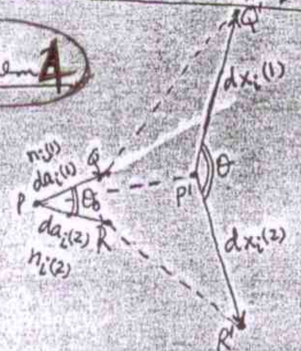
$$\tilde{\omega}_{13} = \frac{1}{2}(u_{1,3} - u_{3,1})\Big|_{a_1=a_2=a_3=0} = -B = 0 \quad , \quad \tilde{\omega}_{23} = \frac{1}{2}(u_{2,3} - u_{3,2})\Big|_{a_1=a_2=a_3=0} = -C = 0 \quad \Rightarrow B = C = 0$$

$$\tilde{\omega}_{12} = \frac{1}{2}(u_{1,2} - u_{2,1})\Big|_{a_1=a_2=a_3=0} = K = 0$$

Working for change of angle between two arbitrary line elements. Result was given in class without details, method was outlined in class.

So $\boxed{u_1 = a_1^2 + a_2^2 , \quad u_2 = 2a_1a_2 + a_1^2 , \quad u}$

Problem 4



Consider $\underline{dx}(1) \cdot \underline{dx}(2)$. We have,

$$\underbrace{dx_{i(1)}\,dx_{i(2)}}_{LHS} = \underbrace{\sqrt{dx_{i(1)}dx_{i(1)}}\;\sqrt{dx_{j(2)}dx_{j(2)}}}_{RHS}\,\cos\theta \,.$$

$$LHS = \left(\frac{\partial x_i}{\partial a_j}\,da_j\right)_{(1)}\left(\frac{\partial x_i}{\partial a_k}\,da_k\right)_{(2)} = \left(\frac{\partial(a_i+u_i)}{\partial a_j}\,da_j\right)_{(1)}\left(\frac{\partial(a_i+u_i)}{\partial a_k}\,da_k\right)_{(2)}$$

$$= \left(da_i + \frac{\partial u_i}{\partial a_j}\,da_j\right)_{(1)}\left(da_i + \frac{\partial u_i}{\partial a_k}\,da_k\right)_{(2)}$$

$$LHS = da_{i(1)}da_{i(2)} + da_{i(1)}da_{k(2)}\,u_{i,k} + da_{j(1)}da_{i(2)}\,u_{i,j} + da_{j(1)}da_{k(2)}\,u_{i,j}\,u_{i,k}$$

$$= da_{i(1)}da_{i(2)} + da_{i(1)}da_{j(2)}\left[u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}\right]$$

$$RHS = \sqrt{da_{i(1)}da_{i(1)} + 2L_{ik}\,da_{i(1)}da_{k(1)}}\;\sqrt{da_{j(2)}da_{j(2)} + 2L_{jm}\,da_{j(2)}da_{m(2)}}\,\cos\theta$$

(used $(dx)^2 - (da)^2$ formula given).

Now $\dfrac{LHS}{da_{(1)}\,da_{(2)}} = \dfrac{RHS}{da_{(1)}\,da_{(2)}}$ (Note $da_{(1)} = |\underline{da}_{(1)}|$ etc.).

$$\Rightarrow \underbrace{n_{i(1)}n_{i(2)}}_{\cos\theta_0} + n_{i(1)}n_{j(2)}\left[2L_{ij}\right] = \sqrt{\underbrace{n_{i(1)}n_{i(1)}}_{1} + 2L_{ik}\,n_{i(1)}n_{k(1)}} \;*\; \sqrt{\underbrace{n_{j(2)}n_{j(2)}}_{1} + 2L_{jm}\,n_{j(2)}n_{m(2)}}\,\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\cos\theta_0 + 2L_{ij}\,n_{i(1)}n_{j(2)}}{\sqrt{1 + 2L_{ik}\,n_{i(1)}n_{k(1)}}\;\sqrt{1 + 2L_{jm}\,n_{j(2)}n_{m(2)}}}$$

T2, P7

P-6

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{m,i}\,u_{m,j}\right) \longrightarrow \text{nonlinear}$$

$$\varepsilon_{ij} = \frac{1}{2}\begin{pmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{nonlinear.}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) \longrightarrow \text{linear}$$

$$\therefore \varepsilon_{ij} = \begin{pmatrix} 0 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(dy)^2 - (da)^2 = 2\varepsilon_{rs}\, da_r\, da_s$$
$$= dy_i\, dy_i - da_i\, da_i$$

Now $dy_1 = \frac{\partial y_1}{\partial a_1} da_1 + \frac{\partial y_1}{\partial a_2} da_2 + \frac{\partial y_1}{\partial a_3} da_3$

$$= da_1 + k\, da_2 + 0$$

$$dy_1 = da_1 + k\, da_2 + 0$$

$$dy_2 = da_2 \;;\quad dy_3 = da_3.$$

For AB :-

$$da_1 = dL\ ;\ da_2 = da_3 = 0.$$

$$\therefore\ dy_1 = da_1 = dL.$$

$$dy_2 = 0\ ;\ dy_3 = 0.$$

$$\therefore (dy)^2 - (da)^2 = da_1\, da_1 - da_1\, da_1 = 0.$$

$$\therefore (dy)^2 - (da)^2 = 0.$$

For AD :- $da_1 = 0;\ da_2 = dL\ ;\ da_3 = 0.$

$$\therefore (dy)^2 - (da)^2 = dy_1\, dy_1 - da_1\, da_1$$

$$= K^2 (dL)^2 - \cancel{(dL)^2}\ 0$$

$$(dy)^2 - (da)^2 = \cancel{(k^2-1)(dL)^2}\ k^2 (dL)^2$$

$$\therefore (dy)^2 - (da)^2 = k^2 (dL)^2$$

For AC :- $da_1 = dL\ ;\ da_2 = dL\ ;\ da_3 = 0.$

$$(dy)^2 - (da)^2 = dy_1\, dy_1 - da_1\, da_1$$

$$= (dL + K dL)^2 - (dL)^2$$

$$= (dL)^2 (1 + k^2 + 2k - 1)$$

$$= (k^2 + 2k)(dL)^2$$

For DB :- $da_1 = dL\ ;\ da_2 = -dL\ ;\ da_3 = 0.$

$$\therefore (dy)^2 - (da)^2 = dy_1\, dy_1 - da_1\, da_1$$

$$= (dL - k dL)^2 - (dL)^2$$

$$= (k^2 - 2k)(dL)^2$$

$\lambda = \frac{dY}{da}$ (5)

$\therefore \frac{(dY)^2-(da)^2}{(da)^2} = \lambda^2 - 1.$

$\therefore \lambda = \left[1 + \frac{(dY)^2-(da)^2}{(da)^2}\right]^{1/2}$

$\therefore \varepsilon = \frac{dY - da}{da} = \lambda - 1. =$ unit extension.

in linear theory $\varepsilon = \varepsilon_{ij} n_i n_j$

<u>For AB</u> :-

$\lambda_{AB} = (1+0)^{1/2} = 1.$

$\therefore \lambda_{AB} = 1$

$\varepsilon_{AB} = 0.$ by both linear & nonlinear theories.

<u>For AD</u> :-

$\lambda_{AD} = \left[1 + \frac{K^2 (dL)^2}{(dL)^2}\right]^{1/2}$

$\therefore \lambda_{AD} = (1+K^2)^{1/2}$

$\varepsilon_{AD} = (1+K^2)^{1/2} - 1. \longrightarrow$ nonlinear theory
$\varepsilon_{AD} = \varepsilon_{22} n_2 n_2 = 0 \longrightarrow$ linear theory

equal if $K \ll 1$

<u>For AC</u> :-

$\lambda_{AC} = \left[1 + \frac{(K^2+2K)(dL)^2}{2(dL)^2}\right]^{1/2}$

$\lambda_{AC} = \left(1 + K + \frac{K^2}{2}\right)^{1/2}$

again they are same if $K \ll 1$ we put in linear or nonlinear ε_{AC}

$\varepsilon_{AC} = \left(1+K+\frac{K^2}{2}\right)^{1/2} - 1. \longrightarrow$ nonlinear.

$\varepsilon_{AC} = \underset{0}{\varepsilon_{11}} n_1 n_1 + \varepsilon_{12} n_1 n_2 + \varepsilon_{21} n_1 n_2 + \underset{0}{\varepsilon_{22}} n_2 n_2, \quad n_1 = \frac{1}{\sqrt{2}}, \; n_2 = \frac{1}{\sqrt{2}}$

$= 2 \times \frac{K}{2} \times \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{K}{2}$

<u>For DB</u>

$\lambda_{DB} = \left(1 + \frac{(K^2-2K)(dL)^2}{2(dL)^2}\right)^{1/2}$

$\lambda_{DB} = \left(1 - K + \frac{K^2}{2}\right)^{1/2}$

$\varepsilon_{DB} = \left(1-K+\frac{K^2}{2}\right)^{1/2} - 1. \longrightarrow$ nonlinear.

$\varepsilon_{DB} = \varepsilon_{ij} n_i n_j = 2 \times \frac{K}{2} \times \frac{1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right) = -\frac{K}{2}$

again same if we put $K \ll 1$ in nonlinear ε_{AC}.

7

T2, P8

P(8) $\mathcal{E}_{11} = u_{1,1} = \frac{a}{4}(x_2+x_3)^2 = a$; $\mathcal{E}_{22} = a$; $\mathcal{E}_{33} = a$

$\mathcal{E}_{12} = \mathcal{E}_{21} = \frac{1}{\cancel{2}}\frac{a}{4}\left(\cancel{2}x_1(x_2+x_3) + \cancel{2}x_2(x_1+x_3)\right) = a$

$\mathcal{E}_{13} = \mathcal{E}_{31} = \mathcal{E}_{23} = \mathcal{E}_{32} = a$

Principal strains.

$$a^3 \det \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(1-\lambda)^2-1\right] - 1\left[(1-\lambda)-1\right] + 1\left[1-(1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)^3 - (1-\lambda) - (1-\lambda) + 1 + 1 - (1-\lambda) = 0$$

$$\Rightarrow \cancel{1} - \cancel{3\lambda} + 3\lambda^2 - \lambda^3 - \cancel{3} + \cancel{3\lambda} + \cancel{2} = 0 \Rightarrow \lambda^2(3-\lambda) = 0 \Rightarrow \lambda = 0, 0, 3$$

$$\Rightarrow \epsilon(1) = 3a, \quad \epsilon(2) = 0, \quad \epsilon(3) = 0$$

When referred to the principal system, $\epsilon'_{11} = \epsilon(1)$, $\epsilon'_{22} = \epsilon(2)$, $\epsilon'_{33} = \epsilon(3)$, all other components are zero.

Then $\mathcal{E} = \epsilon_{ij} n_i n_j = \epsilon'_{ij} n'_i n'_j = \epsilon(1) n_1'^2 + \epsilon(2) n_2'^2 + \epsilon(3) n_3'^2$

Thus $(\mathcal{E})_{max} = \epsilon(1)\, 1^2 = 3a \rightarrow$ occurs along p-direction corresponding to $\epsilon(1)$

$(\mathcal{E})_{min} = \epsilon(2)\, 1^2$ or $\epsilon(3)\, 1^2 = 0 \rightarrow$ " " " " " $\epsilon(2)$ or $\epsilon(3)$

END of P. 3 (b)

extra

So there is no need to find p-directions. Anyway we will do it.

For $\epsilon(1)$: $\left.\begin{array}{l} -2n_1(1) + n_2(1) + n_3(1) = 0 \\ n_1(1) - 2n_2(1) + n_3(1) = 0 \end{array}\right\} \Rightarrow -3n_2(1) + 3n_3(1) = 0 \Rightarrow n_2(1) = n_3(1)$

$\Rightarrow n_1(1) = n_2(1)$

$n_1^2(1) + n_2^2(1) + n_3^2(1) = 1 \Rightarrow n_i(1) \Rightarrow \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

For $\epsilon(2)$: $\left.\begin{array}{l} n_1(2) + n_2(2) + n_3(2) = 1 \\ n_1^2(2) + n_2^2(2) + n_3^2(2) = 1 \end{array}\right\}$ So $n_i(2)$ has some arbitrariness, as expected. Choose as any direction orthogonal to $n_i(1)$.

Thus $n_i(2) \Rightarrow \left(0, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$

For $\epsilon(3)$: Similarly choose $n_i(3)$ to be orthogonal to $n_i(1)$ & $n_i(2)$

So $n_i(3) \Rightarrow \left(\mp\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}\right)$

Now $\mathcal{E}_{max} = \epsilon_{ij} n_i(1) n_j(1) = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) * 3 = 3$

$\mathcal{E}_{min} = \epsilon_{ij} n_i(2) n_j(2) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$

or $\mathcal{E}_{min} = \epsilon_{ij} n_i(3) n_j(3) = \left(\frac{-2}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right)^2 = 0$

as expected

T3, P2

P9

▶ Since u_θ is multi-valued (ie, for the same physical point we can have $\theta = \theta^*, \theta^* + 2n\pi$), the displ field is not compatible for arbitrary constants.

▶ In order to ensure finite displacements, $A = 0$ if the origin is part of the continuum.

▶ In order " " single valued displ's, $B = 0$

P-10 a) $\tilde{w}_i = \frac{1}{2}\epsilon_{ijk}\tilde{w}_{kj} = \frac{1}{2}\frac{1}{2}\epsilon_{ijk}(u_{k,j} - u_{j,k}) = \frac{1}{4}(\epsilon_{ijk}u_{k,j} - \epsilon_{ijk}u_{j,k}) = \frac{1}{4}(\epsilon_{ijk}u_{k,j} + \epsilon_{ikj}u_{j,k})$

$= \frac{1}{4}2\epsilon_{ijk}u_{k,j} = \frac{1}{2}\epsilon_{ijk}u_{k,j}$ ◀ (where ',' is $\frac{\partial}{\partial a}$)

(b) $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$e_{11} = 0,\ e_{22} = 0,\ e_{33} = 0,\ e_{12} = cx_3,\ e_{23} = cx_1,\ e_{13} = cx_2$

$w_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \longrightarrow$ get $w_{ij} = 0$

So this displacement field represents a case of pure straining.

(c) $du_i = \tilde{w}_{ij}\,da_j$, $\tilde{w}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)$

$$\tilde{w}_{ij} \Rightarrow \begin{pmatrix} 0 & \frac{c}{2}(a_1 - a_2) & -ca_3 \\ \frac{c}{2}(a_2 - a_1) & 0 & -ca_3 \\ ca_3 & ca_3 & 0 \end{pmatrix}$$

The two pts PQ lie on the line (in $x_1 x_2$ plane) making equal acute ∠'s with positive x_1 & x_2-axes.

Hence $a_1 = a_2$, $a_3 = 0 \Rightarrow \tilde{w}_{ij} = 0$ @ P (also @ Q)

So $du_i = 0$ ◀

P-11 For infinitesimal strains, $\tilde{D} = \frac{\Delta V}{V} = l_{ii} = u_{1,1} + u_{2,2} + u_{3,3}$

For incompressibility, $\Delta V = 0 \Rightarrow \tilde{D} = 0$

$\therefore l_{ii} = u_{1,1} + u_{2,2} + u_{3,3} = (1 - x_2^2)(b + 2cx_1) + u_{2,2} + 0 = 0$

$\Rightarrow u_2 = \left(\frac{x_2^3}{3} - x_2\right)(b + 2cx_1) + f(x_1, x_3)$

Now $u_2|_{x_2 = \pm\sqrt{3}} = f(x_1, x_3) = 0$ (given) $\Rightarrow u_2 = x_2\left(\frac{x_2^2}{3} - 1\right)(b + 2cx_1)$ ◀

T4, P1

P.13 $\phi = -\frac{A}{6}x_2^3 - \frac{B}{6}x_1^2x_2^3 + \frac{C}{2}x_1^2x_2 - \frac{q}{4}x_1^2 + \frac{B}{30}x_2^5$ (Shames & Dym, pp64, Prob 1.26)

$\sigma_{11} = \phi_{,22} = -Ax_2 - Bx_1^2x_2 + \frac{2B}{3}x_2^3$

$\sigma_{22} = \phi_{,11} = -\frac{B}{3}x_2^3 + Cx_2 - \frac{q}{2}$

$\sigma_{12} = -(-Bx_1x_2^2 + Cx_1)$

(a) $\nabla^4\phi = 0 \Rightarrow$ Inplane compatibility eqn satisfied. (i.e., the $i=j=1$, $i=j=2$ B-M compat. eqns)
$i=1, j=2$ BM (compat) eqn not satisfied
$i=3, j=3$ " " satisfied ($\because f_i = 0$)

(b) B.C's

$(\sigma_{12} = \sigma_{21})_{x_2=\pm h/2} = 0 \Rightarrow -x_1\left(-\frac{Bh^2}{4} + C\right) = 0 \longrightarrow$ ①

$(\sigma_{22})_{x_2=+h/2} = -q \Rightarrow -\frac{Bh^3}{24} + \frac{Ch}{2} - \frac{q}{2} = -q$
$(\sigma_{22})_{x_2=-h/2} = 0 \Rightarrow -\left(-\frac{Bh^3}{24} + \frac{Ch}{2}\right) - \frac{q}{2} = 0$
$\longrightarrow$ ② (Both eqns identical)

$\left(\int_{-h/2}^{h/2}\sigma_{11}x_2\,dx_2\right)_{x_1=\pm\frac{L}{2}} = 0$ (ie zero applied moment at left/right faces)

$\Rightarrow -\frac{Ah^3}{12} - \frac{BL^2}{4}\frac{h^3}{12} + \frac{2B}{3}\frac{h^5}{80} = 0 \longrightarrow$ ③

①, ②, ③ $\Rightarrow A = \frac{6q}{h^3}\left(\frac{L^2}{4} - \frac{h^2}{10}\right)$, $B = -\frac{6q}{h^3}$, $C = -\frac{3}{2}\frac{q}{h}$

NOTE: $\left(\int_{-h/2}^{h/2}\sigma_{11}\,dx_2\right)_{x_1=\pm\frac{L}{2}} = 0$ (i.e, zero axial force condt. identically satisfied $\because \sigma_{11}$ has only odd powers of x_2)

$\left(\int_{-h/2}^{h/2}\sigma_{12}\,dx_2\right)_{x_1=\pm\frac{L}{2}} = \pm q\frac{L}{2}$ (i.e, shear force is $\pm\frac{qL}{2}$ at left/right faces, identically satisfied
$\longrightarrow$ (in fact this condt. gives eqn identical to ②)

(c) Stress Distribution (subst A, B, C into σ_{11}, σ_{22}, σ_{12} above)

$\sigma_{11} = -\frac{q}{2I}\left[\left(\frac{L}{2}\right)^2 - \frac{h^2}{10} - x_1^2 + \frac{2}{3}x_2^2\right]x_2$

$\sigma_{22} = \frac{q}{2I}\left[\frac{x_2^3}{3} - \left(\frac{h}{2}\right)^2 x_2 - \frac{h^3}{12}\right]$

$\sigma_{12} = \frac{-q}{2I}\left[x_1x_2^2 - \frac{h^2}{4}x_1\right]$

where, $I = \frac{th^3}{12} = \frac{h^3}{12}$

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ (plane stress assumed)

Strain Distribution (Hookes law used, $\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$

$\epsilon_{11} = -\frac{q}{2EI}\left[(2+\nu)\frac{x_2^3}{3} + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_2 - x_1^2x_2 - \frac{\nu h^3}{12}\right]$

(10)

$$\left.\begin{aligned} l_{22} &= -\frac{q}{2EI}\left[-(1+2\nu)\frac{x_2^3}{3} + \left(-\nu\frac{L^2}{4} + \frac{\nu h^2}{10} + \frac{h^2}{4}\right)x_2 + \nu x_1^2 x_2 + \frac{h^3}{12}\right] \\ l_{12} &= -\left(\frac{1+\nu}{E}\right)\frac{q}{2I}\left(x_1 x_2^2 - \frac{h^2}{4}x_1\right) \end{aligned}\right\}$$

stress distribution not valid on left/right faces ∵ σ_{11}, σ_{12} non-zero on these faces.

(d) Displacements

$$l_{11} = \frac{\partial u_1}{\partial x_1} \Rightarrow u_1 = -\frac{q}{2EI}\left[(2+\nu)\frac{x_1 x_2^3}{3} + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_1 x_2 - \frac{x_1^3 x_2}{3} - \frac{\nu h^3}{12}x_1 + f[x_2]\right]$$

$$l_{22} = \frac{\partial u_2}{\partial x_2} \Rightarrow u_2 = -\frac{q}{2EI}\left[-(1+2\nu)\frac{x_2^4}{12} + \left(-\nu\frac{L^2}{4} + \frac{\nu h^2}{10} + \frac{h^2}{4}\right)\frac{x_2^2}{2} + \nu\frac{x_1^2 x_2^2}{2} + \frac{h^3}{12}x_2 + g[x_1]\right]$$

subst u_1, u_2 into l_{12}:

$$2l_{12} = (u_{1,2} + u_{2,1}) \Rightarrow -\frac{q}{\cancel{2EI}}\left[2(1+\nu)\left\{\cancel{x_1 x_2^2} - \frac{h^2}{4}x_1\right\}\right] = -\frac{q}{\cancel{2EI}}\left[\cancel{(2+\nu)x_1 x_2^2} + \left(\frac{L^2}{4} - \frac{h^2}{10} - \nu\frac{h^2}{4}\right)x_1 - \frac{x_1^3}{3} + f' + \cancel{\nu x_1 x_2^2} + g'\right]$$

$$\Rightarrow g' + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_1 - \frac{x_1^3}{3} + 2(1+\nu)\frac{h^2}{4}x_1 = -f' = K \text{ (const)}$$

$$\Rightarrow f = -Kx_2 + C_1 \quad , \quad g = Kx_1 + \frac{x_1^4}{12} - \left(\frac{L^2}{4} + \frac{2}{5}h^2 + \frac{1}{4}\nu h^2\right)\frac{x_1^2}{2} + C_2$$

Displacement BC's

$$\left.\begin{aligned} u_2 &= 0 \text{ at } x_1 = \tfrac{L}{2},\ x_2 = -\tfrac{h}{2} \\ u_2 &= 0 \text{ at } x_1 = -\tfrac{L}{2},\ x_2 = -\tfrac{h}{2} \end{aligned}\right\} \Rightarrow K = 0, \quad C_2 = \frac{5}{192}L^4 + \left(\frac{1}{20} + \frac{\nu}{32}\right)h^2 L^2 + h^4\left(\frac{1}{64} - \frac{\nu}{480}\right)$$

→ prevent R.B. transl in x_2 dir. and RB rot. in $x_1 x_2$ plane. Further to prevent RB transl in x_1 dir, put $u_1|_{x_1 = \frac{L}{2}, x_2 = -\frac{h}{2}} = 0$, find C_1.

$$\Rightarrow u_2[x_1, 0] = -\frac{qL^4}{24EI}\left[\left(\frac{x_1}{L}\right)^4 - \frac{3}{2}\left(\frac{x_1}{L}\right)^2 + \frac{5}{16} + \left(\frac{h}{L}\right)^2\left\{\left(\frac{12}{5} + \frac{3\nu}{2}\right)\left(\frac{1}{4} - \left\{\frac{x_1}{L}\right\}^2\right) + \left(\frac{3}{16} - \frac{\nu}{40}\right)\left(\frac{h}{L}\right)^2\right\}\right]$$

$24u_2^*[x_1, 0]$

(e) Slender-beam

put put $\frac{h}{L} \simeq 0$ in $u_2^*[x_1, 0]$ and get,

$$u_2^*[x_1] = -\frac{1}{24}\left[\left(\frac{x_1}{L}\right)^4 - \frac{3}{2}\left(\frac{x_1}{L}\right)^2 + \frac{5}{16}\right]$$

T3, P4

(P. 12) Hookes law after inverting → $l_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$ ⇒ $l_{kk} = \frac{1-2\nu}{E}\sigma_{kk}$

$\Rightarrow l_{kk} = 1.869 \times 10^{-4} = \mathfrak{D}$

Now $\mathfrak{D} = \lim_{V \to 0} \frac{\Delta V}{V}$ but stress/strain distr uniform throughout cube ⇒ $\mathfrak{D}$ constant throughout cube. ⇒ $\Delta V = \mathfrak{D} V = 2.336 \times 10^{-8}\,m^3$

(R17)

$\tilde{f}_1 = \tilde{f}_1[x_1, x_2]$, $\tilde{f}_2 = \tilde{f}_2[x_1, x_2]$, $\tilde{f}_3 = 0$

T4, P2. Do by using strain compat eqns directly. Here it is done by Beltrami Michell compat eqns, which was not done in class

ext loading $\perp^{ar}$ to longitudinal axis & indep of x_3 $\Rightarrow \dfrac{\partial}{\partial x_3} = 0$ $\Big\} \Rightarrow u_i = u_i(x_1, x_2)$. Assume : $u_3 = 0$ $\Big\} \Rightarrow e_{i3} = 0$

Hookes Law (linear case) $\rightarrow$ $\sigma_{ij} = \dfrac{E}{1+\nu}\left[e_{ij} + \dfrac{\nu}{1-2\nu} e_{kk}\delta_{ij}\right] \Rightarrow \sigma_{13} = \sigma_{23} = 0$

B-M compatibility (includes equilibrium eqn) :

$i=j=1$: $\nabla^2\sigma_{11} + \frac{1}{1+\nu}(\sigma_{11,11} + \sigma_{22,11} + \sigma_{33,11}) + \tilde{f}_{1,1} + \tilde{f}_{1,1} + \frac{\nu}{1-\nu}(\tilde{f}_{1,1} + \tilde{f}_{2,2} + \cancelto{0}{\tilde{f}_{3,3}}) = 0 \longrightarrow (1)$

$i=j=2$: $\nabla^2\sigma_{22} + \frac{1}{1+\nu}(\sigma_{11,22} + \sigma_{22,22} + \sigma_{33,22}) + \tilde{f}_{2,2} + \tilde{f}_{2,2} + \frac{\nu}{1-\nu}(\tilde{f}_{1,1} + \tilde{f}_{2,2} + \cancelto{0}{\tilde{f}_{3,3}}) = 0 \rightarrow (2)$

$i=1, j=2$: $\nabla^2\sigma_{12} + \frac{1}{1+\nu}(\sigma_{11,12} + \sigma_{22,12} + \sigma_{33,12}) + \tilde{f}_{1,2} + \tilde{f}_{2,1} = 0 \longrightarrow (3)$

$i=j=3$: $\nabla^2\sigma_{33} + \frac{1}{1+\nu}(\cancelto{0}{\sigma_{11,33}} + \cancelto{0}{\sigma_{22,33}} + \cancelto{0}{\sigma_{33,33}}) + \cancelto{0}{f_{3,3}} + \cancelto{0}{f_{3,3}} + \frac{\nu}{1-\nu}(f_{1,1} + f_{2,2} + \cancelto{0}{f_{3,3}}) = 0 \rightarrow (4)$

$i=1, j=3$: $i=2, j=3$: $\Big\}$ identically satisfied $(0=0)$

Hookes law $\rightarrow$ $\sigma_{33} = \frac{E}{1+\nu}\left[\cancelto{0}{e_{33}} + \frac{\nu}{1-2\nu}(e_{11} + e_{22} + \cancelto{0}{e_{33}})\right] = \frac{\cancel{E}\nu}{(1+\nu)(1-2\nu)}\left[\frac{1+\nu}{\cancel{E}}\{\sigma_{11}+\sigma_{22}\} - \frac{2\nu}{E}\{\sigma_{11}+\sigma_{22}+\sigma_{33}\}\right]$

$\Rightarrow \sigma_{33}\left(1 + \frac{2\nu^2}{(1+\nu)(1-2\nu)}\right) = \frac{(1-\nu)\nu}{(1+\nu)(1-2\nu)}\{\sigma_{11} + \sigma_{22}\}$

$\Rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$

Subst. σ_{33} into eqn (3), and get,

$\nabla^2\sigma_{12} + (\sigma_{11} + \sigma_{22})_{,12} + \tilde{f}_{1,2} + \tilde{f}_{2,1} = 0 \rightarrow (3)^*$

EOM's are, $\sigma_{11,1} + \sigma_{12,2} + \tilde{f}_1 = 0 \rightarrow (A)$
$\sigma_{12,1} + \sigma_{22,2} + \tilde{f}_2 = 0 \rightarrow (B)$ $\Big\}$ Thus $(3)^* \equiv \dfrac{\partial (A)}{\partial x_2} + \dfrac{\partial (B)}{\partial x_1}$

$0 = 0$

So we can replace (3) hence $(3)^*$ by the EOM's, ie, (A), (B).

Adding (1), (2) & (4), use Hookes law for σ_{33} and get,

$(1+\nu)\nabla^2(\sigma_{11}+\sigma_{22}) + \frac{1}{1+\nu}\left[(1+\nu)\nabla^2(\sigma_{11}+\sigma_{22})\right] + 2\left(1 + \frac{3\nu}{2(1-\nu)}\right)(\tilde{f}_{1,1} + \tilde{f}_{2,2}) = 0$

$\Rightarrow \nabla^2(\sigma_{11}+\sigma_{22}) = -\frac{1}{1-\nu}(\tilde{f}_{1,1} + \tilde{f}_{2,2}) \rightarrow (C)$

Thus (A), (B), (C) give exact sol. of p-strain problem.

Now for conservative body forces, the solution which satisfies equil eqns (A), (B) is $\rightarrow$ $\sigma_{11} = \phi_{,22} - \psi$, $\sigma_{22} = \phi_{,11} - \psi$, $\sigma_{12} = -\phi_{,12}$,

where $f_3 = 0$, $f_i = \psi[x_1, x_2]_{,i}$ for $i = 1, 2$ (Mid-term test Result)

This solution when substituted into (c) yields,

$$\nabla^4 \phi = \left(\frac{1-2\nu}{1-\nu}\right) \nabla^2 p$$

NOTE: only this need to be solved in conjunction with appropriate BC's on ϕ, for the p-strain problem. Thus p-strain problems exact sol. reduces to solving this eqn.

P.14 For p-stresses & corresponding p-axes we solve,

$$(\sigma_{ij} - \sigma\delta_{ij})n_j = 0 \longrightarrow (1)$$

Subst. constitutive law for isotropic body, we get,

$$\left(\frac{E}{1+\nu}\left[l_{ij} + \frac{\nu}{1-2\nu}\delta_{ij} l_{mm}\right] - \sigma\delta_{ij}\right)n_j = 0$$

$$\Rightarrow \left(l_{ij} - \left\{\frac{1+\nu}{E}\sigma - \frac{\nu}{1-2\nu} l_{mm}\right\}\delta_{ij}\right) n_j = 0 \longrightarrow (2)$$

Now for p-strains & corresponding p-axes we solve,

$$(l_{ij} - l\delta_{ij})n_j^* = 0 \rightarrow (3)$$

Now (3) & (2) are of the same form. Their solution is obtained by solving either evalue problem (1) or (3). Hence e-vectors n_j & n_j^* must coincide (ie, p-axes of stress (n_j) coincide with p-axes of strain (n_j^*)).

P.15 Thin plate, inplane edge loads only. Since plate is thin it implies that edge loads do not vary in the thickness direction. Furthermore $\bar{f}_3 = 0$ (in fact $\bar{f}_1 = \bar{f}_2 = 0$ is given). Thus this can be approximated as a plane stress problem.

T3, P5

Hence $\sigma_{i3} = 0$, $\frac{\partial}{\partial x_3} = 0$

$l_A = -100 * 10^{-6} = l_{ij} n_i n_j \quad (n_i = [1,0,0]) \Rightarrow l_A = -100*10^{-6} = l_{11}$

$l_C = 400 * 10^{-6} = l_{ij} n_i n_j \quad (n_i = [0,0,1]) \Rightarrow l_C = 400*10^{-6} = l_{22}$

$l_B = -200*10^{-6} = l_{ij} n_i n_j \quad \left(n_i = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]\right) \Rightarrow l_B = -200*10^{-6} = \frac{l_{11}}{2} + \frac{l_{22}}{2} + l_{12}$

$\Rightarrow l_{12} = -350*10^{-6}$

From constitutive law $l_{13} = l_{23} = 0 \quad (\because \sigma_{13} = \sigma_{23} = 0)$

Also, $\sigma_{33} = 0 = \frac{E}{1+\nu}\left[l_{33} + \frac{\nu}{1-2\nu}(l_{11} + l_{22} + l_{33})\right] \Rightarrow l_{33} = -128.6*10^{-6}$

Thus $\sigma_{11} = \frac{E}{1+\nu}\left[l_{11} + \frac{\nu}{1-2\nu}(l_{11} + l_{22} + l_{33})\right] = 658.85 \text{ psi}$

$\sigma_{22} = \frac{E}{1+\nu}\left[l_{22} + \frac{\nu}{1-2\nu}(l_{11} + l_{22} + l_{33})\right] = 12197.3 \text{ psi}$

$\sigma_{12} = \frac{E}{1+\nu} l_{12} = -8076.9 \text{ psi}.$

For p-stresses,

$$|\sigma_{ij} - \sigma\delta_{ij}| = 0 \Rightarrow -\sigma\left[(\sigma-\sigma_{11})(\sigma-\sigma_{22}) - \sigma_{12}^2\right] = 0$$

$$\Rightarrow \sigma(3) = 0, \ \& \ \sigma\substack{(1)\\(2)} = \frac{(\sigma_{11}+\sigma_{22}) \pm \sqrt{(\sigma_{11}+\sigma_{22})^2 - 4(\sigma_{11}\sigma_{22}-\sigma_{12}^2)}}{2}$$

$$\Rightarrow \sigma(1) = 16353.8 \text{ psi}, \ \sigma(2) = -3497.7 \text{ psi}, \ \sigma(3) = 0$$

T3, P6

P.16 (a) $l_{11} = c, \ l_{22} = -c\nu, \ l_{33} = -c\nu, \ l_{12} = l_{13} = l_{23} = 0$

$\Rightarrow l_{ij}$ are const $\Rightarrow \sigma_{ij}$ are const $\Rightarrow \tilde{f}_i = 0$ (from equil eqns).

(b) $\mu = \dfrac{E}{2(1+\nu)}$ & $K = \dfrac{E}{3(1-2\nu)}$

$$\left.\begin{array}{l}\text{For } E>0, \ \& \ \mu>0, \ \nu \geq -1 \\ \text{For } E>0, \ \& \ K>0, \ \nu \leq \frac{1}{2}\end{array}\right\} \Rightarrow -1 \leq \nu \leq 1/2$$