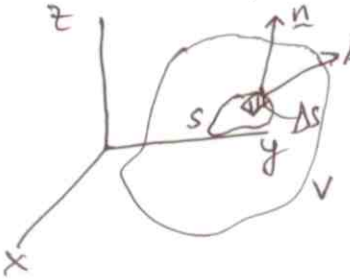
ELASTICITY FUNDAMENTALS.

(I) Analysis of Stress.

Surface and body forces/moments.

Surface forces act across a surface (external or internal) of the body. These occur due to direct contact.



$S \rightarrow$ close surface embedded in V. To study interaction of material inside & outside S, we consider small surface element ΔS oriented at $\underline{n}$. $\Delta \underline{F}$ is force exerted by outside material on inside material, & acts across ΔS, & depends on position ($\underline{x}$) and $\Delta \underline{S}$.

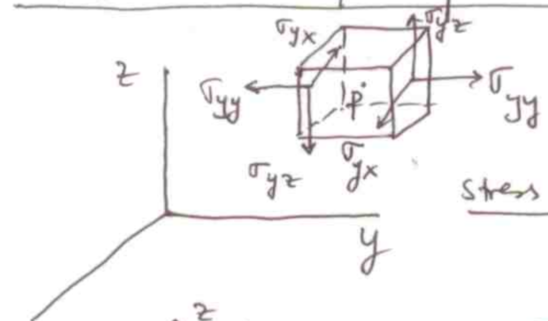
Define $\boxed{\underline{\sigma} = \lim_{\Delta S \to 0} \frac{\Delta \underline{F}}{\Delta S}}$ = stress vector ($\underline{t}$ in some books).

As $\Delta S \rightarrow 0$, moments due to $\Delta \underline{F}$ about any pt on ΔS vanishes $\Rightarrow$ surface moments dont exist.

Body forces/moments act at a distance (ie not due to direct contact) due to external field (gravity, magnetism) and are defined as;

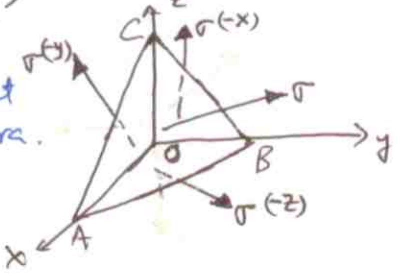
$$\underline{f} = \lim_{\Delta m \to 0} \frac{\Delta \underline{F}}{\Delta m} = \frac{d\underline{F}}{\rho dV} = \frac{\tilde{f}}{\rho}$$

Stress Tensor/components of stress. and its relation with stress vector



stress tensor at P.

$$\underline{T} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}.$$



h = height of tetra.

$$\underline{\sigma}^{(-x)} = -\sigma_{xx}\underline{i} - \sigma_{xy}\underline{j} - \sigma_{xz}\underline{k}$$

$$\underline{\sigma}^{(-y)} = -\sigma_{yx}\underline{i} - \sigma_{yy}\underline{j} - \sigma_{yz}\underline{k}$$

$$\underline{\sigma}^{(-z)} = -\sigma_{zx}\underline{i} - \sigma_{zy}\underline{j} - \sigma_{zz}\underline{k}$$

$$l = \frac{\Delta OCB}{\Delta ABC}, \quad m = \frac{\Delta(OAC)}{\Delta(ABC)}, \quad n = \frac{\Delta(OAB)}{\Delta(ABC)}$$

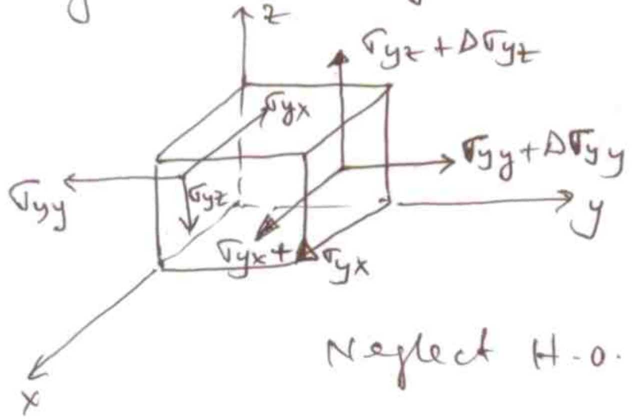
Equilibrium of forces ⇒ (incl. body forces, put lim $h \to 0$)

Cauchy rel. →

$$\boxed{\begin{aligned} \overset{n}{T}_x:\quad \underline{\sigma} = {} & (l\sigma_{xx} + m\sigma_{yx} + n\sigma_{zx})\,\underline{i} \\ \overset{n}{T}_y:\quad & + (l\sigma_{xy} + m\sigma_{yy} + n\sigma_{yz})\,\underline{j} \\ \overset{n}{T}_z:\quad & + (l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz})\,k \end{aligned} \quad = \sigma_{ji} n_j \underline{e}_i \text{ in tensorial index notation}}$$

where $\underline{n}$ = normal to ABC plane = $l\underline{i} + mj + n\underline{k}$

Absence of body moments, when considered with moment equilibrium of an element yields,



$$\underset{\sim}{k}: (\sigma_{yx} + \Delta\sigma_{yx} + \sigma_{yx})\frac{\Delta y}{2} * \Delta x \Delta z - (\sigma_{xy} + \Delta\sigma_{xy} + \sigma_{xy})\frac{\Delta x}{2} * \Delta y \Delta z$$

$$= \underset{=0}{\mathcal{M}_z} \quad (\because \text{no body moment})$$

Neglect H.O.T's, ⇒ $\boxed{\sigma_{yx} = \sigma_{xy}}$ etc.

Normal & Shearing stresses on arbitrary plane.

$$N = \sigma_N = \underline{\sigma} \cdot \underline{n} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\,\sigma_{xy} + 2ln\,\sigma_{xz} + 2mn\,\sigma_{yz} \longrightarrow ①$$

$$S = \sigma_S = \sqrt{|\underline{\sigma}|^2 - \sigma_N^2} \longrightarrow ②$$

σ_N and σ_S are important in failure/design criteria.

Stress transformation.

l_1, m_1, n_1 denote dir. cosines between X and x, y, z, resp'ly.
l_2, m_2, n_2 " " " " Y " " "
l_3, m_3, n_3 " " " " Z " " "

From ①,

$$\sigma_{XX} = l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + n_1^2\sigma_{zz} + 2l_1m_1\sigma_{xy} + 2l_1n_1\sigma_{xz} + 2m_1n_1\sigma_{yz}$$

$|||^{ly}$ for σ_{YY}, σ_{ZZ}

Let $\underline{\sigma}^{(X)}$ = stress vector on plane having X-axis direction, ie. $\underline{I}$, as its normal.

Also $\underline{J} = l_2\underline{i} + m_2\underline{j} + n_2\underline{k}$

③

So, $\sigma_{XY} = \underline{\sigma}^{(X)} \cdot \underset{\sim}{\underline{J}} = \underline{\sigma}^{(Y)} \cdot \underline{I}$

$$= l_1 l_2 \sigma_{xx} + m_1 m_2 \sigma_{yy} + n_1 n_2 \sigma_{zz} + (m_1 l_2 + l_1 m_2)\sigma_{xy}$$
$$+ (n_1 l_2 + l_1 n_2)\sigma_{xz} + (m_1 n_2 + n_1 m_2)\sigma_{yz}$$

$|||^{\text{arly}}$ for σ_{XZ}, σ_{YZ}

In compact form, using index notation,

$$\left.\begin{aligned} \sigma'_{ij} &= a_{im} a_{jn} \sigma_{mn} \\ \text{or } \underline{\underline{\sigma}}' &= \underline{\underline{a}}\, \underline{\underline{\sigma}}\, \underline{\underline{a}}^T \end{aligned}\right\} \longrightarrow \underline{\text{③}}$$

where

	x	y	z
X	a_{11}	a_{12}	a_{13}
Y	a_{21}	a_{22}	a_{23}
Z	a_{31}	a_{32}	a_{33}

We see that $a a^T = \underline{I}$

$$= \begin{bmatrix} \cancelto{1}{\underset{\sim}{X}\cdot\underline{X}} & \cancelto{0}{\underline{X}\cdot\underline{Y}} & \cancelto{0}{\underline{X}\cdot\underline{Z}} \\ \cancelto{0}{\underline{Y}\cdot\underline{X}} & \cancelto{1}{\underline{Y}\cdot\underline{Y}} & \cancelto{0}{\underline{Y}\cdot\underline{Z}} \\ \cancelto{0}{\underline{Z}\cdot\underline{X}} & \cancelto{0}{\underline{Z}\cdot\underline{Y}} & \cancelto{1}{\underline{Z}\cdot\underline{Z}} \end{bmatrix}$$

Principal stresses and axes:

There exist 3 mutually $\perp^{\text{ar}}$ planes at any point for which σ_S vanishes on each of these planes, ie $\underline{\sigma} = \sigma_N \underline{n}$ on each of these planes, ie $|\underline{\sigma}| = \sigma_N$. Further (it may be shown (not here)) σ_N also has stationary values on each of these planes. Thus

$$\underline{\sigma} = \sigma_{ji} n_j \underline{e}_i = (l\sigma_{xx} + m\sigma_{yx} + n\sigma_{zx})\,\underline{i} + (l\sigma_{xy} + m\sigma_{yy} + n\sigma_{yz})\,\underline{j}$$
$$+ (l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz})\,\underline{k}$$
$$= \sigma_N(l\underline{i} + m\underline{j} + n\underline{k}) \quad \text{since } (l, m, n) \text{ is principal plane.}$$

compare coeffs, get 3 linear eqns in l, m, n, then note non-triviality of solution

$$\Rightarrow \begin{vmatrix} \sigma_{xx} - \sigma_N & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_N & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_N \end{vmatrix} = 0 \longrightarrow \text{solve for } \sigma_N.$$

rename as σ.

Then solve for (l, m, n) from two indep or one indep simultaneous equations and $l^2 + m^2 + n^2 = 1$.

The cubic is, (rename $\sigma_N \to \sigma$)

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0.$$

$$\left.\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz}, \quad I_3 = \det(\text{stress tensor}) \\ I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2 \end{aligned}\right\} \longrightarrow \underline{④}$$

P-stresses & axes depend only on stress tensor at the point, ie on state of stress at the point which in turn depends on applied loads on the body. Hence I_1, I_2, I_3 are invariants.

If $\sigma_1 = \sigma_2$ (ie 2 equal p-stresses) then ∞ set of two $\perp^{ar}$ axes contained in plane normal to third p-axes are the other two p-axes.

If $\sigma_1 = \sigma_2 = \sigma_3$, then ∞ set of three $\perp^{ar}$ axes are p-axes.

Octahaedral stress

If (x^*, y^*, z^*) is a p-axes coord system (i.e it coincides with p-axes at a particular pt P), then referring to this system the eight planes satisfying $l^2 = m^2 = n^2 = \frac{1}{3}$ are octahaedral planes.

$$\left.\begin{aligned} (\sigma_N)_{oct} &= \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) = \frac{I_1}{3} \textcircled{=} \frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) \\ 9(\sigma_S)^2_{oct} = 9\tau^2_{oct} &= (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2I_1^2 - 6I_2 \\ &\textcircled{=} (\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 \\ &\quad + 6\sigma_{xy}^2 + 6\sigma_{xz}^2 + 6\sigma_{yz}^2 \end{aligned}\right] \rightarrow ⑤$$

Since I_1, I_2, I_3 are invariant, we substitute back their general-coord-system forms.

Octahaedral stresses are required in failure criteria.

Pure shear state of stress.

If $\exists$ system x, y, z for which $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ vanish at a point, then the state of stress at that Point is a Pure Shear state. Neccessary + suff condt for Pure Shear state of stress to exist is $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$.

Deviatoric stress.

$$\underset{\underline{\underline{T}} \equiv \underline{\underline{\sigma}}}{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ \text{symm} & & \sigma_{zz} \end{bmatrix}} = \underset{= \hat{\underline{\underline{\sigma}}}}{\overset{\text{deviatoric part.}}{\underline{\underline{T}}_d}} + \underset{\underline{\underline{\sigma}}_m}{\overset{\text{hydrostatic or spherical part}}{\underline{\underline{T}}_m}}, \text{ where } \underline{\underline{T}}_m = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$\text{where } \sigma_m = \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} = \frac{I_1}{3}.$$

$$\Rightarrow \sigma_{ij} = \hat{\sigma}_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Note that $\underline{\underline{T}}_d$ represents a pure shear state of stress.

Expts indicate that yielding & plastic deformation depend on $\underline{\underline{T}}_d$ and independent of $\underline{\underline{T}}_m$

Now evp for $\underline{\underline{T}}$ in index notation is $\underline{\sigma} = \sigma n_i \underline{e}_i = \sigma_{ij} n_j \underline{e}_i$

$$\left(\sigma_{ij} - \sigma\delta_{ij}\right) n_j = 0. \quad , \text{ where } n_1 = l,\ n_2 = m,\ n_3 = l.$$

So evp for $\underline{\underline{T}}_d$ is

$$\left(\hat{\sigma}_{ij} - \hat{\sigma}\delta_{ij}\right) n_j = 0 \Rightarrow \left(\sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} - \hat{\sigma}\delta_{ij}\right) n_j = 0$$

$$\Rightarrow \left(\sigma_{ij} - \underbrace{\left[\hat{\sigma} + \tfrac{1}{3}\sigma_{kk}\right]}_{=\sigma}\delta_{ij}\right) n_j = 0.$$

$$\Rightarrow \hat{\sigma} = \sigma - \tfrac{1}{3}\sigma_{kk} = \sigma - \sigma_m$$

Plane stress.

$$\sigma_{XX} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta$$

$$\sigma_{YY} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) - \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\theta - \sigma_{xy}\sin 2\theta$$

$$\sigma_{XY} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) = -\tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\theta + \sigma_{xy}\cos 2\theta$$

$$\Rightarrow \left[\sigma_{XX} - \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})\right]^2 + \sigma_{XY}^2 = \tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xy}^2$$

$$\left.\begin{array}{l} \text{Principal stresses occur for } \tan 2\theta = \dfrac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} \\ \text{and are equal to } \sigma_{1,2} = \dfrac{\sigma_{xx}+\sigma_{yy}}{2} \pm \tfrac{1}{4}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xy}^2} \end{array}\right\} \longrightarrow \underline{\underline{⑥}}$$

Maximum shearing stress.

$$(\sigma_s)_{max} = \max\left(\frac{|(\sigma_1-\sigma_2)|}{2}, \frac{|(\sigma_1-\sigma_3)|}{2}, \frac{|(\sigma_2-\sigma_3)|}{2}\right) \longrightarrow \underline{\underline{⑦}}$$

and acts on ±45° planes (in the p-axes system) in the 1-2, 1-3, 2-3 plane, respectively.

MAXIMUM SHEAR STRESS.

We wish to find max (S). Refer to p-coord system:

$$\underset{\sim}{\Sigma} = (\sigma(1) n_1 , \sigma(2) n_2 , \sigma(3) n_3) \quad \Bigg| \quad S^2 = |\underset{\sim}{\Sigma}|^2 - N^2$$

$$N = \sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2$$

where $\underset{\sim}{n} = (n_1, n_2, n_3)$ is plane containing max(S). Note that n_1, n_2, n_3 are w.r.t. p-coord system. Thus,

$$n_1^2 + n_2^2 + n_3^2 = 1 \longrightarrow ①$$

$$S^2 = \sigma^2(1) n_1^2 + \sigma^2(2) n_2^2 + \sigma^2(3) n_3^2 - \left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2\right]^2 \longrightarrow \textcircled{*}$$

Doing $\frac{\partial S^2}{\partial n_1^2} = \frac{\partial S^2}{\partial n_2^2} = \frac{\partial S^2}{\partial n_3^2} = 0$ to get n_1, n_2, n_3 wont work due to constrain ①. Instead you can use ① to eliminate (say) n_3^2 from $\textcircled{*}$ by putting $n_3^2 = 1 - n_1^2 - n_2^2$ in $\textcircled{*}$, and then do $\frac{\partial S^2}{\partial n_1} = \frac{\partial S^2}{\partial n_2} = 0$, since now n_1, n_2 are independent

However, this is somewhat messy. So use the Lagrange Multiplier approach for constrained optimization. Form augmented function G:

$$G = S^2 + \underset{\hookrightarrow \text{Lagrange multiplier.}}{\lambda} (n_1^2 + n_2^2 + n_3^2 - 1)$$

Now n_1, n_2, n_3, λ can be treated as independent. Note that constraint is included in G, ie., if constraint satisfied then $G = S^2$.

Do $\frac{\partial G}{\partial \lambda} = \frac{\partial G}{\partial n_1} = \frac{\partial G}{\partial n_2} = \frac{\partial G}{\partial n_3} = 0$

$$\frac{\partial G}{\partial \lambda} = 0 = n_1^2 + n_2^2 + n_3^2 - 1 \longrightarrow ①$$ (retrieved again!).

$$\frac{\partial G}{\partial n_1} = 0 = n_1 \left\{ \sigma^2(1) - 2\left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2\right]\sigma(1) + \lambda \right\} \rightarrow ②$$

$$\frac{\partial G}{\partial n_2} = 0 = n_2 \left\{ \sigma^2(2) - 2\left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2\right]\sigma(2) + \lambda \right\} \rightarrow ③$$

$$\frac{\partial G}{\partial n_3} = 0 = n_3 \left\{ \sigma^2(3) - 2\left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2\right]\sigma(3) + \lambda \right\} \rightarrow ④$$

Solutions:

(i) Case I : only one component of $\underline{n}$ is non-zero.

$n_1 = 1,\ n_2 = n_3 = 0 \Rightarrow \lambda = \sigma^2(1),\quad S^2 = 0$

or, $n_2 = 1,\ n_1 = n_3 = 0 \Rightarrow \lambda = \sigma^2(2),\quad S^2 = 0$

or, $n_3 = 1,\ n_1 = n_2 = 0 \Rightarrow \lambda = \sigma^2(3),\quad S^2 = 0$

This is the well-known solution that $S = 0$ on principal planes. Here $S = 0$ is a local extremum value but not the max(S) that we seek – so NOT INTERESTING

(ii) Case II : only one component of $\underline{n}$ is zero.

Say $n_1 = 0,\ n_2 \neq 0,\ n_3 \neq 0$

$$①, ③, ④ \Rightarrow \left.\begin{aligned} &\sigma^2(2) - 2\left[\sigma(2) n_2^2 + \sigma(3) - \sigma(3) n_2^2\right]\sigma(2) + \lambda = 0 \\ &\sigma^2(3) - 2\left[\sigma(2) n_2^2 + \sigma(3) - \sigma(3) n_2^2\right]\sigma(3) + \lambda = 0 \end{aligned}\right\}$$

$$\Downarrow$$

$$\left[\sigma(2) - \sigma(3)\right]\left[\sigma(2) + \sigma(3) - 2\left\{\sigma(2) n_2^2 + \sigma(3) - \sigma(3) n_2^2\right\}\right] = 0$$

$\Rightarrow$ If $\sigma(2) = \sigma(3) \Rightarrow \lambda = \sigma^2(2),\quad S^2 = 0,\ n_2 = \text{arb},\ n_3 = \text{arb}$

If $\sigma(2) \neq \sigma(3) \Rightarrow n_2 = \pm\frac{1}{\sqrt{2}},\ n_3 = \pm\frac{1}{\sqrt{2}},\ \lambda = \sigma(2)\sigma(3),$

$$S^2 = \left(\frac{\sigma(2) - \sigma(3)}{2}\right)^2 \Rightarrow S = \left|\frac{\sigma(2) - \sigma(3)}{2}\right|$$

Thus possible solutions are,

$$n_1 = 0, \quad \underbrace{n_2 = \pm\sqrt{\tfrac{1}{2}}, \quad n_3 = \pm\sqrt{\tfrac{1}{2}}}_{\sigma(2) \neq \sigma(3)}, \quad \underbrace{n_2 = \text{arb}, \quad n_3 = \text{arb}}_{\sigma(2) = \sigma(3)}, \quad \lambda = \sigma(2)\sigma(3), \quad S = \tfrac{1}{2}|\sigma(2) - \sigma(3)|$$

or

$$n_2 = 0, \quad \underbrace{n_1 = \pm\sqrt{\tfrac{1}{2}}, \quad n_3 = \pm\sqrt{\tfrac{1}{2}}}_{\sigma(1) \neq \sigma(3)}, \quad \underbrace{n_1 = \text{arb}, \quad n_3 = \text{arb}}_{\sigma(1) = \sigma(3)}, \quad \lambda = \sigma(1)\sigma(3), \quad S = \left|\frac{\sigma(1) - \sigma(3)}{2}\right|$$

or

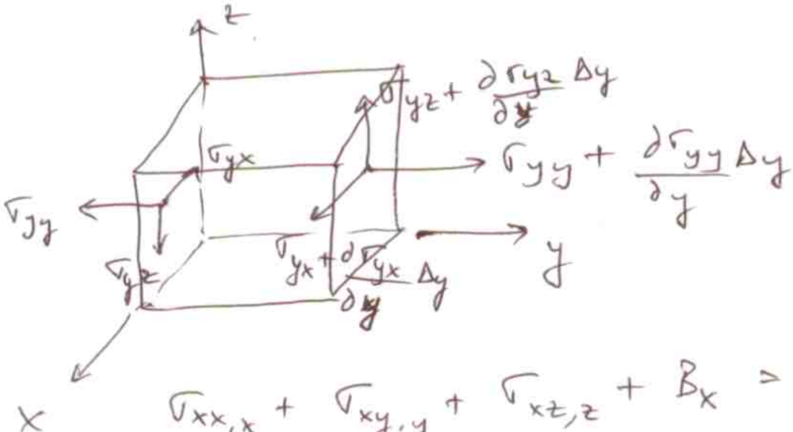
$$n_3 = 0, \quad \underbrace{n_2 = \pm\sqrt{\tfrac{1}{2}}, \quad n_1 = \pm\sqrt{\tfrac{1}{2}}}_{\sigma(1) \neq \sigma(2)}, \quad \underbrace{n_1 = \text{arb}, \quad n_2 = \text{arb}}_{\sigma(1) = \sigma(2)}, \quad \lambda = \sigma(1)\sigma(2), \quad S = \left|\frac{\sigma(1) - \sigma(2)}{2}\right|$$

- So if all p-stresses are distinct then max(S) acts on a plane containing one p-axis & equally inclined (at ±45°) to other two p-axes.
- If any two p-stresses are same, say $\sigma(1) = \sigma(2)$, then $S_{max} = \frac{|\sigma(1) - \sigma(3)|}{2} = \frac{|\sigma(2) - \sigma(3)|}{2}$, acts on plane with $n_3 = \pm\sqrt{\frac{1}{2}}$ and n_1, n_2 arbitrary.

Case III : All components of $\underline{n}$ are non-zero.

Here ②, ③, ④ give an inconsistent system of equations if two or more p-stresses are distinct, i.e, determinant of coefficient matrix of $[n_1^2 \ n_2^2 \ n_3^2]^T$ vanishes. Hence solution is only possible if $\sigma(1) = \sigma(2) = \sigma(3)$ which yields $S = 0$ on all planes. Thus, this value of S is not an extremum — so NOT INTERESTING.

EQUILIBRIUM EQUATIONS



$$\left.\begin{aligned} &\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + B_x = 0 \\ &\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{yz,z} + B_y = 0 \\ &\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + B_z = 0 \end{aligned}\right\} \quad \sigma_{ij,j} + B_i = 0. \longrightarrow$$ ⑧

In general curvilinear orthogonal coordinates we have,

$$\left.\begin{aligned} &\frac{\partial(\beta\gamma\sigma_{xx})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{xy})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{xz})}{\partial z} + \gamma\sigma_{xy}\frac{\partial\alpha}{\partial y} \\ &\quad + \beta\sigma_{xz}\frac{\partial\alpha}{\partial z} - \gamma\sigma_{yy}\frac{\partial\beta}{\partial x} - \beta\sigma_{zz}\frac{\partial\gamma}{\partial x} + \alpha\beta\gamma B_x = 0 \\ &\frac{\partial(\beta\gamma\sigma_{xy})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yy})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{yz})}{\partial z} \\ &\quad + \gamma\frac{\partial\beta}{\partial x}\sigma_{xy} + \alpha\frac{\partial\beta}{\partial z}\sigma_{yz} - \gamma\frac{\partial\alpha}{\partial y}\sigma_{xx} - \alpha\frac{\partial\gamma}{\partial y}\sigma_{zz} + \alpha\beta\gamma B_y = 0 \\ &\frac{\partial(\beta\gamma\sigma_{xz})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yz})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{zz})}{\partial z} \\ &\quad + \beta\frac{\partial\gamma}{\partial x}\sigma_{xz} + \alpha\frac{\partial\gamma}{\partial y}\sigma_{yz} - \alpha\frac{\partial\beta}{\partial z}\sigma_{yy} - \beta\frac{\partial\alpha}{\partial z}\sigma_{xx} + \alpha\beta\gamma B_z = 0 \end{aligned}\right\} \longrightarrow$$ ⑨

Here (α, β, γ) are metric coefficients defined by

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2 = (\text{diagonal of element})^2$$

Cartesian — $\alpha = \beta = \gamma = 1$;: $ds^2 = dx^2 + dy^2 + dz^2$

Polar/cylindrical $ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \Rightarrow \alpha = 1,\ \beta = r,\ \gamma = 1$

Spherical — $ds^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2 \Rightarrow \alpha = 1,\ \beta = r,$
$\gamma = r\sin\theta$

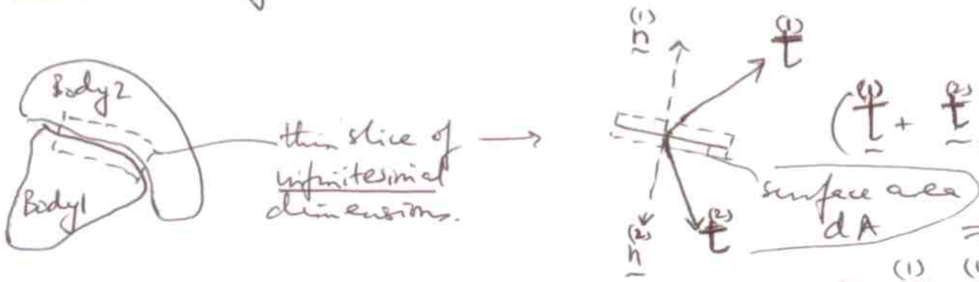
Cylindrical coords :

$$\left.\begin{aligned} &\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \sigma_{rz,z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + B_r = 0 \\ &\sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \sigma_{\theta z,z} + \frac{2\sigma_{r\theta}}{r} + B_\theta = 0 \\ &\sigma_{rz,r} + \frac{\sigma_{\theta z,\theta}}{r} + \sigma_{zz,z} + \frac{\sigma_{rz}}{r} + B_z = 0 \end{aligned}\right\} \longrightarrow ⑩$$

Spherical coords :

$$⑪ \longleftarrow \left\{\begin{aligned} &\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \frac{\sigma_{r\phi,\phi}}{r\sin\theta} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta) + B_r = 0 \\ &\sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{\sigma_{\theta\phi,\phi}}{r\sin\theta} + \frac{1}{r}\left[(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}\right] + B_\theta = 0 \\ &\sigma_{r\phi,r} + \frac{\sigma_{\theta\phi,\theta}}{r} + \frac{\sigma_{\phi\phi,\phi}}{r\sin\theta} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta) + B_\phi = 0 \end{aligned}\right.$$

Stress Boundary Conditions.



$$\left(\overset{(1)}{\underset{\sim}{t}} + \overset{(2)}{\underset{\sim}{t}}\right) \cancel{dA} = 0.$$

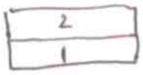
$$\Rightarrow \overset{(1)}{\underset{\sim}{t}} + \overset{(2)}{\underset{\sim}{t}} = 0$$

$$\Rightarrow \overset{(1)}{\sigma}_{ij}\,\overset{(1)}{n}_j = \overset{(2)}{\sigma}_{ij}\,\overset{(1)}{n}_j$$

$$\Rightarrow \boxed{\overset{(1)}{\underset{\approx}{\sigma}} \cdot \overset{(1)}{\underset{\sim}{n}} = \overset{(2)}{\underset{\approx}{\sigma}} \cdot \overset{(1)}{\underset{\sim}{n}}} \longrightarrow ⑫$$

$$\left\{\begin{aligned} &\overset{(1)}{\sigma}_{xx} l + \overset{(1)}{\sigma}_{xy} m + \overset{(1)}{\sigma}_{xz} n = \overset{(2)}{\sigma}_{xx} l + \overset{(2)}{\sigma}_{xy} m + \overset{(2)}{\sigma}_{xz} n \\ &\cdot \;\ldots \\ &\cdot \;\ldots \end{aligned}\right.$$

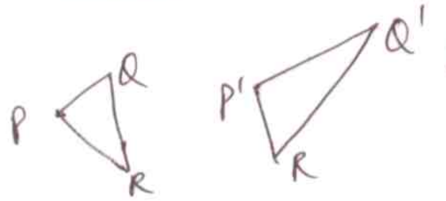
(eg)



$$\underset{\sim}{n} = (0,0,1) \Longrightarrow \overset{(1)}{\sigma}_{xz} = \overset{(2)}{\sigma}_{xz}, \quad \overset{(1)}{\sigma}_{yz} = \overset{(2)}{\sigma}_{yz}, \quad \overset{(1)}{\sigma}_{zz} = \overset{(2)}{\sigma}_{zz}$$

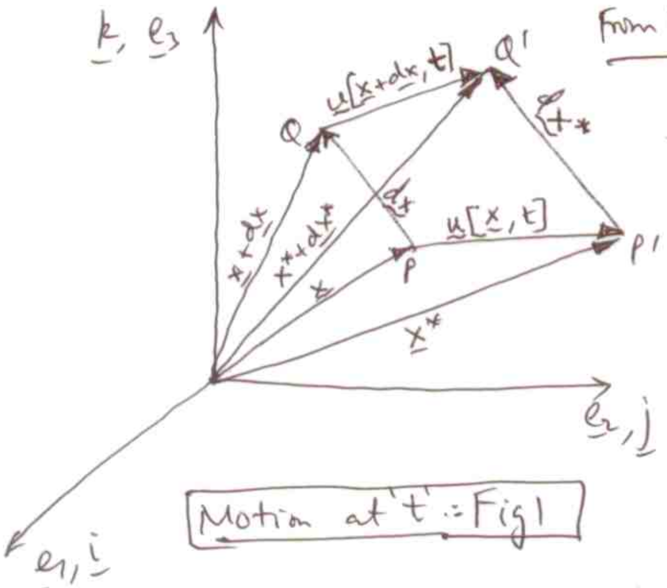
No b.c's involving other stress components. Further, other stress components (ie those not involving 'z' are not necessarily continuous across boundary).

(2) Deformation & Strain Analysis.



total motion = deformation + RB motion

$$\boxed{\begin{array}{l} x_i^* = x_i + u_i \\ x_i^* = x_i^*[x_1, x_2, x_3, t], \; u_i = u_i[x_1, x_2, x_3, t] \end{array}}$$



Motion at 't' :- Fig1

From Fig1 → $x_i^* + dx_i^* = x_i + dx_i + u_i[\underline{x}+d\underline{x}, t]$

$$du_i = u_i[Q] - u_i[P] = u_i[\underline{x}+d\underline{x}, t) - u_i(\underline{x}, t)$$

Note that du_i is evaluated at 't', ie it is a snapshot at 't'.

$$\Rightarrow \boxed{du_i = \left.\frac{\partial u_i}{\partial x_j}\right|_{P, t} dx_j} \rightarrow ①$$

(ie no change in 't' while finding du_i, ie $\Delta t = 0$).

$$\overline{PQ} = |d\underline{x}| = dx_i dx_i = (dx)^2, \quad \overline{P'Q'} = dx_i^* dx_i^* = (dx^*)^2$$

$$\boxed{dx_i^* = \frac{\partial x_i^*}{\partial x_j} dx_j} \rightarrow ②$$

at given instant 't'.

$$\text{stretch ratio} \triangleq (dx^*)^2 - (dx)^2 = dx_i^* dx_i^* - dx_i dx_i = \left[\frac{\partial x_r^*}{\partial x_i}\frac{\partial x_r^*}{\partial x_j} - \delta_{ij}\right] dx_i dx_j$$

$$= \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_r}{\partial x_i}\frac{\partial u_r}{\partial x_j}\right] dx_i dx_j$$

$$= 2\epsilon_{ij} dx_i dx_j$$

$$\Rightarrow M \overset{\text{= magnification factor}}{=} \frac{1}{2}\left[\frac{(dx^*)^2 - (dx)^2}{(dx)^2}\right] = \frac{1}{2}\left[\left(\frac{dx^*}{dx}\right)^2 - 1\right] = \frac{1}{2}\left[(1+\epsilon_E)^2 - 1\right] = \epsilon_E + \frac{\epsilon_E^2}{2}$$

$$= \epsilon_{ij} n_i n_j \longrightarrow$$

Summary →

$$\boxed{\begin{array}{l} M \triangleq \frac{1}{2}\left[\frac{(dx^*)^2 - (dx)^2}{(dx)^2}\right] = \epsilon_E + \frac{\epsilon_E^2}{2} = \epsilon_{ij} n_i n_j \rightarrow ③ \\ \text{where } \underset{\text{Ext Engg strain}}{\epsilon_E} \triangleq \frac{dx^* - dx}{dx}, \quad \epsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{r,i} u_{r,j}\right] \rightarrow ④ \end{array}}$$

<u>Notes:</u> (1) ϵ_{ij} is a measure of straining. If $dx \neq da$ then $\epsilon_{ij} \neq 0$ (ie all components cannot be zero

(2) $\epsilon_{ij} = \epsilon_{ji}$

(3) R.B. motion does not affect M.

(4) M is analogous to N (normal stress on a plane) → compare their expressions.

Geometric (Physical) interpretation of ϵ_{ij} (nonlinear).

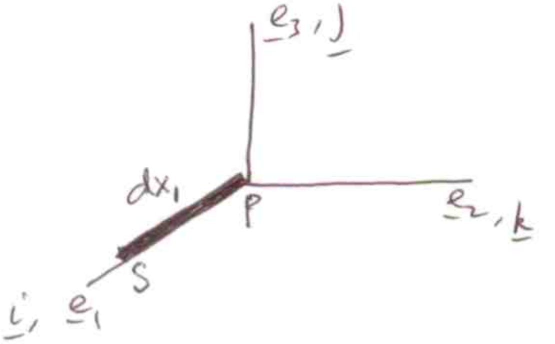


Fig 2

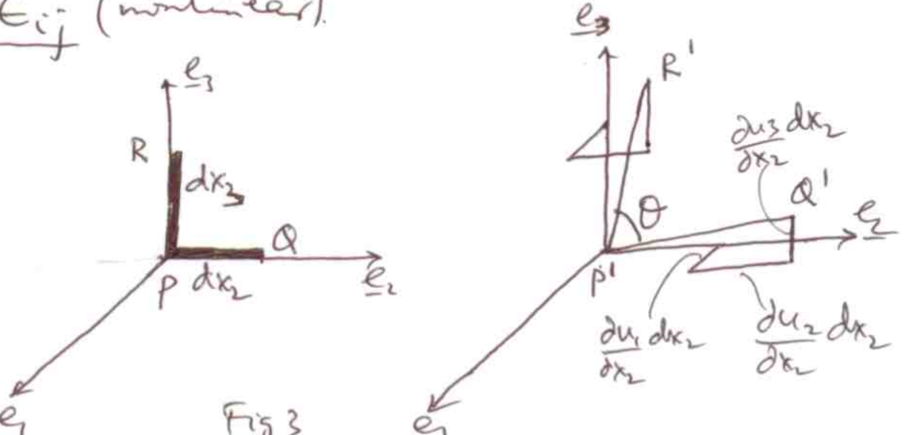


Fig 3

P, Q, R displace to P', Q', R'
(Note P's displ is not necessarily horizontal as is shown here).

Normal strains :

Ref. Fig 2. Engg. strain for element PS, having $n_i = (1,0,0)$, is

$$\epsilon_{E1} = \frac{dx^* - dx}{dx} = \frac{(dx^*)^2 - (dx)^2}{(dx)^2(\epsilon_E + 2)} = \frac{2\epsilon_{11}}{2+\epsilon_{E1}}$$

$$\Rightarrow \boxed{\epsilon_{E1} = \sqrt{1+2\epsilon_{11}} - 1} \text{ , } |||^{ly} \text{ for } \epsilon_{22}, \epsilon_{33}. \longrightarrow ⑤$$

$$\boxed{M_1 = \epsilon_{E1} + \frac{\epsilon_{E1}^2}{2} = \epsilon_{11}} \text{ , } |||^{ly} \text{ for } M_2, M_3. \longrightarrow ⑥$$

Shear strains

Ref Fig 3.

$$\overline{P'Q'} = dx^*\Big|_{PQ} = \sqrt{1+2\epsilon_{22}}\, dx_2$$

$$\overline{P'R'} = dx^*\Big|_{PR} = \sqrt{1+2\epsilon_{33}}\, dx_3$$

$$\underline{P'Q'} \cdot \underline{P'R'} = \cos\theta \sqrt{1+2\epsilon_{22}} \sqrt{1+2\epsilon_{33}}\, dx_2\, dx_3 \longrightarrow ⑦$$

Also

$$\underline{P'Q'} \cdot \underline{P'R'} = \underline{dx}^*\Big|_{PQ} \cdot \underline{dx}^*\Big|_{PR} = dx_i^*\Big|_{PQ} dx_i^*\Big|_{PR}$$

$$= \left(\frac{\partial x_i^*}{\partial x_j}\right)_P dx_j\Big|_{PQ} \left(\frac{\partial x_i^*}{\partial x_k}\right)_P dx_k\Big|_{PR} = \left(\frac{\partial x_i^*}{\partial x_2}\frac{\partial x_i^*}{\partial x_3}\right)_P dx_2\, dx_3$$

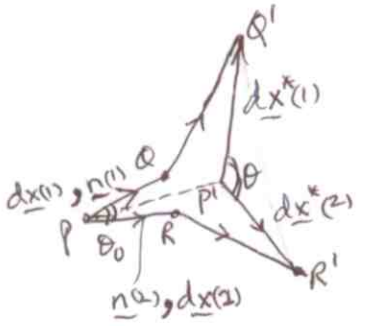
$$= 2\epsilon_{23}\, dx_2\, dx_3. \longrightarrow ⑧$$

From ⑦, ⑧ P.9.,

$$2\epsilon_{23} = \sqrt{1+2\epsilon_{22}}\,\sqrt{1+2\epsilon_{33}}\,\cos\theta = \left(1+\epsilon_{E2}\right)\left(1+\epsilon_{E3}\right)\cos\theta. \rightarrow ⑨$$

Generalization of Shear Strain interpretation

Consider two line elements with directions $\underline{n}(1)$, $n(2)$, resply initially being θ_0 apart, as shown below.



Now,

$$d\underline{x}^*(1) \cdot d\underline{x}^*(2) = \underset{\text{LHS}}{\underbrace{dx_i^*(1)\,dx_i^*(2)}} = \underset{\text{RHS}}{\underbrace{\sqrt{dx_i^*(1)\,dx_i^*(1)}\,\sqrt{dx_j^*(1)\,dx_j^*(1)}\cos\theta}}$$

$$\text{LHS} = \left(\frac{\partial x_i^*}{\partial x_j}dx_j\right)_{(1)}\left(\frac{\partial x_i^*}{\partial x_k}dx_k\right)_{(2)} = \left(dx_i + \frac{\partial u_i}{\partial x_j}dx_j\right)_{(1)}\left(dx_i + \frac{\partial u_i}{\partial x_k}dx_k\right)_{(2)}$$

$$= dx_i(1)\,dx_i(2) + dx_i(1)\,dx_j(2)\left[u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}\right]_P$$

Using ③, p.8,

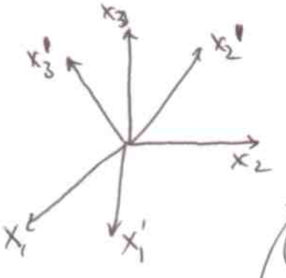
$$\text{RHS} = \sqrt{(dx^*(1))^2}\,\sqrt{(dx^*(2))^2}\cos\theta = \sqrt{dx_i(1)\,dx_i(1) + 2\epsilon_{ij}\,dx_i(1)\,dx_j(1)} \;*\; \sqrt{dx_p(2)\,dx_p(2) + 2\epsilon_{pq}\,dx_p(2)\,dx_q(2)} \;*\cos\theta$$

Divide LHS & RHS by $dx(1)\,dx(2)$, note that $n_i = dx_i/dx$, and equate (LHS=RHS) and note that $n_i(1)n_i(1) = n_i(2)n_i(2) = 1$, and $n_i(1)\,n_i(2) = \cos\theta_0$, you get,

$$\cos\theta = \frac{\cos\theta_0 + 2n_i(1)\,n_j(2)\,\epsilon_{ij}}{\sqrt{1+2n_i(1)\,n_j(1)\,\epsilon_{ij}}\,\sqrt{1+2n_p(2)\,n_q(2)\,\epsilon_{pq}}} \longrightarrow ⑩$$

Thus we have angle between two line elements after straining, in terms of ϵ_{ij} and angle before straining.

Strain Transformations.



Let the element lie along x_1' axis. Then from ③ & ⑥ we have

$$M_1' = \epsilon_{11}' \quad : \text{from ⑥}$$

&

$$M_1' = \epsilon_{ij}\,n_i n_j \;(\text{from ③}) = \epsilon_{ij}\,a_{1i}\,a_{1j}$$

$(\because n_i = \text{unit vector along } x_1' \text{ axis} = (a_{11}, a_{12}, a_{13}))$

$$\Rightarrow \epsilon_{11}' = \epsilon_{ij}\,a_{1i}\,a_{1j} \quad (|||^{ly} \text{ for } \epsilon_{22}', \epsilon_{33}') \rightarrow \text{(11a)}$$

For transf of shear strains, use ⑩ for two line elements originally along x'_2 & x'_3 axes, respectively, with unit vectors denoted as $\underline{n}(2)$, $\underset{\sim}{n}(3)$, resply. From ⑨ and ⑩ we have,

from ⑨ → in $\underset{\sim}{x}'$ system → $$\frac{2\epsilon'_{23}}{\sqrt{1+2\epsilon'_{22}}\sqrt{1+2\epsilon'_{33}}} = \cos\theta \longrightarrow \circledast$$

from ⑩ → in $\underset{\sim}{x}$ system → $$\cos\theta = \left[\frac{\cos\underset{90^\circ}{\phi_0} + 2\epsilon_{ij}\,\overset{a_{2i}}{\cancel{n_i(2)}}\,\overset{a_{3j}}{\cancel{n_j(3)}}}{\underbrace{\sqrt{1+2\underset{a_{2i}}{n_i(2)}\,\underset{a_{2j}}{n_j(2)}\,\epsilon_{ij}}}_{2\epsilon'_{22}}\;\underbrace{\sqrt{1+2\underset{a_{3p}}{n_p(3)}\,\underset{a_{3q}}{n_q(3)}\,\epsilon_{pq}}}_{2\epsilon'_{33}}}\right].$$

using transf law (11(a)) for normal strains

$$\Rightarrow \cos\theta = \frac{2\epsilon_{ij}a_{2i}a_{3j}}{\sqrt{1+2\epsilon'_{22}}\sqrt{1+2\epsilon'_{33}}} \longrightarrow (**)$$

From $\circledast$ & $(**)$ → $\epsilon'_{23} = \epsilon_{ij}a_{2i}a_{3j}$ ($|||^{aly}$ for $\epsilon'_{12}, \epsilon'_{13}$) → $\underline{\underline{\text{(11b)}}}$

In compact form, (11(a, b)) read,

$$\epsilon'_{ij} = a_{im}a_{jn}\epsilon_{mn} \longrightarrow \underline{\underline{⑪}}$$

This is same as stress transf. law.

Principal strains & directions.

Seek mutually ⊥ar directions $\underline{n}(1)$, $\underline{n}(2)$, $\underline{n}(3)$, ie a coordinate transf from $\underline{x}$ to $\underline{x}'$ system, such that ϵ_{ij} gets diagonalized, ie $\epsilon'_{ij} = 0$ for $i \neq j$. We know, (from analogy with stress ~~that~~) this transf exists, since transf ⑪ above is same as transf for stresses (ie, ③, p.3). Thus $\underline{n}(1) \equiv x'_1$ axis and so on. So we solve the evalue problem

$$(\epsilon_{ij} - \underset{M(1),\,M(2),\,M(3)}{\cancel{\lambda}}\,\delta_{ij})\,n_j = 0. \longrightarrow \underline{\underline{⑫}}$$

M(1), M(2), M(3), due to analogy with N(1), N(2), N(3).

Again, from analogy with stress (ref ①, p.2 & ③, p.8), we know that M (N) is stationary at the p-axes of strain (stress). Further, since $\epsilon'_{ij} = 0$ for $i \neq j$, the p-axes retain their orthogonality during deformation. So from analogy betwn M & N, we get p-strains

M_1 M_2 M_3 in book of Boresi – but not adopted here due to conflict with Eq ⑪a

as ~~$M(1)$~~, ~~$M(2)$~~, ~~$M(3)$~~ where principal extensional engg strains are $\epsilon'_{E1} + \frac{1}{2}\epsilon'^2_{E1} = M(1)$, etc (for $\epsilon'_{E2}, \epsilon'_{E3}$). Now from ③,

$$dM = d\epsilon_E(1+\epsilon_E) \Rightarrow d\epsilon_E = 0 \text{ if } dM = 0 \because (1+\epsilon_E) > 0 \text{ by}$$

definition of $\epsilon_E \left(= \frac{dx^*}{dx} - 1\right)$.

Thus $\epsilon'_{E1}, \epsilon'_{E2}, \epsilon'_{E3}$ are stationary values of ext engg strains.

Analogous to stress, we have,

$$M^3 - J_1 M^2 + J_2 M - J_3 = 0 \longrightarrow ⑬$$

$J_1 = \epsilon_{ii}$, $J_2 = \sum \det(\text{cofactors of diag's of } [\epsilon_{ij}])$.

$J_3 = \det(\epsilon_{ij})$

$\Rightarrow J_1 = M(1) + M(2) + M(3)$, $J_2 = M(1)M(2) + M(2)M(3) + M(3)M(1)$

$J_3 = M(1)M(2)M(3)$

So we solve $M(1), M(2), M(3)$ from ⑬, then $\underline{n}(1), \underline{n}(2), \underline{n}(3)$ from ⑫ subject to $|\underline{n}| = 1$.

(Next: see p. 13 for linear p-strain theory – a different presentation.)

Small displacement gradient theory – Linearizations

Assume $\partial u_i/\partial x_j \ll 1$.

$$\Rightarrow \epsilon_{ij} \rightarrow \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \xrightarrow{\text{small (infinitesimal)}} ⑭$$

Thus $du_i = u_{i,j}dx_j$ is small $\Rightarrow dx + dx^* \simeq 2dx$

$$\Rightarrow M = \frac{1}{2}\frac{(dx^*)^2 - (dx)^2}{(dx)^2} = \frac{(dx^* - dx)\overset{\simeq 2dx}{\cancel{(dx + dx^*)}}}{2(dx)^2} \simeq \frac{dx^* - dx}{dx} = \epsilon_E$$

ie $M \simeq \epsilon_E$ (can see from putting $\epsilon_E \ll 1$ in ③ as well).

$$\text{Now } M = \varepsilon_{ij} n_i n_j \simeq \epsilon_E \longrightarrow ⑮$$

Geometric interpretations

Put ϵ_{ij} $\ll 1$, i.e. infinitesimal thruout (retain only linear terms in ϵ_{ij}).

From ⑤, ⑥ $\rightarrow$ $M_1 = \epsilon_{E1} \cong \epsilon_{11}$, $|||^{\text{ly}}$ for $\epsilon_{22}, \epsilon_{33}$ $\longrightarrow$

From ⑨ (put $\epsilon_{ij} \ll 1$ or $\epsilon_E \ll 1$) $\rightarrow$ $2\epsilon_{23} \simeq \cos\theta$, $|||^{\text{ly}}$ for $\varepsilon_{12}, \varepsilon_{13}$ $\rightarrow$ ⑯

Note: that $u_{i,j} \ll 1 \Rightarrow \varepsilon_{ij} \ll 1 \Rightarrow M \ll 1 \Rightarrow \varepsilon_E \ll 1$

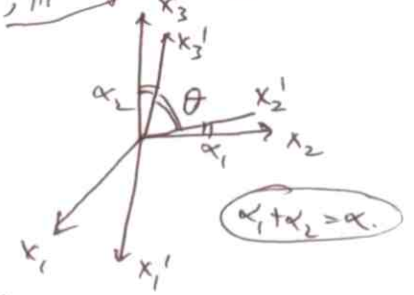
Note on Engg shear strains:

$$\gamma_{ij} \triangleq 2\,\epsilon_{ij} \text{ or } 2\,\varepsilon_{ij} \quad \text{for } i \neq j.$$

This is done to effect tensor transformations of tensorial shear strains (ie ϵ_{ij} or ε_{ij}). For linear theory (ie $u_{i,j} << 1$),

for eg, $\gamma_{23} = \cos\theta = \sin\left(\frac{\pi}{2}-\theta\right) = \frac{\pi}{2}-\theta = \alpha$, $|||^{\text{ly}}$ for γ_{12}, γ_{13}

since $\frac{\pi}{2}-\theta$ is small due to $u_{i,j} << 1$.



$\alpha_1+\alpha_2=\alpha$.

Other analogies with stress tensor (Nonlinear theory).

(i) Deviatoric strain : $\underbrace{\hat{\epsilon}_{ij}}_{\text{deviatoric}} = \overset{\text{total.}}{\epsilon_{ij}} - \overset{\text{spherical.}}{\frac{1}{3}\delta_{ij}\,\epsilon_{mm}}$

(ii) Pure shear state of strain: $\exists$ iff $\epsilon_{ii} = 0$.
Physically it means that there exists three mutually $\perp^{ar}$ directions $\underset{\sim}{n}^{(1)}, \underset{\sim}{n}^{(2)}, \underset{\sim}{n}^{(3)}$ along which $\epsilon_{E1}, \epsilon_{E2}, \epsilon_{E3}$ (ie engg ext strains) are zero, resply.

Infinitesimal Rotation & Analysis of Relative displ / Linear P-strain theory.

(to do after general nonlinear p-strain theory as a different treatment).

For $u_{i,j} << 1$ we have,

$$du_i = u_{i,j}\,dx_j = \frac{1}{2}\Big[\underbrace{(u_{i,j}+u_{j,i})}_{\varepsilon_{ij}} + \underbrace{(u_{i,j}-u_{j,i})}_{\tilde{\omega}_{ij}\ (\text{linear rot tensor})}\Big]dx_j = [\varepsilon_{ij} + \tilde{\omega}_{ij}]\,dx_j \quad ⑰$$

RB motion $\Rightarrow \varepsilon_{ij} = 0 \Longrightarrow du_i = \tilde{\omega}_{ij}\,dx_j$

Note that by defn $\tilde{\omega}_{ij} = -\tilde{\omega}_{ji}$ (antisymm).

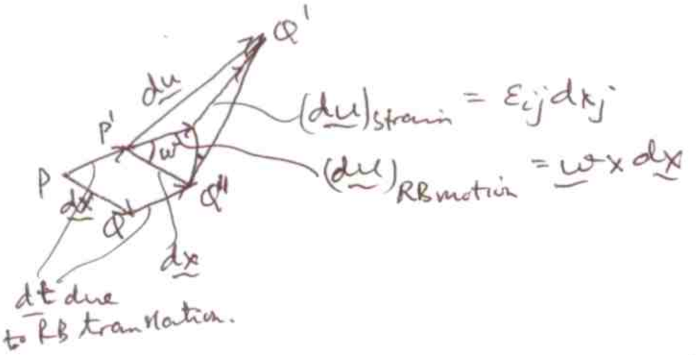
$$\text{Now } du_i = \begin{bmatrix} 0 & \tilde{\omega}_{12} & \tilde{\omega}_{13} \\ -\tilde{\omega}_{12} & 0 & \tilde{\omega}_{23} \\ -\tilde{\omega}_{13} & -\tilde{\omega}_{23} & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 & -\tilde{\omega}_{21} & \tilde{\omega}_{13} \\ \tilde{\omega}_{21} & 0 & -\tilde{\omega}_{32} \\ -\tilde{\omega}_{13} & \tilde{\omega}_{32} & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$= \left[(-\tilde{\omega}_{21}\,dx_2 + \tilde{\omega}_{13}\,dx_3),\ (\tilde{\omega}_{21}\,dx_1 - \tilde{\omega}_{32}\,dx_3),\ (-\tilde{\omega}_{13}\,dx_1 + \tilde{\omega}_{32}\,dx_2)\right]^T$$

$$⑱ \longleftarrow = \underset{\sim}{\tilde{\omega}} \times \underset{\sim}{dx} \quad \text{where } \underset{\sim}{\tilde{\omega}} \triangleq [\tilde{\omega}_{32}, \tilde{\omega}_{13}, \tilde{\omega}_{21}]^T = \text{linear rot vector}$$

Thus $\omega_{ij}\,dx_j = \underset{\sim}{\omega} \times \underline{dx} \left(= \overset{\text{permutation symbol}}{\epsilon_{ijk}}\,\omega_j\,dx_j\right) \rightarrow$ not done in class

Thus $\underline{\omega} \times \underline{dx}$ is the rotation component of du_i, the rel displ..



$\longrightarrow$ so relative motion = RBM + STRAINING.

So we can study their effects separately & superpose..

<u>Another</u> <u>explanation</u> of <u>p-strains</u> <u>for</u> <u>infinitesimal</u> <u>theory</u>

Consider du_i due to straining only.

$$⑲ \longleftarrow \boldsymbol{\varepsilon}_i \triangleq \frac{du_i}{dx} = \frac{\varepsilon_{ij}\,dx_j}{dx} = \varepsilon_{ij} n_j \text{ = rel displ per unit original length.}$$

In general ε_i is neither directed along PQ or P'Q' (see above fig). In fact ε_i is along the $(\underset{\sim}{du})_{strain}$ vector.

another explanation

Now seek orientation n_i of element PQ (after deformation) for which the element stays perpendicular to plane of particles to which it was originally $\perp^{ar}$ before deformation. Since RBM has been excluded, this means that ε_i directed along PQ, ie, along n_i. This does not mean that direction of PQ is invariant, since in general RBM is present. All it means is that filtering out the RBM you get direction of PQ as invariant. Thus,

$$\varepsilon_i = \varepsilon^* n_i = \varepsilon^* \delta_{ij} n_j \longrightarrow ⑳$$

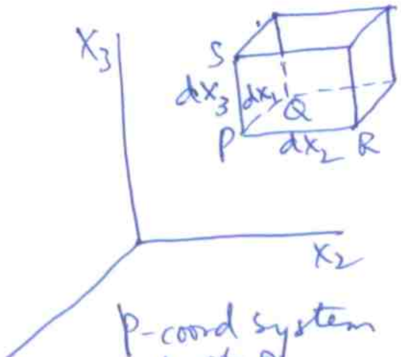
From ⑲ & ⑳,

$$(\varepsilon_{ij} - \varepsilon^* \delta_{ij}) n_j = 0 \longrightarrow \text{p-strain/axes eqn.}$$

Volumetric Strain (Cubical Dilatation).

Its defined as

$$e \ (\text{or } \bar{\Delta}) \triangleq \frac{dV^* - dV}{dV} \longrightarrow \textcircled{21}$$



p-coord system at pt P.

Consider a rect. parallelopiped & p-coord system at pt. P. Thus side stay ∥ after deformation (since they must remain mutually ⊥ar as they lie along p-axes).

Thus,

$$dV^* = \left(\overset{\left[\underset{\sim}{dx}\right|_{PQ} \text{ etc.}}{dx_1^*}\right)\left(dx_2^*\right)\left(dx_3^*\right) = (1+\epsilon_{E1})(1+\epsilon_{E2})(1+\epsilon_{E3})$$

$$\Rightarrow e\,(=\bar{\Delta}) = \epsilon_{E1} + \epsilon_{E2} + \epsilon_{E3} + \epsilon_{E1}\epsilon_{E2} + \epsilon_{E2}\epsilon_{E3} + \epsilon_{E3}\epsilon_{E1} + \epsilon_{E1}\epsilon_{E2}\epsilon_{E3}$$

$$= \epsilon_{E1} + \epsilon_{E2} + \epsilon_{E3} \quad (\underline{\text{linearized}})$$

$$= \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii} \quad \Big\}(\underline{\text{linearized}})$$

$$= \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)}$$

㉒ ← (only this for linear case)

An analogous magnification factor is defined as,

$$M_V = \frac{1}{2}\left(\left(\frac{dV^*}{dV}\right)^2 - 1\right) = e + \frac{e^2}{2} \longrightarrow \textcircled{23} \rightarrow M_V \simeq e \ (\underline{\text{linearized}})$$

$$\text{Also } M_1 = \frac{1}{2}\left(\frac{(dx_1^*)^2}{(dx_1)^2} - 1\right), \quad M_2 = \frac{1}{2}\left(\frac{(dx_2^*)^2}{(dx_2)^2} - 1\right), \quad M_3 = \frac{1}{2}\left(\frac{(dx_3^*)^2}{(dx_3)^2} - 1\right)$$

where M_1, M_2, M_3 are principal strains (i.e. nonlinear case).

$$\Rightarrow M_V = \frac{1}{2}\left[(2M_1+1)(2M_2+1)(2M_3+1) - 1\right] = 4M_1M_2M_3 + 2(M_1M_2 + M_2M_3 + M_3M_1) + M_1 + M_2 + M_3$$

$$= 4J_3 + 2J_2 + J_1 \longrightarrow \textcircled{24}$$

Conservation of mass. (Lagrangian continuity eqn)

Consider above figure but x_i are not p-axes. Then deformed differential element is skewed parallelopiped.

$$dV^* = \underset{\sim}{dx}^*_{P'Q'} \cdot \left(\underset{\sim}{dx}^*_{P'R'} \times \underset{\sim}{dx}^*_{P'S'}\right) = \text{vol of skewed. parallelopiped.}$$

(Triple product for vol of skewed ∥piped) (Since we can easily see that opp faces and sides remain parallel.)

$$\Rightarrow dV^* = \left(\frac{\partial \underset{\sim}{x}^*}{\partial x_1} dx_1\right)_{P'Q'} \cdot \left\{ \left(\frac{\partial \underset{\sim}{x}^*}{\partial x_2} dx_2\right)_{P'R'} \times \left(\frac{\partial \underset{\sim}{x}^*}{\partial x_3} dx_3\right)_{P'S'} \right\}.$$

$$= \underbrace{\left(\frac{\partial \underset{\sim}{x}^*}{\partial x_1}\right)_P \cdot \left(\frac{\partial \underset{\sim}{x}^*}{\partial x_2} \times \frac{\partial \underset{\sim}{x}^*}{\partial x_3}\right)_{P.}}_{J|_P = \det\left[\frac{\partial x_i^*}{\partial x_j}\right] = \det\begin{bmatrix} \frac{\partial x_1^*}{\partial x_1} & \frac{\partial x_1^*}{\partial x_2} & \frac{\partial x_1^*}{\partial x_3} \\ x^*_{2,1} & x^*_{2,2} & x^*_{2,3} \\ x^*_{3,1} & x^*_{3,2} & x^*_{3,3} \end{bmatrix}} dx_1\, dx_2\, dx_3$$

$dV^* = J\,dV \longrightarrow \underline{\underline{(25)}}$ (Algebraic form of Lagrangian continuity eqn).

Incompressible $\Rightarrow e$ (or $\hat{D}$) $= 0$ and $J = 1$. (ie $dV^* = dV$).

Thus, consv of mass $\Rightarrow$ $dm^* = dm$

$$\rho^* dV^* = \rho\, dV$$

$$\rho^* J|_P = \rho$$

If $\rho = \text{const} = \rho^*$ (ie incompr) $\Rightarrow$ $J = 1$.

Note that cubical dilatation (linearized) is zero for strain deviatoric tensor (ie $\hat{\varepsilon}_{ii} = 0$) by its definition. So strain deviatoric part causes no volumetric strain but only distortion shear. Similarly the spherical part causes only volumetric strain.

③ Constitutive Equations.

For 3-D anisotropic solid we have

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

From symmetry of stress and strain tensors you get C_{ijkl} comprises 36 indep coefficients (elastic constants).

Strain Displacement Relations in orthogonal Curvilinear coord.

Results (see Boresi)

$$\varepsilon_{xx} = \frac{1}{\alpha}\left[u_{,x} + \frac{v}{\beta}\alpha_{,y} + \frac{w}{\gamma}\alpha_{,z}\right]$$

$$\varepsilon_{yy} = \frac{1}{\beta}\left[v_{,y} + \frac{w}{\gamma}\beta_{,z} + \frac{u}{\alpha}\beta_{,x}\right]$$

$$\varepsilon_{zz} = \frac{1}{\gamma}\left[w_{,z} + \frac{u}{\alpha}\gamma_{,x} + \frac{v}{\beta}\gamma_{,y}\right]$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{1}{\beta}u_{,y} + \frac{1}{\alpha}v_{,x} - \frac{v}{\alpha\beta}\beta_{,x} - \frac{u}{\alpha\beta}\alpha_{,y}\right]$$

$$\varepsilon_{xz} = \frac{1}{2}\left[\frac{1}{\gamma}u_{,z} + \frac{1}{\alpha}w_{,x} - \frac{w}{\alpha\gamma}\gamma_{,x} - \frac{u}{\alpha\gamma}\alpha_{,z}\right]$$

$$\varepsilon_{yz} = \frac{1}{2}\left[\frac{1}{\gamma}v_{,z} + \frac{1}{\beta}w_{,y} - \frac{w}{\beta\gamma}\gamma_{,y} - \frac{v}{\beta\gamma}\beta_{,z}\right]$$

where u, v, w are projections of displ vector onto the x, y, z coordinates at a point

④ Compatibility Equations. (Linear strain theory).

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \longrightarrow \underline{\underline{①}}$$

If strains are given (or determined first) then to get displacements we need to solve 6 strain-displ equations to obtain the 3 displ components. Thus to get a unique solution, strains (chosen or determined) must satisfy compatibility equations. Another way of looking at it is that u_i should be single-valued, continuous, and unique (ie a particle before def cannot occupy two pts after def – no voids or tearing, and two particles before def cannot coalesce after def). So ε_{ij} are not all indep and cannot be chosen arbitrarily.

Geometric explanation: medium composed of small cubes. Assign arbitrary strains. After def there should be no gaps or overlaps, ie deformed cubes must fit together.

Derivation: Consider two pts denoted (1), (2) in the medium. Then

$$u_i(2) = u_i(1) + \int_c du_i$$

where C is any path from 1 to 2 within a simply connected domain. Thus

$$u_i(2) = u_i(1) + \int_c \left(\varepsilon_{ij} + \tilde{\omega}_{ij}\right) dx_j = u_i(1) + \int_c \varepsilon_{ij}\, dx_j + \int_c \tilde{\omega}_{ij}\, d\left[x_j - x_j(2)\right]$$

Int by parts,

$$u_i(2) = u_i(1) + \int_c \varepsilon_{ij}\, dx_j - \tilde{\omega}_{ij}(1)\left[x_j(1) - x_j(2)\right] - \int_c \left(x_j - x_j(2)\right) \underbrace{\tilde{\omega}_{ij,k}\, dx_k}_{``d\tilde{\omega}_{ij}}$$

$$\text{Now } \tilde{\omega}_{ij,k} = \frac{1}{2}\left(u_{i,jk} - u_{j,ik} + u_{k,ij} - u_{k,ij}\right) = \left(\varepsilon_{ik,j} - \varepsilon_{jk,i}\right)$$

$$\Rightarrow u_i(2) = u_i(1) - \tilde{\omega}_{ij}(1)\left[x_j(1) - x_j(2)\right] + \int_c F_{ik}\, dx_k$$

$$\text{where } F_{ik} \triangleq \varepsilon_{ik} - \left(x_j - x_j(2)\right)\left(\varepsilon_{ik,j} - \varepsilon_{jk,i}\right)$$

So for $u_i(2)$ to be single valued, $F_{ik}dx_k$ must be a perfect differential, i.e,

$$F_{ik}dx_k = dG_i = G_{i,k}dx_k$$

$$\Rightarrow F_{ik} = G_{i,k} \Rightarrow \boxed{F_{ik,l} = F_{il,k}.}$$

$$\Rightarrow \cancel{\varepsilon_{ik,l}} - (x_j - x_j(2))(\varepsilon_{ik,jl} - \varepsilon_{jk,il}) - (\cancel{\varepsilon_{ik,l}} - \cancel{\varepsilon_{lk,i}})$$

$$= \cancel{\varepsilon_{il,k}} - (x_j - x_j(2))(\varepsilon_{il,jk} - \varepsilon_{jl,ik}) - (\cancel{\varepsilon_{il,k}} - \cancel{\varepsilon_{kl,i}})$$

$$\Rightarrow \varepsilon_{ik,jl} - \varepsilon_{jk,il} + \varepsilon_{jl,ik} - \varepsilon_{il,jk} = 0$$

($\because x_j - x_j(2)$ is arbitrary).

$$k \rightleftarrows j \Rightarrow \varepsilon_{ij,kl} - \varepsilon_{jk,il} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} = 0$$

$$i \rightleftarrows j \text{ or } k \rightleftarrows l \Rightarrow \boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{jl,ik} + \varepsilon_{ik,jl}} \longrightarrow ②$$

Only 6 of these 3^4 eqns are independent. The list is

	1st eq	2nd	3rd	4th	5th	6th
i	1	3	2	1	2	3
j	1	3	2	1	2	3
k	2	1	3	2	3	1
l	2	1	3	3	1	2

Writing in full form in classical notation, you get,

$$\boxed{\begin{array}{ll} \varepsilon_{yy,xx} + \varepsilon_{xx,yy} = 2\varepsilon_{xy,xy} & \text{and other two } (x\to y, y\to z, z\to x) \\ \varepsilon_{zz,xy} + \varepsilon_{xy,zz} = \varepsilon_{yz,zx} + \varepsilon_{zx,yz} & \text{and other two } (x\to y, y\to z, z\to x) \end{array}}$$

<u>Compat eqns in terms of stresses</u>. (Isotropic case)

We had Const law: $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{mm}$ $\left(\Rightarrow \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk}\right)$. $\longrightarrow ③$

From ② & ③,

$$(1+\nu)\left[\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik}\right] = \nu\left[\delta_{ij}\sigma_{mm,kl} + \delta_{kl}\sigma_{mm,ij} - \delta_{jl}\sigma_{mm,ik} - \delta_{ik}\sigma_{mm,jl}\right]$$

So from above you can use same table for indep eqns and obtain them (ie 6 indep eqns). However an easier and compact way is as follows:

Perform contraction $k=l$, to get,

$$(1+\nu)\left[\sigma_{ij,kk}+\sigma_{kk,ij}-\sigma_{ik,jk}-\sigma_{jk,ik}\right]=\nu\left[\delta_{ij}\sigma_{mm,kk}.+3\sigma_{mm,ij}-\delta_{jk}\sigma_{mm,ik}-\delta_{ik}\sigma_{mm,jk}\right]$$

Subst equil eqn $\sigma_{ij,j}+B_i=0$ in above, to get,

$$(1+\nu)\left[\sigma_{ij,kk}+\sigma_{kk,ij}+\tilde{B}_{i,j}+\tilde{B}_{j,i}\right]=\nu\left[\delta_{ij}\sigma_{mm,kk}+\sigma_{mm,ij}\right]$$

From here itself, due to symmetry of σ_{ij}, we see that there are only 6 indep eqns. However, going a step further, contracting $i=j$, solving for $\sigma_{mm,kk}$, you get,

$$\sigma_{mm,kk}=-\frac{1+\nu}{1-\nu}\tilde{B}_{i,i}$$

Subst this in above, yields,

$$\boxed{\sigma_{ij,kk}+\frac{1}{1+\nu}\sigma_{kk,ij}+\tilde{B}_{i,j}+\tilde{B}_{j,i}+\frac{\nu}{1-\nu}\delta_{ij}\tilde{B}_{k,k}=0}$$

↳ B.M. compat eqns.

PLANE STRESS.

(eg) Thin plate subject to thicknesswise uniformly distributed in-plane loading along edges. Further $\bar{B}_3 = 0$ assumed. Thus $\sigma_{i3} = 0$ on top & bottom face, & due to thinness we assume $\sigma_{i3} = 0$ throuout and $\frac{\partial}{\partial x_3} = 0$.

Equil : $\sigma_{11,1} + \sigma_{12,2} + \bar{B}_1 = 0 \rightarrow$ Ⓘ

$\sigma_{12,1} + \sigma_{22,2} + \bar{B}_2 = 0 \rightarrow$ Ⓘⓘ (3rd equil is $0=0$).

BM compat :

① ← $i=1, j=1$: $\nabla^2\sigma_{11} + \frac{1}{1+\nu}(\sigma_{11,11} + \sigma_{22,11}) + \bar{B}_{1,1} + \bar{B}_{1,1} + \frac{\nu}{1-\nu}(\bar{B}_{1,1} + \bar{B}_{2,2}) = 0$

② ← $i=2, j=2$: $\nabla^2\sigma_{22} + \frac{1}{1+\nu}(\sigma_{11,22} + \sigma_{22,22}) + \bar{B}_{2,2} + \bar{B}_{2,2} + \frac{\nu}{1-\nu}(\bar{B}_{1,1} + \bar{B}_{2,2}) = 0$

③ ← $i=1, j=2$: $\nabla^2\sigma_{12} + \frac{1}{1+\nu}(\sigma_{11,12} + \sigma_{22,12}) + \bar{B}_{1,2} + \bar{B}_{2,1} = 0$

④ ← $i=3, j=3$: $\frac{\nu}{1-\nu}(\bar{B}_{1,1} + \bar{B}_{2,2}) = 0$

⑤,⑥ ← $i=1, j=3$ & $i=2, j=3$: $0=0$

X From $\frac{\partial Ⓘ}{\partial x_2} + \frac{\partial Ⓘⓘ}{\partial x_1} = 0$ we infer that ③ is violated unless $\nu = 0$ (even if $\bar{B}_1 = \bar{B}_2 = \text{const}$)

Also ④ is violated unless $\bar{B}_1 = \bar{B}_2 = \text{const}$ assumption.

Due to $\partial/\partial z = 0$

① + ② $\Rightarrow$ $\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{2(1+\nu)}{(2+\nu)(1-\nu)}\left(\cancelto{\nabla\cdot\bar{B}}{\bar{B}_{1,1} + \bar{B}_{2,2}}\right) \longrightarrow$ ⑦

Strain compat : (eqs numbered according to table on p.19).

$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \rightarrow$ ①'

$\varepsilon_{33,11} = 0 \rightarrow$ ②'

$\varepsilon_{33,22} = 0 \rightarrow$ ③'

$0=0 \rightarrow$ ④'

$0=0 \rightarrow$ ⑤'

$\varepsilon_{33,12} = 0 \rightarrow$ ⑥'

where $\frac{\partial}{\partial x_3}$ and $\varepsilon_{13} = \varepsilon_{23} = 0$ have been used.

Const law is,

$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\sigma_{22}$, $\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}\sigma_{22}$, $\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$,

$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$

Insert const law in ①' :

$$① ' \rightarrow \frac{1}{\cancel{E}}\left\{ (\sigma_{11}-\nu\sigma_{22})_{,22} + (\sigma_{22}-\nu\sigma_{11})_{,11} \right\} = \frac{2(1+\nu)}{\cancel{E}}\sigma_{12,12} \rightarrow (A)$$

Also $$\frac{\partial Ⓘ}{\partial x_1} + \frac{\partial Ⓘⓘ}{\partial x_2} \longrightarrow \sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12} + \tilde{B}_{1,1} + \tilde{B}_{2,2} = 0 \rightarrow (B)$$

Eliminate $\sigma_{12,12}$ from (A), (B) $\longrightarrow$. $\boxed{\nabla^2(\sigma_{11}+\sigma_{22}) = -(1+\nu)\underset{\sim}{\nabla}\cdot\underset{\sim}{\tilde{B}}}$ $\longrightarrow$ ⑦'

III^{ly} Insert CL in ②', ③', ⑥' :

$$②' \rightarrow -\frac{\nu}{\cancel{E}}(\sigma_{11}+\sigma_{22})_{,11} = 0 \rightarrow (C)$$

$$③' \rightarrow -\frac{\nu}{\cancel{E}}(\sigma_{11}+\sigma_{22})_{,22} = 0 \rightarrow (D)$$

$$⑥' \rightarrow -\frac{\nu}{\cancel{E}}(\sigma_{11}+\sigma_{22})_{,12} = 0 \rightarrow (E)$$

-X- For (C), (D), (E) to be satisfied $\rightarrow$ $\underset{= -\frac{\nu}{E}\varepsilon_{33}}{(\sigma_{11}+\sigma_{22})} = ax_1 + bx_2 + c$

This places a severe restriction on σ_{11}, σ_{22} (hence ε_{33}), and hence the remaining strain compat eqns (ie, ②', ③', ⑥') are not satisfiable in general. Again this arises due to $\frac{\partial}{\partial x_3} = 0$ assumption.

Note that ⑦ & ⑦' defer by a factor. This is due to fact that BM compat & strain compat do not have one-to-one correspondence, and since some equations in each case are non satisfiable, we cannot expect the remaining eqn (which we intend to satisfy) to be identical by the two approaches. The restrictions placed on the solution (or problem data) when attempting to satisfy ③, ④ (ie $\nu=0$ or $\tilde{B}_1$ & $\tilde{B}_2$ being constant) are very severe as compared to those placed when attempting to satisfy (C), (D), (E) (ie $\sigma_{11}+\sigma_{22}$ = linear f^n of x_1, x_2). So we choose strain compat approach for plane stress & attempt to satisfy ⑦' only (as in Timoshenko etc.).

PLANE STRAIN.

(eg) Long prismatic body loaded $\perp^{ar}$ to longitudinal axis by forces that don't vary along longitudinal coordinate. Assume ends restrained from longitudinal motion. Thus $u_3 = 0$ at ends, and by symmetry $u_3 = 0$ thruout (first infer $u_3 = 0$ at center and extend symmetry arguement). Further $\frac{\partial}{\partial x_3} = 0$ due to loading. (Example is a dam or retaining wall.) Thus $\varepsilon_{i3} = 0$.

Thus $\sigma_{13} = \sigma_{23} = 0$ here also and 3rd equil eqn is $0 = 0$ and other two are Ⓘ, Ⓘⓘ p. 21 (as in plane stress)

Strain compat

$②^2 - ②^6$ p. 20 identically satisfied.

$$CL: \ \varepsilon_{33} = 0 \Rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \Rightarrow \varepsilon_{11} = \frac{1-\nu^2}{E}\left(\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right)$$

$$\varepsilon_{22} = \frac{1-\nu^2}{E}\left(\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right)$$

Insert CL in $②^1$ p. 21 $\Rightarrow$

$$\frac{1-\nu^2}{E}\left[\left(\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right)_{,22} + \left(\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right)_{,11}\right] = 2\,\frac{1+\nu}{E}\,\sigma_{12,12}$$

Eliminate $\sigma_{12,12}$ using equil, ie, (B) p. 22 (ie $\frac{\partial Ⓘ}{\partial x_1} + \frac{\partial Ⓘⓘ}{\partial x_2} = 0$) $\Rightarrow$

$$\boxed{\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{1}{1-\nu}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{\tilde{B}}} \longrightarrow ①$$

Note that all other compat eqns are satisfied identically. So we expect the BM compat approach to yield same result as ① above. (see following page).

Let $\phi(x_1, x_2)$, $\psi(x_1, x_2)$ be such that

$$\sigma_{11} = \phi_{,22} - \psi, \quad \sigma_{22} = \phi_{,11}, \quad \sigma_{12} = \phi_{,12} - \psi \longrightarrow Ⓘ^*$$

$$\bar{B}_1 = \psi_{,1}, \quad \bar{B}_2 = \psi_{,2} \longrightarrow Ⓘⓘ^*$$

$Ⓘ^* \Rightarrow$ Equil eqns Ⓘ & Ⓘⓘ p. 21 identically satisfied.

$$Ⓘ^* \text{ in } ① \Rightarrow \boxed{\nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2\psi}$$

P25 BM + plane strain
xerox

Discussion on validity of plane stress solution

$\frac{\partial}{\partial x_3} = 0$ is the cause of non-satisfaction of compat eqn which causes a severe restriction to be posed on ε_{33} (hence $\sigma_{11}+\sigma_{22}$) as discussed on p. 23. So we relax this assumption and seek exact 3-D elasticity solutions for which $\sigma_{13}=\sigma_{23}=\sigma_{33}=0$ given $\bar{B}_3=0$, $\bar{B}_1$ & $\bar{B}_2$ are constant w.r.t (x_1,x_2).

$\bar{B}_1, \bar{B}_2$ const in (x,y) plane $\Rightarrow \psi[x_1,x_2] = p x_1 + q x_2 + \underset{\downarrow}{\boxed{\alpha[x_3]}}$

inconsequential. Can drop w/o loss of generality, as you can see in the following & the final soln for stresses.

Equil $\rightarrow$

$$\sigma_{11,1} + \sigma_{12,2} + \psi_{,1} = 0 \quad — \text{(1a)}$$

$$\sigma_{12,1} + \sigma_{22,2} + \psi_{,2} = 0 \quad — \text{(1b)}$$

$$0 = 0 \quad — \text{(1c)}$$

BM compat $\rightarrow$

$$(1+\nu)\nabla^2\sigma_{11} + \theta_{,11} = 0 \quad — \text{(2a)}$$

$$(1+\nu)\nabla^2\sigma_{22} + \theta_{,22} = 0 \quad — \text{(2b)}$$

$$(1+\nu)\nabla^2\cancelto{0}{\sigma_{33}} + \theta_{,33} = 0 \quad — \text{(2c)}$$

where $\theta = \sigma_{11}+\sigma_{22}+\sigma_{33}$

$$(1+\nu)\nabla^2\sigma_{12} + \theta_{,12} = 0 \quad — \text{(2f)}$$

$$(1+\nu)\nabla^2\cancelto{0}{\sigma_{13}} + \theta_{,13} = 0 \quad — \text{(2e)}$$

$$(1+\nu)\nabla^2\cancelto{0}{\sigma_{23}} + \theta_{,23} = 0 \quad — \text{(2d)}$$

$$2(c,e,d) \Rightarrow \theta_{,3} = k\ (\text{const}) \Rightarrow \theta = kx_3 + \theta_0[x_1,x_2] \longrightarrow ③$$

Satisfaction of ① $\Rightarrow$

$$\sigma_{11} = \phi_{,22} - \psi, \rightarrow \text{(4a)}$$

$$\sigma_{22} = \phi_{,11} - \psi \rightarrow \text{(4b)}$$

where $\psi = \psi[x_1,x_2,x_3]$

$$\sigma_{12} = -\phi_{,12} \rightarrow \text{(4c)}$$

$$\text{(2a)}+\text{(2b)}+\text{(2c)}=0 \Rightarrow \nabla^2\theta = 0 \rightarrow ⑤ \qquad \left(\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^3}{\partial x_3^2}\right)$$

$$\Rightarrow \nabla_1^2\theta_0 = 0 \rightarrow ⑥ \qquad \left(\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

$$\text{(4a)}+\text{(4b)} \Rightarrow \nabla_1^2\phi = \theta + 2\psi = kx_3 + \theta_0 + 2\psi \longrightarrow ⑦.$$

$$③, \text{(4a)} \text{ in } \text{(2a)} \Rightarrow (1+\nu)\nabla^2\phi_{,22} + \theta_{0,11} = 0 \longrightarrow ⑧$$

$$\text{Now } \nabla^2\phi_{,22} = (\nabla^2\phi)_{,22} = (\nabla_1^2\phi + \phi_{,33})_{,22} \underset{\substack{\uparrow \\ \text{use ⑦}}}{=} (kx_3 + \theta_0 + 2\psi + \phi_{,33})_{,22}$$

$$\Rightarrow \nabla^2\phi_{,22} = (\theta_0 + \phi_{,33})_{,22} \rightarrow ⑨$$

⑨ in ⑧ using ⑥ ⇒ $\nabla^2\phi_{,22} + \frac{\Theta_{0,11}}{1+\nu} = \left(\phi_{,33} + \frac{\nu}{1+\nu}\Theta_0\right)_{,22} = 0 \longrightarrow$ (10a)

$|||^{arly}$ ③, (4b) in (2b) & using ⑦ ⇒ $\left(\phi_{,33} + \frac{\nu}{1+\nu}\Theta_0\right)_{,11} = 0 \longrightarrow$ (10b)

③, (4c) in (2f) & using ⑦ ⇒ $\left(\phi_{,33} + \frac{\nu}{1+\nu}\Theta_0\right)_{,12} = 0 \longrightarrow$ (10c)

So ⑩ is reduced BM compat eqns (since they represent 2a, 2b, 2f with 2c, 2d, 2e satisfied vide ③).

⑩ ⇒ $\phi_{,33} + \frac{\nu}{1+\nu}\Theta_0 = a[x_3]\,x_1 + b(x_3]\,x_2 + C[x_3]$

⇒ (integrate twice wrt x_3) $\phi = A[x_3]\,x_1 + B[x_3]\,x_2 + C[x_3] - \frac{\nu}{1+\nu}\Theta_0\frac{x_3^2}{2} + \phi_1(x_1, x_2]\,x_3 + \phi_0[x_1, x_2] \longrightarrow$ ⑪

Now we can set $Ax_1 + Bx_2 + C = 0$ since it won't affect stress distribution obtained via ④ ∵ Airy stress fn non-unique upto linear terms in x_1, x_2. Further, considering loading as symmetric about midplane implies stress is symm about midplane ⇒ $\phi_1 = 0$ in ⑪ and $k = 0$ in ③

Thus, ⑪ ⇒ $\phi = -\frac{1}{2}\frac{\nu}{1+\nu}\Theta_0 x_3^2 + \phi_0 \longrightarrow$ (11a)

⑪ in ⑦ using ⑤, ⑥ ⇒ $\nabla_1^2\phi = \nabla_1^2\phi_0 = \Theta_0 + 2\psi \longrightarrow$ ⑫

$\Rightarrow \nabla_1^4\phi_0 = 0 \longrightarrow$ ⑬

Algorithm: solve ⑬ for ϕ_0 → get Θ_0 from ⑫ → ϕ from (11a)
→ $\sigma_{11}, \sigma_{22}, \sigma_{12}$ from ④

Thus stresses consists of two parts, first one due to ϕ_0 (obt from $\nabla_1^4\phi_0 = 0$) and second one varying quadratically thru thickness. So if thickness is small, $\phi \simeq \phi_0$ and $\nabla^4\phi = 0$ gives good enough approximation (ie all BM compat not satisfied but good enough approx) for thin plates with $\bar{B}_3 = 0$, $\bar{B}_1, \bar{B}_2$ linear wrt (x_1, x_2), and loading symm about $X_3 = 0$. If not thin plate then use above algorithm and expect quadratic variation thru thickness for the stresses. Note that if edge loads are symm wrt midplane, but not necessarily quadratic, the solution predicts quadratic variation in interior & at edges. This is acceptable by virtue of Saint Venant's Principle.

TWO DIMENSIONAL PROBLEMS – CARTESIAN COORDINATES.

POLYNOMIAL SOLUTIONS.

Consider thin plate, ie., plane stress w/o body forces (for convenience – w/o loss of generality).

Elementary solutions:-

(i) $\phi_1 = a_1 x + b_1 y \rightarrow \sigma_{xx} = \sigma_{yy} = 0 = \sigma_{xy} \Rightarrow \nabla^4\phi \equiv 0$.

If const b.f., e.g. gravity, $\psi = px + qy = -\rho g y$, $\sigma_{xx} = \sigma_{yy} = -\psi$.

(ii) $\phi_2 = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$, $\Rightarrow \nabla^4\phi \equiv 0$.

$$\boxed{\sigma_{xx} = c_2,\ \sigma_{yy} = a_2,\ \sigma_{xy} = -b_2} \longrightarrow$$

Constant stresses.

(iii) $\phi_3 = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3 \Rightarrow \nabla^4\phi \equiv 0$

$$\boxed{\sigma_{xx} = c_3 x + d_3 y,\ \sigma_{yy} = a_3 x + b_3 y,\ \sigma_{xy} = -b_3 x - c_3 y} \rightarrow$$

linearly varying stresses

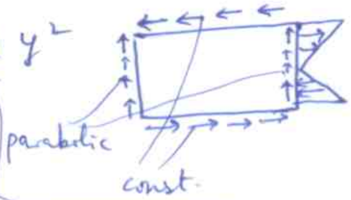
(iv) $\phi_4 = \frac{a_4}{12}x^4 + \frac{b_4}{6}x^3 y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{6}xy^3 + \frac{e_4}{12}y^4$

For $\nabla^4\phi = 0 \Rightarrow e_4 = -(2c_4 + a_4)$ ← ie, coeff of x^0y^0 is zero

$$\boxed{\begin{aligned} \sigma_{xx} &= c_4 x^2 + d_4 xy - (2c_4 + a_4)y^2 \\ \sigma_{yy} &= a_4 x^2 + b_4 xy + c_4 y^2 \\ \sigma_{xy} &= -\frac{b_4}{2}x^2 - 2c_4 xy - \frac{d_4}{2}y^2 \end{aligned}} \longleftarrow$$

quadratic variation.

(eg) only $d_4 \neq 0 \rightarrow \sigma_{xx} = d_4 xy$, $\sigma_{yy} = 0$, $\sigma_{xy} = -\frac{d_4}{2}y^2$



(v) $\phi_5 = \frac{a_5}{20}x^5 + \frac{b_5}{12}x^4 y + \frac{c_5}{6}x^3y^2 + \frac{d_5}{6}x^2y^3 + \frac{e_5}{12}xy^4 + \frac{f_5}{20}y^5$

$\nabla^4\phi = 0 \Rightarrow e_5 = -(2c_5 + 3a_5)$ ← setting x^1 coeff in $\nabla^4\phi$ to zero

$f_5 = -\frac{1}{3}(b_5 + 2d_5)$ ← " y^1 " " $\nabla^4\phi$ to zero.

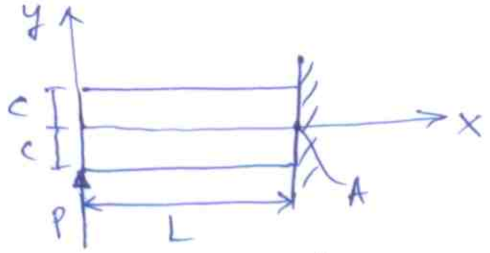
$$\Rightarrow \boxed{\begin{aligned} \sigma_{xx} &= \frac{c_5}{3}x^3 + d_5 x^2 y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3 \\ \sigma_{yy} &= a_5 x^3 + b_5 x^2 y + c_5 xy^2 + \frac{d_5}{3}y^3 \\ \sigma_{xy} &= -\frac{1}{3}b_5 x^3 - c_5 x^2 y - d_5 xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3 \end{aligned}}$$

↑ cubic variation.

Similarly for ϕ_6 and higher order polynomials.

Although $\nabla^4\phi = 0$ for plane stress as well a plane strain problems, in the absence of body forces, the displacements wont be same since constitutive law is not same, ie strains are not same (due to presence of σ_{zz} in plane strain).

(Eg) <u>Tip Loaded Cantilever.</u>



$t=1,\ t \ll L,\ t \ll c\ ,\ I = \frac{2c^3}{3}$

$$\phi = \phi_2 + \phi_3 + \phi_4$$

<u>BC's</u>

$\sigma_{xx}\big|_{x=0} = 0 \Rightarrow c_2 = 0 \longrightarrow$ ①

(ie set y^0, y^1, y^2 coeff to zero)

$d_3 = 0 \longrightarrow$ ②

$2c_4 + a_4 = 0 \longrightarrow$ ③ $\xrightarrow{w/⑥} (c_4 = 0)$

$\sigma_{yy}\big|_{y=\pm c} = 0 \Rightarrow a_2 \pm b_3 c + c_4 c^2 = 0 \rightarrow$ ④ $\xrightarrow{w/⑧} (a_2 = 0)$

(set x^0, x^1, x^2 coeff to zero)

$a_3 \pm b_4 c = 0 \longrightarrow$ ⑤ $\rightarrow (a_3 = 0,\ b_4 = 0)$

$a_4 = 0 \longrightarrow$ ⑥

$\sigma_{xy}\big|_{y=\pm c} = 0 \Rightarrow -b_2 \mp c_3 c - \frac{d_4}{2} c^2 = 0 \rightarrow$ ⑦ $\rightarrow c_3 = 0$

(set x^0, x^1, x^2 coeff to zero) $-b_3 \mp 2c_4 c = 0 \longrightarrow$ ⑧ $\rightarrow (b_3 = 0,\ c_4 = 0)$

$b_4 = 0 \longrightarrow$ ⑨

$\sigma_{xy}\big|_{x=0} = 0 \Rightarrow b_2 = 0 \longrightarrow$ ⓐ

(set coeff y^0, y^1, y^2 to zero)

$c_3 = 0 \longrightarrow$ ⓑ

$d_4 = 0 \longrightarrow$ ⓒ

} cant satisfy (see below).

ⓐ-ⓒ, ①-⑨ $\Rightarrow a_2 = c_2 = a_3 = b_3 = c_3 = d_3 = a_4 = b_4 = c_4 = 0$

So if you satisfy ⓐ, ⓒ then all coeffs vanish & $\phi = 0$

ie, cant satisfy ⓐ, ⓑ.

So satisfy $\int_{-c}^{c} \sigma_{xy}\big|_{x=0} dy = -P \Rightarrow 2\left[-b_2 c - \frac{d_4}{6} c^3\right] = -P$ ⑩

From ⑦ ($w/c_3 = 0$) and ⑩ $\rightarrow b_2 = \frac{3P}{4c}\ ,\ d_4 = -\frac{3P}{2c^3}$

$$\Rightarrow \sigma_{xx} = -\frac{3}{2}\frac{P}{c^3}xy = \frac{-P}{I}xy \; ; \sigma_{yy} = 0 \; ; \sigma_{xy} = -\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right) = -\frac{P}{I}\frac{1}{2}(c^2-y^2)$$

So solution is exact if P applied via parabolic σ_{xy} distribution — otherwise St. Venant's effect.

$$\Rightarrow \varepsilon_{xx} = u_{,x} = \frac{\sigma_{xx}}{E} = -\frac{P}{EI}xy \longrightarrow Ⓐ$$

$$\varepsilon_{yy} = v_{,y} = -\frac{\nu}{E}\sigma_{xx} = \frac{\nu P}{EI}xy \longrightarrow Ⓑ$$

$$\varepsilon_{xy} = \frac{1}{2}u_{,x} + v_{,y} = \frac{1+\nu}{E}\sigma_{xy} = -\frac{P}{4IG}(c^2-y^2) \longrightarrow Ⓒ.$$

$$Ⓐ \rightarrow u = -\frac{P}{2EI}x^2y + f(y) \longrightarrow Ⓓ$$

$$Ⓑ \rightarrow v = \frac{\nu P}{2EI}xy^2 + g(x) \longrightarrow Ⓔ$$

$$Ⓓ, Ⓔ \text{ in } Ⓒ \rightarrow g' - \frac{P}{2EI}x^2 = k_1 \text{ (const)} \longrightarrow Ⓕ$$

$$f' + \frac{\nu P}{2EI}y^2 - \frac{P}{2IG}y^2 = k_2 \text{ (const)} \longrightarrow Ⓖ$$

$$\text{where, } k_1 + k_2 = -\frac{Pc^2}{2IG} \longrightarrow Ⓗ$$

$$(D, E, F, G) \rightarrow u = -\frac{P}{2EI}x^2y - \frac{\nu P}{6EI}y^3 + \frac{P y^3}{6IG} + k_2 y + k_3 \rightarrow Ⓘ$$

$$v = \frac{\nu P}{2EI}xy^2 + \frac{Px^3}{6EI} + k_1 x + k_4 \longrightarrow Ⓙ.$$

For k_1, k_2, k_3, k_4 set RB motion to zero. Thus, for no RB translation,

$$u\Big|_{\substack{y=0\\x=L}} = v\Big|_{\substack{y=0\\x=L}} = 0 \longrightarrow k_3 = 0, \quad k_4 = -\frac{PL^3}{6EI} - k_1 L$$

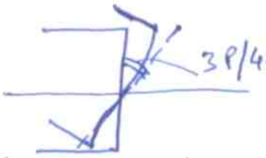
For no RB rotation, we have several possibilities, as follows:

(i) Longitudinal-axis element fixed at A.

$$\Rightarrow v_{,x}\Big|_{\substack{x=L\\y=0}} = 0 \xrightarrow{Ⓗ} k_1 = -\frac{PL^2}{2EI} \longrightarrow k_2 = \frac{PL^2}{2EI} - \frac{Pc^2}{2IG}$$

$$\Rightarrow v\Big|_{y=0} = \frac{Px^3}{6EI} - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} \longleftarrow \text{matches elementary beam theory at } x=0.$$

$$\Rightarrow u_{,y}\Big|_{\substack{y=0\\x=L}} = -\frac{3}{4}\frac{P}{cG} \longrightarrow$$



→ i.e, ε_{xy} due to σ_{xy} at $x=L$, $y=0$

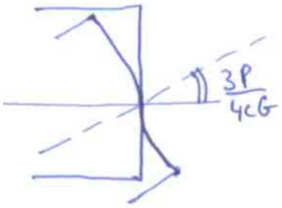
Distortion of section at x=L

(ii) Vertical-axis element fixed at A

$$\Rightarrow \quad u_{,y}\Big|_{\substack{x=L\\ y=0}} = 0 \quad \xrightarrow{Ⓗ} \quad k_2 = \frac{PL^2}{2EI} \longrightarrow k_1 = -\frac{PL^2}{2EI} - \frac{Pc^2}{2IG}$$

$$v\Big|_{\substack{y=0\\ \cancel{x=L}}} = \frac{P}{6EI}x^3 - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} + \underbrace{\frac{Pc^2}{2IG}(L-x)}_{\text{due to rotation of } \mathbb{C} \text{ ie effect of } \sigma_{xy}.}$$

$$v_{,x}\Big|_{\substack{y=0\\ x=L}} = -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{cG}$$



Distortion at x=L

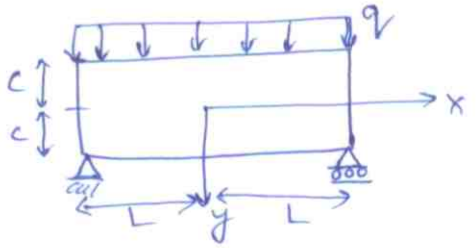
(iii) $$v_{,x} - u_{,y}\Big|_{\substack{x=L\\ y=0}} = 0 \quad \Rightarrow \quad \frac{PL^2}{2EI} + k_1 + \frac{PL^2}{2EI} - k_2 = 0$$

$$\Rightarrow (\text{using } Ⓗ) \quad \left.\begin{matrix}k_1\\ k_2\end{matrix}\right\} = \mp\frac{PL^2}{2EI} - \frac{Pc^2}{4IG}$$

$$v\Big|_{y=0} = \frac{Px^3}{6EI} - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} + \frac{Pc^2}{4IG}(L-x).$$

$$u\Big|_{x=L} = -\frac{Pc^2}{4IG}y + \frac{P}{6IG}\left(\frac{2+\nu}{2(1+\nu)}\right)y^3.$$

(Eg) Bending of beam by uniform load.



$$\phi = \phi_2 + \phi_3 + \phi_4 + \phi_5$$

$$\sigma_{xx} = c_2 + c_3 x + d_3 y + c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2 + \frac{c_5}{3} x^3 + d_5 x^2 y - (2c_5 + 3a_5) xy^2 - \frac{1}{3}(b_5 + 2d_5) y^3$$

$$\sigma_{yy} = a_2 + a_3 x + b_3 y + a_4 x^2 + b_4 xy + c_4 y^2 + a_5 x^3 + b_5 x^2 y + c_5 xy^2 + \frac{d_5}{3} y^3$$

$$\sigma_{xy} = -b_2 - b_3 x - c_3 y - \frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2 - \frac{1}{3} b_5 x^3 - c_5 x^2 y - d_5 xy^2 + \frac{1}{3}(2c_5 + 3a_5) y^3$$

BC's $\quad \sigma_{yy}\big|_{y=-c} = -q$

(set x^0, x^1, x^2, x^3 coeffs to zero) $\Rightarrow$

$$a_2 - b_3 c + c_4 c^2 - \frac{d_5}{3} c^3 = -q \rightarrow ①$$
$$a_3 - b_4 c + c_5 c^2 = 0 \rightarrow ②$$
$$a_4 - b_5 c = 0 \rightarrow ③$$
$$a_5 = 0 \rightarrow ④$$

$$\sigma_{yy}\big|_{y=c} = 0$$

(set x^0, x^1, x^2, x^3 coeffs to zero) $\Rightarrow$

$$a_2 + b_3 c + c_4 c^2 + \frac{d_5}{3} c^3 = 0 \rightarrow ⑤$$
$$a_3 + b_4 c + c_5 c^2 = 0 \rightarrow ⑥$$
$$a_4 + b_5 c = 0 \rightarrow ⑦$$

$$\int_{-c}^{c} \sigma_{xy}\big|_{x=\pm L} dy = \mp qL$$

(set $x^0 y^0$ coeff to zero, odd & even power of x terms separate out) $\Rightarrow$

$$2\left(-b_2 c - \frac{b_4}{2} L^2 c - \frac{d_4}{6} c^3\right) = 0 \rightarrow ⑧$$
$$2\left(-b_3 Lc - \frac{1}{3} b_5 L^3 c - \frac{d_5}{3} Lc^3\right) = -qL \rightarrow ⑨$$

$$\sigma_{xx}\big|_{x=\pm L} = 0$$

(set y^0, y^1, y^2, y^3 coeffs to zero) $\Rightarrow$

$$c_2 \pm c_3 L + c_4 L^2 \pm \frac{c_5}{3} L^3 = 0 \rightarrow ⑩$$

$d_3 \pm d_4 L + d_5 L^2 = 0 \longrightarrow$ ⑪

$-(2c_4 + a_4) \mp (2c_5 + 3a_5)L = 0 \longrightarrow$ ⑫

$b_5 + 2d_5 = 0 \longrightarrow$ ⑬

$\tau_{xy}\big|_{y=\pm c} = 0$

(set x^0, x^1, x^2, x^3 coeff to zero) $\Longrightarrow$ $-b_2 - \frac{d_4}{2} c^2 = 0 \longrightarrow$ ⑭

$-b_3 - d_5 c^2 = 0 \longrightarrow$ ⑮

$-\frac{b_4}{2} = 0 \longrightarrow$ ⑯

$-\frac{b_5}{3} = 0 \longrightarrow$ ⑰

$-c_3 c + \frac{1}{3}(2c_5 + 3a_5) c^3 = 0 \longrightarrow$ ⑱

$-2c_4 c = 0 \longrightarrow$ ⑲

$-c_5 c = 0. \longrightarrow$ ⑳.

solution of ① - ⑳.

④ $\rightarrow \underline{a_5 = 0}$

②, ⑥ $\rightarrow \underline{b_4 = 0}$

③, ⑦ $\rightarrow \underline{a_4 = 0, b_5 = 0}$

⑪ $\rightarrow \underline{d_4 = 0}$

④, ⑫ $\rightarrow \underline{c_5 = 0}$

②, ⑥, $c_5 = 0 \rightarrow \underline{a_3 = 0}$

⑧, $b_4 = 0, d_4 = 0 \rightarrow \underline{b_2 = 0}$

⑩, $c_5 = 0 \rightarrow \underline{c_3 = 0}$

⑬, $b_5 = 0 \rightarrow \underline{d_5 = 0}$

⑫, $a_4 = 0 \rightarrow \underline{c_4 = 0}$

⑪, $d_5 = 0 \rightarrow \underline{d_3 = 0}$

⑮, $d_5 = 0 \rightarrow \underline{b_3 = 0}.$

⑤, $b_3 = 0, c_4 = 0, d_5 = 0 \rightarrow \underline{a_2 = 0}$

⑩, $c_4 = 0 \rightarrow c_2 = 0.$

$\Longrightarrow \phi = 0.$

Trivial.

So satisfying $\sigma_{xx}\big|_{x=\pm L}=0$ yields $\phi=0$ and also that ①, ⑨ cannot be satisfied. Hence we discard $\sigma_{xx}\big|_{x=\pm L}=0$ in favor of

$$\int_{-c}^{c}\sigma_{xx}\,dy\Big|_{x=\pm L}=0$$

(coeff x^0y^0 to zero, odd & even powers of x separate out) $\Rightarrow$

$$c_2 c + c_4 L^2 c - \frac{2}{3}(c_4+a_4)c^3 = 0 \longrightarrow ㉑$$

$$c_3 L c + \frac{c_5}{3} L^3 c - (2c_5+3a_5)\frac{Lc^3}{3} = 0 \longrightarrow ㉒$$

and $\displaystyle\int_{-c}^{c}\sigma_{xx}\Big|_{x=\pm L} y\,dy = 0$

(same here) $\Rightarrow$

$$d_3\frac{c^3}{3} + d_5 L^2\frac{c^3}{3} - \frac{1}{3}(b_5+2d_5)\frac{c^5}{5} = 0 \longrightarrow ㉓$$

$$d_4 L\frac{c^3}{3} = 0 \longrightarrow ㉔$$

So discard ⑩ – ⑬, solve ①–⑨ & ⑭ – ㉔, get,

$$b_2 = c_2 = c_3 = a_3 = b_4 = c_4 = d_4 = a_4 = b_5 = c_5 = a_5 = 0$$

and,

$$⑤-① \rightarrow 2\left[b_3 c + \frac{d_5}{3}c^3\right] = q \longrightarrow (a)$$

$$⑤+①,\ c_4=0 \rightarrow 2a_2 = -q \longrightarrow (b)$$

$$⑮ \rightarrow b_3 + d_5 c^2 = 0 \longrightarrow (c)$$

$$㉓,\ b_5=0 \rightarrow d_3 + d_5 L^2 - 2d_5\frac{c^2}{5} = 0 \longrightarrow (d)$$

Solution of (a)–(d) gives,

$$a_2 = -\frac{q}{2},\quad d_5 = -\frac{3q}{4c^3},\quad b_3 = \frac{3q}{4c},\quad d_3 = \frac{3q}{4c}\left[\frac{L^2}{c^2}-\frac{2}{5}\right]$$

$\therefore$ Stresses are,

$$\sigma_{xx} = \phi_{,yy} = \frac{q}{2I}(L^2-x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3-\frac{2}{5}c^2y\right)$$

→ first term same as Euler Bernoulli
→ second term is correction due to $\sigma_y \neq 0$, it is small when $c<<L$

$$\sigma_{yy} = \phi_{,xx} = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right)$$

→ $\sigma_{yy} << \sigma_{xx}$ when $c<<L$
Note $\sigma_{yy}=0$ for Euler Bernoulli beam

$$\sigma_{xy} = -\phi_{,xy} = -\frac{q}{2I}(c^2-y^2)x$$

→ Same as in Euler Bernoulli beam

Deflections are obtained in the usual manner by integrating strain-displ eqns (see solution of HW#2, prob 13). We get for $v|_{y=0}$,

---

$$v\Big|_{y=0} = \frac{5}{24}\frac{qL^4}{EI}\left[\underbrace{1-\frac{6}{5}\left(\frac{x}{L}\right)^2+\frac{1}{5}\left(\frac{x}{L}\right)^4}_{\text{Term I}} + \underbrace{\frac{12}{5}\left(\frac{c}{L}\right)^2\left\{\left(\frac{4}{5}+\frac{\nu}{2}\right)\left(1-\left(\frac{x}{L}\right)^2\right)+\underbrace{\left(\frac{1}{64}-\frac{\nu}{480}\right)\left(\frac{h}{L}\right)^2}_{\text{Term II(a)}}\right\}}_{\text{Term II}}\right.$$

Term I → Classical Euler Bernoulli beam, ie for $\frac{c}{L} \simeq 0$

Term II → correction due to shear deformation — non-negligible if $\frac{c}{L} \cancel{\ll} 1$

Note : Timoshenko & Shames & Dym use b.c $v\Big|_{\substack{y=0\\x=\pm L}}=0$ in which case Term II(a) Vanishes. In HW#2, P.13 we have used $v\Big|_{\substack{y=-c\\x=\pm L}}=0$. which yields the additional Term II (a)

Thus Timoshenko/Shames assume beam supported at two ends at $\overset{(y=0)}{\text{℄}}$ whereas we have assumed supports at $y=-c$

Note: $v\Big|_{\substack{y=-c\\x=\pm L}}=0$ (or alternatively $v\Big|_{\substack{y=0\\x=\pm L}}=0$) prevents RB translation in y-dir and RB rot in x-y plane. Additionally, $u\Big|_{\substack{y=-c\\x=-L}}=0$ (or alternatively $u\Big|_{\substack{y=0\\x=0}}=0$) is also required to be imposed to prevent RB translation in x-dir (see HW2, P.13)

Note: can also impose, instead, $u, v, (u_{,y}-v_{,x})$ as zero at $y=-c$, $x=-L$ for zero RB transl in x, y dir & zero RB rot of infinitesimal element at left bottom.

Note: For all 3 alternatives of bc's noted above the constants of integration will be different, yielding different solutions.

Fourier series solutions.

Polynomial solutions restricted to simple loading, and they cannot handle point loads. Thus we use Fourier series which can handle more general loading, including discontinuities also.

Choose $\phi = \sin\alpha x \, f(y)$ and $\phi = \cos\alpha x \, f(y) \longrightarrow 1(a,b)$
(where α = arbitrary as yet)
which satisfy $\nabla^4 \phi = 0. \longrightarrow (2)$.

(Note: This choice is appropriate when loading applied only on faces parallel to x-axis)

Subst 1(a) in (2) or 1(b) in (2), get

$$f^{IV} - 2\alpha^2 f'' + \alpha^4 f = 0$$

which has solution

$$f = A\sinh\alpha y + B\cosh\alpha y + C\alpha y\sinh\alpha y + D\alpha y\cosh\alpha y.$$

$$\Rightarrow \phi(x,y) = \sum_{m=1}^{\infty}\Big[\sin\alpha_m x\left(A_m\sinh\alpha_m y + B_m\cosh\alpha_m y + C_m\alpha_m y\sinh\alpha_m y + D_m\alpha_m y\cosh\alpha_m y\right)$$
$$+\cos\alpha'_m x\left(A'_m\sinh\alpha'_m y + B'_m\cosh\alpha'_m y + C'_m\alpha'_m y\sinh\alpha'_m y + D'_m\alpha'_m y\cosh\alpha'_m y\right)\Big]$$

∴ Stresses are,

$$\sigma_{xx} = \sum_{m=1}^{\infty}\Big\{\alpha_m^2\sin\alpha_m x\Big[(A_m+2D_m)\sinh\alpha_m y + (B_m+2C_m)\cosh\alpha_m y + C_m\alpha_m y\sinh\alpha_m y + D_m\alpha_m y\cosh\alpha_m y\Big]$$
$$+\alpha_m'^2\cos\alpha'_m x\begin{bmatrix}A_m\to A'_m, B_m\to B'_m\\ C_m\to C'_m, D_m\to D'_m\\ \alpha_m\to\alpha'_m\end{bmatrix}$$

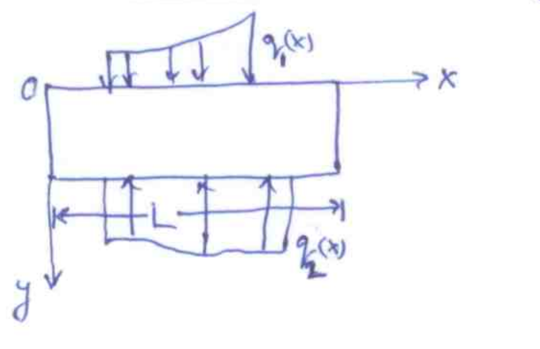
$$\sigma_{yy} = -\sum_{m=1}^{\infty}\alpha_m^2\sin\alpha_m x\Big[A_m\sinh\alpha_m y + B_m\cosh\alpha_m y + C_m\alpha_m y\sinh\alpha_m y + D_m\alpha_m y\cosh\alpha_m y\Big]$$
$$+\alpha_m'^2\cos\alpha'_m x\left[A_m\to A'_m, \text{etc}\right]$$

$$\sigma_{xy} = -\sum_{m=1}^{\infty}\alpha_m^2\cos\alpha_m x\Big[(B_m+C_m)\sinh\alpha_m y + (A_m+D_m)\cosh\alpha_m y + D_m\alpha_m y\sinh\alpha_m y + C_m\alpha_m y\cosh\alpha_m y\Big]$$
$$+\alpha_m'^2\sin\alpha'_m x\left[A_m\to A'_m \text{ etc}\right].$$

$$\longrightarrow (*)$$

Above stresses satisfy equilibrium & compatibility. BC's satisfied by choosing A_m, B_m etc.

Consider beam with arbitrary transverse loads.



BC's

$$\sigma_{xx}\big|_{x=0} = \sigma_{xx}\big|_{x=L} = 0 \longrightarrow (3)$$

$$\sigma_{xy}\big|_{y=0} = \sigma_{xy}\big|_{y=h} = 0 \longrightarrow (4)$$

$$\sigma_{yy}\big|_{y=0} = -q_1(x), \quad \sigma_{yy}\big|_{y=h} = -q_2(x) \longrightarrow (5)$$

$$(3) \rightarrow \quad A_m' = B_m' = C_m' = D_m' = 0, \quad \alpha_m = \frac{m\pi}{L} \longrightarrow (6)$$

$$(4) \rightarrow \begin{cases} \sum_{i=1}^{m} m^2 \cos\frac{m\pi x}{L}\left(A_m + D_m\right) = 0 \longrightarrow (7) \\ \sum_{i=1}^{m} m^2 \cos\frac{m\pi x}{L}\left[(B_m + C_m)\sinh\frac{m\pi h}{L} + (A_m + D_m)\cosh\frac{m\pi h}{L} + D_m\frac{m\pi h}{L}\sinh\frac{m\pi h}{L} + C_m\frac{m\pi h}{L}\cosh\frac{m\pi h}{L}\right] = 0 \rightarrow (8) \end{cases}$$

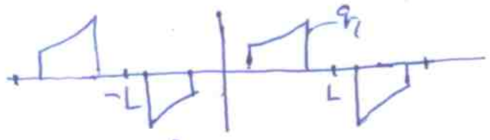
$$(5) \rightarrow \begin{cases} \frac{\pi^2}{L^2}\sum_{i=1}^{m} m^2 \sin\frac{m\pi x}{L}(B_m) = q_1(x) \longrightarrow (9) \\ \frac{\pi^2}{L^2}\sum_{i=1}^{m} m^2 \sin\frac{m\pi x}{L}\left[A_m\sinh\frac{m\pi h}{L} + B_m\cosh\frac{m\pi h}{L} + C_m\frac{m\pi h}{L}\sinh\frac{m\pi h}{L} + D_m\frac{m\pi h}{L}\cosh\frac{m\pi h}{L}\right] = q_2(x). \rightarrow (10) \end{cases}$$

In (7), (8), equating coeffs of $\cos\frac{m\pi x}{L}$ to zero (or alternatively multiply series $_\wedge^{(7),(8)}$ by $\cos\frac{k\pi x}{L}$ and do $\int_0^L (\ )dx = 0$) we get,

$$A_m + D_m = 0 \longrightarrow (11)$$

$$A_m\cosh\frac{m\pi h}{L} + B_m\sinh\frac{m\pi h}{L} + C_m\left(\sinh\frac{m\pi h}{L} + \frac{m\pi h}{L}\cosh\frac{m\pi h}{L}\right) + D_m\left(\cosh\frac{m\pi h}{L} + \frac{m\pi h}{L}\sinh\frac{m\pi h}{L}\right) = 0 \rightarrow (12)$$

Expand q_1 and q_2 in Fourier sine series (ie, consider q_1 (and q_2) to be a piece of a function that is odd & periodic), we get



$$q_1(x) = \underbrace{\left\{\sum_{m=1}^{\infty}\left[\frac{2}{L}\int_0^L q_1(x)\sin\frac{m\pi x}{L}dx\right]\sin\frac{m\pi x}{L}\right\}}, \quad q_2(x) = \{\ \} \text{ w/ } q_1 \rightarrow q_2$$

Comparing coeffs of $\sin\frac{m\pi x}{L}$ (or eqvtly multiplying $_\wedge^{(9),(10)}$ by $\sin\frac{k\pi x}{L}$ and $\int_0^L (\ ) = 0$) we get from series (9), (10),

$$B_m = \frac{2L}{m^2\pi^2}\int_0^L q_1(x)\sin\frac{m\pi x}{L}dx \longrightarrow (13)$$

$$A_m\sinh\frac{m\pi h}{L} + B_m\cosh\frac{m\pi h}{L} + C_m\frac{m\pi h}{L}\sinh\frac{m\pi h}{L} + D_m\frac{m\pi h}{L}\cosh\frac{m\pi h}{L} = \frac{2L}{m^2\pi^2}\int_0^L q_2(x)\sin\frac{m\pi x}{L}dx. \rightarrow (14)$$

Evaluate definite integrals in (13), (14) & then solve (11)–(14) for A_m – D_m, and get stress solution, then displ solution.

Note that we don't need to satisfy explicitly the conditions,

$$R_1 = \int_0^h \tau_{xy}\big|_{x=0}\,dy \quad , \quad R_2 = -\int_0^L \tau_{xy}\big|_{x=L}\,dy \longrightarrow (15)$$

where R_1 & R_2 are given from external (static) equilibrium as

$$R_1 = \frac{1}{L}\int_0^L (q_1 - q_2)(L-x)\,dx \quad , \quad R_2 = \frac{1}{L}\int_0^L (q_2 - q_1)\,x\,dx \longrightarrow (16)$$

since the τ_{xy} with the series form ⊛ satisfy $\nabla^4\phi = 0$ (ie equil & compat).

However, you can use (15) & (16) as a check to see that coeffs $A_m - D_m$ were correctly obtained. (ie, evaluate R_1, R_2 from (15), (16) and check they are the same).

For loading on all four side, the above ie 1(a,b) solution is not general. In that case we consider $\phi = \phi_x + \phi_y$

where $\phi_x = \sum \left(\cos\alpha'_m x\, f'_m(y) + \sin\alpha_m x\, f_m(y) \right)$

$\phi_y = \sum \left(\cos\beta'_m y\, g'_m(x) + \sin\beta_m x\, g_m(x) \right)$

where (') does not denote derivative but primed coeffs A_m', etc and $f'_m(y)$, $f_m(y)$, $g'_m(x)$, $g_m(x)$ have same functional form. This is the double Fourier series method.

2-D Problems in Polar Coordinates.

Transformation of equilibrium equations

$$\boxed{\begin{aligned}&\sigma_{r,r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\sigma_{r\theta,\theta}}{r} + B_r = 0\\ &\sigma_{r\theta,r} + \frac{2\sigma_{r\theta}}{r} + \frac{\sigma_{\theta,\theta}}{r} + B_\theta = 0\end{aligned}}\qquad\begin{aligned}&(\Sigma F_r = 0) \rightarrow ①\\ &(\Sigma F_\theta = 0) \rightarrow ②.\end{aligned}$$

Transformation of derivatives & Compatibility Eqn.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = c\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = s\theta, \quad \frac{\partial \theta}{\partial x} = \frac{-y}{r^2} = \frac{-s\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{c\theta}{r}$$

$$\frac{\partial}{\partial x} = c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}\right)\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= c^2\theta\frac{\partial^2}{\partial r^2} + s^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) - 2s\theta c\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \quad ③$$

For $\frac{\partial^2}{\partial y^2}$ & $\frac{\partial}{\partial y}$ do $\left\{\begin{matrix} s\theta \rightarrow -c\theta \\ c\theta \rightarrow s\theta\end{matrix}\right\}$ in $\frac{\partial^2}{\partial x^2}$, & $\frac{\partial}{\partial x}$, respectively. → ④

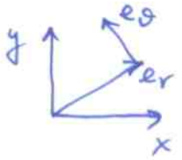
$$\frac{\partial^2}{\partial x \partial y} = -s\theta c\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial r^2}\right) + (c^2\theta - s^2\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \quad ⑤$$

Thus $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2_{xy} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} = \nabla^2_{r\theta}$

$$\nabla^4_{xy} = \nabla^2_{xy}\nabla^2_{xy} \longrightarrow \boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2\phi = 0} \rightarrow ⑥$$

COMPATIBILITY EQN.

Transformation of Stress & stress function relations.



$$\Rightarrow. \boxed{\begin{aligned}\sigma_r &= \sigma_{xx}c^2\theta + \sigma_{yy}s^2\theta + 2\sigma_{xy}s\theta c\theta\\ \sigma_\theta &= \sigma_{xx}s^2\theta + \sigma_{yy}c^2\theta - 2\sigma_{xy}s\theta c\theta\\ \sigma_{r\theta} &= -(\sigma_{xx} - \sigma_{yy})s\theta c\theta + \sigma_{xy}(c^2\theta - s^2\theta)\end{aligned}} \longrightarrow ⑦.$$

Substitute $\sigma_{xx} = \phi_{,yy}$, $\sigma_{yy} = \phi_{,xx}$, $\sigma_{xy} = -\phi_{,xy}$ in ⑦ and use ③-⑤ to transform second-partials from x-y-z to r-θ system. we get,

$$\boxed{\begin{aligned} \sigma_r &= \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} \\ \sigma_\theta &= \phi_{,rr} \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\phi_{,\theta}\right) \end{aligned}} \longrightarrow ⑧$$

(in absence of body forces).

Transformation of strain-displacement relations:

Let (u, v) & (u_r, u_θ) be Cartesian & Polar comp's of displacement, respectively.

$$\Rightarrow \left.\begin{aligned} u &= c\theta u_r - s\theta u_\theta \\ v &= s\theta u_r + c\theta u_\theta \end{aligned}\right\} \rightarrow ⑨$$

Now, using transf of strain comp's, transf. of displ. comp's, & transf. of derivatives,

$$\varepsilon_{xx} = u_{,x} \underset{\substack{\uparrow \\ \text{(used transf. of strains)}}}{=} c^2\theta\,\varepsilon_r + s^2\theta\,\varepsilon_\theta - 2\varepsilon_{r\theta}\,s\theta c\theta = c^2\theta\,u_{r,r} - c\theta s\theta\,u_{\theta,r} - \frac{s\theta}{r}\left(-s\theta\,u_r - c\theta\,u_\theta + c\theta\,u_{r,\theta} - s\theta\,u_{\theta,\theta}\right) \rightarrow \text{(i)}$$

(used transf of displ's, and derivatives.).

$$\left.\begin{aligned} s\theta &\to -c\theta \\ c\theta &\to s\theta \end{aligned}\right\} \rightarrow \varepsilon_{yy} = v_{,y} = s^2\theta\,\varepsilon_r + c^2\theta\,\varepsilon_\theta + 2\varepsilon_{r\theta}\,s\theta c\theta = s^2\theta\,u_{r,r} + s\theta c\theta\,u_{\theta,r} + \frac{c\theta}{r}\left(c\theta u_r - s\theta u_\theta + s\theta\,u_{r,\theta} + c\theta\,u_{\theta,\theta}\right) \rightarrow \text{(ii)}$$

$$\varepsilon_{xy} = \frac{1}{2}(u_{,y} + v_{,x}) = (\varepsilon_r - \varepsilon_\theta)\,s\theta c\theta + \varepsilon_{r\theta}(c^2\theta - s^2\theta) = \frac{1}{2}\left(2s\theta c\theta\left\{u_{r,r} - \frac{1}{r}u_{\theta,\theta} - \frac{u_r}{r}\right\} + [c^2\theta - s^2\theta]\left\{u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r}\right\}\right) \rightarrow \text{(iii)}$$

Solve (i)-(iii) [details on reverse] and get,

$$\boxed{\begin{aligned} \varepsilon_r &= u_{r,r} \\ \varepsilon_\theta &= \frac{u_r}{r} + \frac{1}{r}u_{\theta,\theta} \\ \varepsilon_{r\theta} &= \frac{1}{2}\left(u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r}\right) \end{aligned}} \rightarrow ⑩$$

Transformation of Constitutive Relations.

They remain same with $\varepsilon_{xx} \to \varepsilon_r$, $\sigma_{xx} \to \sigma_r$, $\varepsilon_{yy} \to \varepsilon_\theta$, $\sigma_{yy} \to \sigma_\theta$, $\varepsilon_{xy} \to \varepsilon_{r\theta}$, $\sigma_{xy} \to \sigma_{r\theta}$.

$$\boxed{\begin{aligned} \varepsilon_r &= \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \\ \varepsilon_\theta &= \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \\ \varepsilon_{r\theta} &= \frac{1+\nu}{E}\sigma_{r\theta} \\ \varepsilon_z &= -\frac{\nu}{E}(\sigma_r + \sigma_\theta) \end{aligned}} \xrightarrow[\text{(Plane stress)}]{} ⑪$$

$$\boxed{\begin{aligned} \varepsilon_r &= \frac{1-\nu^2}{E}\left[\sigma_r - \frac{\nu}{1-\nu}\sigma_\theta\right] \\ \varepsilon_\theta &= \frac{1-\nu^2}{E}\left[\sigma_\theta - \frac{\nu}{1-\nu}\sigma_r\right] \\ \varepsilon_{r\theta} &= \frac{1+\nu}{E}\sigma_{r\theta} \end{aligned}} \rightarrow ⑫$$

(Plane strain)

ie $E \to \frac{E}{1-\nu^2}$

$\nu \to \frac{\nu}{1-\nu}$

Details of solution of (i) – (iii).

Let $P = u_{r,r}$, $Q = \frac{u_{\theta,\theta}}{r} - \frac{u_r}{r}$, $R = u_{\theta,r} + \frac{u_{r,\theta}}{r} - \frac{u_\theta}{r}$

$(i)+(ii) \rightarrow \varepsilon_r + \varepsilon_\theta = P + Q \longrightarrow (iv)$

$c^2\theta * (ii) - s^2\theta * (i) \rightarrow 2c\theta s\theta\, \varepsilon_{r\theta} + (c^2\theta - s^2\theta)\,\varepsilon_\theta = c\theta s\theta\, R + (c^2\theta - s^2\theta)\,Q \rightarrow (v)$

(iii) remains same → (vi).

Now solve (iv) – (vi), ie,

$$\frac{s2\theta}{2} * (vi) - c2\theta * (v) \xrightarrow{2\theta \equiv \alpha} s^2\alpha\, \varepsilon_r - (1 + c^2\alpha)\,\varepsilon_\theta = s^2\alpha\,(P - Q) - 2c^2\alpha\, Q \quad (vii)$$

Then put (iv) in (vii) and get (10a), then from (iv) get (10 b),
and from (v) get (10 c).

Axisymmetric Problems.

Loading, hence stresses, are invariant about an axis.

$\Rightarrow \phi = \phi(r)$. $\Rightarrow$ stresses, strains indep of θ, but not so for displacements.

$$\nabla^4\phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 = \phi^{IV} + \frac{2}{r}\phi^{III} - \frac{1}{r^2}\phi'' + \frac{1}{r^3}\phi' = 0.$$

put $r = e^t$ to convert to ODE with const coeffs, ie,

details only
$$\left\{\begin{array}{l} \frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = e^{-t}\frac{d\phi}{dt}, \quad \frac{d^2\phi}{dr^2} = e^{-2t}\left[-\frac{d\phi}{dt} + \frac{d^2\phi}{dt^2}\right] \\ \frac{d^3\phi}{dr^3} = e^{-3t}\left[2\frac{d\phi}{dt} - 3\frac{d^2\phi}{dt^2} + \frac{d^3\phi}{dt^3}\right], \quad \frac{d^4\phi}{dt^4} = e^{-4t}\left[-6\frac{d\phi}{dt} + 11\frac{d^2\phi}{dt^2} - 6\frac{d^3\phi}{dt^3} + \frac{d^4\phi}{dt^4}\right] \end{array}\right.$$

$$\Rightarrow \nabla^4\phi = \frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

$$\phi = e^{\lambda t} \rightarrow \lambda^4 - 4\lambda^3 + 4\lambda^2 = 0 \rightarrow \lambda = 0, 0, 2, 2$$

$$\Rightarrow \phi = A^* + B^*t + C^*e^{2t} + D^*te^{2t} = \boxed{A\ln r + Br^2\ln r + Cr^2 + D = \phi.} \rightarrow ⑬$$

Stresses are (use ⑧)

$$\boxed{\begin{array}{l} \sigma_r = \frac{A}{r^2} + B(1 + 2\ln r) + 2C \quad \left(\frac{1}{r}\phi_{,r}\right) \\ \sigma_\theta = \phi_{,rr} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \\ \sigma_{r\theta} = 0 \end{array}} \rightarrow ⑭$$

Note: If $r=0$ is part of domain, then finite stresses $\Rightarrow A = B = 0 \Rightarrow$ stresses are constant, i.e, for plate w/o hole we have only one possible axisymm stress distribution wherein $\sigma_r = \sigma_\theta = 2C =$ const, ie representing plate under uniform tension or compression in all directions in its plane.

Displacements:

$$(14, 11^a, 10^a) \rightarrow u_{r,r} = \frac{1}{E}\left[\frac{(1+\nu)}{r^2}A + 2(1-\nu)B\ln r + (1-3\nu)B + 2(1-\nu)C\right]$$

$$\int \rightarrow u_r = \frac{1}{E}\left[-\frac{(1+\nu)}{r}A + 2(1-\nu)Br\ln r - B(1+\nu)r + 2C(1-\nu)r\right] + f(\theta)$$

$$(14, 11^b, 10^b, 2_{?}) \rightarrow u_{\theta,\theta} = \frac{4Br}{E} - f(\theta)$$

$$u_{r,r} = \varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{1}{E}\left(\frac{1}{r}\phi_{,r} - \nu\phi_{,rr}\right)$$

$$u_r = \frac{1}{E}\left(A\left[-\frac{1}{r} - \frac{\nu}{r}\right] + B\left[r + 2r(\ln r - 1) - \nu 2r\ln r - \nu r\right] + 2C\left[r - \nu r\right]\right)$$

$$= \frac{1}{E}\left(-\frac{A}{r}(1+\nu) + B(1-\nu)2r\ln r - B(1+\nu)r + 2C(1-\nu)r\right) + f(\theta)$$

$$u_{\theta,\theta} = r\varepsilon_\theta - u_r = r\frac{1}{E}(\sigma_\theta - \nu\sigma_r) - u_r$$

$$= \frac{1}{E}\left(-\frac{A}{r}(1+\nu) + B(1-\nu)2r\ln r + B(3-\nu)r + 2Cr(1-\nu)\right) - u_r$$

$$= \frac{1}{E}(4Br) - f(\theta)$$

$$\int \rightarrow \quad u_\theta = \frac{4Br\theta}{E} - \int f(\theta)\,d\theta + g(r)$$

$$(u_r, u_\theta, 10^c, 14^c, 11^c) \rightarrow \frac{1}{r} f_{,\theta} + g_{,r} + \frac{1}{r}\int f\,d\theta - \frac{1}{r} g = 0.$$

$$\Rightarrow \quad g - r g_{,r} = Z \text{ (const)}.$$

$$f_{,\theta} + \int f\,d\theta = Z$$

$$\Rightarrow f'' + f = 0 \rightarrow \boxed{f = H\sin\theta + K\cos\theta} \rightarrow Z = H\cancel{c\theta} - K\cancel{s\theta} - H\cancel{c\theta} + K\cancel{s\theta} = 0$$

$$\rightarrow r\frac{dg}{dr} = g \rightarrow \boxed{g = Gr} \; (G = \text{const})$$

$$\Rightarrow \boxed{\begin{aligned} u_r &= \frac{1}{E}\left[-\frac{(1+\nu)}{r}A + 2(1-\nu)Br\ln r - B(1+\nu)r + 2C(1-\nu)r\right] \\ &\qquad + H\sin\theta + K\cos\theta \\ u_\theta &= \frac{4Br\theta}{E} + Gr + H\cos\theta - K\sin\theta \end{aligned}} \rightarrow (15)$$

(I) Thick Walled Cylinder.

Case A Ends open, unconstrained $\rightarrow \sigma_{zz} = 0$

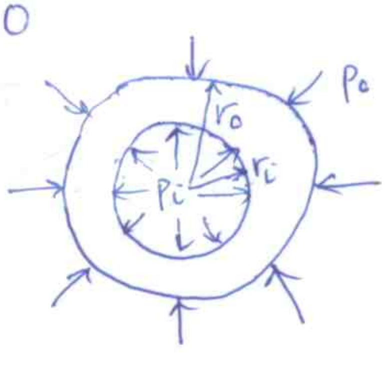
Symmetry along z-axis $\rightarrow \sigma_{\theta z} = \sigma_{rz} = 0$ $\Big\}\Rightarrow$ Plane Stress.

Axisymmetry $\rightarrow \sigma_{r\theta} = 0$ (from 14^c, or physical symmetry).

Symmetry $\rightarrow u_\theta = 0 \Rightarrow B = 0$ (also from compatibility – single valuedness). $\Rightarrow Gr = K\sin\theta - H\cos\theta = \text{const}.$

$$u_r \neq u_r[\theta] \Rightarrow K = H = 0 \rightarrow G = 0$$

Stress BC's $\rightarrow \sigma_r\big|_{r=r_o} = -p_o$, $\sigma_r\big|_{r=r_i} = -p_i$



$$(14) \rightarrow A = \left(\frac{r_i^2 r_o^2}{r_o^2 - r_i^2}\right)(p_o - p_i)$$

$$2C = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

Thus,

$$\boxed{\begin{aligned} \sigma_{r/\theta} &= \pm \frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2}\frac{1}{r^2} + \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \\ u_\theta = 0, \quad u_r &= \left(\frac{1+\nu}{E}\right)\frac{r_i^2 r_o^2 (p_i - p_o)}{(r_o^2 - r_i^2)r} + \left(\frac{1-\nu}{E}\right)\left(\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right)r \end{aligned}} \rightarrow (16)$$

indep of E, ν

- $\sigma_{r/\theta}$ indep of E, ν.
- $$\varepsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{E} = \underbrace{\frac{u_r}{r} + \frac{\cancel{u_{\theta,\theta}}}{r}}_{\text{checks out from } (15^c)}$$
- $u_\theta = 0 \not\Rightarrow \varepsilon_\theta = 0$
- $\sigma_r + \sigma_\theta = \text{const} \longrightarrow \varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = \text{const} \longrightarrow$ sections $\perp^a$ to z-axis remain plane.
- If $p_o = 0 \rightarrow \sigma_r < 0$ (compressive), $\sigma_\theta > 0$ (tensile).

$$\left.\begin{aligned} (\sigma_\theta)_{max} &= (\sigma_\theta)_{r=r_i} = p_i \frac{r_i^2 + r_o^2}{r_o^2 - r_i^2} > p_i \\ (\sigma_r)_{min} &= (\sigma_r)_{r=r_i} = -p_i \end{aligned}\right\} \text{max shear stress at } r = r_i = \frac{\sigma_\theta - \sigma_r}{2}$$

- Thin walled $\rightarrow r_o - r_i = t, \ \frac{t}{r_i} \ll 1 \rightarrow \sigma_\theta \cong \frac{p_i r_i}{t}, \ \sigma_r \simeq 0$ (use $r_o \simeq r_i \simeq r$ appropriately.)
  ($p_o = 0$)

  Same result from SOM for closed thw cyl wherein $\sigma_z = \frac{pr}{2t}$ additionally.

Case B. • Ends restrained $\Rightarrow \varepsilon_{zz} = 0$

also $\sigma_{\theta z} = \sigma_{rz} = 0$ as before from symm^x argument $\rightarrow \varepsilon_{rz} = \varepsilon_{\theta z} = 0$

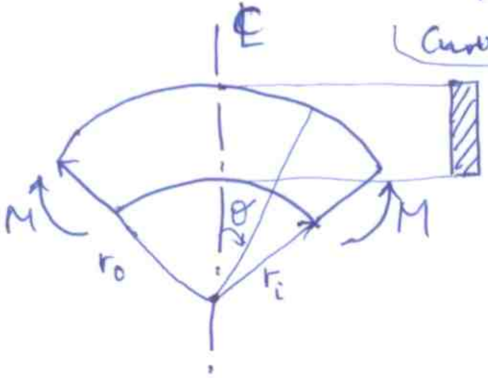
So Plane strain case.

So results are same as (15) w/ $\nu \rightarrow \frac{\nu}{1-\nu}$, $E \rightarrow \frac{E}{(1+\nu)(1-\nu)}$

$$\sigma_{zz} = \nu(\sigma_r + \sigma_\theta) = 2\nu \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} = \text{const thru } 'r'$$

Case C. • Ends capped : (see over)

(II) Pure Bending of Curved Beams.



Curved Beam with narrow rectangular section, and axis being a circular arc. Loaded by end couples M acting in plane of beam axis. Hence bending moment is same at each section. Since $M \neq M[\theta]$, problem is axisymmetric & stresses dont vary across θ.

BC's: $\sigma_r = 0$ for $r = r_i$ and $r = r_o$ $\longrightarrow$ (1)

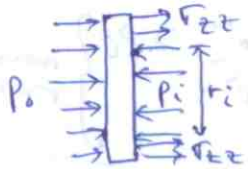
$$\int_{r_i}^{r_o} \sigma_\theta \, dr = 0 \longrightarrow (2)$$

$$\int_{r_i}^{r_o} \sigma_\theta \, r \, dr = -M \longrightarrow (3)$$

$\sigma_{r\theta} = 0$ on entire boundary. $\longrightarrow$ (4)

Case C: Ends capped. (Thick walled cylinder):

For restrained ends we got $\sigma_{zz} = \text{const}$. Thus if capped rigidly, but ends unrestrained, we still expect $\sigma_{zz} = \text{const}$. This is true for sections away from end-caps. Thus external equilibrium for end caps yields,



$$\sigma_{zz}\pi(r_o^2-r_i^2)+p_o\pi r_o^2=p_i\pi r_i^2$$

$$\sigma_{zz}=\frac{p_i r_i^2-p_o r_o^2}{r_o^2-r_i^2}$$

σ_r, σ_θ same as for previous two cases.

$$① \Rightarrow \left.\begin{aligned} &\frac{A}{r_i^2} + B(1 + 2\ln r_i) + 2C = 0 \\ &\frac{A}{r_0^2} + B(1 + 2\ln r_0) + 2C = 0 \end{aligned}\right\} \longrightarrow ⑤$$

② → (w/ ⑧ p. 40)

$$\int_{r_i}^{r_0} \sigma_\theta \, dr = \int_{r_i}^{r_0} \frac{\partial^2 \phi}{\partial r^2} dr = \left.\frac{\partial \phi}{\partial r}\right|_{r_i}^{r_0} = \left. r\sigma_r \right|_{r_i}^{r_0} = 0 \text{ if ① satisfied.}$$

ie ② satisfied identically if ① satisfied. (ie, ① ⇒ ②).

(This was expected ∵ usage of stress fn means equil. satisfied).

③ →

$$\int_{r_i}^{r_0} \sigma_\theta r \, dr = -M = \int \phi_{,rr} r \, dr = \left.\cancel{\phi_{,r} r}\right|_{r_i}^{r_0} - \left.\phi\right|_{r_i}^{r_0} = \phi[r_i] - \phi[r_0]$$

=0 from ① being satisfied

$$\overset{13, p.41}{\Longrightarrow} M = A \ln \frac{r_0}{r_i} + B(r_0^2 \ln r_0 - r_i^2 \ln r_i) + C(r_0^2 - r_i^2) \longrightarrow ⑥$$

$$5^a, 5^b, 6 \longrightarrow A = -\frac{4M}{N} r_0^2 r_i^2 \ln \frac{r_0}{r_i}, \quad B = -\frac{2M}{N}(r_0^2 - r_i^2), \quad C = \frac{M}{N}\left(r_0^2 - r_i^2 + 2\{r_0^2 \ln r_0 - r_i^2 \ln r_i\}\right)$$

$$N \triangleq (r_0^2 - r_i^2)^2 - 4 r_0^2 r_i^2 \left(\ln \frac{r_0}{r_i}\right)^2$$

(14) p. 41 →

$$\sigma_r = -\frac{4M}{N}\left(\frac{r_0^2 r_i^2}{r^2} \ln \frac{r_0}{r_i} + r_0^2 \ln \frac{r}{r_0} + r_i^2 \ln \frac{r_i}{r}\right)$$

$$\sigma_\theta = -\frac{4M}{N}\left(-\frac{r_0^2 r_i^2}{r^2} \ln \frac{r_0}{r_i} + r_0^2 \ln \frac{r}{r_0} + r_i^2 \ln \frac{r_i}{r} + b^2 - a^2\right)$$

$$\sigma_{r\theta} = 0$$

(see over for fig)

Soln valid if M applied via σ_θ as above, else St. Venant effects.

Comparing stresses with S.o.M. solution, which assumes plane sections plane & $\sigma_r = 0$, we see that stresses compare well (max, min values comparison done in Timoshenko & Goodier p. 74 table, see Popov p. 362-363 for SoM solution). The discrepancy is due to $\sigma_r = 0$ assumption in the SoM solution.

Displacements:

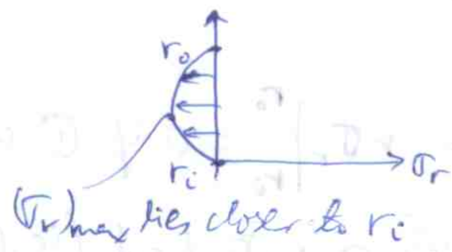
$\theta = 0$ is ℄ which is fixed due to symmetry considerations.

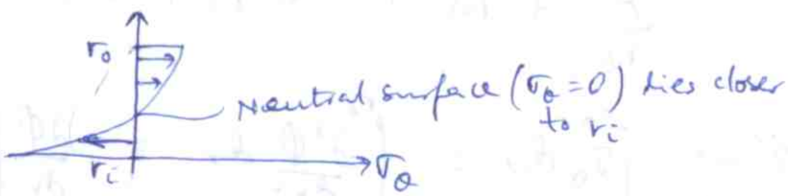
$$\Rightarrow u_r = 0, \ u_\theta = 0, \ \frac{\partial u_\theta}{\partial r} = 0 \text{ for } r = \frac{r_0 + r_i}{2} = \bar{r} \ \& \ \theta = 0$$

(15) p. 42 →

$$\left. u_r \right|_{\substack{\theta=0 \\ r=\bar{r}}} = 0 \longrightarrow K = -\frac{1}{E}\left(\frac{-(1+\nu)A}{\bar{r}} + 2(1-\nu) B \bar{r} \ln \bar{r} - B(1+\nu)\bar{r} + 2C(1-\nu)\bar{r}\right)$$

Stress distr in curved beam bending





$$u_\theta\Big|_{\substack{r=\bar{r}\\ \theta=0}}=0 \rightarrow G\bar{r}+H=0$$

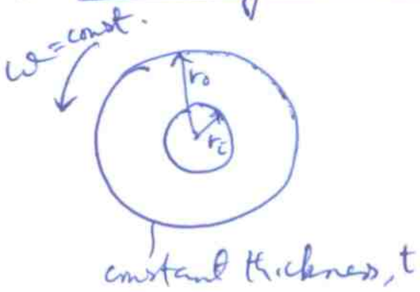
$$\frac{\partial u_\theta}{\partial r}\Big|_{\substack{r=\bar{r}\\ \theta=0}} \rightarrow G=0 \rightarrow H=0$$

$$\Rightarrow u_r=\frac{1}{E}\left[-\frac{(1+\nu)}{r}A+2(1-\nu)Br\ln r-B(1+\nu)r+2C(1-\nu)r\right]+K\cos\theta$$

function of 'r'

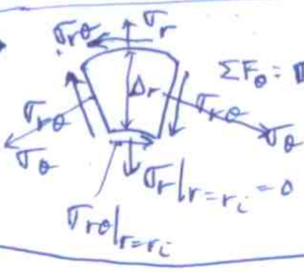
$$u_\theta=\frac{4Br\theta}{E}-K\sin\theta$$ → ie transl $(-K\sin\theta)$ + rotation thru angle $\frac{4B\theta}{E}$ → ie plane sections remain plane (so Popov assumption OK except $\sigma_r\neq 0$).

(III) Rotating disks.



constant thickness, t.

Assume thin (closed) disk, $t \ll r_i \Rightarrow \sigma_z=\sigma_{rz}=\sigma_{\theta z}=0$ ⇒ plane stress. Problem is obviously axisymmetric (∵ closed disk) $\Longrightarrow \sigma_{r\theta}=0$. (Explanation FBD of slice also shows $\sigma_{r\theta}=0$ if you start with slice at $r=r_i$ or r_o for which $\sigma_r\big|_{\substack{r=r_i\\ \text{or } r=r_o}}=0$ and $\sigma_{r\theta}\big|_{\substack{r=r_i\\ \text{or } r=r_o}}=0$)



ΣF_θ: $\sigma_{r\theta}+2\sigma_{r\theta}\frac{\Delta\theta}{2}=0$ → $\boxed{\sigma_{r\theta}=0}$

Put $B_r=\rho\omega^2 r$, $B_\theta=0$

θ equil eqn ((2) p. 39) → identically satisfied.

r equil eqn ((1) p. 39) → $\frac{d}{dr}(r\sigma_r)-\sigma_\theta+\rho\omega^2r^2=0$.

strain displ ((10) p 40) → $\varepsilon_r=\frac{du_r}{dr}$

$$\varepsilon_\theta=\frac{u_r}{r}+\frac{1}{r}\overset{=0 \text{ due to axisymm}}{\frac{du_\theta}{d\theta}}$$

stresses (plane stress constitutive law). → $\sigma_r=\frac{E}{1-\nu^2}(\varepsilon_r+\nu\varepsilon_\theta)=\frac{E}{1-\nu^2}\left(\frac{du_r}{dr}+\nu\frac{u_r}{r}\right)$

$$\sigma_\theta=\frac{E}{1-\nu^2}\left(\frac{u_r}{r}+\nu\frac{du_r}{dr}\right)$$

Put stress-displ relations in r-equil eqn (Navier formulation),

$$\rightarrow r^2\frac{d^2u_r}{dr^2}+r\frac{du_r}{dr}-u=-\frac{1-\nu^2}{E}\rho\omega^2r^3 \longrightarrow ①$$

Solve ① → $u_r=\overset{\text{put } r=e^t}{(u_r)_h}+\overset{\text{proportional to } r^3 \text{ (easily seen)}}{(u_r)_p}$.

① $\xrightarrow{r=e^t}$ $$\frac{d^2u_r}{dt^2} - u_r = -\left(\frac{1-\nu^2}{E}\right)\rho\omega^2 e^{3t}$$

$$u_r = \underbrace{C_1 e^t + C_2 e^{-t}}_{(u_r)_h} - \underbrace{\left(\frac{1-\nu^2}{E}\right)\frac{\rho\omega^2}{8} e^{3t}}_{(u_r)_p}$$

$$u_r = C_1 r + \frac{C_2}{r} - \frac{1-\nu^2}{E}\frac{\rho\omega^2}{8} r^3$$

(C.L, ie stress-displ relations) $\Rightarrow$ $$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2} - (3+\nu)\left(\frac{1-\nu^2}{8E}\right)\rho\omega^2 r^2\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2} - (1+3\nu)\left(\frac{1-\nu^2}{8}\right)\rho\omega^2 r^2\right]$$

Solid Disk ($r_i=0$)

Stress BC's : Finite σ_r, σ_θ at $r=0 \Rightarrow C_2=0$ (also get it from finite u_r at $r=0$).

$$(\sigma_r)_{r=r_0} = 0 \rightarrow C_1 = \frac{(3+\nu)(1-\nu)}{8E}\rho\omega^2 r_0^2$$

$$\rightarrow \sigma_r = \frac{(3+\nu)}{8}\rho\omega^2(r_0^2 - r^2)$$

$$\sigma_\theta = \frac{(3+\nu)}{8}\rho\omega^2 r_0^2 - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

$\rightarrow \sigma_r, \sigma_\theta$ max at $r = r_i = 0$, $(\sigma_r)_{max} = (\sigma_\theta)_{max} = \frac{3+\nu}{8}\rho\omega^2 r_0^2$

Disk with hole. ($r_i \neq 0$)

Stress BC's : $(\sigma_r)_{r=r_i} = (\sigma_r)_{r=r_0} = 0 \rightarrow$ get C_1 C_2

Stresses are $\rightarrow$ $$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(r_0^2 + r_i^2 - \frac{r_0^2 r_i^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(r_0^2 + r_i^2 + \frac{r_0^2 r_i^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2\right).$$

Results $\rightarrow$ $\frac{d(\sigma_r)}{dr} = 0 \rightarrow (\sigma_r)_{max}$ at $r = \sqrt{r_0 r_i}$, $(\sigma_r)_{max} = \frac{3+\nu}{8}\rho\omega^2(r_0 - r_i)^2$

$(\sigma_\theta)_{max}$ at $r = r_i$, $(\sigma_\theta)_{max} = \frac{3+\nu}{4}\rho\omega^2\left(r_0^2 + \frac{1-\nu}{3+\nu} r_i^2\right)$

$(\sigma_\theta)_{max} > (\sigma_r)_{max}$ (by mere observation $\because r_0^2 > (r_0 - r_i)^2$)

$r_i \rightarrow 0 \Rightarrow (\sigma_\theta)_{max} = \frac{3+\nu}{4}\rho\omega^2 r_0^2 = 2 * (\sigma_\theta)_{max}$ of solid disk.

i.e, indicates stress concentration around hole.

Displacements

u_r : put C_1, C_2 in u_r eqn or use $u_r = r\varepsilon_\theta$.

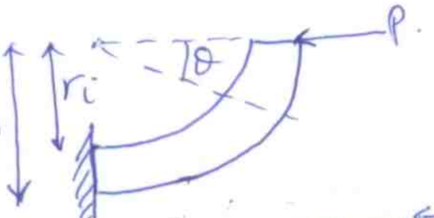
$u_r = u_r[r]$.

u_θ : $\sigma_{r\theta} = 0 \rightarrow \varepsilon_{r\theta} = 0 = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\overset{\nearrow 0}{\cancel{\frac{\partial u_r}{\partial \theta}}} - \frac{u_\theta}{r}\right)$

$\Rightarrow u_\theta = kr$, $k = $ wdt, i.e RB displ.

NON-AXISYMMETRIC PROBLEMS.

(I) Bending of curved bar with force at the end.



Circular arc with narrow rectangular cross section. Force applied in radial direction. Narrow section ⇒ plane stress. assumption.

External equilibrium ⇒ $M \propto \sin\theta$

$\Rightarrow \sigma_\theta \propto \sin\theta$

$\Rightarrow \phi_{,rr} \propto \sin\theta$

So try $\phi = f(r)\sin\theta$

$$\nabla^4\phi = 0 \rightarrow \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2}\right) = 0$$

$$f^{IV} + \frac{2}{r}f^{III} - \frac{3}{r^2}f^{II} + \frac{3}{r^3}f^{I} - \frac{3}{r^4}f = 0$$

$$r = e^t \rightarrow e^{-4t}\left[\frac{d^4 f}{dt^2} - 4\frac{d^3 f}{dt^3} + 2\frac{d^2 f}{dt^2} + 4\frac{df}{dt} - 3f\right] = 0$$

$$f = e^{st} \rightarrow s^4 - 4s^3 + 2s^2 + 4s - 3 = 0 \rightarrow s = 3, -1, +1, +1.$$

$$f(r) = Ar^3 + B\frac{1}{r} + Cr + Dr\ln r$$

stresses $\rightarrow$ $\sigma_r = \left(\frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta}\right) = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$

$\sigma_\theta = \phi_{,rr} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$

$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\phi_{,\theta}\right) = -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta$

Stress BC's : $\sigma_r = \sigma_{r\theta} = 0$ at $r = r_o$, $r = r_i$ → gives only two indep eqns, i.e, (see form of $\sigma_r, \sigma_{r\theta}$).

$$2Ar_i - \frac{2B}{r_i^3} + \frac{D}{r_i} = 0 \longrightarrow (a)$$

$$2Ar_o - \frac{2B}{r_o^3} + \frac{D}{r_o} = 0 \longrightarrow (b)$$

Also, $\displaystyle\int_{r_i}^{r_o} \sigma_{r\theta}\Big|_{\theta=0} dr = -\int_{r_i}^{r_o} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\Big|_{\theta=0} dr = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r_o}^{r_i}\Big|_{\theta=0} = P$

$$\Rightarrow -A(r_o^2 - r_i^2) + \frac{B(r_o^2 - r_i^2)}{r_i^2 r_o^2} - D\ln\frac{r_o}{r_i} = P \rightarrow (c)$$

$$(a, b, c) \rightarrow A = \frac{P}{2N}, \quad B = \frac{-P r_i^2 r_o^2}{2N}, \quad D = -\frac{P}{N}(r_i^2 + r_o^2),$$

$$\underline{N \triangleq r_i^2 - r_o^2 + (r_i^2 + r_o^2)\ln\left(\frac{r_o}{r_i}\right)}$$

Note: $\sigma_\theta\big|_{\theta=0} = 0$ identically satisfied.

Soln is exact only if P applied via $\sigma_{r\theta}$ as above, else St. Venant's effect.

Displacements

$$\varepsilon_r = u_{r,r} = \frac{\sin\theta}{E}\left[2Ar(1-3\nu) - \frac{2B}{r^3}(1+\nu) + \frac{D}{r}(1-\nu)\right] \longrightarrow \underline{(d)}$$

$$u_{\theta,\theta} = r\varepsilon_\theta - u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r) - u_r = \frac{r}{E}\sin\theta\left[Ar(6-2\nu) + \frac{2B}{r^3}(1+\nu) + \frac{D}{r}(1-\nu)\right] - u \rightarrow \underline{(e)}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right] \longrightarrow \underline{(f)}$$

$$\int(d) \longrightarrow u_r = \frac{\sin\theta}{E}\left[Ar^2(1-3\nu) + \frac{B}{r^2}(1+\nu) + D(1-\nu)\ln r\right] + f(\theta) \rightarrow \underline{\underline{(g)}}$$

$$\int(e) \rightarrow u_\theta = -\frac{\cos\theta}{E}\left[Ar^2(5+\nu) + \frac{B}{r^2}(1+\nu) - D\log r(1-\nu) + D(1-\nu)\right] - \int f\,d\theta + F(r) \longrightarrow \underline{\underline{(h)}}$$

$$(g),(h) \text{ in } (f) \rightarrow \int f\,d\theta + f' + rF' - F = -\frac{4D\cos\theta}{E}$$

$$\Rightarrow \int f\,d\theta + f' + \frac{4D\cos\theta}{E} = F - rF' = \overset{\alpha}{\cancel{\text{const}}}$$

$$f + f'' = \frac{4D\sin\theta}{E} \Rightarrow f = \underbrace{K\sin\theta + L\cos\theta}_{f_h} \underbrace{-\frac{2D}{E}\theta\cos\theta}_{f_p}$$

$$\Rightarrow \alpha = 0. \Rightarrow F = rF' \Rightarrow F = Hr.$$

Thus u_r, u_θ are obtained (they contain A, B, D, H, K, L).

Displ. BC's: $u_\theta = 0$, $\frac{\partial u_\theta}{\partial r} = 0$ at $\theta = \frac{\pi}{2}$, $\forall r$. .

gives $H = 0$, $L = \frac{D\pi}{E}$

$u_r = 0$ at $\theta = \frac{\pi}{2}$, $r = \bar{r} = \frac{r_o + r_i}{2}$ $\longrightarrow$ gives K.

$$u_r\Big|_{\theta=0} = L = \frac{D\pi}{E} = -\frac{P\pi(r_i^2+r_0^2)}{E\left[(r_i^2-r_0^2)+(r_i^2+r_0^2)\ln(r_0/r_i)\right]}$$

When $r_0 \simeq r_i$, $\ln(r_0/r_i) = \ln\left(1+\frac{h}{r_i}\right) \simeq \frac{h}{a} + O\left(\frac{h}{a}\right)^2$, where $h \triangleq r_0 - r_i$

$$\Rightarrow u_r\Big|_{\theta=0} = -\frac{3\pi a^3 P}{Eh^3}$$

- Take $\phi = f(r)\cos\theta$ to get case where vertical force P and couple are applied (ie $BM = M + P\bar{r} - P\bar{r}\cos\theta$). Then subtract solution due to pure couple (p. 43 – pure bending of curved bars) to get soln due to pure vertical load. Then do combination of soln due to vertical & horizontal load to get solution for arbitrary inclined load.

- Can also use this approach when σ_r, $\sigma_{r\theta}$ on curved faces are non-zero. However, it would apply only when loading on curved face is $(\sigma_{rr})_{\text{applied}} \propto \sin\theta$ and $(\sigma_{r\theta})_{\text{applied}} \propto \cos\theta$ ($\because$ b.m at section, M is $\propto \sin\theta$ only for such loading and we started (see p. 47) with this fact.

II Plate with circular hole.

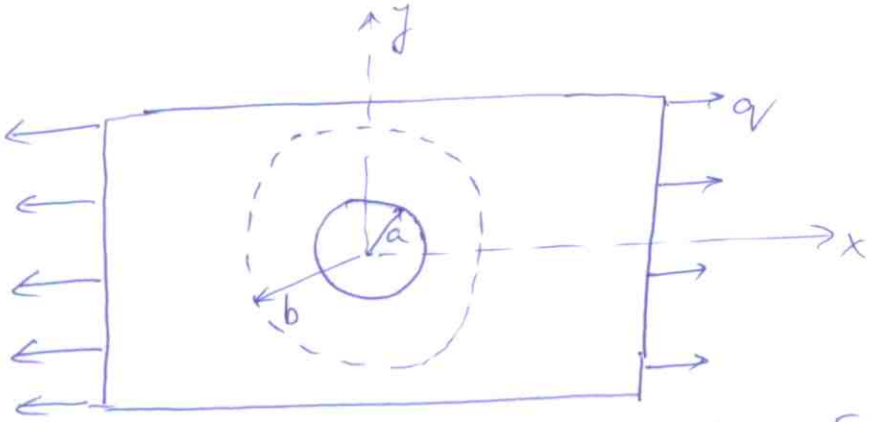


Plate uniformly loaded in x-direction. Fictitious boundary at b where $b \gg a$, a = hole radius

$\because b \gg a$, $\sigma_x\big|_{r=b} = q$, $\sigma_y\big|_{r=b} = 0$, $\sigma_{xy}\big|_{r=b} = 0$

$$\Rightarrow \sigma_r\big|_{r=b} = \frac{q}{2} + \frac{q}{2}\cos 2\theta, \quad \sigma_{r\theta}\big|_{r=b} = -\frac{q}{2}\sin 2\theta.$$

$$\sigma_\theta\big|_{r=b} = \frac{q}{2} - \frac{q}{2}\cos 2\theta$$

Equivalent problem is a annular disk loaded with $\sigma_r\big|_{r=b}$ and $\sigma_{r\theta}\big|_{r=b}$ on its outer boundary.

Part(A) : Axisymm loading, $\sigma_r\big|_{r=b} = \frac{q}{2}$ applied

$\Rightarrow$ put $p_o = -\frac{q}{2}$, $p_i = 0$ in Thick-walled Cylinder soln, and use $b \gg a$, get

$$\left.\begin{aligned} \sigma_r &= \frac{q}{2}\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_\theta &= \frac{q}{2}\left(1 + \frac{a^2}{r^2}\right) \\ \sigma_{r\theta} &= 0. \end{aligned}\right\} \text{Part(A) solution.}$$

Part(B) $\sigma_r\big|_{r=b} = \frac{q}{2}\cos 2\theta$, $\sigma_{r\theta}\big|_{r=b} = -\frac{q}{2}\sin 2\theta$ applied.

semi inverse method — try $\phi = f(r)\cos 2\theta$.

$$\nabla^4\phi = \cos 2\theta\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0.$$

$$\nabla^4\phi = 0 \longrightarrow \cos 2\theta\left[f^{IV} + \frac{2}{r}f^{III} - \frac{9}{r^2}f^{II} + \frac{9}{r^3}f^{I}\right] = 0.$$

$$r = e^t \longrightarrow \frac{d^4f}{dt^4} - 4\frac{d^3f}{dt^3} - 4\frac{d^2f}{dt^2} + 16\frac{df}{dt} = 0.$$

$$s^4 - 4s^3 - 4s^2 + 16 = 0, \quad s = 0, 2, -2, 4.$$

$$f(r) = Ar^4 + Br^2 + C + \frac{D}{r^2}$$

$$\sigma_r = -\cos 2\theta\left[2B + \frac{4C}{r^2} + \frac{6D}{r^4}\right]$$

$$\sigma_\theta = \cos 2\theta\left[12Ar^2 + 2B + \frac{6D}{r^4}\right]$$

$$\sigma_{r\theta} = \sin 2\theta\left[6Ar^2 + 2B - \frac{2C}{r^2} - \frac{6D}{r^4}\right]$$

BC's : $\sigma_r\big|_{r=a} = \sigma_{r\theta}\big|_{r=a} = 0$

and $\sigma_r\big|_{r=b} = \frac{q}{2}\cos 2\theta$, $\sigma_{r\theta}\big|_{r=b} = -\frac{q}{2}\sin 2\theta$

$$\Rightarrow 2B + \frac{4C}{b^2} + \frac{6D}{b^4} = -\frac{q}{2}$$

$$6Ab^2 + 2B - \frac{2C}{b^2} - \frac{6D}{b^4} = -\frac{q}{2}$$

$$2B + \frac{4C}{a^2} + \frac{6D}{a^4} = 0$$

$$6Aa^2 + 2B - \frac{2C}{a^2} - \frac{6D}{a^4} = 0$$

Solution (use $\frac{a}{b} \cong 0$) $\rightarrow$ $A = 0$, $B = -\frac{q}{4}$, $C = \frac{qa^2}{2}$, $D = -q\frac{a^4}{4}$

Part(A) + (B)) solution: $\sigma_r = \frac{q}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{q}{2}\cos 2\theta\left(1 - \frac{a^2}{r^2}\right)\left(1 - 3\frac{a^2}{r^2}\right)$

$\sigma_\theta = \frac{q}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{q}{2}\cos 2\theta\left(1 + 3\frac{a^4}{r^4}\right)$

$\sigma_{r\theta} = -\frac{q}{2}\sin 2\theta\left(1 - \frac{a^2}{r^2}\right)\left(1 + 3\frac{a^2}{r^2}\right)$

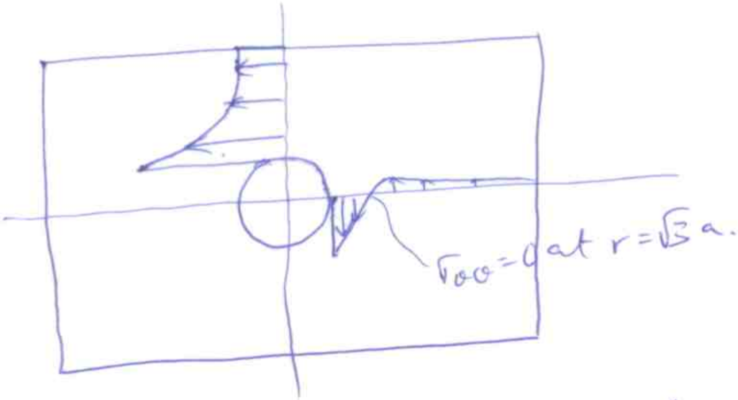
at $r = a$, $\sigma_r = 0$, $\sigma_\theta = q(1 - 2\cos 2\theta)$

$$= 3q, \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$= -q, \quad \theta = 0, \pi.$$

at $\theta = \frac{\pi}{2}$, $\sigma_\theta = q\left(1 + \frac{1}{2}\frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)$

$\theta = 0$, $\sigma_\theta = -\frac{q}{2}\frac{a^2}{r^2}\left(\frac{3a^2}{r^2} - 1\right)$



If q_2 (uniform) acts in y-direction, superpose (q_1 soln) + (q_1 soln with $q_1 \to q_2$, $\theta \to \theta - \pi/2$).

In general for arbitrary shaped plate with arbitrary inplane loads, approximate algorithm is:

(i) Find stresses for plate w/o hole.

(ii) Find stress comp's for plate w/o hole at coordinates of hole.

(iii) Find principal stresses corresponding to solution in (ii) above. Let them be q_1, q_2.

(iv) Put origin at hole center, and superpose solutions due to q_1 & q_2.

2D PROBLEMS IN CURVILINEAR COORDINATES USING COMPLEX POTENTIALS & CONFORMAL MAPPING

(I) Stress function in terms of complex potentials.

$\phi(x,y)$ = real potential function satisfying $\nabla^4\phi=0$

(i.e, body forces neglected).

$z=x+iy \rightarrow$ complex variable., $\bar{z}=x-iy$.

Thus $(x,y)\leftrightarrow(z,\bar{z})$ is transformation from real to complex independent variable

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial\phi}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial x}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right)\phi\\ \frac{\partial\phi}{\partial y} &= i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar{z}}\right)\phi\end{aligned}\right\}\longrightarrow \text{①}$$

$$\Rightarrow \frac{\partial\phi}{\partial x}+i\frac{\partial\phi}{\partial y}=2\frac{\partial\phi}{\partial\bar{z}}\ ,\quad \frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}=2\frac{\partial\phi}{\partial z}\longrightarrow \text{②}$$

$$\Rightarrow \nabla^2_{xy}\phi = 4\frac{\partial^2\phi}{\partial z\,\partial\bar{z}}\ ,\quad \nabla^4_{xy}\phi=16\frac{\partial^4\phi}{\partial z^2\partial\bar{z}^2}=0.\longrightarrow \text{③}$$

Solution to biharmonic is,

$$\phi=f_1(z)+\bar{z}f_2(z)+f_3(\bar{z})+zf_4(\bar{z})$$

∵ ϕ is real, $f_1=\bar{f}_3$ and $f_2=\bar{f}_4$.

Re-naming $f_1\equiv\frac{\chi(z)}{2}$, $f_2\equiv\frac{\psi(z)}{2}$ we obtain,

$$\boxed{\phi=\tfrac{1}{2}(\bar{z}\psi+z\bar{\psi}+\chi+\bar{\chi})=Re\left[\bar{z}\psi(z)+\chi(z)\right]}$$ Goursats formula → ④

Here ψ and χ are analytic functions, ie, they satisfy CR eqns.

Aside on CR eqns

Let $f(z)=\alpha(x,y)+i\beta(x,y)$ be analytic, ie possessing partial derivatives wrt x & y.

$$\Rightarrow \frac{df(z)}{dx}=f'(z)\ ,\quad \frac{\partial f(z)}{\partial y}=if'(z)\longrightarrow \text{⑤}$$

Also, $$\frac{\partial f(z)}{\partial x}=\left(\frac{\partial\alpha}{\partial x}+i\frac{\partial\beta}{\partial x}\right),\quad \frac{\partial f}{\partial y}=\left(\frac{\partial\alpha}{\partial y}+i\frac{\partial\beta}{\partial y}\right)\longrightarrow \text{⑥}$$

⑤ & ⑥ → $\frac{\partial\alpha}{\partial x}=+\frac{\partial\beta}{\partial y}$, $\frac{\partial\alpha}{\partial y}=-\frac{\partial\beta}{\partial x}$ → ⑦ CR eqns

⑦ → $\nabla^2\alpha = 0$, $\nabla^2\beta = 0$ → ie α, β are Harmonic

(II) Stress & displacement in terms of ψ & χ.

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

(3a, 4) → $$\boxed{\sigma_y + \sigma_x = \nabla^2\phi = 4\frac{\partial^2\phi}{\partial z\partial \bar z} = 4\,\mathrm{Re}\,\psi'(z) \longrightarrow (7a)}$$

(2b)~~(4)~~ → $$\boxed{\begin{aligned}\sigma_y - \sigma_x + 2i\sigma_{xy} &= \phi_{,xx} - \phi_{,yy} - 2i\phi_{,xy} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^2\phi \\ &= 4\frac{\partial^2\phi}{\partial z^2} = 2\left[\bar z\psi'' + \chi''\right] \longrightarrow (7b)\end{aligned}}$$

Gives Cartesian components of stress in terms of $\psi(z), \chi(z)$

Using strain-displ & CL, for plane stress we have,

$$\left.\begin{aligned} E\frac{\partial u}{\partial x} &= (\sigma_x + \sigma_y) - (1+\nu)\sigma_y = 2(\psi' + \bar\psi') - (1+\nu)\phi_{,xx} \\ E\frac{\partial v}{\partial y} &= (\sigma_x + \sigma_y) - (1+\nu)\sigma_x = 2(\psi' + \bar\psi') - (1+\nu)\phi_{,yy} \\ \frac{E}{2(1+\nu)}(v_{,x} + u_{,y}) &= \sigma_{xy}\end{aligned}\right\} (8)$$

Now $$\frac{\partial}{\partial x}(\psi \pm \bar\psi) = \psi'(z) + \bar\psi'(\bar z), \quad \frac{\partial}{\partial y}(\psi \pm \bar\psi) = i\left(\psi'(z) \mp \bar\psi'(\bar z)\right) \quad (9).$$

(9, 8a,b) → $$\left.\begin{aligned} Eu_{,x} &= 2(\psi + \bar\psi)_{,x} - (1+\nu)\phi_{,xx} \\ Ev_{,y} &= -2i(\psi - \bar\psi)_{,y} - (1+\nu)\phi_{,yy}\end{aligned}\right\} \longrightarrow (10)$$

∫(10) → $$\left.\begin{aligned} Eu &= 2(\psi + \bar\psi) - (1+\nu)\phi_{,x} + U(y) \\ Ev &= -2i(\psi - \bar\psi) - (1+\nu)\phi_{,y} + V(x)\end{aligned}\right\} \longrightarrow (11)$$

(11) in (8c) w/(9) → $-U_{,y} = V_{,x} \Rightarrow U = u_0 - \omega y$, $V = v_0 + \omega x \rightarrow (12)$

(details on reverse)

ie U, V are RB displ, so discard them since we consider only elastic displ.

(11, 2a, 4a) ⇒ $$\boxed{E(u + iv) = 4\psi - (1+\nu)(\phi_{,x} + i\phi_{,y}) = (3-\nu)\psi - (1+\nu)(z\bar\psi' + \bar\chi')} \quad (13)$$

Gives cartesian comp of displ. in terms of $\psi(z), \chi(z)$. (details on reverse)

Details of (11) in (8(c)) w/(9):

$$\frac{E}{2(1+\nu)}(u_{,y}+v_{,x}) = \frac{1}{1+\nu}\left[\cancelto{i(\psi'-\overline{\psi'})}{(\psi+\overline{\psi})_{,y}} - \cancelto{\cancel{i(\overline{\psi'}-\psi')}}{i(\psi-\overline{\psi})_{,x}}\right] \cancel{-\frac{1}{2}2\phi_{,xy}} + U_{,y} + V_{,x} \cancel{= U_{xy}}$$

$$\Rightarrow U_{,y} = -V_{,x}$$

Details of (13) (after discarding U, V)

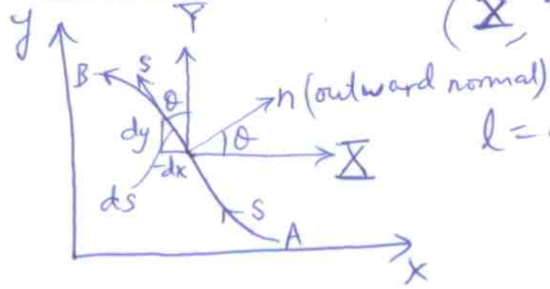
$$E(u+iv) = 4\psi - (1+\nu)[\cancelto{2\phi_{,\bar{z}}}{\phi_{,x}+i\phi_{,y}}] = 4\psi - \frac{\cancel{2}}{\cancel{2}}(1+\nu)(\psi + z\overline{\psi'} + \overline{\chi'})$$

$$= (3-\nu)\psi - (1+\nu)(z\overline{\psi'} + \overline{\chi'})$$

- (7a, 7b) → If $(iCz+\gamma_1)$ & $(\gamma_2 z+\gamma_3)$ are added to $\psi(z)$ & $\chi(z)$, respectively, where C is any real const and γ_1, γ_2, γ_3 are any complex constants, then stresses are not affected.
- (13) → If γ_4 and $\left(\frac{3-\nu}{1+\nu}\bar{\gamma}_4 z+\gamma_5\right)$ are added to ψ & χ, resply, where γ_4, γ_5 are any complx const then displ's. are not affected. Further, (7a, 7b) show that even the stresses are not affected.
- For Plane strain do $E \to E/(1-\nu^2)$, $\nu \to \nu/(1-\nu)$

(III) Boundary conditions in terms of ψ, χ.

STRESS BC'S



$(\mathbf{X}, \mathbf{Y})$ are tractions applied on boundary $\mathcal{S}$.

$$l=\cos\theta=\frac{dy}{ds}, \quad m=\sin\theta=-\frac{dx}{ds}.$$

$$\mathbf{X}=l\,\sigma_x\big|_{\mathcal{S}}+m\,\tau_{xy}\big|_{\mathcal{S}}=l\,\phi_{,yy}\big|_{\mathcal{S}}-m\,\phi_{,xy}\big|_{\mathcal{S}}$$

$$=\frac{dy}{ds}\phi_{,yy}\Big|_{\mathcal{S}}+\frac{dx}{ds}\phi_{,xy}\Big|_{\mathcal{S}}=\frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)\Big|_{\mathcal{S}}$$

$|||^{arly}$, $$\mathbf{Y}=l\,\tau_{xy}\big|_{\mathcal{S}}+m\,\sigma_y\big|_{\mathcal{S}}=-\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)\Big|_{\mathcal{S}}$$

(2) & above → $$\mathbf{X}+i\mathbf{Y}=\frac{d}{ds}(\phi_{,y}-i\phi_{,x})\Big|_{\mathcal{S}}=-2i\frac{d}{ds}\left(\frac{d\phi}{d\bar{z}}\right)\Big|_{\mathcal{S}}$$

$$=-i\frac{d}{ds}\left(\overline{\chi'}+z\overline{\psi'}+\psi\right)$$

$$\Rightarrow \left[\overline{\chi'}+z\overline{\psi'}+\psi\right]\Big|_A^B=i\int_A^B(\mathbf{X}+i\mathbf{Y})\,ds$$

Let B be generic point z. Let $\left(\overline{\chi'}+z\overline{\psi'}+\psi\right)\big|_A=k$, a complx constant. Choose additive terms for ψ & χ as γ_4 and $\frac{3-\nu}{1+\nu}\bar{\gamma}_4 z$, resply, with γ_4 chosen such that $\left(\frac{3-\nu}{1+\nu}\gamma_4+\gamma_4\right)=k$. Then the extra contribution in $\left[\overline{\chi'}+z\overline{\psi'}+\psi\right]\big|_B$ cancels out $(-k)$ arising from $-\left[\overline{\chi'}+z\overline{\psi'}+\psi\right]_A$, ie k drops out.

Thus

$$\boxed{\left(\overline{\chi'(z)} + z\,\overline{\psi'(z)} + \psi(z)\right)_S = i\int\left(\bar{X} + i\bar{Y}\right)ds} \rightarrow (14)$$

- Eqn (14) represents stress b.c's in terms of ψ, χ.
- It relates the resultant of surface forces between generic pt & datum (i-e, surface integral term) to the bracketed term containing complex potentials evaluated at a generic pt on S.

DISPLACEMENT BC'S

When $\hat{u}$ and $\hat{v}$ are applied displ's on S, they must satisfy (13), i.e,

$$\boxed{E\left(\hat{u} + i\hat{v}\right) = \left[(3-\nu)\,\psi(z) - (1+\nu)\left(z\,\overline{\psi'(z)} + \overline{\chi'}\right)\right]_S} \longrightarrow (15)$$

(IV) Conformal Mapping & Curvilinear Coordinates.

- Used for problems with irregular (e.g. non-circular) boundaries.

Consider mapping from ζ plane (non-physical) to z-plane (physical):

$$(16) \longleftarrow \boxed{z = f(\zeta)}\ ,\quad z = x + iy,\ \zeta = \xi(x,y) + i\eta(x,y)$$

- $f(\zeta)$ is analytic $\Rightarrow$ ξ & η satisfy C.R. eqns.
- $f(\zeta)$ suitably chosen to map regular boundary (eg. circle) in ζ-plane to irregular (i-e, less simple) one in z-plane (eg. ellipse). This makes the application of b.c's more manageable.

Use polar coords in ζ–plane,

$$\boxed{\zeta = \xi + i\eta = \rho e^{i\theta}} \longrightarrow (17)$$

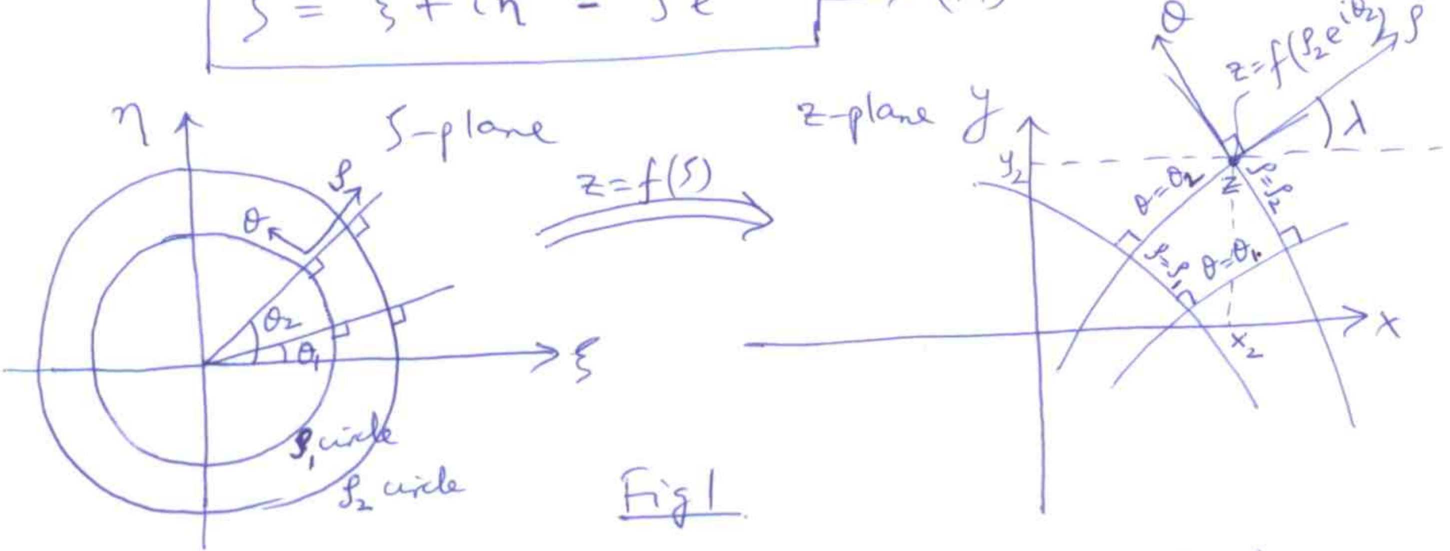


Fig 1

- ρ = const are θ coordinate curves, while θ = const are ρ coordinate curves. Hence (ρ, θ) are curvilinear coordinates.
- $\because f$ is analytic the transformation across ζ and z planes is conformal (ie, angle preserving – see eg. Kreyszig).
- (ρ_2, θ_2) are curvilinear coordinates and (x_2, y_2) are cartesian coordinates of same point in physical plane.

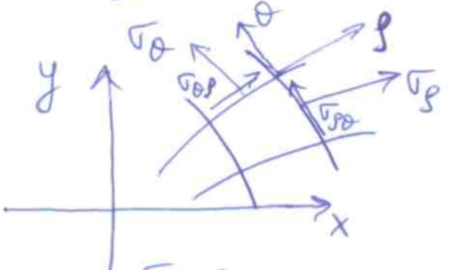


Fig 2

$$\left.\begin{aligned} \sigma_{\rho/\theta} &= \frac{\sigma_x+\sigma_y}{2} \pm \frac{\sigma_x-\sigma_y}{2}\cos 2\lambda \pm \sigma_{xy}\sin 2\lambda \\ \sigma_{\rho\theta} &= \frac{\sigma_x-\sigma_y}{2}\sin 2\lambda + \sigma_{xy}\cos 2\lambda . \end{aligned}\right\} (18)$$

$$\text{Fig 1} \Rightarrow \left.\begin{aligned} dz &= |dz| e^{i\lambda} \\ d\zeta &= |d\zeta| e^{i\theta} \end{aligned}\right\} \text{for } \theta = \text{const curve.} \longrightarrow (19)$$

$$(16, 17, 19) \longrightarrow e^{i\lambda} = \frac{\zeta}{\rho}\frac{f'(\zeta)}{|f'(\zeta)|} \Rightarrow e^{2i\lambda} = \frac{\zeta^2}{\rho^2}\frac{f'(\zeta)}{\overline{f'(\zeta)}} \longrightarrow (20)$$

$$(18, 7, 20) \longrightarrow \boxed{\begin{aligned} \sigma_\theta + \sigma_\rho &= 4 \,\mathrm{Re}\, \psi'(z) \\ \sigma_\theta - \sigma_\rho + 2i\sigma_{\rho\theta} &= 2\left[\bar{z}\psi''(z) + \chi''(z)\right]\frac{\zeta^2}{\rho^2}\frac{f'(\zeta)}{\overline{f'(\zeta)}} \end{aligned}} \rightarrow \underline{\underline{(21a)}}$$

Gives curvilinear components of stress as function of Cartesian coords + curvilinear coords

$$\left.\begin{aligned} &\text{Defining, } \psi(z) = \psi(f(\zeta)) \triangleq \tilde{\psi}(\zeta) \\ &\underbrace{\chi'(z) \triangleq \chi_1(z)}, \ \chi_1(z) = \chi_1(f(\zeta)) \triangleq \tilde{\chi}(\zeta) \\ &\psi'(z) = \tilde{\psi}'(\zeta)/f'(\zeta) \triangleq \hat{\psi}(\zeta) \\ &\chi''(z) = \chi_1'(z) = \tilde{\chi}'(\zeta)/f'(\zeta) \triangleq \hat{\chi}(\zeta) \\ &\psi''(z) = \hat{\psi}'(\zeta)/f'(\zeta) \end{aligned}\right\} (21b)$$

can introduce this from Eqn (7) onwards ∵ only $\chi'(z)$ & higher derivatives appear.

(21a,b) →

$$\boxed{\begin{aligned} \sigma_\theta + \sigma_\rho &= 2\left[\hat{\psi}(\zeta) + \overline{\hat{\psi}(\zeta)}\right] = 4\,\mathrm{Re}\left[\hat{\psi}(\zeta)\right] \\ \sigma_\theta - \sigma_\rho + 2i\,\sigma_{\rho\theta} &= \frac{2\zeta^2}{\rho^2\,\overline{f'(\zeta)}}\left[\overline{f(\zeta)}\,\hat{\psi}'(\zeta) + f'(\zeta)\,\hat{\chi}(\zeta)\right] \end{aligned}} \rightarrow (21c)$$

Gives curvilinear components of stresses of a point in physical plane as a function of its curvilinear coordinates (ie its coords in non-physical plane).

Curvilinear components of displacements are,

$$u_\rho = u\cos\lambda + v\sin\lambda \quad , \quad u_\theta = -u\sin\lambda + v\cos\lambda \rightarrow (22)$$

(22, 13, 20, 21b) →

$$\boxed{\begin{aligned} E(u_\rho + i u_\theta) &= \left[(3-\nu)\psi(z) - z\,\overline{\psi'(z)} - \overline{\chi'(z)}\right]\frac{\bar{\zeta}}{\rho}\,\frac{\overline{f'(\zeta)}}{|f'(\zeta)|} \\ E(u_\rho + i u_\theta) &= \left[(3-\nu)\tilde{\psi}(\zeta) - f(\zeta)\,\overline{\hat{\psi}(\zeta)} - \overline{\tilde{\chi}(\zeta)}\right]\frac{\bar{\zeta}}{\rho}\,\frac{\overline{f'(\zeta)}}{|f'(\zeta)|} \end{aligned}} \rightarrow (23a,b)$$

(V) Solution method for Infinite body with non-circular holes.

The method of Mushkelishvili is outlined below.

Here we map a unit circle in the ζ-plane (centered at $\zeta=0$) to the desired shaped hole in z-plane using,

$$\boxed{z = f(\zeta) = R\left(\frac{1}{\zeta} + c_0 + c_1\zeta + \cdots + c_n\zeta^n\right).} \longrightarrow (24)$$

where n is +ve integer, R = real constant, $c_0 \cdots c_n$ are cmplx consts and $|c_1| + \cdots |c_n| < 1$.

(* lengthy & non-trivial)

From conditions of finite & single-valued stress & displ. solutions, *it can be shown (ref. eg. Sokolnikoff) using (24) that $\tilde{\psi}(\zeta)$ and $\tilde{\chi}(\zeta)$ should be of the form,

see (29) for B, B^*, C^* definitions.

$$\boxed{\begin{aligned}\tilde{\psi}(\zeta) &= \frac{1+\nu}{8\pi}(X+iY)\ln\zeta + B f(\zeta) + \tilde{\psi}_0(\zeta)\\ \tilde{\chi}(\zeta) &= -\frac{3-\nu}{8\pi}(X-iY)\ln\zeta + (B^*+iC^*)f(\zeta) + \tilde{\chi}_0(\zeta)\\ \text{where } \tilde{\psi}_0(\zeta) &= \sum_{k=1}^{\infty}\alpha_k\zeta^k, \quad \tilde{\chi}_0(\zeta) = \sum_{k=1}^{\infty}\beta_k\zeta^k\end{aligned}} \rightarrow (25)$$

(25) $\Rightarrow \tilde{\psi}, \tilde{\chi}$ analytic inside unit circle $|\zeta| = 1$.

Here

$$\boxed{X = \sum_{k=1}^{m} X_k, \quad Y = \sum_{k=1}^{m} Y_k, \quad X_k = \oint \underline{X}\, ds, \quad Y_k = \oint \underline{Y}\, ds} \rightarrow (26)$$

i.e, X_k, Y_k represent total applied surface force components on k^{th} hole and X, Y are total applied surface force components (ie, summed over all holes).

The boundary conditions are assumed as stress bc's, ie (14). This transforms to (use 16, 21(b), 25)

$$\boxed{\tilde{\psi}_0(\sigma) + \frac{f(\sigma)}{\overline{f'(\sigma)}}\,\overline{\tilde{\psi}_0'(\sigma)} + \overline{\tilde{\chi}_0(\sigma)} = f_0} \rightarrow (27)$$

where $\sigma = \zeta|_{\rho=1} = 1.e^{i\theta}$ and,

$$\boxed{f_0 = i\int(\bar{X} + i\bar{Y})\,ds - \frac{X+iY}{2\pi}\ln\sigma - \frac{1+\nu}{8\pi}(X-iY)\frac{f(\sigma)}{\overline{f'(\sigma)}}\sigma - 2Bf(\sigma) - (B^* - iC^*)\overline{f(\sigma)}} \quad (28)$$

$$\boxed{B = \frac{1}{4}\left(\sigma(1) + \sigma(2)\right), \quad B^* + iC^* = -\frac{1}{2}\left(\sigma(1) - \sigma(2)\right)e^{-2i\alpha}} \quad (29)$$

- $\sigma(1)$, $\sigma(2)$ are principal stresses in far field and α is angle of $\sigma(1)$ with x-axis.
- Thus f_0 is known f^n of σ, i.e, known from the applied load (bc's) at hole and the applied far-field load and the mapping function.
- $\bar{X} = \bar{Y} = 0$, $X = Y = 0$ if hole is not loaded.
- $B = B^* = C^* = 0$ when far-field is not loaded.
  $\because$ in that case loading only the hole won't much affect the far-field which will remain almost stress free.
- (28) and onwards assumes a single hole.

Now recall Cauchy integral formulas (as applied to our unit circle):

(i) If $F(\zeta)$ is analytic inside the unit circle then

$$\boxed{\frac{1}{2\pi i}\oint_{\sigma}\frac{F(\sigma)}{\sigma-\zeta}\,d\sigma = F(\zeta)} \longrightarrow (30a)$$

(ii) If $F(\zeta)$ is analytic outside the unit circle,

$$\boxed{\frac{1}{2\pi i}\oint\frac{F(\sigma)}{\sigma-\zeta}\,d\sigma = F(\infty)} \longrightarrow (30b)$$

Analytic $\Longleftrightarrow$ $F(\zeta)$ should be defined and have a derivative at every pt. inside unit circle.

(eg) $\tilde{\Psi}_0$ (see (25)) analytic inside unit circle, while $\overline{\tilde{\Psi}}_0$ analytic outside unit circle.

Thus (27) and its conjugate yield (using 25(c,d) & 30(a,b))

$$\boxed{\begin{aligned} \tilde{\Psi}_0(\zeta) &= \frac{1}{2\pi i}\oint_{\sigma}\left(f_0 - \frac{f(\sigma)}{f'(\sigma)}\,\overline{\tilde{\Psi}'_0}(\sigma)\right)\frac{d\sigma}{\sigma-\zeta} \\ \tilde{X}_0(\zeta) &= \frac{1}{2\pi i}\oint_{\sigma}\left(\bar{f}_0 - \frac{\bar{f}(\sigma)}{f'(\sigma)}\,\tilde{\Psi}'_0(\sigma)\right)\frac{d\sigma}{\sigma-\zeta} \end{aligned}} \longrightarrow (31)$$

Algorithm:

① Choose mapping $f(\zeta)$ to map $|\zeta|=1$ to physical hole.

②. Find f_0 using (28) (also (29)).

③ Find $\tilde{\Psi}_0(\zeta)$ using (31a). Here for $\tilde{\Psi}'_0(\sigma)$ required in integral, use (25c).

④. Find $\tilde{X}_0(\zeta)$ using (31b)

⑤ Find $\tilde{\Psi}(\zeta)$, $\tilde{X}(\zeta)$ using (25a,b)

⑥ Find stresses and displacements using (21b) (21c), (23).

Application to plate with elliptic hole.

Step① Choose $z = f(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right)$

/ $R = \frac{a+b}{2}$, $m = \frac{a-b}{a+b}$, a, b are semi-axes and $m<1$ (obvious)

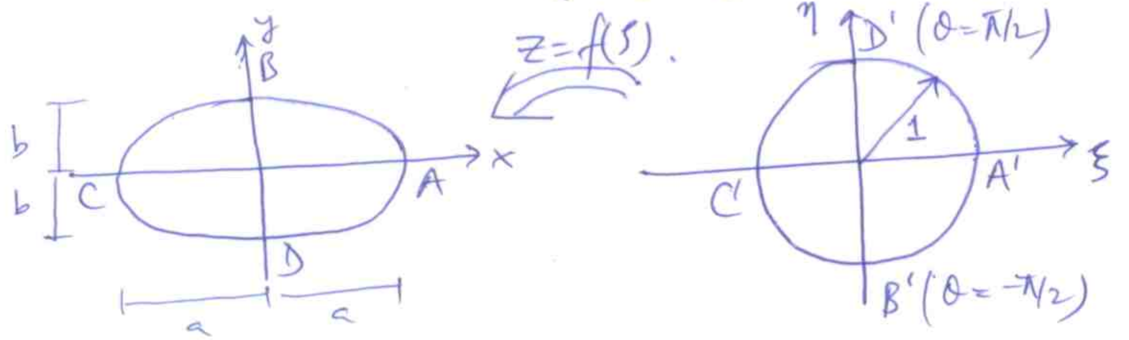
$z = x+iy$, $\zeta = \rho e^{i\theta}$ yields,

$$x = R\left(\frac{1}{\rho} + m\rho\right)\cos\theta, \quad y = -R\left(\frac{1}{\rho} - m\rho\right)\sin\theta$$

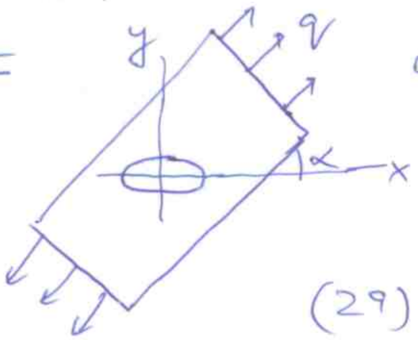
$$\Rightarrow \frac{x^2}{R^2\left(\frac{1}{\rho}+m\rho\right)^2} + \frac{y^2}{R^2\left(\frac{1}{\rho}-m\rho\right)^2} = 1 \rightarrow \text{family of ellipses for different } \rho\text{'s.}$$

$$\frac{x^2}{4R^2 m\cos^2\theta} - \frac{y^2}{4R^2 m\sin^2\theta} = 1 \rightarrow \text{family of hyperbolae for different } \theta\text{'s.}$$

Take $\rho = 1$, θ varying (ie elliptical coords).



Step 2



as Plate with elliptic hole loaded at angle α wrt major axis.

$$\sigma(1) = q, \quad \sigma(2) = \bar{X} = \bar{Y} = X = Y = 0$$

$$(29) \Rightarrow B = \frac{q}{4}, \quad B^* - iC^* = -\frac{q}{2}e^{2i\alpha} \rightarrow \underline{(a)}$$

$$(30) \Rightarrow f_0 = -\frac{qR}{2}\left(\frac{1}{\sigma} + m\sigma\right) + \frac{qR}{2}\left(\sigma + \frac{m}{\sigma}\right)e^{2i\alpha} \searrow \underline{(b)}$$

Step 3. (31a) & (25c) yield,

integral = 0.

$$\tilde{\varphi}_0(\zeta) = \frac{1}{2\pi i}\oint \frac{f_0}{\sigma-\zeta}\,d\sigma - \frac{1}{2\pi i}\oint_{\sigma} \underbrace{\left(-\frac{1}{\sigma}\,\frac{m\sigma^2+1}{\sigma^2-m}\right)\left(\bar{\alpha}_1+\frac{2\bar{\alpha}_2}{\sigma}+\frac{3\bar{\alpha}_3}{\sigma^2}+\dots\right)}\frac{d\sigma}{\sigma-\zeta}$$

corresponding $F(\zeta)$ is analytic outside unit circle so evaluate $F(\infty)$ which is zero.

$$= -\frac{qR}{2}\left[\left(1-me^{-2i\alpha}\right)\zeta+\left(m-e^{2i\alpha}\right)\frac{m+\zeta^2}{1-m\zeta^2}\zeta\right] \rightarrow \underline{\underline{(c)}}$$

Step 4. $\underline{\underline{(c)}}$ & (31b) yield $\tilde{\chi}_0(\zeta)$

Step 5 (25a, b), (c), $\tilde{\chi}_0(\zeta)$ yield

$$\underline{\underline{(d)}} \leftarrow \begin{cases} \tilde{\psi}(\zeta) = \frac{qR}{4}\left[\frac{1}{\zeta}+\left(2e^{2i\alpha}-m\right)\zeta\right] \\ \tilde{\chi}(\zeta) = -\frac{qR}{2}\left[\frac{1}{\zeta}e^{-2i\alpha}+\frac{\zeta^3e^{2i\alpha}+\left(me^{2i\alpha}-m^2-1\right)\zeta}{m\zeta^2-1}\right] \end{cases}$$

Step 6. $\underline{\underline{(d)}}$, (21b, c), yield σ_θ, σ_ρ, $\sigma_{\rho\theta}$ as a function of ζ (ie real function of ρ, θ after separating real & imaginary parts in (21c)).

At the hole boundary, ie putting $\zeta=\sigma$, get,

$$\sigma_\theta = q\left(\frac{1-m^2+2m\cos 2\alpha-2\cos 2(\theta+\alpha)}{1+m^2-2m\cos 2\theta}\right)$$

$\sigma_{\rho\theta} = \sigma_\rho = 0$ (ie bc satisfied)

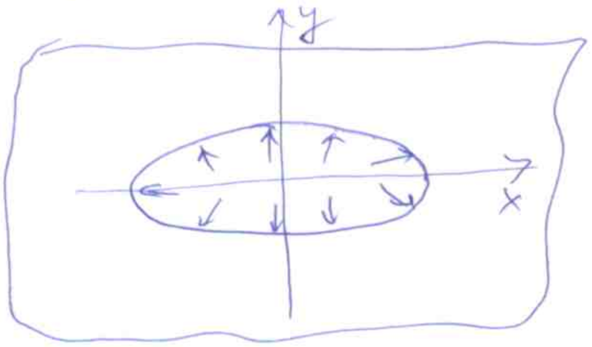
For $\underline{\alpha=0}$, $(\sigma_\theta)_{max} = q\left(1+\frac{2b}{a}\right)$, occurs at $\theta=\pm\frac{\pi}{2}$

$(\sigma_\theta)_{min} = -q$, occurs at $\theta=0,\pi$.

$\alpha=\frac{\pi}{2}$, $(\sigma_\theta)_{max} = q\left(1+\frac{2a}{b}\right)$, occurs at $\theta=0,\pi$

$(\sigma_\theta)_{min} = -q$, occurs at $\theta=0, \pi/2$

If only hole loaded with q,



$$\overline{X} + i\overline{Y} = -lq - imq$$

$$(\overline{X} + i\overline{Y})ds = -q(l\,ds + im\,ds)$$
$$= -q(dy - i\,dx)$$
$$= iq(dx + i\,dy) = iq\,dz$$

∵ resultant surface forces on hole are zero (due to symmetry), $X = Y = 0$. Also no applied far field load $\Rightarrow$ $B = B^* = C^* = 0$.

$$\Rightarrow f_0(\zeta) = i\int(\overline{X} + i\overline{Y})ds = -q\int dz = -qz = -qR\left(\frac{1}{\zeta} + m\zeta\right)$$

ie $$f(\sigma) = -qR\left(\frac{1}{\sigma} + m\sigma\right)$$

Results: $$\tilde{\varphi}_0(\zeta) = \tilde{\varphi} = -qRm\zeta$$

$$\tilde{\chi}_0(\zeta) = \tilde{\chi} = -qR(1+m^2)\frac{\zeta}{1-m\zeta^2}$$

$$\sigma_\theta = q\left(\frac{1-3m^2+2m\cos2\theta}{1+m^2-2m\cos2\theta}\right)$$

$$\sigma_\rho = -q$$

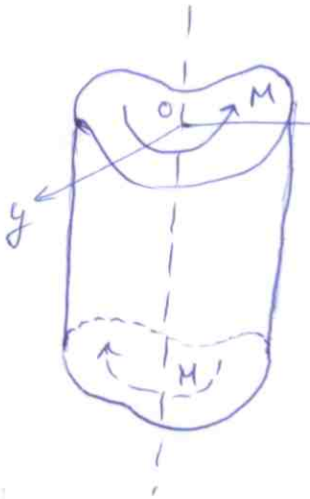
$(\sigma_\theta)_{max} = q\left(\frac{2a}{b} - 1\right)$ at $\theta = 0, \pi$

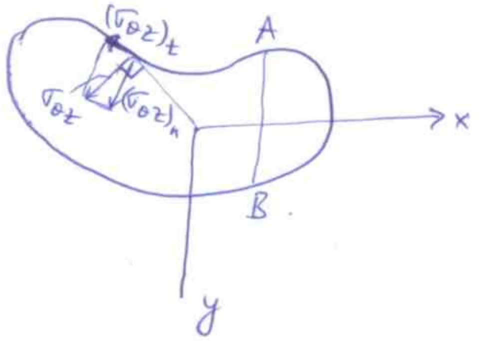
$(\sigma_\theta)_{min} = q\left(-\frac{2b}{a} - 1\right)$ at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

$$(|\sigma|)_{max} = (\sigma_\theta)_{max}$$

$$(|\sigma|)_{min} = |\sigma_\rho| = q$$

TORSION OF NONCIRCULAR SECTION MEMBERS.





Note that unlike circular sections, here you cant assume that $\sigma_{\theta z}$ is the only shear stress in the z-plane. If you did so, you would get a non-zero $(\sigma_{\theta z})_n$ (ie normal-to-boundary component of $\sigma_{\theta z}$) which remains unbalanced & hence violates bc on longitudinal surface which are stress free. You can also see it in the following manner. If only $\sigma_{\theta z}$ exists as the shear stress, then

$$\sigma_{xz} = -\frac{y}{r}\sigma_{\theta z}, \quad \sigma_{yz} = \frac{x}{r}\sigma_{\theta z}.$$

lateral face bc $\Rightarrow$ $\sigma_{xz} \underset{dy/ds}{l} + \sigma_{yz} \underset{-dx/ds}{m} = 0 \Rightarrow \frac{\sigma_{\theta z}}{r}\left(-y\,dy - x\,dx\right) = 0$

can be zero for circle only.

(I) Solution by Prandtl stress function (ϕ) formulation

Guided by Solid Mechanics, assume σ_{xz} and σ_{yz} are only non-zero stresses.

Equilibrium (zero bodyforces) $\longrightarrow$

$$\left.\begin{aligned} \sigma_{zx,z} &= 0 \\ \sigma_{zy,z} &= 0 \end{aligned}\right\} \longrightarrow \sigma_{xz}, \sigma_{yz} \text{ f}^{\text{n}} \text{ of } (x,y).$$

$$\sigma_{zx,x} + \sigma_{zy,y} = 0$$

The stress function ϕ defined by

$$\boxed{\sigma_{xz} = \phi_{,y}, \quad \sigma_{yz} = -\phi_{,x}}, \quad \phi = \phi(x,y) \longrightarrow ①$$

satisfies 3rd equil eqn.

BM compatibility eqns (see p.21, $\sigma_{ij,kk}=0$) reduce to

$$\nabla^2\sigma_{xz}=0\,,\quad \nabla^2\sigma_{yz}=0\quad,\quad \nabla^2=\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\quad\longrightarrow ②$$

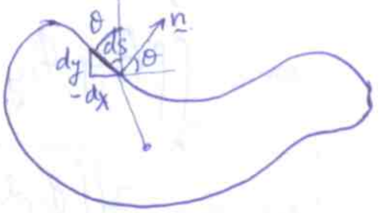
(can also get them from strain compatibility as on reverse).

$$①,②\longrightarrow\ \frac{\partial}{\partial x}(\nabla^2\phi)=0\quad,\quad \frac{\partial}{\partial y}(\nabla^2\phi)=0$$

$$\Longrightarrow \boxed{\nabla^2\phi=K\text{(const)}}\quad\xrightarrow[\text{Poisson's eqn}]{}\ ③$$

BC's on lateral face:

$$\underline{n}=(l,m,0)\quad,\quad l=\frac{dy}{ds}\,,\ m=-\frac{dx}{ds}$$



$(\sigma_{ij}n_j)_s=\sigma_i^*$ (applied tractions)

$i=1$, $i=2$ bc's are i.s. (identically satisfied)

$i=3$: $(l\,\sigma_{xz}+m\,\sigma_{yz})_s=0$

$$\Rightarrow \frac{dy}{ds}\left(\frac{\partial\phi}{\partial y}\right)_s+\frac{dx}{ds}\left(\frac{\partial\phi}{\partial x}\right)_s=\left(\frac{d\phi}{ds}\right)_s=0\ \Rightarrow \boxed{(\phi)_s=\text{const}}\longrightarrow \underline{(4a)}$$

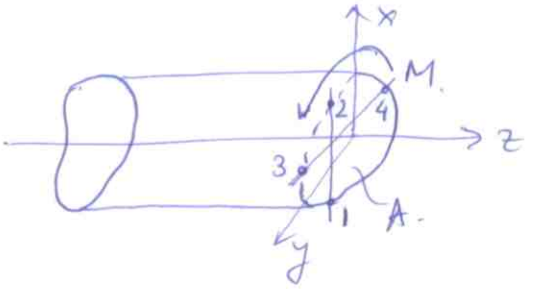
$\because$ addition of constant to ϕ does not affect stresses

so take $\boxed{(\phi)_s=0}$ — valid for simply connected domain. $\rightarrow$ (4b)

[For multiply connected domain, take $\phi=0$ along one boundary and find the non-zero constant value on other boundaries by single valuedness of displacements].

BC's on end faces.

$$\underline{n}=(0,0,\pm1)$$



$$\iint_A \sigma_{xz}\,dx\,dy=0\longrightarrow \underline{(5a)}$$

$$\iint_A \sigma_{yz}\,dx\,dy=0\longrightarrow \underline{(5b)}$$

$$\iint_A(-y\,\sigma_{xz}+x\,\sigma_{yz})\,dx\,dy=M\longrightarrow \underline{(5c)}$$

$$(5a)\rightarrow \iint_A\frac{\partial\phi}{\partial y}\,dx\,dy=\int(\phi(3)-\phi(4))\,dx=0$$ (ie $\underline{5(a)}$ is i.s., and $|||^{\text{aly}}$ so is $\underline{5(b)}$)

Strain compat reduce to :

$0 = 0$ —(i) , $0=0$ —(ii), $0=0$ —(iii)

$\varepsilon_{yz,xx} = \varepsilon_{xz,xy}$ —(iv), $\varepsilon_{xz,yy} = \varepsilon_{yz,xy}$ —(v) , $0=0$ —(vi).

(iv) + 3rd equil $\rightarrow \varepsilon_{yz,xx} = -\varepsilon_{zy,yy} \rightarrow \nabla^2 \varepsilon_{yz} = 0 \rightarrow \nabla^2 \sigma_{yz} = 0$

$\|^{ly}$ (v) + 3rd equil $\rightarrow \nabla^2 \sigma_{xz} = 0$

Alternative derivation for Eq ⑥ :

$$M = \iint_A (-y\,\phi_{,y} - x\,\phi_{,x})\,dx\,dy = -\iint \left((y\phi)_{,y} + (x\phi)_{,x} - 2\phi\right) dx\,dy$$

$$\boxed{M = 2\iint_A \phi\,dx\,dy - \oint_S (xl + ym)\,\underset{=0 \text{ on } S}{\cancel{\phi}}\,ds} \rightarrow ⑥^*$$

w/o, $\phi = 0$ on S

$$= 2\iint_A \phi\,dx\,dy$$

$$\underline{(5c)} \rightarrow \iint_A \left(-y\frac{\partial \phi}{\partial y} - x\frac{\partial \phi}{\partial x}\right)dx\,dy = \int\left(-y(3)\underset{0}{\cancel{\phi}(3)} + y(4)\underset{0}{\cancel{\phi}(4)}\right)dx$$

$$+\iint \phi\, dx\,dy + \int\left(-x(2)\underset{0}{\cancel{\phi}(2)} + x(1)\underset{0}{\cancel{\phi}(1)}\right)dy$$

$$+\iint \phi\, dx\,dy$$

alternative (more elegant) for Eq ⑥ in p.67 b.

$$\Rightarrow \boxed{2\iint_A \phi\, dx\,dy = M.} \rightarrow ⑥ \rightarrow \underline{\text{for simply connected only}} \quad (\because \phi(3) = \phi(4) = 0 \text{ implied}).$$

<u>Summary</u>: solve ③ subject to ④(b) & ⑥ for $\phi(x, y)$. Then use ① to get stresses. Then get displacements.

<u>Displacements</u>:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 0\\
&\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = -\frac{1}{G}\frac{\partial \phi}{\partial x}, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{1}{G}\frac{\partial \phi}{\partial y}\\
&\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0
\end{aligned}\right\} ⑦.$$

$$7(a, b, c) \rightarrow u = f_1(y, z), \quad v = f_2(x, z), \quad w = f_3(x, y) \rightarrow ⑧$$

Subst this in $7(d, e, f) \rightarrow$

$$\left.\begin{aligned}
\frac{\partial f_3}{\partial y} + \frac{\partial f_2}{\partial z} &= -\frac{1}{G}\frac{\partial \phi}{\partial x}\\
\frac{\partial f_1}{\partial z} + \frac{\partial f_3}{\partial x} &= \frac{1}{G}\frac{\partial \phi}{\partial y}\\
\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} &= 0
\end{aligned}\right\} ⑨$$

$$\frac{\partial (9b)}{\partial z} \Rightarrow \frac{\partial^2 f_1}{\partial z^2} = 0, \quad \frac{\partial (9c)}{\partial y} \Rightarrow \frac{\partial^2 f_1}{\partial y^2} = 0$$

$$\left.\begin{aligned}
f_1 &= ayz + by + cz + d\\
\text{III}^{\text{ly}} \quad f_2 &= exz + fx + gz + h
\end{aligned}\right\} \rightarrow ⑪$$

$$(11, 9c) \rightarrow (e + a)z + f + b = 0 \Rightarrow e = -a, \; f = -b.$$

$$\Rightarrow u = f_1 = ayz + by + cz + d, \quad v = f_2 = -axz - bx + gz + h$$

The constant & linear terms in u, v represent rigid body displacements & rotations, respectively. Assuming that the end z=0 is restrained to prevent RB motions, ie $b=c=d=g=h=0$,

$$\boxed{u = ayz, \qquad v = -axz.} \longrightarrow (12)$$

$$\left.\begin{aligned} u_r &= v\sin\theta + u\cos\theta = az(-\underset{r\cos\theta}{\cancel{x}}\sin\theta + \underset{r\sin\theta}{\cancel{y}}\cos\theta) \doteq 0 \\ u_\theta &= v\cos\theta - u\sin\theta = \alpha z r = \beta r \end{aligned}\right\} \rightarrow (13).$$

(where $\boxed{a=-\alpha}$ is used, $\beta = \alpha z$ = twist of section, $\alpha = \frac{d\beta}{dz}$ = twist per length = constant).

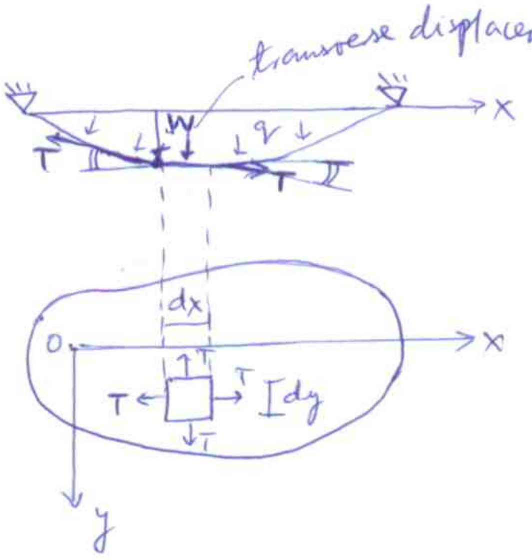
$$7(c,d), (12) \rightarrow \boxed{\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{G}\frac{\partial \phi}{\partial y} + \alpha y \\ \frac{\partial w}{\partial y} &= -\frac{1}{G}\frac{\partial \phi}{\partial x} - \alpha x \end{aligned}} \longrightarrow (14) \longrightarrow \text{use to determine } w \text{ — } \underline{\text{warping displ.}}$$

$$14(a,b) \rightarrow \boxed{\nabla^2\phi = -2G\alpha.} \rightarrow (15) \rightarrow \text{use to determine } \alpha \text{ from } \phi.$$

This implies that kinematics is inplane kinematics (ie u, v) resulting from rotation of points in a section thru angle β, and then out-of-plane kinematics (ie w) superposed. So plane sections dont stay plane.

MEMBRANE ANALOGY

Prandtl observed analogy with static deformation of a uniformly tensioned membrane subject to uniform (but small) transverse pressure.



Equilibrate vertical components of T acting on edges of element dxdy with the applied pressure.

$$\Rightarrow -\cancel{T dy \frac{\partial w}{\partial x}} + T dy \left(\cancel{\frac{\partial w}{\partial x}} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) dx\right)$$

$$-\cancel{T dx \frac{\partial w}{\partial y}} + T dx \left(\cancel{\frac{\partial w}{\partial y}} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) dy\right)$$

$$+ q\, dx dy = 0$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{q}{T} \Rightarrow \nabla^2\left(\frac{T}{q} w\right) = -1 \\ &\underline{\text{b.c.}} \quad (w)_S = 0 \end{aligned}\right\} \rightarrow (16)$$

Compare with Prandtl formulation ie (~~8~~ 15, 4b, 6), i.e.,

$$\nabla^2\left(\frac{\phi}{2G\alpha}\right) = -1, \quad \left(\frac{\phi}{2G\alpha}\right)_S = 0. \longrightarrow (17)$$

$$(16, 17) \longrightarrow \phi = \frac{2G\alpha T}{q} w.$$

$$(6) \rightarrow M = 2\iint \phi\, dx dy = \frac{2G\alpha T}{q}\, 2 \overset{\text{vol displaced by membrane}}{\overset{\parallel}{\iint_A w\, dx dy}} = \frac{2G\alpha T}{q} 2V$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = \frac{2G\alpha T}{q} \frac{\partial w}{\partial y}$$

If $G, \alpha, T, q,$ adjusted : $\frac{2G\alpha T}{q} = 1$, then

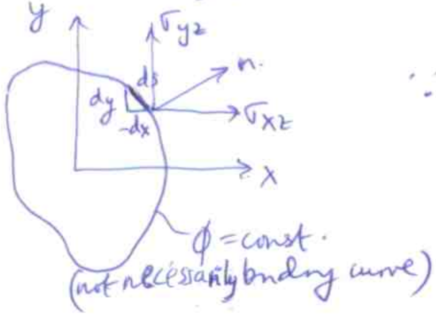
$$\boxed{\phi = w, \quad M = 2V, \quad \tau_{zx} = \frac{\partial w}{\partial y}, \quad \tau_{zy} = -\frac{\partial w}{\partial x}} \rightarrow (18).$$

Lines of shearing stress.

$$\underset{\sim}{\nabla}\phi \cdot (\tau_{xz}, \tau_{yz}) = (\phi_{,x}, \phi_{,y}) \cdot (\phi_{,y}, -\phi_{,x}) = 0$$

Thus on ϕ = const curves, total shear stress (τ_{xz}, τ_{yz}) lies along tangent to curves ($\because \underset{\sim}{\nabla}\phi$ is along normal).

Thus, $l = \frac{dy}{ds} = \frac{\partial x}{\partial n} = \frac{\sigma_{yz}}{\sqrt{\sigma_{xz}^2+\sigma_{yz}^2}}$, $m = -\frac{dx}{ds} = \frac{\partial y}{\partial n} = \frac{\sigma_{xz}}{\sqrt{\sigma_{xz}^2+\sigma_{yz}^2}}$



$\because$ resultant shear $\tau = (\sigma_{xz}, \sigma_{yz})$ lies tangential to contour we have

$$\sigma_{nz} = 0$$

$$\sigma_{sz} = \tau = \sigma_{yz}\frac{dy}{ds} - \sigma_{xz}\left(-\frac{dx}{ds}\right).$$

$$= -\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n} - \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n} = -\frac{\partial \phi}{\partial n}$$

$\Rightarrow$ magnitude of shearing stress is $\frac{\partial \phi}{\partial n}$

$\because$ τ is tangential to $\phi = \text{const}$,

$$\boxed{\tau = -\frac{\partial \phi}{\partial n} = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}} \longrightarrow (19)$$

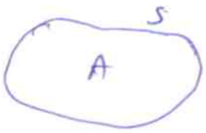
Now, $$\nabla^2 \tau^2 = 2\big(\phi_{,xx}^2 + \underline{\phi_{,x}\phi_{,xxx}} + \phi_{,xy}^2 + \underline{\phi_{,y}\phi_{,xxy}} + \phi_{,yy}^2 + \underline{\phi_{,x}\phi_{,xyy}} + \underline{\phi_{,y}\phi_{,yyy}}\big)$$

($\because$ $\nabla^2\phi = \text{const}$ the underlined pair of terms vanish)

$\Rightarrow \nabla^2\tau^2 \geq 0 \Rightarrow \tau^2$ is subharmonic, and its maximum occurs on the boundary.

aside from Maths.

Note: (i) $\nabla^2\tau^2 = 0$ & $\tau^2 \neq \text{const}$ inside region A, then max and min of τ^2 occur on boundary & not inside A



(ii) $\nabla^2\tau^2 \geq 0$ & $\tau^2 \neq \text{const}$ inside A, then τ^2 is subharmonic & max (τ^2) occurs on boundary & not inside A

(iii) $\nabla^2\tau^2 \leq 0$ & $\tau^2 \neq \text{const}$ inside A, then τ^2 is superharmonic and min (τ^2) occurs on bndry & not inside A.

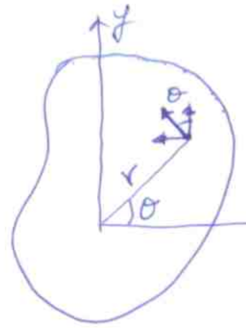
※ Thus max shear stress occurs on boundary, so shear failure, if it occurs, will occur on lateral face.

(II) St. VENANT's semi-inverse method & formulation (ψ, g)

Guided by kinematics of circular shaft, St. Venant assumed

$$\left.\begin{aligned} u &= -u_\theta \sin\theta = -r\beta\sin\theta = \boxed{-y\alpha z = u} \\ v &= u_\theta\cos\theta = r\beta\cos\theta = \boxed{x\alpha z = v} \xrightarrow{} (12) \\ &\boxed{w = \alpha\,\psi(x,y)} \end{aligned}\right\} \rightarrow (20)$$

$\psi(x,y)$ → WARPING FUNCTION



Observe that (20 a, b) match (12).

$$\left.\begin{aligned} \Rightarrow \gamma_{xz} &= \alpha\left(\frac{\partial\psi}{\partial x} - y\right), \quad \gamma_{yz} = \alpha\left(\frac{\partial\psi}{\partial y} + x\right) \\ \gamma_{xy} &= \varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \end{aligned}\right\} \rightarrow (21)$$

$$\Rightarrow \boxed{\sigma_{xz} = G\alpha(\psi_{,x} - y), \quad \sigma_{yz} = G\alpha(\psi_{,y} + x)} \rightarrow (22)$$

$\therefore \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0 \Longrightarrow$ no distortion in plane of ⊗ section.

Equilibrium → 1st, 2nd are i.s. 3rd yields

$$\boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0} \longrightarrow (23)$$

$$(1), (22) \longrightarrow \boxed{\phi_{,y} = G\alpha(\psi_{,x} - y), \quad \phi_{,x} = -G\alpha(\psi_{,y} + x)} \rightarrow (24)$$

So eliminating ϕ from (24) or from (14), you can get back (23).

Lateral face BC's:

$$(l\sigma_{xz} + m\sigma_{yz})\big|_S = 0 \Rightarrow \left(\overset{\partial x/\partial n}{\frac{dy}{ds}}\right)\left(\frac{\partial\psi}{\partial x} - y\right) + \left(\overset{\partial y/\partial n}{-\frac{dx}{ds}}\right)\left(\frac{\partial\psi}{\partial y} + x\right) = 0$$

$$\Rightarrow \boxed{\underbrace{\frac{\partial\psi}{\partial n}}_{n\cdot\nabla\psi} = y\frac{\partial x}{\partial n} - x\frac{\partial y}{\partial n} = yl - xm \quad \text{on } S \text{ (lateral face's closed curve).}} \rightarrow (25)$$

(23),(24) represent Neumann problem for ψ. Soln of ψ unique upto a constant (since bc involves $\nabla\psi$).

aside. Now (23) alone $\Rightarrow \oint_S \frac{\partial\psi}{\partial n}\,ds = 0$. The details are:

$$\oint_S \frac{\partial\psi}{\partial n}\,ds = \oint_S n\cdot\nabla\psi\,ds \underset{\text{2D div thm.}}{=} \iint_A \underbrace{\nabla\cdot\nabla\psi}_{=0 \text{ from (23)}}\,dA = 0. \longrightarrow (A).$$

Does (25) satisfy condition (A). Yes it does! (details are:)

aside. ↑

$$\text{(25)} \Rightarrow \oint_S \frac{\partial \psi}{\partial n} ds = \oint_S \Big(y\underset{\frac{dy}{ds}}{\underset{\downarrow}{l}} - x\underset{\left(-\frac{dx}{ds}\right)}{\underset{\downarrow}{m}}\Big) ds = \oint_S \cancel{(ydy + xdx)} \rightarrow d\left(\frac{x^2+y^2}{2}\right) = 0.$$

End-face bc's:

The force resultant bc's will be i.s. as was the case in the stress-function approach. Details are:

$$\iint_A \sigma_{xz}\, dA \overset{\text{add 3rd equil eqn}}{=} \iint_A \left[\frac{\partial}{\partial x}(x\sigma_{xz}) + \frac{\partial}{\partial y}(x\sigma_{yz})\right] dA \overset{\text{2D Div Thrm}}{=} \int_S x\underbrace{(l\sigma_{xz} + m\sigma_{yz})}_{=0 \text{ from lateral face bc}} = 0.$$

and similarly for $\iint_A \sigma_{yz}\, dA = 0$.

Moment resultant BC becomes,

$$\text{(26)} \quad M = \iint_A (-y\sigma_{xz} + x\sigma_{yz})\, dx\,dy = G\alpha \underbrace{\iint_A \left(x^2+y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dx\,dy}_{\triangleq C/G} = C\alpha$$

ie $\boxed{M = C\alpha}$

where $\boxed{C = \text{torsional rigidity} = G\iint_A (x^2+y^2+x\psi_{,y} - y\psi_{,x})\, dx\,dy} \rightarrow \text{(27a)}$

Now consider

$$I = \iint_A \left(x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dx\,dy = \iint_A \left(\frac{\partial(x\psi)}{\partial y} - \frac{\partial(y\psi)}{\partial x}\right) dx\,dy$$

$$\overset{\text{Div Thrm}}{=} \oint_S (-ly + mx)\psi\, ds = -\oint_S \left(l\frac{\partial \psi}{\partial x} + m\frac{\partial \psi}{\partial y}\right)\psi\, ds$$

$$\overset{\text{Div Thrm + (23)}}{=} -\iint_A \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] dx\,dy \le 0.$$

$$\Rightarrow \frac{C}{G} \le \iint_A (x^2+y^2)\, dA, \text{ ie } \boxed{C \le GJ}$$

For $\psi_{,x} = \psi_{,y} = 0$ in A, ie $\psi = \text{const}$ in A, $C = GJ$

$$\text{Now } \psi = \underset{(\text{in } A)}{\text{const}} \Rightarrow \left(\frac{\partial \psi}{\partial n}\right)_S = 0 \overset{\text{bc (25)}}{=} \Big(y\underset{\left(\frac{dy}{ds}\right)}{\underset{\downarrow}{l}} - x\underset{\left(-\frac{dx}{ds}\right)}{\underset{\downarrow}{m}}\Big)_S = \left(\frac{d\left(\frac{x^2+y^2}{2}\right)}{ds}\right)_S$$

⇒ S is a circle.

∴ ⇒ circle has max torsional rigidity when compared to another section having same J

$$\Rightarrow \boxed{C = GJ - G\iint\left[(\psi_{,x})^2+(\psi_{,y})^2\right]dx\,dy} \longrightarrow (27b)$$

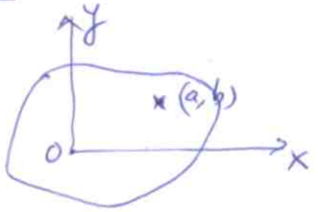
(27(a))×2 − (27b)

$$\Rightarrow C = G\iint_A\left(x^2+y^2+(\psi_{,x})^2+(\psi_{,y})^2+2x\psi_{,y}-2y\psi_{,x}\right)dA$$

$$= G\iint\left((\psi_{,y}+x)^2+(\psi_{,x}-y)^2\right)dA = \frac{1}{\alpha^2 G}\iint_A\left(\sigma_{yz}^2+\sigma_{xz}^2\right)dA.$$

ie $\boxed{C = \frac{1}{\alpha^2 G}\iint\left(\sigma_{yz}^2+\sigma_{xz}^2\right)dA} \longrightarrow (27c)$

Change of torsion axis (ie reference origin)



NEW Axis passes thru (a,b), old one thru O

(28a)

$$u = -\alpha(y-b)z,\quad v = \alpha(x-a)z,\quad w = \alpha\psi_1(x,y)$$

$$\sigma_{xz} = G\alpha\left(\frac{\partial\psi_1}{\partial x} - y + b\right),\quad \sigma_{yz} = G\alpha\left(\frac{\partial\psi_1}{\partial y} + x - a\right)$$

$$\sigma_{xz,x} + \sigma_{yz,y} = 0 \Rightarrow \nabla^2\psi_1 = 0. \longrightarrow (28b)$$

Lateral bc : $l\sigma_{xz} + m\sigma_{yz} = 0 \Rightarrow \frac{\partial}{\partial n}(\psi_1 + bx - ay) = ly - mx$ on S (28c)

So $\boxed{\psi_1 = \psi - bx + ay + \text{const}}$ solves (28b,c) if ψ solves (23, 25) $\longrightarrow$ (29)

(28a) $\Rightarrow \sigma_{xz} = G\alpha(\psi_{,x} - y)$, $\sigma_{yz} = G\alpha(\psi_{,y} + x)$ → i.e., stresses remain invariant to change of torsion axis. So only effect is that displ's differ by rigid body comp.

These results are to be expected ∵ origin won't affect stresses. In fact we presumed invariance by keeping α the same

Hence C is also invariant (see 27c).

X Now ∵ $C \le GJ \Rightarrow C \le GJ_{min}$ for $\uparrow J_c$ (centroidal axis).

Conjugate warping function

$$\nabla^2\psi = 0 \Rightarrow \exists\, g : \boxed{\nabla^2 g = 0} \rightarrow (30a) \quad \text{where,}$$

$$\psi_{,x} = g_{,y}\,, \quad \psi_{,y} = -g_{,x} \rightarrow \text{CR eqns.}$$

lateral bc: $l\psi_{,x} + m\psi_{,y} = yl - xm$ on S

$$\Rightarrow \frac{\partial g}{\partial y}\frac{dy}{ds} + \frac{\partial g}{\partial x}\frac{dx}{ds} = y\frac{dy}{ds} + x\frac{dx}{ds}.$$

$$\Rightarrow \boxed{g = \frac{x^2+y^2}{2} + C \text{ (const)}} \rightarrow (30b)$$

(30a,b) is the Dirichlet problem – unique solution for chosen constant. Stresses are,

$$\sigma_{xz} = G\alpha(g_{,y} - y)\,, \quad \sigma_{yz} = G\alpha(-g_{,x} + x)$$

They are unaffected by constant. Displacements are affected upto rigid body motions only.

Summary of formulations.

(i) Prandtl's stress function (ϕ)

$\nabla^2\phi = -2G\alpha$, $\phi = 0$ on S (simply-connected).

$M = 2\iint \phi\, dA$ (simply connected), $\sigma_{xz} = \phi_{,y}$, $\sigma_{yz} = -\phi_{,x}$

(ii) St Venants Warping function (ψ)

$\nabla^2\psi = 0$, $\frac{\partial\psi}{\partial n} = yl - xm$ on S, $M = C\alpha$,

$\sigma_{xz} = G\alpha(\psi_{,x} - y)$, $\sigma_{yz} = G\alpha(\psi_{,y} + x)$

$C = G\iint_A \left(x^2+y^2-(\psi_{,x})^2-(\psi_{,y})^2\right) dA$

$u = -\alpha yz$, $v = \alpha xz$, $w = \alpha\psi(x,y)$

(iii) Conjugate Warping fn (g)

$\nabla^2 g = 0$, $g = \frac{x^2+y^2}{2} + C$ on S, $M = C\alpha$,

$\sigma_{xz} = G\alpha(g_{,y} - y)$, $\sigma_{yz} = G\alpha(-g_{,x} + x)$

$C = G\iint_A \left(x^2+y^2-(g_{,x})^2-(g_{,y})^2\right) dA.$

Solution Methods.

3 main methods exist.

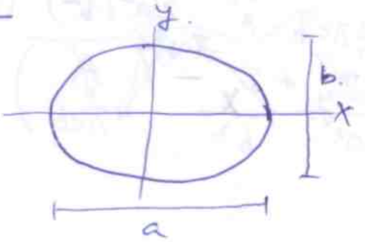
(a) Semi-inverse method: Assume ϕ or ψ or g satisfying b.c's & check whether governing eqn is satisfied.

(b) Separation of variables based solution.

(c) Complex variable method.

Torsion of Elliptic sections

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$



Prandtl's ϕ fn approach.

③ & BC (4(a)) satisfied by choosing $\phi = m\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$
or ⑮

③/⑮ $\Rightarrow \quad m = \frac{a^2b^2}{2(a^2+b^2)}\underset{\searrow -2G\alpha}{K}$

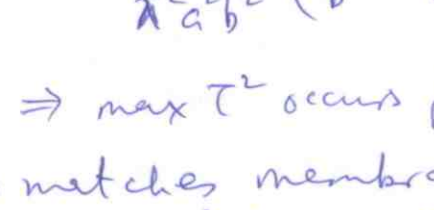
(over for details) – routine

BC ⑥ $\Rightarrow \quad K = -2G\alpha = -2M\frac{(a^2+b^2)}{\pi a^3 b^3}$

$\Rightarrow \phi = -\frac{M}{ab}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$

ϕ, ① $\Rightarrow \quad \boxed{\sigma_{xz} = -\frac{2My}{\pi ab^3}, \quad \sigma_{yz} = \frac{2Mx}{\pi a^3 b}}$

$\Rightarrow \frac{\sigma_{xz}}{\sigma_{yz}} \propto \frac{y}{x}$ so direction of τ is constant along a radial line and coincides with the tangent at the boundary since bndry is a ϕ = const curve (recall lines of shearing stress).



$\sigma_{yz}=0$ on vertical

$\sigma_{xz}=0$ on horizontal axis.

on ∂, $\tau^2 = \frac{4M^2}{\pi^2a^2b^2}\left(\frac{y^2}{b^4}+\frac{x^2}{a^4}\right) \overset{\text{on } \partial \; y^2/b^2 = 1 - x^2/a^2}{=} \frac{4M^2}{\pi^2a^2b^2}\left(\frac{1}{b^2} - x^2\underbrace{\left[\frac{1}{a^2b^2}-\frac{1}{a^4}\right]}_{>0 \because a>b}\right)$

$\Rightarrow$ max τ^2 occurs for $x=0$ on bndry, ie pts A, B

[This matches membrane analogy which indicates max slope $\frac{\partial w}{\partial y}$, analogous to max σ_{xz}, at A & B.

Further, since $\tau = -\frac{\partial\phi}{\partial n}$ on the boundary (since it is a ϕ = const curve), (see p. 70) & since we are certain that $\frac{\partial w}{\partial n}$ membrane is max at A & B, then τ is also max at A & B

Details for elliptic section

$$\nabla^2 \phi = m\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = K \Rightarrow m = K\frac{a^2b^2}{2(a^2+b^2)}$$

$$M = 2\iint_A \phi \, dx\,dy = 2\iint_A K\frac{a^2b^2}{2(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)dx\,dy$$

$$= K\frac{a^2b^2}{(a^2+b^2)}\left(\frac{\overset{\frac{1}{4}\pi a^3 b}{I_y}}{a^2}+\frac{\overset{\pi a b^3/4}{I_x}}{b^2} - \underset{\pi ab}{A}\right) = -\frac{\pi a^3 b^3 K}{2(a^2+b^2)}$$

Thus $\boxed{\tau_{max} = \frac{2M}{\pi a b^2}}$ (at $x=0$, $y=\pm b/2$) with direction tgt to bndry.

Now $\boxed{\underset{\text{torsional rigidity}}{C} = \frac{M}{\alpha} = \frac{G\pi a^3 b^3}{a^2+b^2}}$ (from BC ⑥).

Displacements, from ⑫, ⑭ are:

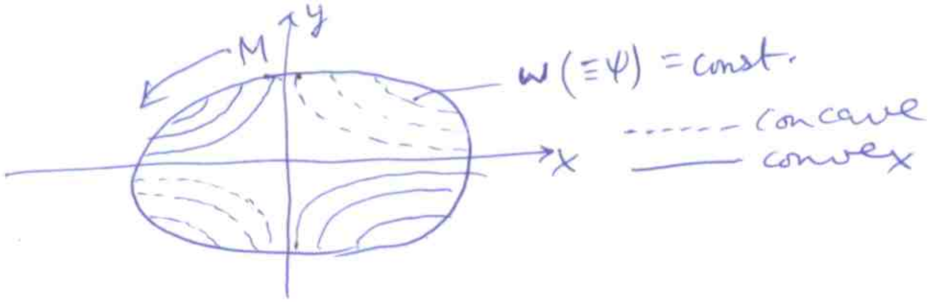
$$u = -\frac{(a^2+b^2)M}{\pi a^3 b^3 G} yz, \quad v = -\frac{(a^2+b^2)M}{\pi a^3 b^3 G} xz$$

$$\frac{\partial w}{\partial x} = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G} y, \quad \frac{\partial w}{\partial y} = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G} x$$

$$\Rightarrow \boxed{w = -\frac{(a^2-b^2)M\,xy}{\pi a^3 b^3 G}} + w_0$$

(w_0 const, RB motion so neglect since we will constrain appropriately).

∴ w = const represent hyperbolas.



Complex variable approach:

Mapping function $w(z) = i(c^2 z^2 + k^2)$,

$z = x+iy$, $w = \psi + ig$.

$\Rightarrow \psi, g$ satisfy C.R. eqns so if $\nabla^2\psi = 0 \Rightarrow \nabla^2 g = 0$.

Seperating Re and Im parts,

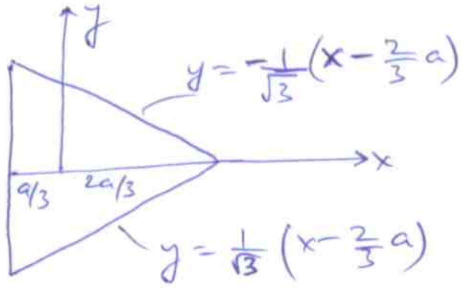
$$\psi = -2xyc^2$$

$$g = (x^2-y^2)c^2 + k^2 \overset{bc}{=} \frac{1}{2}(x^2+y^2) \text{ on } S$$

$\because \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on S we get,

$$c^2 = \frac{(a^2-b^2)}{2(a^2+b^2)}, \quad k^2 = \frac{a^2b^2}{(a^2+b^2)} \rightarrow \text{then get stresses and displ.}$$

Torsion of Triangular section



Eqn of boundary is,

$$F(x,y)=\left(x+\frac{a}{3}\right)\left(x-\frac{2}{3}a+\sqrt{3}y\right)\left(x-\frac{2}{3}a-\sqrt{3}y\right)=0$$

Try,

$$\phi = mF(x,y) = m\left(x^3-ax^2-3y^2x+\frac{4}{27}a^3-ay^2\right)$$

$$\nabla^2\phi=-2GK \Longrightarrow m=\frac{G\alpha}{2a}$$

$$\tau_{xz}=\frac{\partial\phi}{\partial y}=-\frac{G\alpha y}{a}(3x+a)$$

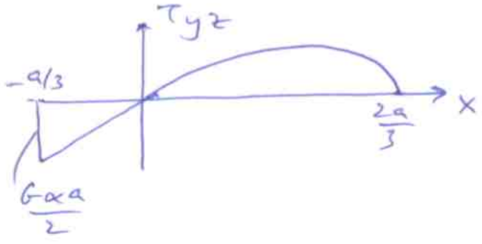
$$\tau_{yz}=-\frac{\partial\phi}{\partial x}=-\frac{G\alpha}{2a}\left(3x^2-2ax-3y^2\right)$$

$\tau_{xz}=0$ on $x=-\frac{a}{3}$ (as it should be ∵ $x=-\frac{a}{3}$ is a stress curve so $\tau_{xz}=0$ along it)

$\tau_{xz}=\tau_{yz}=0$ at corners (as expected ∵ τ which lies along boundary curve cant have two directions)

$\tau_{xz}=\tau_{yz}=0$ at $(x,y)=(0,0)$, ie centroid.

$\boxed{M=\frac{3}{5}G\alpha J_0}$, $J_0=\frac{\sqrt3}{27}a^4$ = polar MI about centroid.



on $y=0$, $\tau_{xz}=0$, $\tau_{yz}=-\frac{G\alpha x}{a}\left(\frac{3}{2}x-a\right)$

$$\boxed{\tau_{max}=\frac{G\alpha a}{2} \text{ at } x=-\frac{a}{3}}$$

Complex variable method

Try $w(z)=i\left(cz^3+k\right)$

$\Rightarrow g=c\left(x^3-3xy^2\right)+k=\frac{1}{2}\left(x^2+y^2\right)$ on S

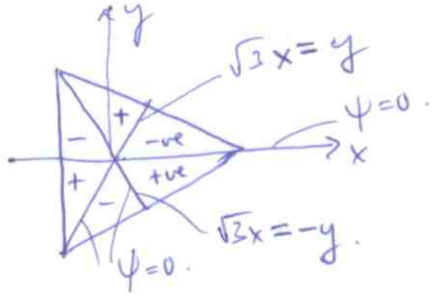
Choosing $c=\frac{1}{2a}$, $k=\frac{2}{27}a^2$, get

$$g-\frac{(x^2+y^2)}{2}=x^3-3xy^2-a\left(x^2+y^2\right)+\frac{4}{27}a^3=0=\phi$$

↳ ie eqn of triangle.

$$\Rightarrow g=\frac{1}{2a}\left(x^3-3xy^2\right)+\frac{2}{27}a^2$$

$$\boxed{\psi=-\frac{1}{2a}\left(3x^2y-y^3\right)}$$



Circular shaft with cutout

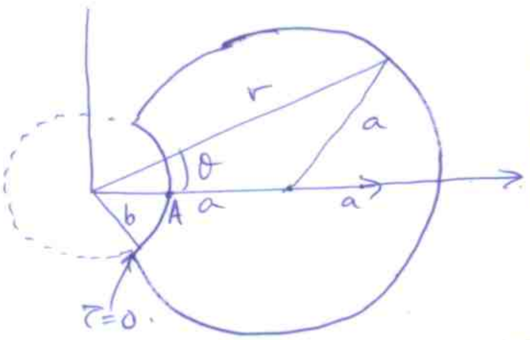
Effect of grooves or slots or key ways, on the max shearing stress can be gauged from this example.

Try $g = a\left(x - b^2\frac{x}{x^2+y^2}\right) + \frac{1}{2}b^2 = a\left(r\cos\theta - \frac{b^2\cos\theta}{r}\right) + \frac{1}{2}b^2$

On S, $g = \frac{1}{2}(x^2+y^2) = \frac{1}{2}r^2$

$$(r^2-b^2)\left(1-\frac{2a\cos\theta}{r}\right)=0$$

Thus boundary comprises two circles, $r=b$, and $r=2a\cos\theta$ $\left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$



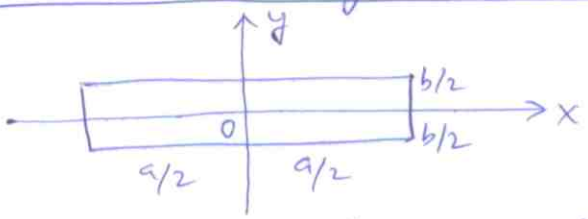
$r = 2a\cos\theta$
$\therefore r^2 = 2ax$
$\therefore x^2+y^2 = 2ax$
$\therefore (x-a)^2+y^2 = a^2$

τ_{max} at A $\longrightarrow$ $\boxed{\tau_{max} = G\alpha(2a-b)}$ $\simeq 2G\alpha a$ if $b \ll a$
= 2 × stress on periphery of circular shaft of radius a

Thus stress concentration due to cutout is evident.

→ Equivalently can choose $\phi = (x^2+y^2-b^2)([x-a]^2+y^2-a^2)$

Narrow Rectangular Bar



$a \gg b.$

membrane analogy ⇒ cylindrical bending ⇒ $w = w(y)$

$\Rightarrow \tau_{zy} = 0, \quad \phi = \phi(y)$

$$\Rightarrow \nabla^2\phi = \frac{d^2\phi}{dy^2} = K$$

Integrating & using b.c's $\phi|_{y=\pm b/2} = 0$, get

$$\phi = \frac{K}{2}\left(y^2 - \frac{b^2}{4}\right)$$

$$M = 2\iint_A \phi\,dx\,dy = 2\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} \frac{K}{2}\left(y^2-\frac{b^2}{4}\right)dx\,dy = -\frac{ab^3}{6}\underset{-2G\alpha}{\cancel{K}}$$

$$\phi = \frac{3M}{ab^3}\left(\frac{b^2}{4}-y^4\right)$$

(31)

$$\boxed{\tau_{zx} = \phi_{,y} = -\frac{6M}{ab^3}y = 2G\alpha y}$$

Membrane analogy ⇒ τ_{max} at $y = \pm\frac{b}{2}$

$$\boxed{\tau_{max} = \frac{3M}{ab^2} = bG\alpha} = 2\times(\max \tau \text{ for circular shaft of diameter } b)$$

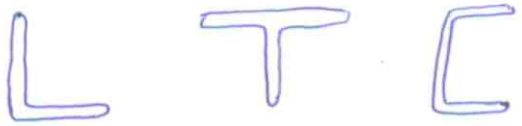
$$\boxed{\alpha = -\frac{\nabla^2\phi}{2G} = \frac{3M}{ab^3 G}} \Rightarrow \boxed{C = G\frac{ab^3}{3}}$$

Calculating M from approximate shear stresses, ie $\tau_{zy}=0$,

$$M = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}(x\underset{\cong 0}{\cancel{\tau_{yz}}} - y\,\tau_{xz})\,dx\,dy \cong \frac{1}{6}G\alpha ab^3 \rightarrow \text{ie half of actual } M \text{ (see above).}$$

Thus we conclude that although $\tau_{yz} \simeq 0$ (ie small), it contributes to half of M since the lever arm ($a/2$) is large.

Rolled Sections – Open Thin Walled members.



Membrane analogy: If narrow rectangular membrane is loaded and then bent, the volume ~~bounded~~ displaced and the slopes will remain unchanged (physical arguement). Hence the narrow rectangular bar, when bent into a curved cross-section one will have the same torsional moment and shear stresses. So treat all curved narrow rectangles as straight ones when solving such problems.

$a_i, b_i \rightarrow$ length & width of i^{th} narrow rectangle in the cross section

$M_i \rightarrow$ twisting moment on i^{th} narrow rectangle, $\underset{\text{total moment.}}{M} = \sum M_i$.

$\tau_i \rightarrow$ max stress on i^{th} rectangle

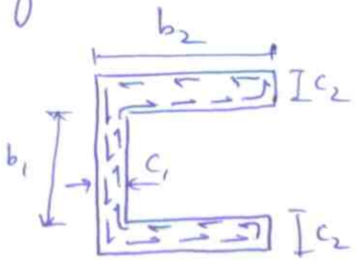
$\alpha \rightarrow$ twist per unit length of bar.

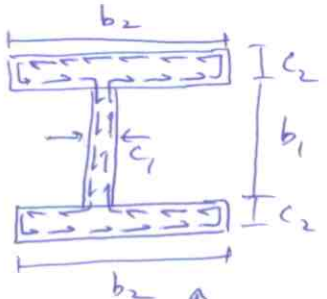
$$\Rightarrow \tau_i = \frac{3M_i}{a_i b_i^2}, \quad \alpha = \frac{3M_i}{a_i b_i^3 G}, \quad M = \sum M_i = \frac{G\alpha}{3}\sum a_i b_i^3$$

boxed eqns. (32) ←

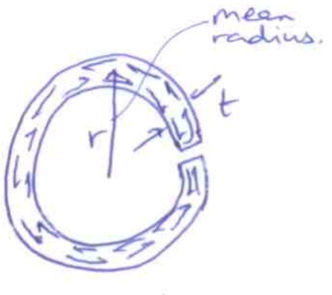
$$M_i = \frac{a_i b_i^3}{\sum a_i b_i^3} M, \quad \boxed{\tau_i = \frac{3Mb_i}{\sum a_i b_i^3}}, \quad \boxed{\alpha = \frac{3M}{G\sum a_i b_i^3}}$$

The formula for $\overset{(\text{ie } \tau_i)}{\tau_{max}}$ does not hold at a corner where high stress concentrations exist.

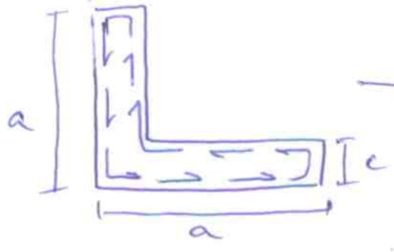




$$\alpha = \frac{3M}{(b_1c_1^3 + 2b_2c_2^3)G}$$



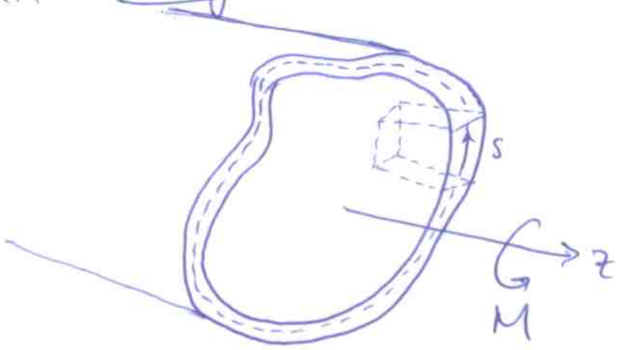
$$\alpha = \frac{3M}{2\pi r t^3 G}$$

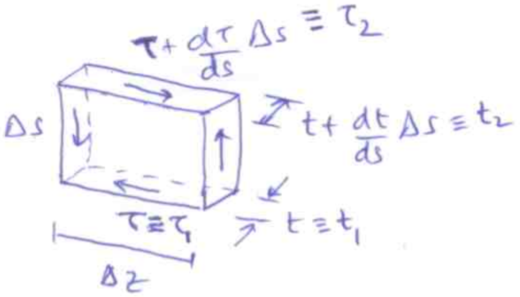


$$\rightarrow \alpha = \frac{3M}{(2a-c)c^3 G}$$

Thin Walled Tubes.

(A) Single celled. TW Tube



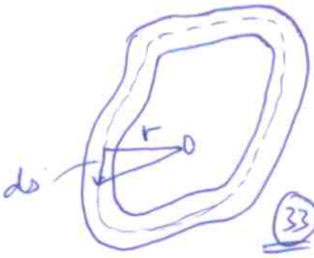


Assume: $\tau_{zs} = \tau$ = const thru thickness and directed along ℄, even when $t = t(s)$.

$$\sum F_z : \; -\overset{\tau_1 t_1}{\tau t}\,\Delta z + \overbrace{\left(\tau + \frac{d\tau}{ds}\Delta s\right)}^{\tau_2}\overbrace{\left(t + \frac{dt}{ds}\Delta s\right)}^{t_2}\Delta z = 0$$

$$\Rightarrow \frac{d}{ds}(\tau t) = 0 \Rightarrow q = \tau t = \text{constant.} = \tau_1 t_1 = \tau_2 t_2$$

(shear flow is const).



$$M\underline{k} = \oint \underline{r} \times \tau t\, \underset{\sim}{ds} = q \oint \underline{r} \times \underset{\sim}{ds} = q\int_A 2\, d\underline{A} = 2qA\underline{k}$$

$$㉝ \leftarrow \boxed{M = 2qA.} \longrightarrow \text{Bredt's formula.}$$

$$㉞ \leftarrow \boxed{\tau = \sigma_{zs} = \frac{M}{2tA}}$$

$\boxed{A = \text{area enclosed by ℄ curve} \approx \text{area encl by inner perimeter}}$

$$U_T = \text{Strain energy in torsion} = \frac{1}{2}\int_V \frac{\tau^2}{G}\, dV = \frac{1}{2}\int_0^z \oint \frac{M^2}{4t^2A^2G}\, t\, ds\, dz$$

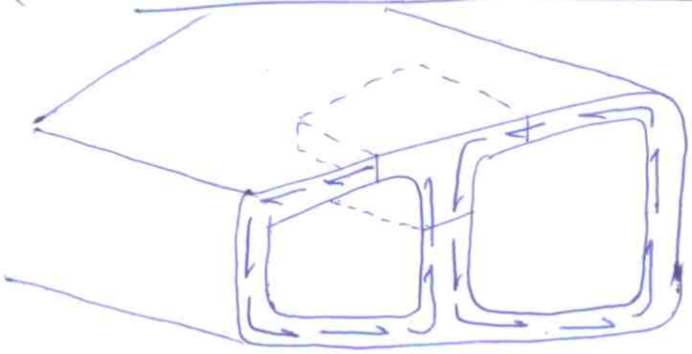
$$= \frac{1}{2}\frac{M^2 z}{4GA^2}\oint \frac{1}{t}\, ds$$

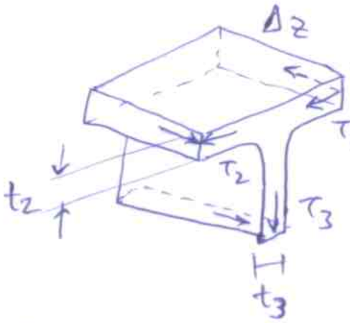
Using Castigliano's theorem,

$$\frac{1}{z}\frac{\partial U_T}{\partial M} = \boxed{\alpha = \frac{M}{4GA^2}\oint \frac{1}{t}\, ds = \frac{q}{2GA}\oint \frac{1}{t}\, ds} \longrightarrow ㉟$$

If q changes over ℄ of $^{a}_{i}$cell then $\boxed{\alpha_i = \frac{1}{2GA}\oint \frac{q}{t}\, ds} \rightarrow ㉟^*$ for use in multicelled case. Arises from compatibility of α for all cells.

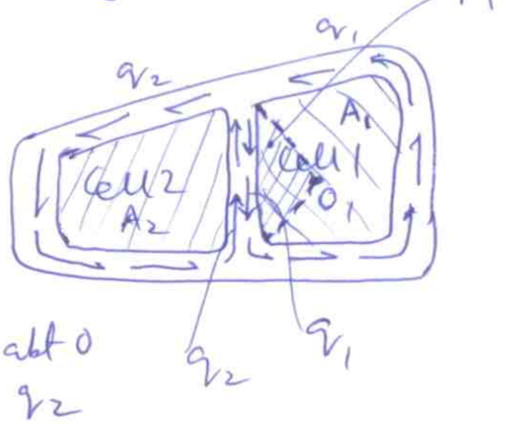
(B) Multi celled. thin walled tube





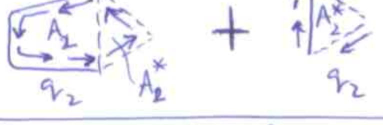
$$\sum F_z : \; (\tau_2 t_2 + \tau_3 t_3 - \tau_1 t_1)\cancel{\Delta z} = 0$$

$$q_2 + q_3 = q_1$$



$M_1 = 2q_1A_1$ = moment abt O due to q_1

$M_2 = \underbrace{2q_2(A_2 + A_1^*)} - \underbrace{2q_2A_1^*}$ = moment abt O due to q_2



$$\boxed{M = M_1 + M_2 = 2q_1A_1 + 2q_2A_2} \longrightarrow ㊱$$

Now rate of twist (α) is same for all cells (ie compatibility condition). So using ㉟*,

For cell 1,

For cell 2,

$$\boxed{\begin{array}{l} 2G\alpha = \frac{1}{A_1}\left(a_1 q_1 - a_{12} q_2\right) \\ 2G\alpha = \frac{1}{A_2}\left(a_2 q_2 - a_{12} q_1\right) \end{array}} \longrightarrow$$ ㊲

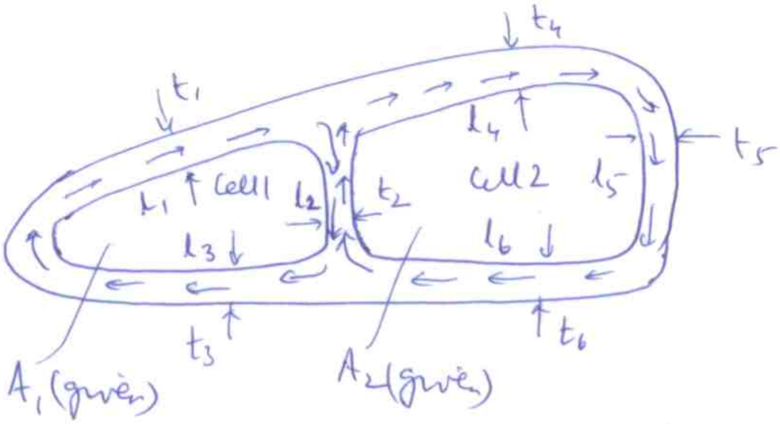
where, $a_1 = \oint \frac{ds}{t}$ for cell 1 (including web)

$a_2 = \oint \frac{ds}{t}$ for cell 2 (including web).

$a_{12} = \oint \frac{ds}{t}$ for web.

} contour integrals over ℄.

(Eg)



~~M~~ (given) applied CW.

$a_1 = \frac{l_1}{t_1} + \frac{l_2}{t_2} + \frac{l_3}{t_3}$

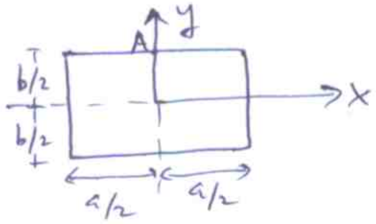
$a_2 = \frac{l_4}{t_4} + \frac{l_5}{t_5} + \frac{l_6}{t_6} + \frac{l_2}{t_2}$

$a_{12} = \frac{l_2}{t_2}$

Solve ㊱, ㊲ for α, q_1, q_2. (ie rate of twist & shear flows).

——— × ———

SOLID RECTANGULAR BAR (using separation of variables approach)



$a \not\gg b.$

$\nabla^2 \phi = -2G\alpha.$

BC's: $\phi_{(x=\pm a/2)} = \phi_{(y=\pm b/2)} = 0$

Consider $\phi(x,y) = \{\phi(y)\}_{\text{narrow rect bar}} + F(x,y) = G\alpha\left(\frac{b^2}{4} - y^2\right) + F$

This transformation from $\phi(x,y)$ to $F(x,y)$ affords homogenization of the DE, ie we get Laplace eqn for $F(x,y)$, as follows.

$\nabla^2\phi = \nabla^2\left\{G\alpha\left(\frac{b^2}{4} - y^2\right) + F\right\} = -2G\alpha$

$\Rightarrow \nabla^2 F = 0$, <u>bc's</u>: $F_{(x=\pm a/2)} = G\alpha\left(y^2 - b^2/4\right) \longrightarrow (a)$

$F_{(y=\pm b/2)} = 0 \longrightarrow (b)$

Assume variable-separable solution

$$F(x,y) = X(x)\, Y(y)$$

$$\nabla^2 F = 0 = X''Y + XY'' \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \overset{\lambda}{\cancel{\text{const}}}$$

From b.c.(b), $Y(y)$ cannot be exponential (hyperbolic) so it must be circular (trigonometric), ie $\lambda > 0$; in which case $X(x)$ is hyperbolic.

From b.c.(a), $F(x,y)$ is even in x, y, i.e. $X(x) = X(-x)$, $Y(y) = Y(-y)$, so $X(x) = \cosh \lambda x$ and $Y(y) = \cos \lambda y$. To satisfy b.c.(a), $\lambda = \frac{(2m+1)\pi}{b} = \frac{n\pi}{b}$, $m = 1, 2, 3, \cdots$, $n = 1, 3, 5, \cdots\cdots$. Thus,

$$F(x,y) = \sum_{n=1,3,5\cdots}^{\infty} A_n \cosh\left(\frac{n\pi}{b}x\right)\cos\left(\frac{n\pi}{b}y\right) \longrightarrow (c)$$

Thus bc (b) becomes,

$$\sum_{n=1,3,5,\cdots}^{\infty} A_n \cosh \frac{n\pi a}{2b} \cos \frac{n\pi y}{b} = G\alpha\left(y^2 - \frac{b^2}{4}\right) \longrightarrow ⊛$$

Expanding the rhs above as a $\cos \frac{n\pi y}{b}$ series,

$$G\alpha\left(y^2 - \frac{b^2}{4}\right) = \sum_{n=1,3,5\cdots}^{\infty} B_n \cos \frac{n\pi y}{b}, \quad B_n = \frac{2}{b}\int_{-b/2}^{b/2} G\alpha\left(y^2 - \frac{b^2}{4}\right)\cos\frac{n\pi y}{b}\,dy \rightarrow (**)$$

Equating coeffs of $\cos \frac{n\pi y}{b}$ from ⊛ & (**),

$$A_n \cosh \frac{n\pi a}{2b} = \frac{2}{b}\int_{-b/2}^{b/2} G\alpha\left(y^2 - \frac{b^2}{4}\right)\cos\frac{n\pi y}{b}\,dy$$

$$= -\frac{8G\alpha b^2}{\pi^3 n^3}(-1)^{\frac{n-1}{2}} \longrightarrow (d)$$

$$\Rightarrow \phi = G\alpha\left[\frac{b^2}{4} - y^2 - \frac{8b^2}{\pi^3}\sum_{n=1,3,5,\cdots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \frac{\cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)}{\cosh\left(\frac{n\pi a}{2b}\right)}\right]$$

From membrane analogy, max torsional shear stress occurs at midpoint of longer side, ie, pt A if $a > b$. Thus,

$$\tau_{max} = (\tau_{zx})_{\substack{x=0,\\ y=-b/2}} = \left(\frac{\partial\phi}{\partial y}\right)_{\substack{x=0,\\ y=-b/2}} = G\alpha b\left[1 - \frac{8}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2 \cosh\frac{n\pi a}{2b}}\right]$$

α in terms of M.

$$M = 2\iint \phi \, dx\,dy = 2\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} \phi \, dx\,dy =$$

$$= G\alpha\, a b^3 \left[\frac{1}{3} - \frac{64}{\pi^5}\frac{b}{a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\tanh \frac{n\pi a}{2b}}{n^5} \right]$$

$$\text{i.e., } \alpha = \frac{M}{ab^3 G\left[\frac{1}{3} - \frac{64}{\pi^5}\frac{b}{a} \sum_{n=1,3,5\cdots\infty}^{\infty} \frac{\tanh \frac{n\pi a}{2b}}{n^5} \right]} = \frac{M}{ab^3 G \beta}$$

Thus,

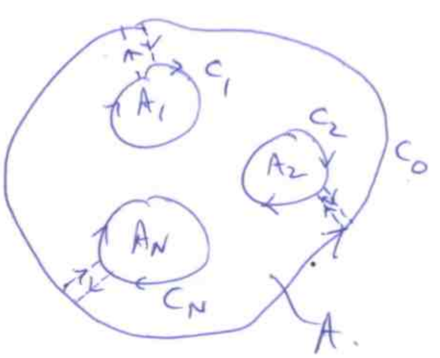
$$\tau_{max} = \frac{M\left[1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2 \cosh \frac{n\pi a}{2b}}\right]}{ab^2\left[\frac{1}{3} - \frac{64}{\pi^5}\frac{b}{a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\tanh \frac{n\pi a}{2b}}{n^5} \right]} = \frac{M}{ab^2 \beta_1}$$

The coefficients β, β_1 are dimensionless and depend on ratio a/b only. For $\frac{a}{b} \to \infty$, $(\beta, \beta_1) \to 1/3$, and we recover result for narrow rectangular section. For other a/b, β, β_1 are tabulated below.

a/b	β	β_1	a/b	β	β_1
1.0	0.141	0.208	3.0	0.263	0.267
1.2	0.166	0.219	4.0	0.281	0.282
1.5	0.196	0.230	5.0	0.291	0.291
2.0	0.229	0.246	10.0	0.312	0.312
2.5	0.249	0.258	∞	0.333	0.333

Torsion of Hollow Shafts (ie multicelled, not thin walled)

Let exterior contour be denoted c_0. Contours $c_1, c_2, \ldots c_N$ correspond to cavities of shaft.



Thus,

$$\nabla^2\phi = -2G\alpha \quad \text{in } A$$

$$\phi = 0 \quad \text{on } C_0$$

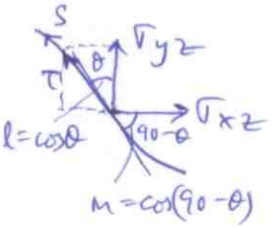
$$\phi = K_m \quad \text{on } C_m, \quad m = 1, \ldots N.$$

The K_m are determined from single-valuedness of ψ (hence w, since u, v are single valued by their definition). Thus,

$$G\alpha\left(\oint_{C_m} d\psi = 0\right)$$

$$\Rightarrow G\alpha\oint_{C_m}(\psi_{,x}dx + \psi_{,y}dy) = \oint_{C_m}\left[(\phi_{,y} + G\alpha y)dx - (\phi_{,x} + G\alpha x)dy\right] = 0$$

$$\oint_{C_m}\left(\underset{\tau_{xz}}{\underset{\downarrow}{\phi_{,y}}}\,\underset{(-m)}{\underset{\downarrow}{\frac{dx}{ds}}} - \underset{-\tau_{yz}}{\underset{\downarrow}{\phi_{,x}}}\,\underset{l}{\underset{\downarrow}{\frac{dy}{ds}}}\right)ds = G\alpha\oint_{C_m}\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)ds$$



Now $l\tau_{yz} - m\tau_{xz} = \tau$ (since ϕ = const on C_m, hence τ is tangential to C_m. Note +ve τ is along +s ∵ CCW +ve for (M)).

{or alternatively, $\phi_{,y}\frac{dx}{ds} - \phi_{,x}\frac{dy}{ds} = -\phi_{,y}\frac{\partial y}{\partial n} - \phi_{,x}\frac{\partial x}{\partial n} = -\frac{\partial\phi}{\partial n} = \tau$] from eq. (19), p. 70)}

So, $\frac{1}{G\alpha}\oint_{C_m}\tau\,ds = \oint(lx + my)\,ds \Rightarrow \frac{1}{G\alpha}\oint_{C_m}\tau\,ds = \iint_{A_m}(1+1)dA = 2A_m$

ie $\boxed{\oint_{C_m}\tau\,ds = 2G\alpha A_m}$ → ㊳ → (gives N equations)

Now $M = 2\iint_A \phi\,dx\,dy - \oint_{\cancel{S}\to \text{or} \equiv C}\phi(xl + ym)\,ds.$

where $\overset{\text{or}\equiv C}{\cancel{S}} = C_0 \cup C_1 \cup \cdots \cup C_N$ traversed in direction shown (ie C_0 traversed CCW, C_m traversed CW).

A = area of hollow section.

Choose $\phi = 0$ on C_0, $\phi = K_m$ on C_m

$$\therefore\ M = 2\iint_A \phi\, dA + \sum_{i=1}^{N} K_i \underbrace{\boxed{\oint_{C_i} (lx+my)\, ds}}_{2A_i}$$

$$\boxed{M = 2\iint_A \phi\, dA + \sum_{i=1}^{N} 2K_i A_i} \xrightarrow{} ⑶⑼$$

where we use $\phi = 0$ on C_0 and $\oint_{C_m} = -\oint_{C_m}$ in obtaining (39).

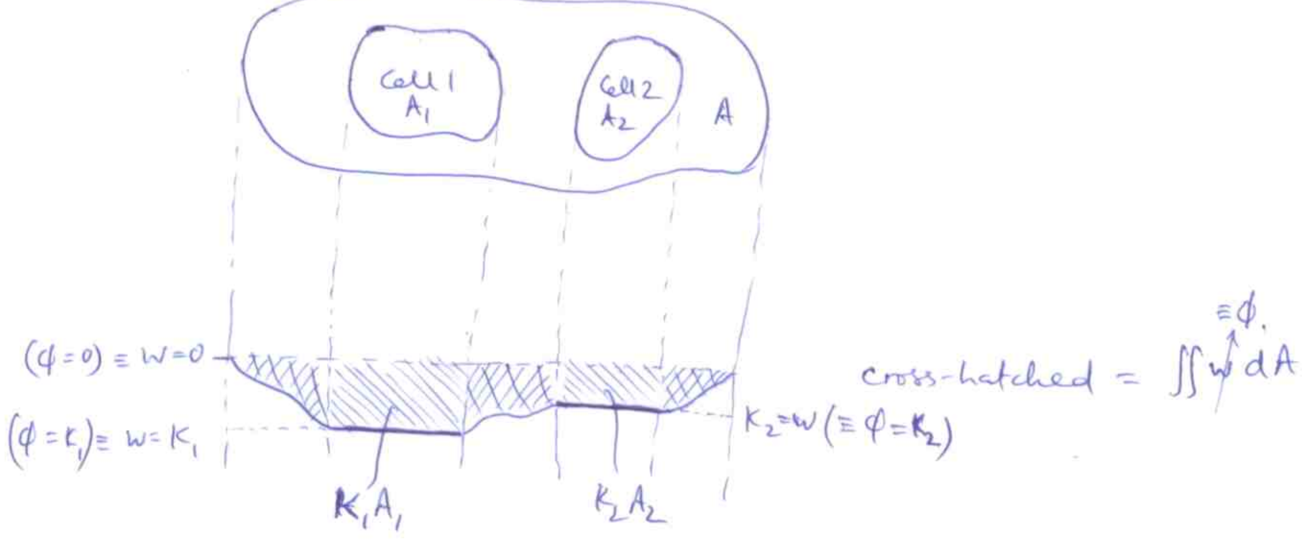
Now solve (38, 39) for α, K_i's. → ie N+1 unknowns

By membrane analogy, $M \equiv$ vol displaced by membrane with flat rigid plates over A_m, ie,

$$2\iint_A \phi\, dA \equiv 2(\text{vol displaced by membrane part})$$

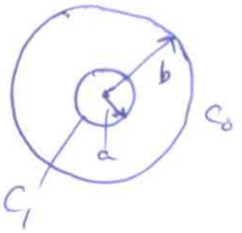
$$2\sum_{i=1}^{N} K_i A_i \equiv 2(\text{vol displaced by flat plates}).$$

(i-e, K_i is analogous to the vertical displ of the i^{th} flat rigid plate.)



(Eg) Hollow cylinder.

$$\phi = -\frac{G\alpha}{2}(x^2+y^2-b^2)$$



satisfies $\nabla^2\phi = -2G\alpha$

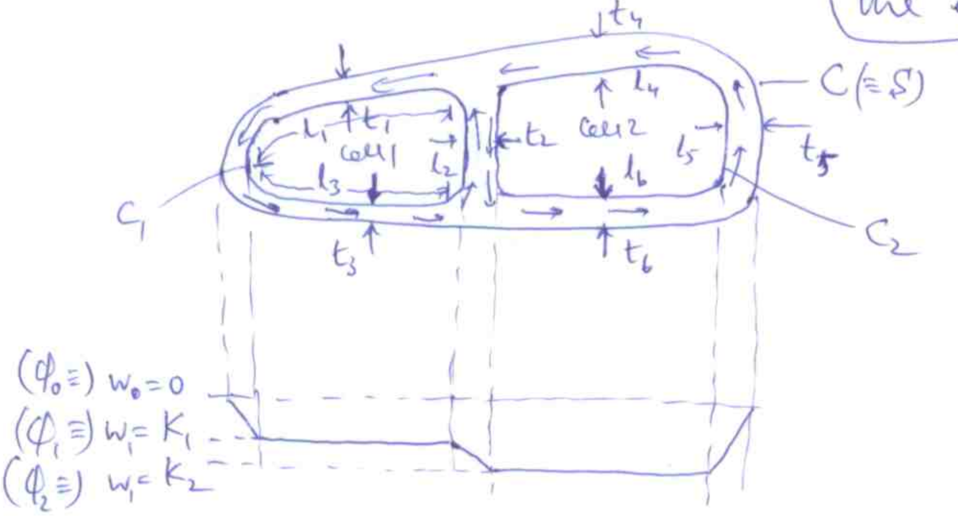
$\phi = 0$ on C_0

$\phi = K_1 = \frac{G\alpha}{2}(b^2-a^2)$ on C_1

$$M = 2\int_0^{2\pi}\int_a^b -\frac{G\alpha}{2}(r^2-b^2)\, r\,dr\,d\theta + 2\frac{G\alpha}{2}(b^2-a^2)\,\pi a^2$$

$$= \frac{G\alpha}{2}\pi(b^4-a^4) \longrightarrow$$ matches with result from basic solid mechanics course.

(Eg) Using (38), (39) obtain (by specializing) the same results for multi-celled TW tube, as were obt by using Eq (36), (37) in the example on p-82.



Since thin walled, assume (from membrane analogy) that $\phi(\equiv w)$ varies linearly over the thickness.

$$\tau = -\frac{\partial\phi}{\partial n} \longrightarrow$$

leg l_1 : $\tau_1 = -\left(\frac{\phi_0-\phi_1}{t_1}\right) = \frac{K_1}{t_1}$

l_2 : $\tau_2 = -\left(\frac{\phi_2-\phi_1}{t_2}\right) = \frac{K_1-K_2}{t_2}$

l_3 : $\tau_3 = -\frac{(\phi_0-\phi_1)}{t_3} = \frac{K_1}{t_3}$

l_4 : $\tau_4 = -\frac{(\phi_0-\phi_2)}{t_4} = \frac{K_2}{t_4}$

l_5 : $\tau_5 = -\frac{(\phi_0-\phi_2)}{t_5} = K_2/t_5$

l_6 : $\tau_6 = -\frac{(\phi_0-\phi_2)}{t_6} = K_2/t_6$

$$C_1 : \oint_{C_1}\tau\,ds = 2G\alpha A_1 \Rightarrow \frac{K_1}{t_1}l_1 + \frac{(K_1-K_2)}{t_2}l_2 + \frac{K_1}{t_3}l_3 = 2G\alpha A_1 \longrightarrow (A)$$

$$C_2 : \oint_{C_2}\tau\,ds = 2G\alpha A_2 \Rightarrow \frac{K_2}{t_4}l_4 + \frac{K_2}{t_6}l_6 + \frac{K_2}{t_5}l_5 + \frac{K_2-K_1}{t_2}l_2 = 2G\alpha A_2 \longrightarrow (B)$$

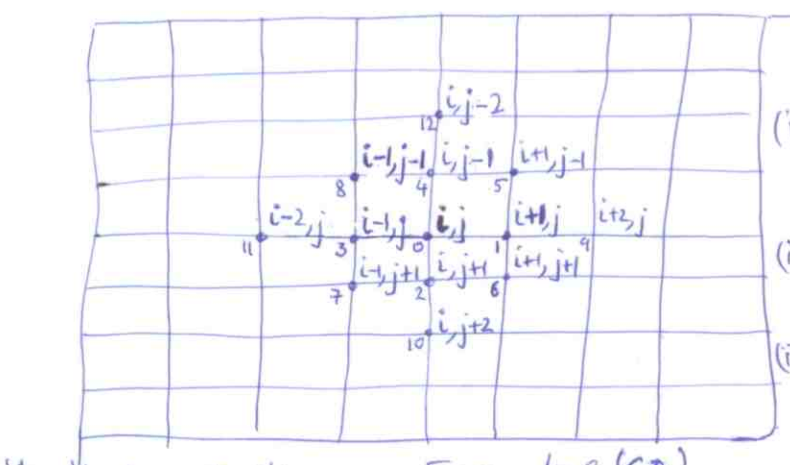
$$M = \underbrace{2\iint_A \phi \, dA}_{\simeq 0 \ \because \ A \text{ is small due to thin wall.}} + 2K_1A_1 + 2K_2A_2 \rightarrow (C)$$

From (A), (B), (C), get K_1, K_2, α. Comparing with (36, 37) see that $K_1 = q_1$, $K_2 = q_2$ are the shear flows.

FINITE DIFFERENCE METHOD FOR TORSION:

Consider simply connected domain only (formulation is more complex for multiply connected domains – ie need to do displacement formulation).

In FDM, the gov eqn and bc's are converted to their finite difference approximations, which are algebraic eqns.



$$(i) \leftarrow f_{i+1,j} = f_{i,j} + h\frac{\partial f}{\partial x}_{i,j} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}_{i,j}$$

$$(ii) \leftarrow f_{i-1,j} = f_{i,j} - h\frac{\partial f}{\partial x}_{i,j} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}_{i,j}$$

$$(iii) \leftarrow f_{i+2,j} = f_{i,j} + 2h\frac{\partial f}{\partial x}_{i,j} + 2h^2\frac{\partial^2 f}{\partial x^2}_{i,j}$$

Central Difference Formulae (CD)

(i), (ii) →

$$\frac{\partial f}{\partial x}_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$

$$\frac{\partial^2 f}{\partial x^2}_{i,j} = \frac{f_{i+1,j} + f_{i-1,j} - 2f_{i,j}}{h^2}$$

(CD)1-5 ←

similarly, $i \rightleftarrows j$

$$\frac{\partial f}{\partial y}_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2h}$$

$$\frac{\partial^2 f}{\partial y^2}_{i,j} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{h^2}$$

not reqd for Torsion ←

$$\frac{\partial^2 f}{\partial x \partial y}_{i,j} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}_{i,j}\right) = \frac{1}{4h^2}\left[\left(f_{i+1,j+1} + f_{i-1,j-1}\right) - \left(f_{i+1,j-1} + f_{i-1,j+1}\right)\right]$$

$$\frac{\partial^4 f}{\partial x^4}{}_{i,j} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}{}_{i,j}\right) = \frac{1}{h^4}\Big[f_{i+2,j} + f_{i,j} - 2f_{i+1,j} + f_{i,j} + f_{i-2,j} - 2f_{i-1,j} - 2f_{i+1,j} - 2f_{i-1,j} + 4f_{i,j}\Big]$$

$$= \frac{1}{h^4}\Big[6f_{i,j} - 4(f_{i+1,j} + f_{i-1,j}) + f_{i+2,j} + f_{i-2,j}\Big]$$

$\boxed{CD_{6-8}}$

$$\frac{\partial^2 f}{\partial x^2 \partial y^2}{}_{i,j} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}{}_{i,j}\right) = \frac{1}{h^4}\Big[f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j} + f_{i-1,j+1} + f_{i-1,j-1} - 2f_{i-1,j} - 2f_{i,j+1} - 2f_{i,j-1} + 4f_{i,j}\Big]$$

$$= \frac{1}{h^4}\Big[f_{i+1,j+1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i-1,j-1} - 2(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 2f_{i,j})\Big]$$

similarly (i⇄j)

$$\frac{\partial^4 f}{\partial y^4} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial y^2}{}_{i,j}\right) = \frac{1}{h^4}\Big[6f_{i,j} - 4(f_{i,j+1} + f_{i,j-1}) + f_{i,j+2} + f_{i,j-2}\Big]$$

Formulae CD_{5-8} are required for eg plane stress probls, ie not for torsion problems.

Forward/Backward difference Formulae

(i),(iii) →
$$\frac{\partial f}{\partial x}{}_{i,j} = \frac{4f_{i+1,j} - 3f_{i,j} - f_{i+2,j}}{2h} \longrightarrow (FD_1)$$

Similarly (replace h by (−h) and (+) by (−) in $(F.D_1)$)

$$\frac{\partial f}{\partial x}{}_{i,j} = \frac{-4f_{i-1,j} + 3f_{i,j} + f_{i-2,j}}{2h} \longrightarrow (BD_1)$$

Similarly (i⇄j)
$$\frac{\partial f}{\partial y}{}_{i,j} = \frac{4f_{i,j+1} - 3f_{i,j} - f_{i,j+2}}{2h} \longrightarrow (FD_2)$$

$$\frac{\partial f}{\partial y}{}_{i,j} = \frac{-4f_{i,j-1} + 3f_{i,j} + f_{i,j-2}}{2h} \longrightarrow (BD_2)$$

(i, iii) →
$$\frac{\partial^2 f}{\partial x^2}{}_{i,j} = \frac{-2f_{i+1,j} + f_{i,j} + f_{i+2,j}}{h^2} \longrightarrow (FD_3)$$

similarly (replace h by (−h) and (+) by (−) in (FD_3))

$$\frac{\partial^2 f}{\partial x^2}{}_{i,j} = \frac{-2f_{i-1,j} + f_{i,j} + f_{i-2,j}}{h^2} \longrightarrow (BD_3)$$

similarly $(i \rightleftarrows j)$
$$\frac{\partial^2 f}{\partial y^2}_{i,j} = \frac{-2f_{i,j+1} + f_{i,j} + f_{i,j+2}}{h^2} \longrightarrow (FD_4)$$

$$\frac{\partial^2 f}{\partial y^2}_{i,j} = \frac{-2f_{i,j-1} + f_{i,j} + f_{i,j-2}}{h^2} \longrightarrow (BD)_4 .$$

In general we use CD formulae ($\because$ more accurate as step size is smaller $\overset{ie}{=}$ h, and derivative is obtained based on function evaluation on either side of node in question). Use FD/BD formulae only when function evaluations on either side of node in question are not easily done – eg. at boundaries.

$$\therefore \nabla^2 \phi(i,j) = -2G\alpha$$

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1}, + \phi_{i,j-1} - 4\phi_{i,j} = -2G\alpha h^2 \longrightarrow \text{(40)}$$

Take nodal value of ϕ on boundary as zero ($\because \phi = 0$ on S)

So (40) can be solved for internal nodal values of ϕ, ie $\phi_{m,n}$.

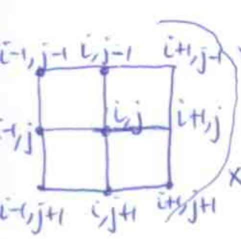
$$\tau_{zx} = \frac{\partial \phi}{\partial y}_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h} \quad \text{for interior node } (i,j)$$

$$= \frac{\pm 4\phi_{i,j\pm1} \mp 3\phi_{i,j} \mp \phi_{i,j\pm2}}{2h}$$

for boundary node (use + or − sign depending on whether boundary node is to left or right).

$$\tau_{zy} = -\frac{\partial \phi}{\partial x}_{i,j} = -\text{ve of above formulae with } i \rightleftarrows j .$$

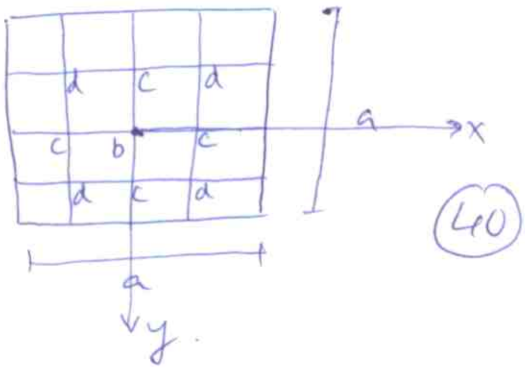
Thus $\phi_{m,n}$ and $\sigma_{xz(m,n)}$, $\sigma_{yz(m,n)}$ can be (solved) expressed in terms of $G\alpha h^2$ and $G\alpha h$, respectively. To express α in terms of applied M consider $M = 2\int_A \phi \, dx\, dy$. Using Simpsons formula in 2-D,



$$\int_{x_i-h}^{x_i+h}\int_{y_j-h}^{y_j+h} \phi \, dx\, dy = \frac{h^2}{9}\Big[16\phi_{i,j} + 4(\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i-1,j} + \phi_{i,j-1}) + \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\Big]$$

Adding the integral over all squares of sides 2h×2h we get $M = \text{number} * G\alpha h^4$ (ie numerical value) from which you solve for α in terms of M

(Fg).



choose $h = a/4$

Symmetry ⇒ for interior nodes, only distinct values are ϕ_b, ϕ_c, ϕ_d

$$㊵ \Rightarrow \quad 4\phi_b - 4\phi_c = 2G\alpha h^2$$
$$4\phi_c - \phi_b - 2\phi_d = 2G\alpha h^2$$
$$4\phi_d - 2\phi_c = 2G\alpha h^2$$

$$\Rightarrow \phi_c = \frac{9}{4} G\alpha h^2, \quad \phi_c = \frac{7}{4} G\alpha h^2, \quad \phi_d = \frac{11}{8} G\alpha h^2$$

$$\frac{M}{2} = 4 * \int_{x_d-h}^{x_d+h} \int_{y_d-h}^{y_d+h} \phi \, dx dy = \left(16\phi_d + 4(2\phi_c) + \phi_b\right) \frac{h^2}{9} * 4 = 17 G\alpha h^4$$

$$\Rightarrow \alpha = \frac{M}{34 G h^4} = \frac{M}{0.133 G a^4}$$ (exact value of coeff in denom is 0.141).

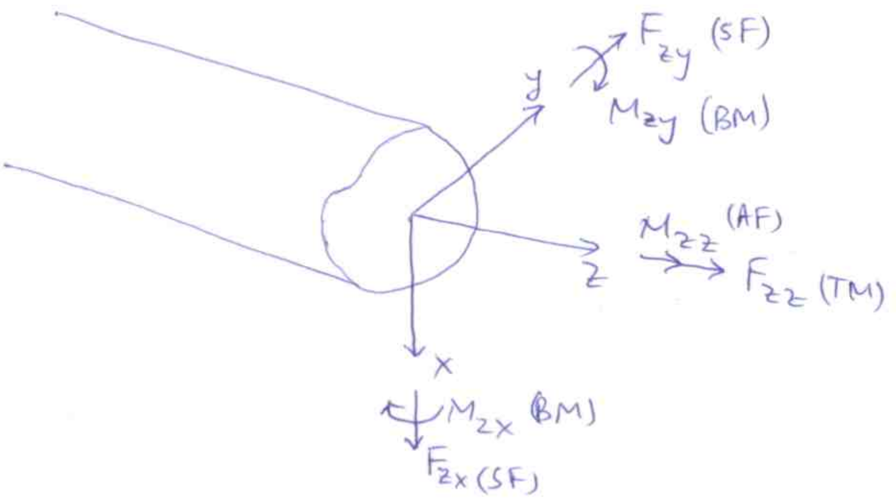
$$\tau_{max} = \sigma_{zy}\Big|_{\substack{x=a/2 \\ y=0}} = -\frac{\partial \phi}{\partial x}\Big|_{\substack{x=a/2 \\ y=0}} = -\frac{-4\phi_c + \phi_b}{2h} = \frac{19}{32} G\alpha a$$

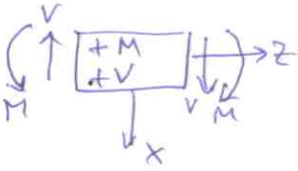
$$\Rightarrow \tau_{max} = \frac{19}{32} \frac{M}{0.133 a^3} = \frac{M}{0.224 a^3}$$ (exact value of coeff is 0.208).

BENDING OF BEAMS.

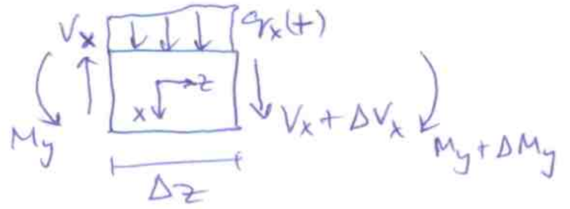
Pages betwn 90–95 are viscoelasticity, Now put at end as p-148–151 so 91–94 dont exist.

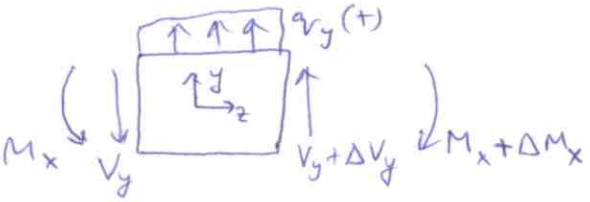
We orient the coordinate system as below (for convenience later on)





Relation between q, V, M.



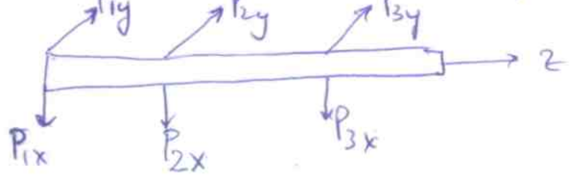


Equilibrium ⇒ $\frac{dV_x}{dz} = -q_x$ $\qquad \frac{dV_y}{dz} = -q_y$

$$\left.\begin{array}{ll} \dfrac{dM_y}{dz} = -V_x & \dfrac{dM_x}{dz} = V_y \\ \downarrow & \downarrow \\ \dfrac{d^2M_y}{dz^2} = q_x & \dfrac{d^2M_x}{dz^2} = -q_y \end{array}\right\} \rightarrow ①$$

M_x and M_y are max/min when V_y and V_x are zero, resply, in case of distributed loading.

When only point loading is applied along the beam length, such that the loads lie in xy plane, i.e,



the SF's (V_x, V_y) are const between loading points and BM's (M_x, M_y) are linearly varying between load points. Hence max M_x and max M_y occur at load pts. We have,

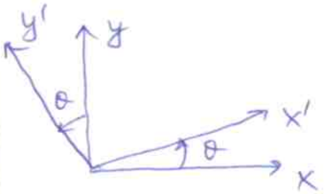
$$M_b^2 = M_x^2 + M_y^2 = (Az+B)^2 + (Cz+D)^2$$

$$\Rightarrow \frac{d^2(M_b^2)}{dz^2} = 2(A^2+C^2) \geq 0$$

ie M_b^2 is sub-harmonic $\Rightarrow (M_b^2)_{max}$ occurs at load pt. (ref. p.70).

Area Moments of Inertia

$$\left.\begin{aligned} I_{x'x'} &= \int_A y'^2\, dA \\ I_{y'y'} &= \int_A x'^2\, dA \\ I_{x'y'} &= \int_A x'y'\, dA \end{aligned}\right\} \rightarrow \underline{\underline{I}}' \triangleq \begin{pmatrix} I_{x'x'} & -I_{x'y'} \\ -I_{x'y'} & I_{yy} \end{pmatrix}$$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{a}} \begin{pmatrix} x \\ y \end{pmatrix}$$

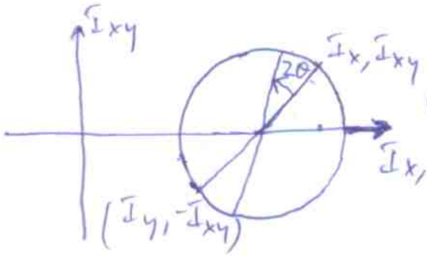
$$\left.\begin{aligned} &\underline{\underline{I}}' = \underline{\underline{a}}\, \underline{\underline{I}}\, \underline{\underline{a}}^T \\ &\text{i.e, } I_{x'x'} / I_{y'y'} = \frac{I_{xx}+I_{yy}}{2} \pm \frac{I_{xx}-I_{yy}}{2} \cos 2\theta \mp I_{xy} \sin 2\theta \\ &I_{x'y'} = \frac{I_{xx}-I_{yy}}{2} \sin 2\theta + I_{xy} \cos 2\theta \end{aligned}\right\} \rightarrow ②$$

Mohr's circle : (i) Take I_{xy} axis as upward +ve ($\because$ off-diagonal comp's are negative).

(ii) Center at $(I_{xx}+I_{yy})/2$

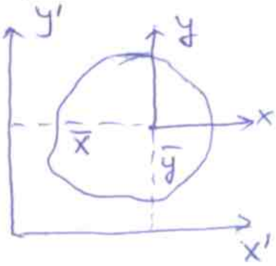
(iii) Radius $= \left[\left(I_{xx} - I_{yy}/2\right)^2 + (I_{xy})^2\right]^{1/2}$

(iv) Rot in physical plane $= +\frac{1}{2}$(Rot in Mohr's plane)



Parallel axis theorem.

$$I_{y'y'} = I_{yy} + \bar{x}^2 A$$
$$I_{x'x'} = I_{xx} + \bar{y}^2 A$$
$$I_{x'y'} = I_{xy} + \bar{x}\bar{y} A$$



(eg) S.T. $I_{x'x'} I_{y'y'} - I_{x'y'}^2 > 0$

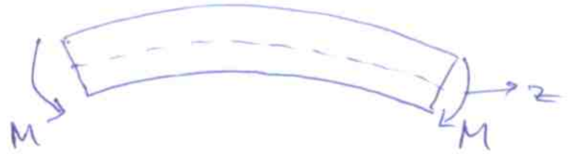
Let x, y be p-axes $\Rightarrow \det(I) = \det\begin{pmatrix} I_{xx} & 0 \\ 0 & I_{yy} \end{pmatrix} = I_{xx} I_{yy} > 0$

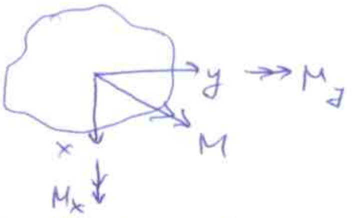
Now $\det(I') = \underbrace{\cancel{\det}(a|}_{=1} \det|I| \underbrace{\cancel{\det(a^T)}}_{=1} = \det(I)$

$\Rightarrow I_{x'x'} I_{y'y'} - I^2_{x'y'} > 0.$

(I) PURE BENDING OF PRISMATIC BEAM

Consider pure bending of prismatic bar by terminal couples M.





Choose the z-axis as line (ie locus) of centroids of sections.

Use semi-inverse method. Hence assume, (guided by introductory S.M).

$$\boxed{\sigma_z = -\frac{E x}{R_x} + \frac{E}{R_y} y, \quad \text{other stresses} = 0}$$

We will show that for appropriate R_x, R_y, the above is an exact solution. Observe that,

i) Above stress distribution satisfies equilibrium ($\because \sigma_z \neq f^n$ of z).

ii) Resulting strains are linear in x, y, hence compatibility satisfied.

iii) Traction free bc's on lateral surface satisfied ($\because \overset{n_3}{\cancel{n_z}} = 0$).
(stress distribution $\Rightarrow \sigma_1 = \sigma_2 = 0$, $n_z = 0 \Rightarrow \sigma_3 = 0$)

iv) On end faces, $F_x = \int_A \sigma_{xz}\, dA = 0$; $F_y = \int_A \sigma_{yz}\, dA = 0$

$$F_z = \int_A \sigma_z\, dA = -\frac{E}{R_x}\underbrace{\int_A x\, dA}_{=0} + \frac{E}{R_y}\underbrace{\int y\, dA}_{=0} = 0$$

($\because$ z is line of centroids)

v) Resultant moments:

$$\left.\begin{aligned} M_x &= \int_A y\,\sigma_z\, dA = -\frac{E}{R_x} I_{xy} + \frac{E}{R_y} I_x \\ M_y &= -\int_A x\,\sigma_z\, dA = \frac{E}{R_x} I_{yy} - \frac{E}{R_y} I_{xy} \end{aligned}\right\} \longrightarrow ③$$

Solve ③ →

$$\boxed{\frac{1}{R_x} = \frac{M_x I_{xy} + M_y I_x}{E(I_x I_y - I_{xy}^2)} \; ; \; \frac{1}{R_y} = \frac{M_x I_y + M_y I_{xy}}{E(I_x I_y - I_{xy}^2)}} \longrightarrow ④$$

④ (Euler-Bernoulli Law).

Thus the stress distribution

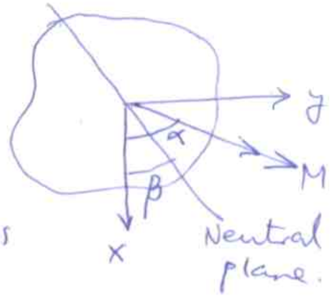
$$\boxed{\sigma_z = -\frac{(M_x I_{xy} + M_y I_x)}{(I_x I_y - I_{xy}^2)}\, x + \frac{(M_x I_y + M_y I_{xy})}{(I_x I_y - I_{xy}^2)}\, y} \longrightarrow ⑤$$

(other stresses zero)

solves the problem of pure bending of a prismatic bar, with applied terminal couples M_x, M_y about x, y axes, resply, and z-axis being line of centroids.

Along the neutral plane we have $\sigma_z = 0$

$$\Rightarrow \boxed{\tan\beta = \frac{y}{x} = \frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}}} \longrightarrow ⑥$$



Where β is ∠ of neutral plane with x-axis

$$\tan\alpha = \frac{M_y}{M_x} \longrightarrow ⑦.$$

So $\alpha = \beta$ iff,

$$M_x M_y (I_x - I_y) + (M_x^2 - M_y^2) I_{xy} = 0$$

If x, y are p-axes,

$$\boxed{\sigma_z = -\frac{M_y}{I_y} x + \frac{M_x}{I_x} y ;} \boxed{\frac{1}{R_x} = \frac{M_y}{E I_y} ; \frac{1}{R_y} = \frac{M_x}{E I_x}} \longrightarrow ⑧$$

Note : $I_{xy} = 0$ for areas possessing one (or two) axes of symmetry.

Displacements :

$$\left.\begin{aligned}
\varepsilon_x &= -\nu \frac{\sigma_z}{E} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} = \frac{\partial u}{\partial x} \\
\varepsilon_y &= -\nu \frac{\sigma_z}{E} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} = \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\sigma_z}{E} = -\frac{x}{R_x} + \frac{y}{R_y} = \frac{\partial w}{\partial z} \\
\gamma_{xz} &= 0 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
\gamma_{yx} &= 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{zy} &= 0 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{aligned}\right\} \longrightarrow ⑩$$

$$(10c) \longrightarrow \quad w = -\frac{xz}{R_x} + \frac{yz}{R_y} + g(x,y) \longrightarrow ⓘ$$

$$(10d, i) \longrightarrow \quad \frac{\partial u}{\partial z} = \frac{z}{R_x} - \frac{\partial g}{\partial x} \longrightarrow u = \frac{z^2}{2R_x} - z\frac{\partial g}{\partial x} + f(x,y) \longrightarrow (ii)$$

$$(10f, i) \longrightarrow \quad \frac{\partial v}{\partial z} = -\frac{z}{R_y} - \frac{\partial g}{\partial y} \longrightarrow v = -\frac{z^2}{2R_y} - z\frac{\partial g}{\partial y} + h(x,y) \longrightarrow (iii)$$

$$(10e, ii, iii) \longrightarrow \quad -2z\frac{\partial^2 g}{\partial x \partial y} + \underbrace{\frac{\partial f}{\partial y} + \frac{\partial h}{\partial x}}_{\text{fn of } (x,y) \text{ only}} = 0$$

$$\Rightarrow \frac{\partial^2 g}{\partial x \partial y} = 0, \quad \frac{\partial f}{\partial y} = -\frac{\partial h}{\partial x} \longrightarrow (iv, v)$$

$$(10a, ii, iii) \longrightarrow \quad -z\frac{\partial^2 g}{\partial x^2} + \underbrace{\frac{\partial f}{\partial x} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y}}_{\text{fn of } (x,y) \text{ only}}$$

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} = 0 \longrightarrow (vi)$$

$$\frac{\partial f}{\partial x} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} \Rightarrow f = \frac{\nu x^2}{2R_x} - \nu\frac{xy}{R_y} + m(y) \longrightarrow (vii)$$

$$(10b, ii, iii) \longrightarrow \quad -z\frac{\partial^2 g}{\partial y^2} + \underbrace{\frac{\partial h}{\partial y} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y}}_{\text{fn of } (x,y) \text{ only}}$$

$$\Rightarrow \frac{\partial^2 g}{\partial y^2} = 0 \longrightarrow (viii)$$

$$\frac{\partial h}{\partial y} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} \Longrightarrow h = \frac{\nu xy}{R_x} - \frac{\nu y^2}{2R_y} + n(x) \longrightarrow (ix)$$

$$(v, vii, ix) \longrightarrow \quad -\nu\frac{x}{R_y} + \underbrace{\frac{dm}{dy} = -\nu\frac{y}{R_x}}_{\text{fn of } y} - \frac{dn}{dx}$$

fn of x

$$\Rightarrow m = -\frac{\nu y^2}{2R_x} + c_1 y + c_2$$

$$n = +\frac{\nu x^2}{2R_y} - c_1 x + c_3$$

Apply bc's $u = v = w = 0$ at $(x, y, z) = (0,0,0)$, and rotations $\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$, $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$ are also zero at $(x,y,z) = (0,0,0)$ (to get rid of RB motions).

Now (iv, vi, viii) $\longrightarrow$ $g = c_4 x + c_5 y$

$$\Rightarrow u = \frac{z^2}{2R_x} - c_4 z + \frac{\nu}{2R_x}x^2 - \frac{\nu}{R_y}xy - \frac{\nu}{2R_x}y^2 + c_1 y + c_2$$

$$v = -\frac{z^2}{2R_y} - z c_5 + \frac{\nu}{R_x}xy - \frac{\nu}{2R_y}y^2 + \frac{\nu x^2}{2R_y} - c_1 x + c_3$$

$$w = -\frac{xz}{R_x} + \frac{yz}{R_y} + c_4 x + c_5 y$$

BC's $\Rightarrow$ $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ (could have got it by noting that const & linear terms in displ represent RB translation & rotation, so drop them).

$$\Rightarrow \left.\begin{aligned} u &= \frac{1}{2R_x}\left[z^2 + \nu(x^2 - y^2)\right] - \frac{\nu}{R_y}xy \\ v &= \frac{1}{2R_y}\left[-z^2 + \nu(x^2 - y^2)\right] + \frac{\nu}{R_x}xy \\ w &= -\frac{xz}{R_x} + \frac{yz}{R_y} \end{aligned}\right\} \longrightarrow ⑪$$

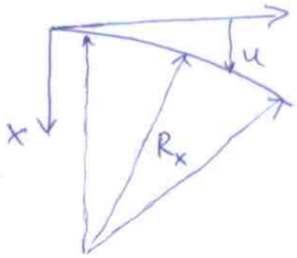
$\longrightarrow$ • add plane sections plane interp from w.

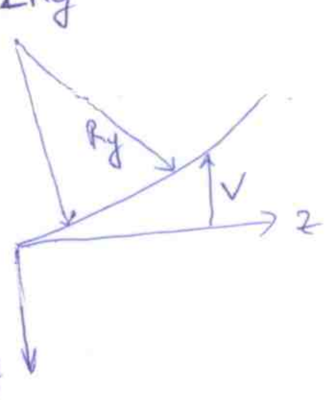
• add anticlastic surface interp

$$\Rightarrow \frac{\partial u}{\partial z^2} = \frac{1}{R_x}, \quad \frac{\partial^2 v}{\partial z^2} = -\frac{1}{R_y} \longrightarrow ⑫$$

Interpretation of R_x, R_y: Deflection of centroidal line $(x = y = 0)$ is,

$$u = \frac{z^2}{2R_x}, \quad v = \frac{-z^2}{2R_y}, \quad w = 0.$$





$$\text{Rad of curvature in xz plane} = \frac{1}{\kappa_x} = \frac{\left[1 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{3/2}}{\partial^2 u/\partial z^2}$$

$$\simeq \frac{1}{\left(\frac{\partial^2 u}{\partial z^2}\right)}, \text{ if } \left(\frac{\partial u}{\partial z}\right) \ll 1$$

This justifies terminology R_x, R_y.

$\Rightarrow R_x \simeq$ rad of curvature of centroidal deformed line in xz plane $\simeq R_x$

Similarly $R_y \simeq$ −ve of rad of curv of centroidal line in yz plane.

Filaments lying in neutral plane undergo no extension (w=0)

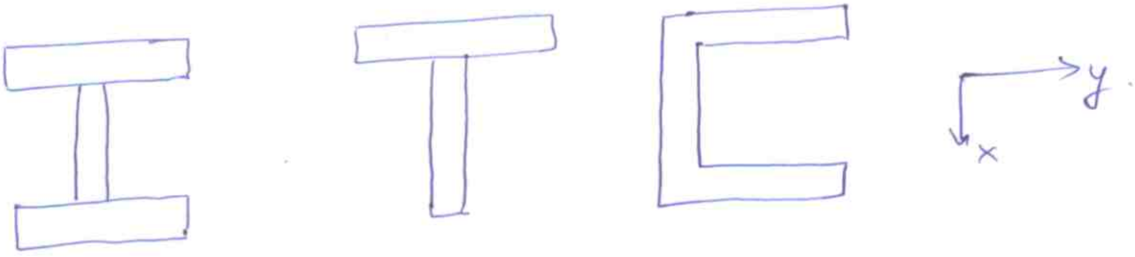
$w=0 \Rightarrow y = \frac{R_y}{R_x} x \rightarrow$ Same condition obtained by $\sigma_z = 0$ (see eqn ⑥).

If x, y are p-axes, ie $I_{xy}=0$,

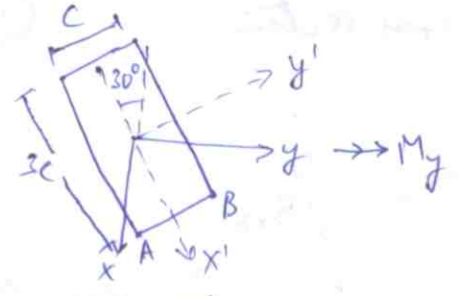
$$⑫, ④ \Rightarrow \frac{1}{R_x} = \frac{\partial^2 u}{\partial z^2} = \frac{M_y}{EI_y} \quad , \quad \frac{1}{R_y} = -\frac{\partial^2 v}{\partial z^2} = \frac{M_x}{EI_x} \longrightarrow ⑬$$

Eqn ⑬ is used in "engineering theory" to find deflection of centerline (which is assumed as deflection of whole beam as well) even when bending moment is not constant. Similarly ⑤ is used to find stresses in 'engineering theory', even when shear forces are present, resulting in a varying BM along the beam length. Experimental evidence, for thin metallic beams, shows this to be a reasonably good approximation.

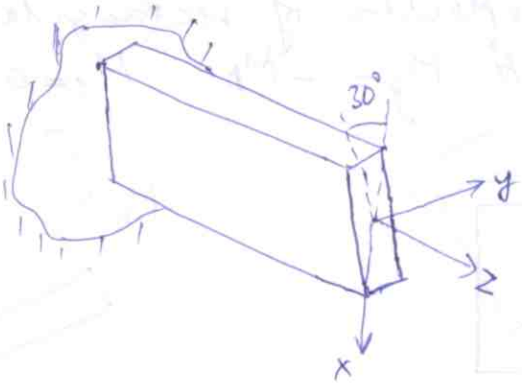
Eqn ⑬ shows that more the 'EI' less are the deflections. Thus 'EI' are moduli of flexural rigidity. To increase rigidity make moment of inertia large. Assuming $M_x = 0$, some crossections that have large I_y are shown below.



(Ex)

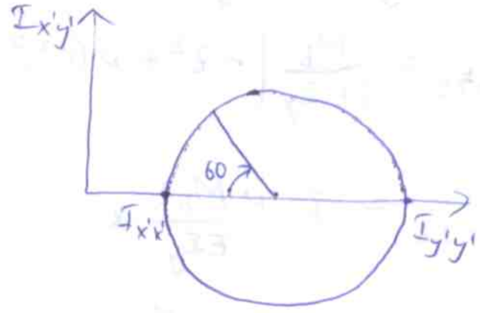


$M_x = 0$



$$I_{y'y'} = \frac{c(3c)^3}{12} = \frac{9c^4}{4}, \quad I_{x'x'} = \frac{3c(c^3)}{12} = \frac{c^4}{4}, \quad I_{x'y'} = 0$$

$$R = \frac{I_{y'y'} - I_{x'x'}}{2} = c^4$$



$I_{xy} = R\sin 60 = 0.87\,c^4$

$I_{xx} = I_{av} - R\cos 60 = 0.75\,c^4$

$I_{yy} = I_{av} + R\cos 60 = 1.75\,c^4$

Alternatively, $\begin{bmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{bmatrix} = \begin{bmatrix} \cos(30) & \sin(-30) \\ -\sin(-30) & \cos(-30) \end{bmatrix} \begin{bmatrix} c^4/4 & 0 \\ 0 & 9c^4/4 \end{bmatrix} \begin{bmatrix} \cos(30) & -\sin(-30) \\ \sin(-30) & \cos(-30) \end{bmatrix}$

gives same result.

Use $M_x = 0$ in ⑤, get,

$$\sigma_z = \frac{-M_y(0.75)c^4}{(0.75)(1.75)c^8 - (0.87)^2 c^8}\,x + \frac{M_y\,(0.87c^4)}{(0.75)(1.75)c^8 - (0.87)^2 c^8}$$

$$= \frac{-1.35\,M_y}{c^4}\,x + \frac{1.566\,M_y}{c^4}\,y$$

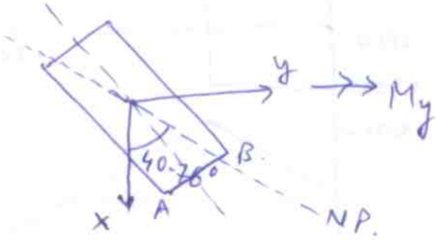
$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 3c/2 \\ -c/2 \end{pmatrix} \Rightarrow x_A = 1.55c, \quad y_A = 0.317c$$

$(\sigma_z)_A = \dfrac{-1.6}{c^3} M_y$ (compressive)

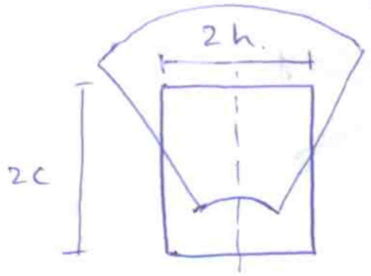
$$\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 3c/2 \\ c/2 \end{pmatrix} = \begin{pmatrix} 1.05c \\ 1.18c \end{pmatrix}$$

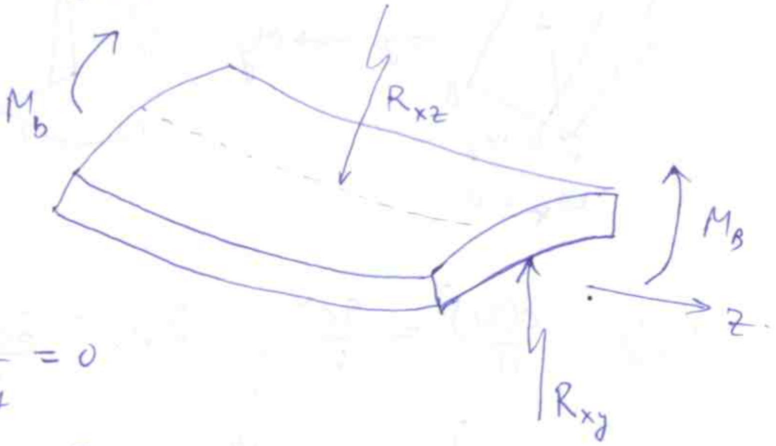
$(\sigma_z)_B = \dfrac{0.43}{c^3} M_y$ (Tensile).

$\tan\beta = \dfrac{0.75}{0.87} \longrightarrow \beta = 40.763°$



(Ex) Analyze deflection of rectangular cross section beam with $M_y = -M_b$, $M_x = 0$





$$\frac{1}{R_x} = \frac{M_y}{EI_y} = \frac{-M_b}{EI_y}, \qquad \frac{1}{R_y} = 0$$

$$u\Big|_{x=\pm c} = \frac{M_b}{2EI_y}\left[-z^2 + \nu\left(-c^2+y^2\right)\right]$$

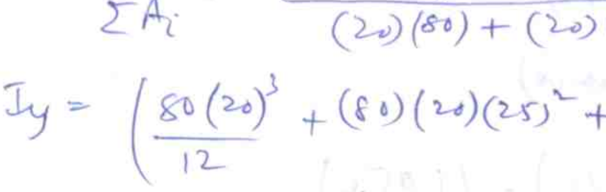
$$v\Big|_{y=\pm h} = \mp \nu \frac{M_b h}{EI_y} x$$

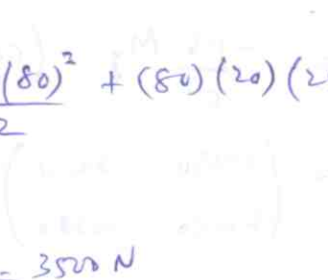
Surfaces originally parallel to yz plane are deformed into saddle-shaped (anticlastic) surfaces. Radius of curvature in xy plane is R/ν where R is rad. of curv in xz plane.

(ie, $\frac{\partial^2 u}{\partial y^2} = \frac{M_b}{EI_y}\nu = \frac{1}{R_{xy}}$; $\frac{\partial^2 u}{\partial z^2} = -\frac{M_b}{EI_y} = -\frac{1}{R_{xz}}$

$\Rightarrow R_{xy} = R_{xz}/\nu$).

(Ex)

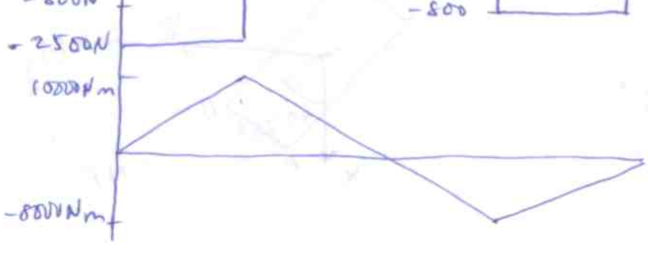




$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{(20)(80)(40) + (20)(80)(90)}{(20)(80) + (20)(80)} = 65$$

$$I_y = \left(\frac{80(20)^3}{12} + (80)(20)(25)^2 + \frac{(20)(80)^2}{12} + (80)(20)(25)^2\right) * 10^{-12}$$

$$= 2.907 * 10^{-6}\ m^4$$



$R_A = 3500$ N

$R_B = 800$ N

From SFD, BMD, you see that critical sections are z = 4m, and 22 m. Critical plane, at each z, are x = 65mm, 0mm, -35mm

(102b)

Assume $\sigma_z = -\frac{Mx}{I_y}$ holds despite shear force being present.

$z=4$ $\quad M = 10000$ N.m.

$$\sigma_z\Big|_{x=0.065} = \frac{-10000(0.065)}{2.9017\times10^{-6}} = -223.6 \text{ MPa}$$

$$\sigma_z\Big|_{x=-0.035} = 120.4 \text{ MPa} = \frac{-10000(-0.035)}{2.9067\times10^{-6}}$$

$$\tau_{xz}\Big|_{x=0} = \frac{VQ}{Ib}, \quad Q = \bar{x}A = \frac{(65)}{2}(65)(20)\times10^{-9} = 4.225\times10^{-5}$$

$$= \frac{-2500(4.225\times10^{-5})}{(2.907\times10^{-6})(0.02)} = -1.816 \text{ MPa}$$

$z=22$ m $\quad M = -8000$ N.m.

$$\sigma_z\Big|_{x=0.065} = \frac{(8000)(0.065)}{2.9067\times10^{-6}} = 178.9 \text{ MPa}$$

$$\sigma_z\Big|_{x=-0.035} = \frac{(8000)(-0.035)}{2.9067\times10^{-6}} = -96.3 \text{ MPa}$$

$$\tau_{xz}\Big|_{\substack{x=0\\z=22}} < \tau_{xz}\Big|_{\substack{z=4m\\x=0}} \quad \text{(in magnitude)}$$

$\therefore$ $\sigma_{max} = 178.9$ MPa, at $z=22$, $x=0.065$ m.

$\sigma_{min} = -223.6$ MPa, at $z=4$, $x=0.065$ m

$\tau_{max} = \frac{223.6}{2} = 111.8$ MPa at $z=4$ m, $x=0.065$ m.

Pure Bending of Composite Beams.

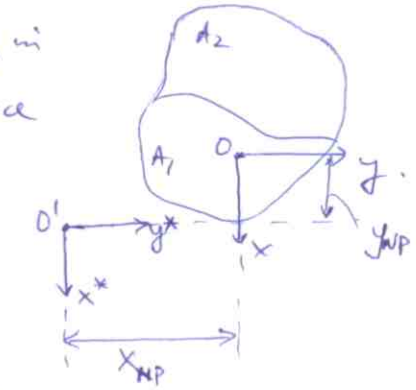
$\nu_1 = \nu_2$

Beam is prismatic and cross sections are uniform (geometry and material-distribution wise) thruout.

We follow semi-inverse procedure as before. So assume stress field of the form,

$$\boxed{\begin{aligned}\sigma_z &= -E_1 K_x x + E_1 K_y y \\ &= -E_2 K_x x + E_2 K_y y\end{aligned}} \quad \begin{aligned}&\text{in } A_1\\ &\text{in } A_2\end{aligned} \longrightarrow (14a)$$

remaining stress components $= 0$.



Note that z-axis is not the line of centroids, in general, now. Locate z-axis by zero axial force condition. ($F_z = 0$ on end face).

$$\int_{A_1+A_2} \sigma_z \, dA = -E_1 K_x \int_{A_1} x\,dA + E_1 K_y \int_{A_1} y\,dA - E_2 K_x \int_{A_2} x\,dA + E_2 K_y \int_{A_2} y\,dA = 0 \longrightarrow \circledast$$

Put $x = x^* - x_{NP}$, $y = y^* - y_{NP}$,

$$-E_1 K_x \bar{x}_1^* A_1 - E_2 K_x \bar{x}_2^* A_2 + (E_1 A_1 + E_2 A_2) K_x x_{NP} + E_1 K_y \bar{y}_1^* A_1 + E_2 K_y \bar{y}_2^* A_2 + (E_1 A_1 + E_2 A_2) K_y y_{NP} = 0.$$

The above should be valid for $K_x \neq 0$, $K_y = 0$ and vice-versa (ie one-plane bending also). Hence,

$$(14b) \longleftarrow \quad x_{NP} = \frac{E_1 A_1 \bar{x}_1^* + E_2 A_2 \bar{x}_2^*}{E_1 A_1 + E_2 A_2}, \quad y_{NP} = \frac{E_1 A_1 \bar{y}_1^* + E_2 A_2 \bar{y}_2^*}{E_1 A_1 + E_2 A_2}$$

So fix z-axis at (x_{NP}, y_{NP}) w.r.t. O' (the arbitrary ref frame). Here $\bar{x}_1^*$, $\bar{x}_2^*$, $\bar{y}_1^*$, $\bar{y}_2^*$ are coordinates of centroid of areas A_1, A_2 measured from O'.

K_x and K_y determined from,

$$M_x = \int_{A_1} y \sigma_z \, dA + \int_{A_2} y \sigma_z \, dA = -E_1 K_x (I_{xy})_1 + E_1 K_y (I_x)_1 - E_2 K_x (I_{xy})_2 + E_2 K_y (I_x)_2 \longrightarrow (**)$$

$$M_y = -\int_{A_1} x\sigma_z\,dA - \int_{A_2} x\sigma_z\,dA = E_1 K_x (I_y)_1 - E_1 K_y (I_{xy})_1 + E_2 K_x (I_y)_2 - E_2 K_y (I_{xy})_2 \longrightarrow (***)$$

Solve (**), (***),

$$K_x = \frac{F_3 M_x + F_1 M_y}{F_1 F_2 - F_3^2}, \quad K_y = \frac{F_2 M_x + F_3 M_y}{F_1 F_2 - F_3^2} \longrightarrow (1.4c)$$

where,

$$F_1 = E_1 (I_x)_1 + E_2 (I_x)_2$$
$$F_2 = E_1 (I_y)_1 + E_2 (I_y)_2$$
$$F_3 = E_1 (I_{xy})_1 + E_2 (I_{xy})_2$$

, I_x, I_y, I_{xy} are about axes thru O.

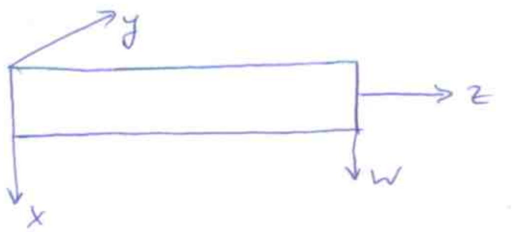
It is readily seen that equilibrium, compatibility, and bc's (lateral, $F_x = F_y = F_z = 0$ on end faces) are satisfied.

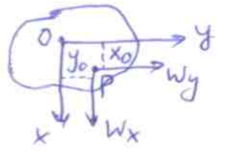
Now $\varepsilon_x = \varepsilon_y = -\nu\sigma_z = -\nu(-xK_x + yK_y)$ if $\nu_1 = \nu_2$. If $\nu_1 \neq \nu_2$ strains will be discontinuous across interface (ie delamination occurs).

(II) BENDING OF BEAMS BY TERMINAL LOADS.

(plane stress solution done on p.29–31).

Consider the problem of a cantilever beam having uniform cross section, and one end ($z=0$) fixed and the other end ($z=l$) loaded by some stess distribution that is statically equivalent to a single force $(W_x, W_y, 0)$ lying in the plane $z=L$ and acting at the load point (x_0, y_0, l). The z-axis is the line of centroids, and x, y are arbitrary axes. Lateral surface is traction free and body forces are negligible.





Using semi-inverse method, assume,

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

Guided by the fact that moment, M_y, produced by W_x acting alone is

$$M_y = W_x(l-z)$$

and that for bending of beams by terminal couple M_y we have,

$$\sigma_z = -\frac{M_y x}{I_y}$$

we assume for the present problem,

$$\boxed{\sigma_z = -E(l-z)(K_x x + K_y y)} \longrightarrow (15)$$

where K_x, K_y are constants to be determined from bc's

$$\int_A \tau_{xz}\, dA = W_x \,, \quad \int_A \tau_{yz}\, dA = W_y \,.$$

Equilibrium equations reduce to

$$(16) \longleftarrow \begin{cases} \dfrac{\partial \tau_{xz}}{\partial z} = 0 \,, \quad \dfrac{\partial \tau_{yz}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = 0 \Rightarrow \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + E(K_x x + K_y y) = 0 \end{cases}$$

BC's on lateral face are

$$\tau_{xz} n_x + \tau_{yz} n_y = 0 \text{ on } S' \text{ (other two i.s)} \longrightarrow (17)$$

Determination of K_x, K_y :

$$W_x = \int_A \tau_{xz}\, dA = \int_A \left[\tau_{xz} + x\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right)^{=0}\right] dA$$

$$= \int_A \left[\frac{\partial}{\partial x}(x\tau_{xz}) + \frac{\partial}{\partial y}(x\tau_{yz}) + \frac{\partial}{\partial z}(x\sigma_z)\right] dA$$

$$= \int_S x\underbrace{(\tau_{xz} n_x + \tau_{yz} n_y)}_{=0 \text{ on } S' \text{ (from (17))}} ds + \int_A x \frac{\partial \sigma_z}{\partial z}\, dA = \int_A x E(K_x x + K_y y)\, dA$$

$$= E K_x I_y + E K_y I_{xy} \longrightarrow (*)$$

Similarly, $W_y = \int_A \tau_{yz}\, dA = E K_x I_{xy} + E K_y I_x \quad (**)$

$$(*), (**) \longrightarrow \boxed{E K_x = \frac{I_x W_x - I_{xy} W_y}{I_x I_y - I_{xy}^2} \,, \quad E K_y = \frac{I_y W_y - I_{xy} W_x}{I_x I_y - I_{xy}^2}} \longrightarrow (18)$$

put $M_y = W_x(l-z)$
$M_x = -W_y(l-z)$ into (15)
and (18) and get (5). So
σ_z has same form
for pure bending or for bend by terminal loads.

Origin at centroid $\Rightarrow \int_A x\,dA = \int_A y\,dA = 0 \Rightarrow \int_A \sigma_z\,dA = 0$

$$\int_A \sigma_z y\,dA = -E(l-z)(K_x I_{xy} + K_y I_x) = -W_y(l-z) = M_x \quad \text{(checks out)}$$

$$-\int_A \sigma_z x\,dA = +E(l-z)(K_x I_y + K_y I_{xy}) = W_x(l-z) = M_y \quad \text{(checks out)}$$

Now ⑯ (Equilibrium) satisfied if stresses expressed as,

$$\left.\begin{aligned}\tau_{xz} &= \frac{\partial\phi}{\partial y} + f(y) - \tfrac{1}{2}EK_x x^2\\ \tau_{yz} &= -\frac{\partial\phi}{\partial x} - g(x) - \tfrac{1}{2}EK_y y^2\end{aligned}\right\} \longrightarrow \text{⑲}$$

where $\phi(x,y)$ is stress function.

Compatibility :

BM compat eqns reduce to,

$$\nabla^2\sigma_{ij} + \frac{1}{(1+\nu)}\Delta_{,ij} = 0, \qquad \Delta = \sigma_x + \sigma_y + \sigma_z = \sigma_z$$

$$\therefore\ \nabla^2\tau_{yz} + \frac{E}{(1+\nu)}K_y = 0 \overset{\text{⑲}}{\Longrightarrow} \frac{\partial}{\partial x}(\nabla^2\phi) = -\frac{E\nu}{(1+\nu)}K_y - \frac{\partial^2 g}{\partial x^2} \to \circledast$$

$$\nabla^2\tau_{xz} + \frac{E}{(1+\nu)}K_x = 0 \overset{\text{⑲}}{\Longrightarrow} \frac{\partial}{\partial y}(\nabla^2\phi) = \frac{E\nu}{(1+\nu)}K_x - \frac{\partial^2 f}{\partial y^2} \to \circledast\circledast$$

$$\circledast\circledast,\ \circledast \Rightarrow \boxed{\nabla^2\phi = -2G\nu K_y x - \frac{\partial g}{\partial x} + 2G\nu K_x y - \frac{\partial f}{\partial y} - 2G\alpha} \to \text{⑳}$$

where constant $(-2G\alpha)$ determined from twisting moment condition, i.e.,

$$\int_A (x\tau_{yz} - y\tau_{xz})\,dA = x_0 W_y - y_0 W_x \longrightarrow \text{㉑}$$

For lateral bc ⑰ to be satisfied,

$$\tau_{xz}n_x + \tau_{yz}n_y = 0 \Rightarrow \underbrace{\frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial x}\frac{dx}{ds}}_{``\frac{d\phi}{ds}"} = \left[\tfrac{1}{2}EK_x x^2 - f(y)\right]\frac{dy}{ds} - \left[\tfrac{1}{2}EK_y y^2 + g(x)\right]\frac{dx}{ds}$$

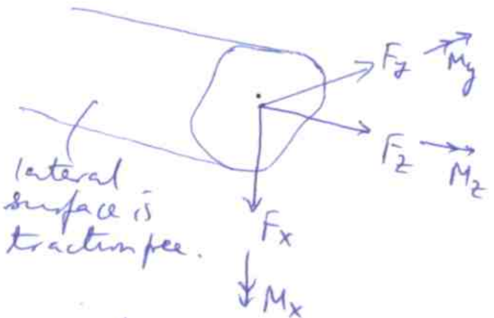
$$\text{(22a)} \longleftarrow \Rightarrow \boxed{\frac{d\phi}{ds} = 0} \text{ on } S \text{ if we choose}$$

$$\text{(23)} \longleftarrow \boxed{g(x) = -\tfrac{1}{2}EK_y y^2} \quad \text{and} \quad \boxed{f(y) = \tfrac{1}{2}EK_x x^2} \text{ on } S$$

Note that $g(x)$ and $f(y)$ can be chosen as any value on that part of S for which $dx/ds = 0$ and $dy/ds = 0$, respectively.

$\Rightarrow$ $\boxed{\phi = \text{const} = 0}$ ($\because$ value of const doesent affect stresses) $\rightarrow$ (22b)
on S if g, f chosen as in (22).

Summarizing, we have verified $F_z = 0$, $F_x = W_x$, $F_y = W_y$,
$M_x = -W_y(l-z)$, $M_y = W_x(l-z)$.



We choose α such that proper twisting moment $M_z = x_0 W_y - y_0 W_x$ is obtained.
We chose φ to automatically satisfy equilibrium.
Compatibility & traction-free bc's will be satisfied if we solve (20) & (23) with g, f chosen as per (22).
Thus a solution of (20) & (23) with (22) gives exact solution to the problem.

Neutral plane : $\varepsilon_z = \sigma_z = 0 \longrightarrow \tan\beta = \frac{y}{x} = -\frac{K_x}{K_y}$

(eg) : $I_{xy} = W_y = 0 \Rightarrow K_y = 0 \Rightarrow \beta = \frac{\pi}{2}$, ie yz is the N.P.

Interpretation of α.

Rotation (infinitesimal) of line/area element, in xy plane is

$$\underset{\substack{\downarrow \\ \omega_{21}}}{\varphi_z} = \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Define local twist at point (x,y) of a cross section as

$$(24) \leftarrow \boxed{\frac{\partial \omega}{\partial z}} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z}\right) = \left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{xz}}{\partial y}\right) \boxed{= \frac{1}{2G}\left(\frac{\partial \tau_{zy}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right)}$$

For pure torsion,

$$\frac{\partial \omega}{\partial z} = \frac{1}{2G}\overbrace{\left(-\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}\right)}^{2G\alpha} = \alpha \qquad \left\{\text{alternatively } \frac{\partial \omega}{\partial z} = \frac{1}{2}\frac{\partial}{\partial z}\left(\overset{\alpha xz}{\frac{\partial v}{\partial x}} - \overset{-\alpha yz}{\frac{\partial u}{\partial y}}\right) = \alpha\right\}$$

So local twist for pure torsion is constant over section (and also over z, if warping restraint is ignored).

For problem at hand, from (19), (24), (20)

$$\frac{\partial \omega}{\partial z} = -\frac{1}{2G}\left[\nabla^2 \phi + \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\right] = \alpha + \nu(K_y x - K_x y)$$

⇒ Mean twist = $\dfrac{\int_A \frac{\partial w}{\partial z}\, dA}{A} = \alpha$ = local twist at origin (centroidal line).

Thus twisting occurs in addition to bending. The general flexure problem can hence be resolved into:

(a) A flexure problem with $\alpha = 0$ (no mean local twist). The position of load point that yields this stress distribution (in which τ_{xz}, τ_{yz} determined for $\alpha = 0$) is obtained by twisting moment condition

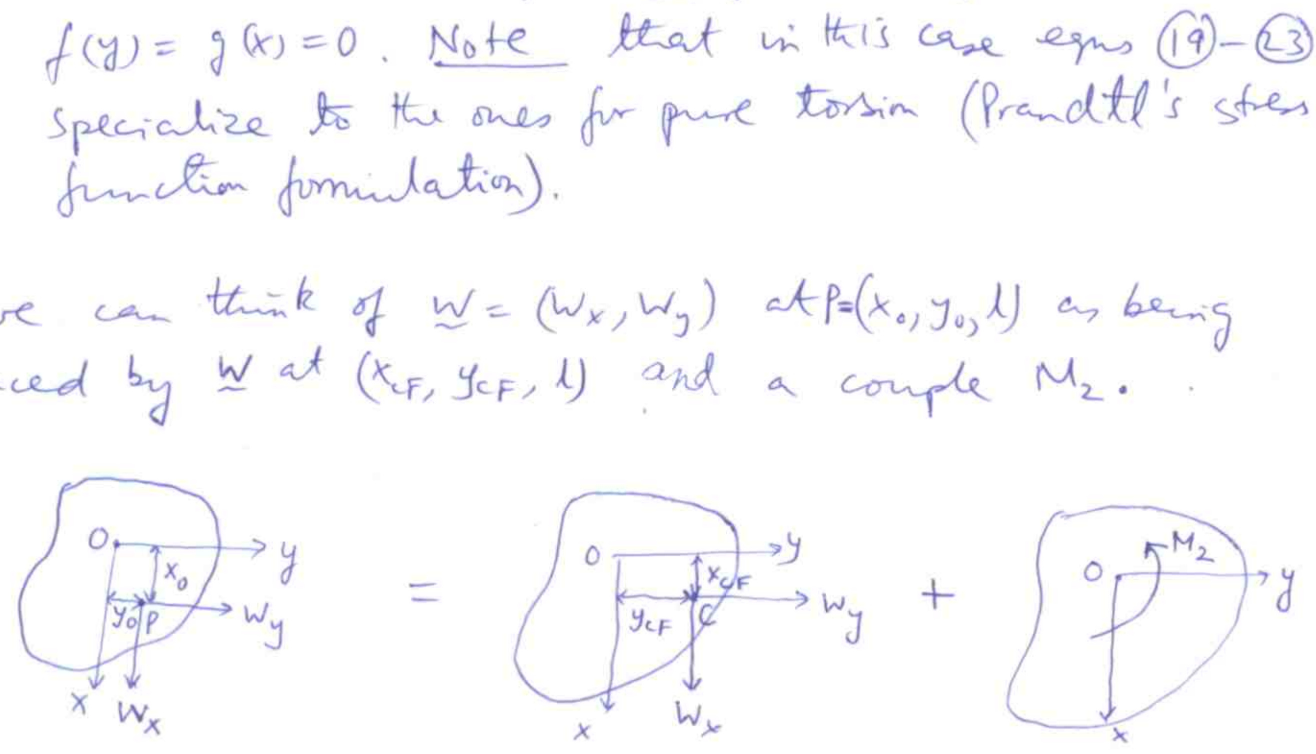
$$\int_A (x\tau_{yz} - y\tau_{xz})\, dA = x_{CF} W_y - y_{CF} W_x \longrightarrow$$ ㉕

(here τ_{yz}, τ_{xz} determined for $\alpha = 0$)

Eqn ㉕ holds for arbitrary loading. The load point (x_{CF}, y_{CF}, l) corresponding to zero mean-twist ($\alpha = 0$) is called 'center of Flexure' or 'Shear Center'.

(b) A torsion problem with mean twist α due to applied couple $M_2 = W_y(x_0 - x_{CF}) - W_x(y_0 - y_{CF})$, and with shear stresses determined (from ⑲) for $k_x = k_y = 0$, and hence $f(y) = g(x) = 0$. Note that in this case eqns ⑲–㉓ specialize to the ones for pure torsion (Prandtl's stress function formulation).

So we can think of $\underset{\sim}{W} = (W_x, W_y)$ at $P = (x_0, y_0, l)$ as being replaced by $\underset{\sim}{W}$ at (x_{CF}, y_{CF}, l) and a couple M_2.



General bending Bending w/o Twist + Torsion w/o Flexure.

Determination of Shear Center (CF)

Use ㉕ to determine (x_{CF}, y_{CF}) as follows. So ㉕, ⑲, (22b), ㉓ yield,

$$x_{CF} W_y - y_{CF} W_x = \int_A \left(x \tau_{yz} - y \tau_{xz} \right) dA$$

$$= \int_A \left[x \left(-\frac{\partial \phi}{\partial x} - g(x) - \frac{1}{2} E K_y y^2 \right) - y \left(\frac{\partial \phi}{\partial y} + f(y) - \frac{1}{2} E K_x x^2 \right) \right] dA$$

$$= \int_A \left[2\phi + \frac{\partial}{\partial x}\left[-x\phi - xy f(y) + \frac{1}{6} E K_x x^3 y \right] - \frac{\partial}{\partial y}\left[y\phi + xy g(x) + \frac{1}{6} E K_y x y^3 \right] \right] dA$$

$$= \int_A 2\phi \, dA + \oint_C \left[y\phi + xy g(x) + \frac{1}{6} E K_y x y^3 \right] dx + \oint_C \left[-x\phi - xy f(y) + \frac{1}{6} E K_x x^3 y \right] dy$$

$$= \int_A 2\phi \, dA - \frac{1}{3} E \left[K_y \oint_C x y^3 dx + K_x \oint_C x^3 y \, dy \right] \qquad \text{(used (22b), (23))}$$

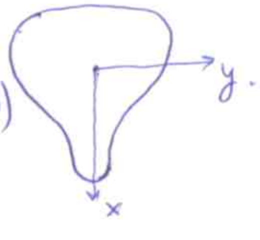
$$= \int_A \left(2\phi - E K_x x^2 y + E K_y x y^2 \right) dA$$

where ϕ is determined from (20), (22b), (23), setting $\alpha = 0$. Inserting (18) in the above & equating coefficients of W_x, W_y ($\because$ they are independent), we obtain

$$\left.\begin{aligned} x_{CF} &= \frac{1}{\cancel{W_y}} \int_A \left[2 \cancel{W_y} \phi_2 + \frac{I_{xy}}{F} \cancel{W_y} x^2 y + \frac{I_y}{F} \cancel{W_y} x y^2 \right] dA \\ &\text{where } F = I_x I_y - I_{xy}^2 \\ \text{and } y_{CF} &= -\frac{1}{\cancel{W_x}} \int_A \left[2 \cancel{W_x} \phi_1 - \frac{I_x}{F} \cancel{W_x} x^2 y - \frac{I_{xy}}{F} \cancel{W_x} x y^2 \right] dA \end{aligned}\right\} \rightarrow ㉖.$$

In the above we have used the fact that since (20) is linear in ϕ, its solution is $\phi = w_y \phi_2 + w_x \phi_1$ where ϕ_2 and ϕ_1 are solutions of (20), (22b), (23) with $(w_y, w_x) = (1,0)$ and $(0,1)$, respectively. Thus ϕ_1 and ϕ_2 depend on the geometry of the cross-section only, and consequently so does (x_{CF}, y_{CF}).

Section having one axis of symmetry:



For e.g. consider x-axis as the axis of symm. (ie $I_{xy} = 0$)
Let $W_x, W_y = 0$ be applied thro CF, so that ($\alpha = 0$) no twisting occurs.

Thus, (see physical interpretation below) $K_y = 0$, ie bending about y-axis (ie in xz plane) only w/o twist. Due to symmetry of cross-section about x-axis, we conclude that τ_{xz} is even in y and τ_{yz} is odd in y (the latter correctly yields $W_y = \int_A \tau_{yz}\,dA = 0$ as a check of this conclusion). From (25),

$$-y_{CF} W_x = \int_A (x\tau_{yz} - y\tau_{xz})\,dA \underset{\text{odd in } y}{=} 0 \quad \Rightarrow \underline{y_{CF} = 0 \text{ when}}$$

x is symm axis and vice versa

Thus a doubly symmetric section has CF as its centroid.

Physical interpretation of K_x, K_y

$$\frac{\partial u}{\partial z} = \frac{\tau_{xz}}{G} - \frac{\partial w}{\partial x} \leftarrow (\overset{\text{used}}{\text{strain}} \text{ displ + const. law})$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{1}{G}\underbrace{\frac{\partial \tau_{xz}}{\partial z}}_{=0\ (\text{equil})} - \frac{\partial^2 w}{\partial x \partial z} = -\frac{1}{E}\frac{\partial \sigma_z}{\partial x} = (l-z)K_x$$

Now, for $\frac{\partial u}{\partial z} \ll 1$, $\frac{\partial^2 u}{\partial z^2} \simeq \frac{1}{R_x} = (l-z)K_x$

Thus K_x is inversely proportional to radius of curvature in xz plane.

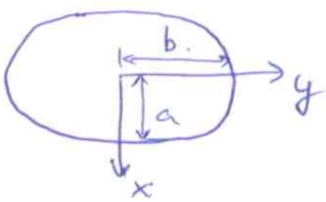
For special case when $W_y = 0$, $I_{xy} = 0$, ie loading parallel to a principal axis, $K_x = \frac{W_x}{EI_y}$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = (l-z)\frac{W_x}{EI_y} = \frac{M_y}{EI_y} \Rightarrow \boxed{M_y = EI_y \frac{\partial^2 u}{\partial z^2}}$$

Thus the Bernoulli Euler law applies even when a transverse load is applied if it is parallel to a p-axes.

(Ex 1) Elliptic ⊗-section, $(W_x, 0)$ applied at centroid. $(0,0)$

$\therefore I_{xy} = 0$, $K_y = 0$, $\sigma_z = -E(\ell - z) K_x x$.



$\because$ CF $= (0,0) \Rightarrow \alpha = 0$ as load applied at shear center.

Thus no twisting occurs.

Boundary eqn : $S \rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$\because K_y = 0$, $g(x) = 0$ on S (and hence choose $g(x) = 0$ on $\dot{A}$).

$$K_x = \frac{W_x}{E I_y} = \frac{4 W_x}{E \pi a^3 b}$$

$$f(y) = \frac{1}{2} E K_x x^2 = \frac{1}{2} \frac{W_x}{I_y} \overset{a^2(1-y^2/b^2)}{\cancel{x^2}} \quad \text{on } S$$

so choose $f(y) = \frac{W_x}{2 I_y} \frac{a^2}{b^2} (b^2 - y^2)$ on A

$$\therefore \nabla^2 \phi = \frac{W_x}{I_y} \left(\frac{\nu}{1+\nu} + \frac{a^2}{b^2} \right) y \quad \text{on } A \quad \text{—— (a)}$$

$$\phi = 0 \quad \text{on } S \quad \text{—— (b)}$$

$\therefore$ solution of the form $\phi = m y \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$ (c) will work.

Subst (c) in (a) to get m. We get,

$$\phi = \underbrace{\frac{a^2 \left[(1+\nu) a^2 + \nu b^2 \right]}{2(1+\nu)(3a^2 + b^2)} \frac{W_x}{I_y}}_{= m} y \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$

$$\therefore \tau_{xz} = \phi_{,y} + f(y) - \frac{1}{2} E K_x x^2 = \frac{2 W_x}{\pi a^3 b} \frac{\left[2(1+\nu) a^2 + b^2 \right]}{(1+\nu)(3a^2 + b^2)} \left[a^2 - x^2 - \frac{(1-2\nu) a^2 y^2}{2(1+\nu) a^2 + b^2} \right]$$

$$\tau_{yz} = -\phi_{,x} - \cancelto{0}{g(x)} - \frac{1}{2} E \cancelto{0}{K_y} y^2 = -\frac{4 W_x}{\pi a^3 b} \frac{(1+\nu) a^2 + \nu b^2}{(1+\nu)(3a^2 + b^2)} x y$$

$$\sigma_z = -\frac{4 W_x}{\pi a^3 b} (\ell - z) x.$$

$\therefore \tau_{yz} = 0$ on $x = 0$, and on $y = 0$.

For $(\tau_{yz})_{max}$ we need $(xy)_{max}$. Obviously $(xy)_{max}$ occurs on boundary S.

$$\underset{x}{x y = (a \cos\theta)} \underset{y}{(b \sin\theta)} = \frac{ab}{2} \sin 2\theta \Rightarrow (xy)_{max} = \frac{ab}{2} \text{ for } \theta = \frac{\pi}{4}$$

(parametric eqn of ellipse)

$$\therefore \boxed{(\tau_{yz})_{max} = \frac{2 W_x}{A} \frac{b}{a} \frac{(1+\nu) a^2 + \nu b^2}{(1+\nu)(3a^2 + b^2)}}, \text{ where } A = \pi a b.$$

consider last term (marked $\longmapsto$) in τ_{xz}.

$$\left(\frac{(1-2\nu)a^2y^2}{2(1+\nu)a^2+b^2}\right)_{max} \overset{?}{\gtrless} a^2 \text{, ie } b^2 \overset{?}{\gtrless} \frac{2(1+\nu)a^2+b^2}{(1-2\nu)}$$ → obviously '<' is correct choice.

$$\therefore \quad a^2-x^2-\frac{(1-2\nu)a^2y^2}{2(1+\nu)a^2+b^2}\begin{cases} > -a^2 \\ \leq a^2 \end{cases}$$ $\Rightarrow (\tau_{xz})_{max}$ occurs at $(x,y)=(0,0)$

$$\boxed{(\tau_{xz})_{max} = \frac{2W_x}{A}\left[\frac{2(1+\nu)a^2+b^2}{(1+\nu)(3a^2+b^2)}\right]}$$

For circular section $(a=b)$, elementary beam theory, which assumes τ_{xz} $\tau_{xz}[y]$, gives

$$(\tau_{xz})_{max} = \frac{4}{3}\frac{W_x}{A} \qquad \left\{\text{from } \tau_{xz} = \frac{\overset{W_x}{V}\overset{\int_x^a x'dA}{Q}}{I_y \underset{2\sqrt{a^2-x^2}}{t}} = \frac{4W_x(a^2-x^2)}{3\pi a^4}\right\}$$

$$\therefore \frac{[(\tau_{xz})_{max}]_{exact}}{[(\tau_{xz})_{max}]_{approx}} = \frac{3}{8}\left(\frac{3+2\nu}{1+\nu}\right)$$ → for $\nu = 0.3$, error $\simeq 4\%$.

(you get same result from elementary beam theory using VQ/It, $t=2b$, $Q=\int_0^a x'dA$)

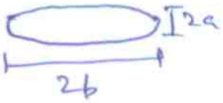
For $b \ll a$, $(\tau_{xz})_{max} \cong \frac{4}{3}\frac{W_x}{A}$, $(\tau_{yz})_{max} \cong \frac{4W_x}{3A}\frac{b}{2a}$

Also, putting $y=\varepsilon$ you get,

$$\tau_{xz} \simeq \frac{4W_x}{3Aa^2}\left[a^2-x^2-O(\varepsilon^2)\right], \quad \tau_{yz} = -\frac{4W_x}{3Aa^2}x\varepsilon = O(\varepsilon)$$

$\Rightarrow \tau_{xz} > \tau_{yz}$ (even when $x-a=\varepsilon$ you get $\tau_{xz} \cong 2\tau_{yz}$).

For $b \gg a$, $(\tau_{xz})_{max} \cong \frac{2}{1+\nu}\frac{W_x}{A}$, $(\tau_{yz})_{max} \cong \frac{2}{(1+\nu)}\frac{W_x}{a}\frac{\nu b}{a} \gg (\tau_{xz})_{max}$



Thus, for $b \gg a$ τ_{yz} is not negligible. Also, for $b \gg a$, $(\tau_{xz})_{max}$ has a large error, compared to elementary beam theory.

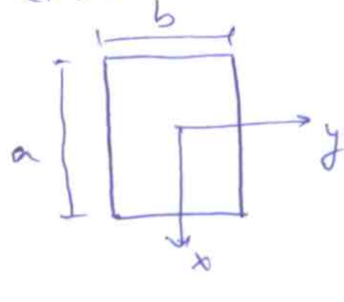
[redacted]

(Ex 2) Rectangular section, $(W_x, 0)$ applied at $(0,0)$.

As in (Ex 1), $g(x)=0$, $K_y=0$, $\alpha=0$.

Choose $f(y) = \frac{1}{2}EK_x\left(\frac{a}{2}\right)^2$ on A.

($\because$ for vertical boundaries, $\frac{dy}{ds}=0$ so on that part $f(y)$ can be any thing — ref p. 106).



$$\Rightarrow \nabla^2\phi = 2G\nu K_x y = \frac{\nu}{(1+\nu)}\frac{W_x y}{I_y} \longrightarrow (a)$$

= even in x, odd in y.

$$\phi = 0 \text{ on } S' \longrightarrow (b)$$

$$\Rightarrow \text{Take } \phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(2m+1)\frac{\pi x}{a}\sin 2n\frac{\pi y}{b} \longrightarrow (c)$$

(satisfies bc & even-odd condt).

Representing y as double Fourier series,

$$\left.\begin{aligned} y &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(2m+1)\frac{\pi x}{a}\sin 2n\frac{\pi y}{b} \\ B_{mn} &= \frac{4}{ab}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} y\cos(2m+1)\frac{\pi x}{a}\sin 2n\frac{\pi y}{b}\,dx\,dy. \end{aligned}\right\} \longleftarrow (d)$$

Subst (c), (d) in (a), and equating coeffs of $\cos(2m+1)\frac{\pi x}{a}\sin 2n\frac{\pi y}{b}$,

$$-A_{mn}\pi^2\left[\left(\frac{2m+1}{a}\right)^2 + \left(\frac{2n}{b}\right)^2\right] = \frac{\nu}{1+\nu}\frac{W_x}{I_y}\frac{4}{ab}\underbrace{\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} y\cos(2m+1)\frac{\pi x}{a}\sin 2n\frac{\pi y}{b}\,dx\,dy}_{\parallel \; \frac{4(-1)^{m+n+1}b}{(2m+1)n\left[\frac{(2m+1)^2}{a^2}+\frac{4n^2}{b^2}\right]}}$$

Thus proceed with soln using this Amn.

$$\tau_{xz} = \phi_{,y} + f(y) - \frac{1}{2}\frac{W_x}{I_y}x^2$$

$$= \phi_{,y} + \frac{1}{2}\frac{W_x}{I_y}\left[\left(\frac{a}{2}\right)^2 - x^2\right]$$

$$\tau_{yz} = -\phi_{,x}$$

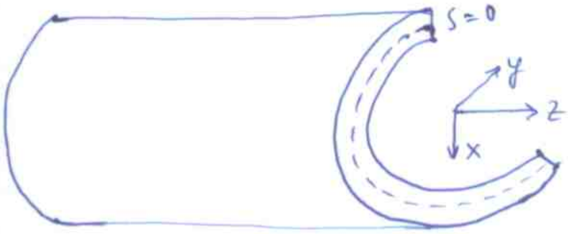
Elementary beam theory gives,

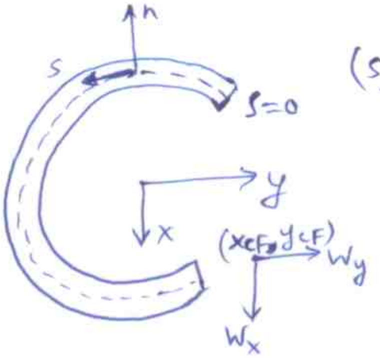
$$\tau_{xz} = \frac{W_x}{2I_y}\left[\left(\frac{a}{2}\right)^2 - x^2\right]$$ based on $\frac{VQ}{It}$ (ref eg. Popov), & $\tau_{yz} = 0$.

So $\phi_{,y}$ and $-\phi_{,x}$ appear to be corrections to elementary beam theory solution

TRANSVERSE SHEAR : 1-DIMENSIONAL SHEAR FLOWS FOR THIN-WALLED BEAMS

Although the expression for σ_z in ⑤ is for pure bending, we assume that it holds for general loading with shear force present (in fact in ⑱ we saw that it exactly holds true for case of terminal loads). So ⑤ is assumed valid when SF present & BM varies along span. We consider only beams whose thickness is small compared to length. In such beams, the shear flow can be assumed to be in the lengthwise direction. The shear stresses are assumed to be uniformly distributed across the thickness. All stresses apart from transverse shear and axial bending stresses are assumed zero. We assume that the point loads W_x, W_y are applied thru the shear center (x_{cF}, y_{cF}) so that no twisting occurs.



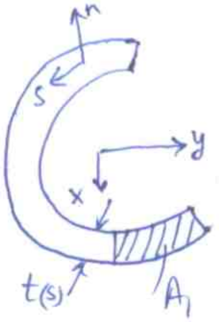


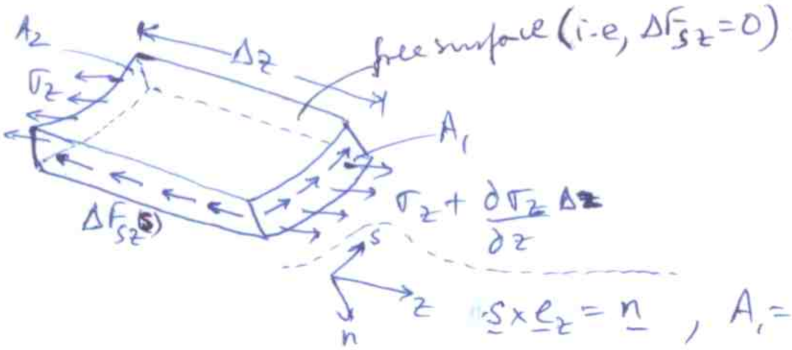
(s : anticlockwise +)

($\underline{s} \times \underline{e}_z = \underline{n}$).

Only σ_z and τ_{sz} are non-zero (assumed) and as per assumption,

$$\sigma_z = -\frac{M_x I_{xy} + M_y I_x}{I_x I_y - I_{xy}^2}\,x + \frac{M_x I_y + M_y I_{xy}}{I_x I_y - I_{xy}^2}\,y \longrightarrow \text{⑤ (repeated)}$$





$\underline{s} \times \underline{e}_z = \underline{n}$, $A_1 = A_2 = A$

$$\Sigma F_z : \quad \Delta F_{sz}^{(s)} - 0 = \int_A \left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\Delta z\right) dA - \int_A \sigma_z \, dA = \int_A \frac{\partial \sigma_z}{\partial z}\Delta z \, dA$$

$$\Rightarrow \quad q_{sz}^{(s)} \triangleq \left.\frac{\Delta F_{sz}^{(s)}}{\Delta z}\right|_{\lim \Delta z \to 0} = \int_A \frac{\partial \sigma_z}{\partial z} dA$$

$\downarrow$ shear flow.

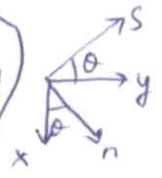
$$\frac{\partial \sigma_z}{\partial z} = -\frac{\overset{V_y}{\cancel{\frac{\partial M_x}{\partial z}}} I_{xy} + \overset{(-V_x)}{\cancel{\frac{\partial M_y}{\partial z}}} I_x}{I_x I_y - I_{xy}^2}\,x + \frac{\frac{\partial M_x}{\partial z} I_y + \frac{\partial M_y}{\partial z} I_{xy}}{I_x I_y - I_{xy}^2}\,y$$

$$\Rightarrow q_{sz} = \left[-\frac{V_y I_{xy}}{F} + \frac{V_x I_x}{F}\right] \underbrace{\int_A x\,dA}_{Q_y} + \left[\frac{V_y I_y}{F} - \frac{V_x I_{xy}}{F}\right] \underbrace{\int_A y\,dA}_{Q_x}$$

$$F = I_x I_y - I_{xy}^2$$

$\tau_{sz} = q_{sz}/t_c$ if we assume shear stress is thru thickness (i.e, uniformly distributed shear flow)

$$\Rightarrow \boxed{\begin{aligned} \tau_{sz} &= \left[\frac{-V_y I_{xy} + V_x I_x}{F t_c}\right] Q_y + \left[\frac{V_y I_y - V_x I_{xy}}{F t_c}\right] Q_x \\ \tau_{xz} &= -\tau_{sz} n_y = \left[\frac{V_y I_{xy} - V_x I_x}{F t_c}\right] Q_x n_y + \left[\frac{V_x I_{xy} - V_y I_y}{F t_c}\right] Q_x n_y \\ \tau_{yz} &= \tau_{sz} n_x = \left[\frac{-V_y I_{xy} + V_x I_x}{F t_c}\right] Q_y n_x + \left[\frac{V_y I_y - V_x I_{xy}}{F t_c}\right] Q_x n_x \end{aligned}} \longrightarrow (27\ a,b,c)$$



Shear Center

Equate moment generated by loads to moment generated by shear stresse distribution arising from bending only (w/o twist), i.e.,

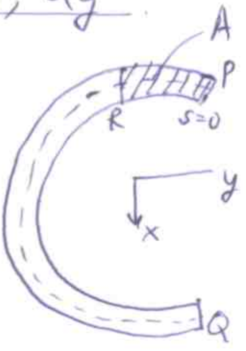
$$x_{CF} W_y - y_{CF} W_x = \int_A (x \tau_{yz} - y \tau_{xz})\, dA \longrightarrow (25) \text{ (repeated)}$$

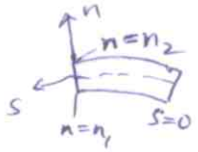
Subst 27(b, c) in (25), put $V_x = W_x$, $V_y = W_y$, equate coefficients of W_x (and of W_y) on either side — equivalent to putting $W_y = 0$, $W_x = 0$ independently, since CF does not depend on loads. We get, on using $dA = t_c ds$,

$$\left.\begin{aligned} x_{CF} &= -\frac{I_{xy}}{F}\left[\int_C Q_y (x n_x + y n_y)\, ds\right] + \frac{I_y}{F}\left[\int_C Q_x (x n_x + y n_y)\, dS\right] \\ &\qquad \left(\text{use } n_x = \frac{dy}{ds},\ n_y = -\frac{dx}{ds}\right) \\ &= -\frac{I_{xy}}{F}\left[\int_C Q_y (x dy - y dx)\right] + \frac{I_y}{F}\left[Q_x (x dy - y dx)\right] \\ y_{CF} &= -\frac{I_x}{F}\left[\int_C Q_y (x n_x + y n_y)\, ds\right] + \frac{I_{xy}}{F}\left[\int_C Q_x (x n_x + y n_y)\, ds\right] \\ &= -\frac{I_x}{F}\left[\int_C Q_y (x dy - y dx)\right] + \frac{I_{xy}}{F}\left[\int_C Q_x (x dy - y dx)\right] \end{aligned}\right\} \rightarrow (28)$$

The contour integrals are evaluated from $s=0$ to $s=s_{max}$. It is important to note that when load W_x, W_y is applied at (x_{CF}, y_{CF}), there is a twisting moment $(x_{CF} W_y - y_{CF} W_x)$ at the centroid, but the mean twist, ie, local twist 'α' at centroid (see p. 107–108) is zero $(\alpha = 0)$. Similarly, if load applied at centroid and CF $\neq$ centroid, then there is no twisting moment about centroid but local twist $\alpha \neq 0$.

Guidelines for calculation of Q_x, Q_y



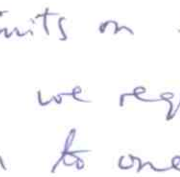


$Q_x = \int_A y\,dA$, $Q_y = \int_A x\,dA$

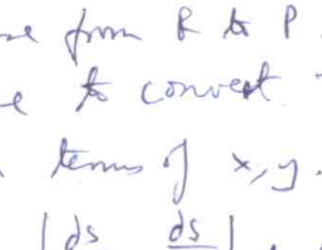
Q_x, Q_y can be either +ve or -ve.

Now, $dA = ds\,dn$ (for thin walled section).

The limits of integration for s are from the value of s at cut section to value of s at free surface. Limits for n are from least value of n (ie n_1) to greatest value of n (ie, n_2). It does not matter if for calculation of shear stresses at section R, above, we consider area A to be section RP or section RQ, as long as the end other than R is free.



(I)



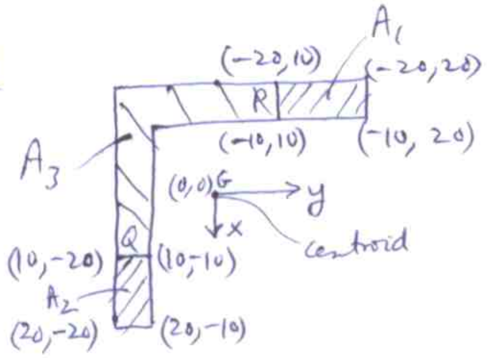
(II)

In case (I) limits on s are from R to P. In (II) it is from R to Q. In many cases we require to convert the integral originally in terms of s, n to one in terms of x, y. For that we use,

$$dA = ds\,dn = \begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial n}{\partial x} & \frac{\partial n}{\partial y} \end{vmatrix} dx\,dy = J\,dx\,dy.$$

The limits of integration for x, y are decided based on those for s, n.

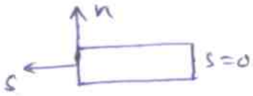
(Ex)



Find : Q_x, Q_y at sections P & Q.

All coordinates in cm.

At P : $s = -y$, $n = -x$



$ds\,dn = dx\,dy$.

$$Q_x = \int_{10}^{20}\int_{-10}^{-20} y\,dx\,dy = \frac{-10\,(20^2-10^2)}{2} = -1500\ \text{cm}^3$$

$$Q_y = \int_{10}^{20}\int_{-10}^{-20} x\,dx\,dy = \frac{10\left((-20)^2-(-10)^2\right)}{2} = 1500\ \text{cm}^3$$

} for direct use in (27)

$n_x = -1$, $n_y = 0 \implies \tau_{xz} = 0$, $\tau_{yz} = -\tau_{sz}$

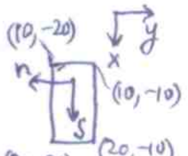
Note: If we take statical first moment of area A_1 you get

$$\boxed{Q_x = \bar{y}A_1 = 15*(10*10) = 1500, \quad Q_y = \bar{x}A_1 = -15*(10*10) = -1500}$$

for use in negative of (27)

Both these are negative of Q_x, Q_y obtained above. If we wish to use these then τ_{sz}, τ_{xz}, τ_{yz} results in (27) will have to be negative of what appears in (27). This arises due to our derivation of $q_{sz}(s) = \int_A \frac{\partial\sigma_z}{\partial z}\,dA$ which implies tacilly that limits are $(s,0)$ and (n_1, n_2). However, if we instead chose limits as $(0,s)$ and (n_1,n_2) for integrating over A, then we would have gotten $q_{sz}(s) = -\int_A \frac{\partial\sigma_z}{\partial z}\,dA$, as in Srinath, in which Q_x, Q_y are statical moments. In our convention, Q_x, Q_y are negative of statical moments.

At Q : $s = x$, $y = -n$, $ds\,dn = -dx\,dy$



$$Q_x = \int_{-10}^{-20}\int_{10}^{20} y\,dx\,dy = (-10)*\left(\frac{20^2-10^2}{2}\right) = -1500\ \text{cm}^3$$

$$Q_y = \int_{-10}^{-20}\int_{10}^{20} x\,dx\,dy = (10)*\frac{\left((-20)^2-(-10)^2\right)}{2} = 1500\ \text{cm}^3$$

Note: If we take statical first moment of A_2, you get same Q_x and Q_y as above. This is because statical moment of A_2 is negative of statical moment of compliment of A_2 $(= A_1 + A_3)$, due to fact that $\int\limits_{\text{Total area}} x\,dA = \int\limits_{\text{Total area}} y\,dA = 0$. Now if we use Srinath's convention then we need to use statical moment of $(A_1 + A_3)$ (ie∵ limits are $(0,s)$, (n_1, n_2)) along with negative of (27).

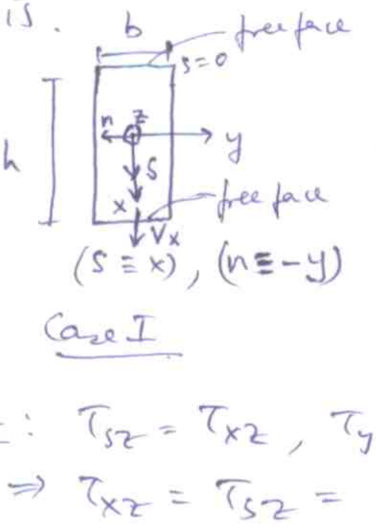
→ So if we use present convention then no need to bother whether our section is A_2 or $A_1 + A_3$, since Q_x, Q_y will come out the same provided we follow the guidelines for limits, ie, s from cut-section to free-surface, n from min to max.

However, if we use Srinath convention with negative of (27), the Q_x, Q_y are based on first statical moment area corresponding to $\int_0^s$, ie, $(A_1 + A_3)$, so care should be taken to reverse sign on Q_x, Q_y in case we compute it based on first statical moment of A_2 ie, corresponding to $\int_s^{s_{end}}$.

For section at Q, $n_x = 0$, $n_y = -1$, ie, $\tau_{xz} = \tau_{sz}$, $\tau_{yz} = 0$.

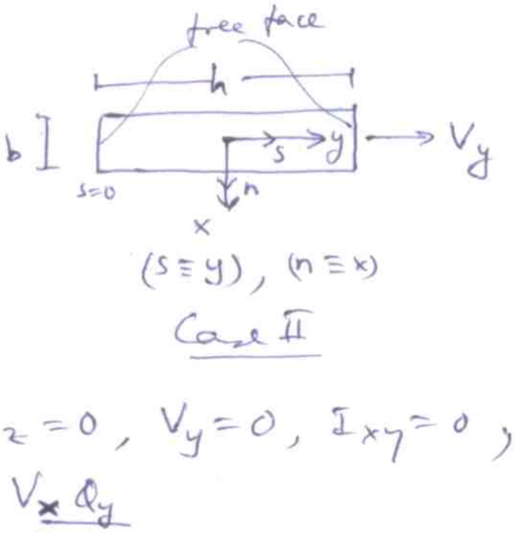
Note: If centerline is straight, then s need not be counterclockwise +ve. Instead, for convenience, we could align s along either the x or the y axis.

(Ex)



$(s \equiv x)$, $(n \equiv -y)$

Case I

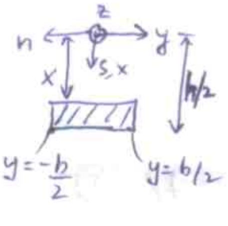


$(s \equiv y)$, $(n \equiv x)$

Case II

Case I: $\tau_{sz} = \tau_{xz}$, $\tau_{yz} = 0$, $V_y = 0$, $I_{xy} = 0$, $ds\,dn = -dx\,dy$

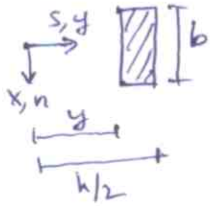
$$\Rightarrow \tau_{xz} = \tau_{sz} = \frac{V_x Q_y}{I_y b}$$



$$Q_y = \int\int x\,ds\,dn = -\int_{+b/2}^{-b/2}\int_x^{h/2} x\,dx\,dy = \frac{b\left(\frac{h^2}{4} - x^2\right)}{2}$$

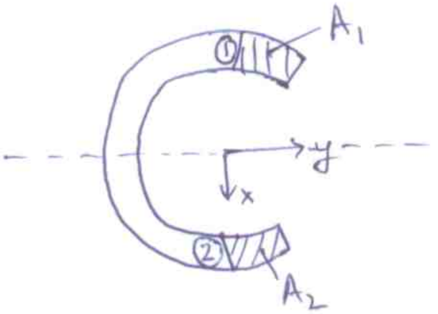
Case II : $\tau_{sz} = \tau_{yz}$, $\tau_{xz} = 0$, $V_x = 0$, $I_{xy} = 0$, $ds\,dn = dx\,dy$.

$$\Rightarrow \tau_{yz} = \tau_{sz} = \frac{V_y Q_x}{I_x b}$$



$$Q_x = \iint y\,ds\,dn = \int_{-b/2}^{b/2}\int_{y}^{h/2} y\,dx\,dy = \frac{b\left(\frac{h^2}{4} - y^2\right)}{2}$$

(Ex) CF for symmetric sections: Already discussed on p. 110 for general thick walled sections. We reconfirm the conclusions for the special case of thin walled sections, as follows.



$I_{xy} = 0$, y-axis is symm axis.
From (28)

$$x_{CF} = \frac{I_y}{F}\int_C (x n_x + y n_y) Q_x\,ds = \frac{1}{I_x}\int_C (x\,dy - y\,dx) Q_x$$

Considering points ① and ② with coordinates (−x, y) & (x, y) resply. Now $(ds\,dn)_{A_1} = -(ds\,dn)_{A_2}$.

$$\Rightarrow (Q_x)_{①} = \iint_{A_1} y\,ds\,dn = -\iint_{A_2} y\,ds\,dn = (-Q_x)_{②}$$

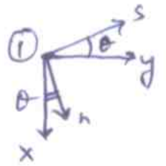
$$(Q_y)_{①} = \iint_{A_1} x\,ds\,dn = \iint_{A_2} (-x)(-ds\,dn) = (Q_y)_{②}$$

can also be seen by statical moment in a more direct manner, ie

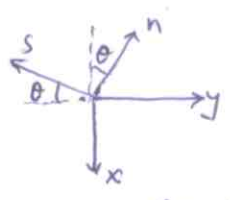
$(Q_x)_{②} = -(Q_x)_{①}$

$(Q_y)_{②} = -(-(Q_y)_{①})$

Also n_x at ① & ② have opposite signs, but n_y has same sign as seen below



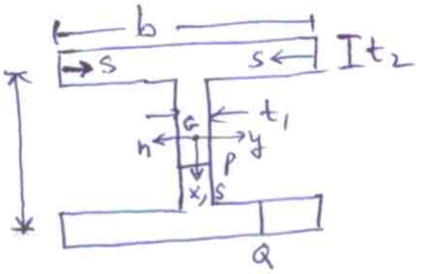
$n_x = \cos\theta$
$n_y = \sin\theta$



$n_x = \cos(2\pi - (\pi + \theta))$
$= -\cos\theta$
$n_y = \sin(2\pi - (\pi + \theta))$
$= \sin\theta$

Thus $x Q_x n_x$ and $y Q_x n_y$ have opposite signs at symmetrical points above and below the y-axis. Considering all such pairs of symmetrical points, we see that $x_{CF} = 0$, in agreement with result on p. 110. Similarly $y_{CF} = 0$ if x-axis is symmetry axis, and for doubly symmetric section (x_{CF}, y_{CF}) = centroid $= (0, 0)$

(Ex)



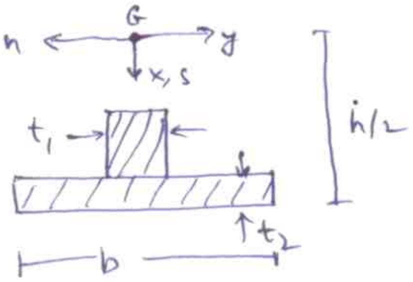
Given: Symm I section, $V_y = 0$.

Find: shear stresses at P & Q.

Doubly symm ⇒ $x_{GF} = y_{CF} = 0$

At section P:

$$\tau_{xz} = -\tau_{sz}\overset{\sin 270^\circ}{\cancel{n_y}} = \tau_{sz} = \frac{V_x Q_y}{I_y t_1}$$



Q_y computed based on shaded ⊗ section as follows:

$$Q_y = \iint x\,dA = \int_{\frac{t_1}{2}}^{-t_1/2}\int_{X}^{h/2} (-dx\,dy)\,x + 2\int_0^{b/2} \overset{x}{\frac{h}{2}}\,\overset{dx}{\frac{t_2}{2}}\,dy = t_1\frac{\left(\frac{h^2}{4}-x^2\right)}{2} + 2\frac{b}{2}\frac{h}{2}t_2$$

Alternatively, first statical moment gives,

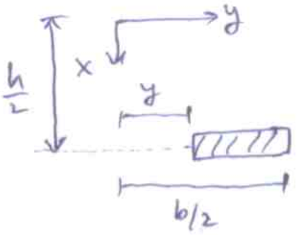
$$Q_y = \underset{\text{for complimentary area}}{-}\left(\underset{\text{for present convention w/ Srinath's convention}}{-}\text{first statical moment}\right) = \text{first statical moment}$$

(They match !!)

$$= b t_2 \frac{h}{2} + \left(\frac{h}{2}-x\right) t_1\left(\frac{\frac{h}{2}-x}{2}+x\right) = b t_2\frac{h}{2} + \frac{t_1}{2}\left(\frac{h^2}{4}-x^2\right)$$

At section Q:

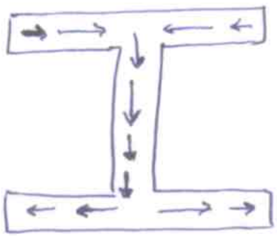
$$\tau_{yz} = \tau_{sz}\overset{\cos 0^\circ}{\cancel{n_x}} = \tau_{sz} = \frac{V_x Q_y}{I_y t_2}$$



Q_y computed based on shaded area, as follows:

$$Q_y = \iint_A x\,dx\,dy = \int_y^{b/2}\frac{h}{2}t_2\,dy = \frac{h}{2}t_2\left(\frac{b}{2}-y\right)$$

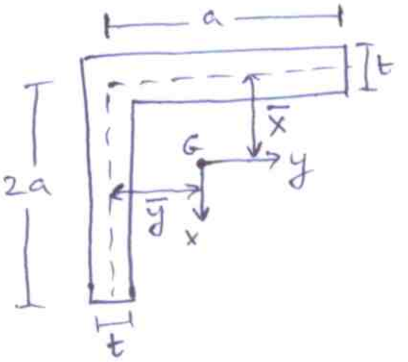
The shear flow is represented as,



If either web or flange are made too thin, we could have buckling of web or flange, respectively, seen as follows:

eg $\tau = \tau_{max}$ at neutral axis

(Ex) Find distribution of shear flows, and the shear center location. $V_y = 0$ given.



Assume $t \ll a$.

$$\bar{x} = \frac{a(2at)}{3at} = \frac{2}{3}a \; ; \; \bar{y} = \frac{\frac{a}{2}(at)}{3at} = \frac{a}{6}$$

$$I_{yy} = \underbrace{\frac{at^3}{12}}_{\text{neglect}} + at\left(\frac{4}{9}a^2\right) + \frac{(2a)^3t}{12} + 2at\left(a - \frac{2}{3}a\right)^2$$

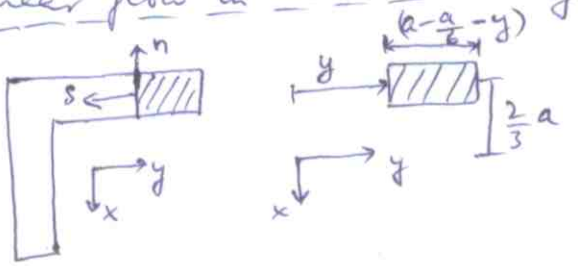
$$I_{yy} \simeq a^3t\left(\frac{4}{9} + \frac{8}{12} + \frac{2}{9}\right) = \frac{4}{3}a^3t$$

$$I_{xx} = \underbrace{\frac{2at^3}{12}}_{\text{neglect}} + 2at\left(\frac{a^2}{36}\right) + \frac{a^3t}{12} + at\left(\frac{a}{2} - \frac{a}{6}\right)^2 \simeq a^3t\left(\frac{1}{18} + \frac{1}{12} + \frac{1}{9}\right) = \frac{a^3t}{4}$$

$$I_{xy} = at\left(-\frac{2}{3}a\right)\left(\frac{a}{2} - \frac{a}{6}\right) + 2at\left(-\frac{a}{6}\right)\left(a - \frac{2}{3}a\right) = \left(-\frac{2}{9} - \frac{1}{9}\right)a^3t = -\frac{a^3t}{3}$$

$$F = I_xI_y - I_{xy}^2 = a^6t^2\left(\frac{1}{3} - \frac{1}{9}\right) = \frac{2}{9}a^6t^2$$

Shear flow in horizontal leg:



$$Q_x = \iint y\,ds\,dn = \int_{(-\frac{2}{3}a+\frac{t}{2})}^{(-\frac{2}{3}a-\frac{t}{2})}\int_{y}^{5a/6} y\,dx\,dy$$

$$= -t\,\frac{\left(\frac{25a^2}{36} - y^2\right)}{2}$$

$$Q_y = \int_{(-\frac{2}{3}a+\frac{t}{2})}^{(-\frac{2}{3}a-\frac{t}{2})}\int_{y}^{5a/6} x\,dx\,dy = \left(\frac{5a}{6} - y\right)\frac{2}{3}at$$

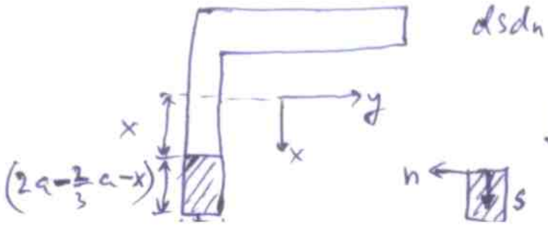
$$\tau_{yz} = -\tau_{sz} = \frac{V_x I_{xy} Q_x}{Ft} - \frac{V_x I_x Q_y}{Ft} = \frac{V_x}{Ft}\left(I_{xy}Q_x - I_xQ_y\right)$$

$$\tau_{yz} = \frac{V_x\,a^3t^2}{\frac{2}{9}a^6t^2t}\left(\frac{25a^2}{6\times36} - \frac{y^2}{6} + \frac{1}{6}ay - \frac{5}{36}a^2\right) = \frac{V_x}{48a^3t}\left(-5a^2 - 36y^2 + 36ay\right)$$

$\tau_{xz} = 0$

You can easily verify that $\int_{-a/6}^{5a/6} \tau_{yz}\,t\,dy$ = nett resultant shear force in y-direction $= 0$.

Shear flow in vertical leg:



$ds\,dn = -dx\,dy$, $Q_x = \iint y\,ds\,dn = -\int_{(-\frac{a}{6}+\frac{t}{2})}^{(-\frac{a}{6}-\frac{t}{2})}\int_{x}^{\frac{4a}{3}} y\,dx\,dy = \frac{at}{6}\left(x - \frac{4a}{3}\right)$

$$Q_y = -\int_{(-\frac{a}{6}+\frac{t}{2})}^{(-\frac{a}{6}-\frac{t}{2})}\int_{x}^{\frac{4a}{3}} x\,dx\,dy = \frac{t}{2}\left(\frac{16a^2}{9} - x^2\right)$$

$$\tau_{xz} = \tau_{sz} = \frac{V_x}{Ft}\left(I_x Q_y - I_{xy} Q_x\right) = \frac{V_x a^3 t^2}{\frac{2}{9} a^6 t^2 t}\left(\frac{2}{9}a^2 - \frac{x^2}{8} + \frac{ax}{18} - \frac{4a^2}{54}\right)$$

$$= \frac{V_x}{48 a^3 t}\left(32a^2 - 27x^2 + 12ax\right)$$

$\tau_{yz} = 0$

Readily verified that $\int_{-\frac{2}{3}a}^{\frac{4}{3}a} \tau_{xz}\, t\, dx = V_x$ = SF in x-direction.

<u>Shear Center</u>:

$$x_{CF} = -\frac{I_{xy}}{F}\int_C Q_y\,(x\,dy - y\,dx) + \frac{I_y}{F}\int_C Q_x\,(x\,dy - y\,dx)$$

$C = C_h + C_v$ = midcontour of (horz + vertical) legs

On C_h, $dx = 0$; on C_v, $dy = 0$.

$$x_{CF} = -\frac{I_{xy}}{F}\overset{(*)}{\left[\int_{C_h} Q_y\left(-\frac{2}{3}a\right)dy + \int_{C_v} Q_y\left(\frac{a}{6}\right)dx\right]} + \frac{I_y}{F}\overset{(**)}{\left[\int_{C_h} Q_x\left(-\frac{2}{3}a\right)dy + \int_{C_v} Q_x\left(\frac{a}{6}\right)dx\right]}$$

Note: contour is from $s=0$ to $s=s_{max}$, so limits on x, y are for increasing s.

$$a^2 x_{CF} = \frac{3}{2}\left[\int_{5a/6}^{-a/6}\left(-\frac{2}{3}\right)\cdot\left(\frac{2}{3}a\right)\left(\frac{5a}{6} - y\right)dy + \int_{-\frac{2}{3}a}^{\frac{4a}{3}}\left(\frac{1}{6}\right)\cdot\left(\frac{1}{2}\right)\left(\frac{16a^2}{9} - x^2\right)dx\right]$$

$$+ 6\left[\int_{5a/6}^{-a/6}\left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)\left(\frac{25a^2}{36} - y^2\right)dy + \int_{-2/3\,a}^{4a/3}\left(\frac{1}{6}\right)\left(\frac{a}{6}\right)\left(x - \frac{4a}{3}\right)dx\right]$$

$$= a^3 \cdot \frac{3}{2}\left[\left(-\frac{20}{54}\right)(-1) - \left(-\frac{4}{9}\right)\left(\frac{1}{2}\right)\left(-\frac{24}{36}\right) + \left(\frac{16}{108}\right)(2) - \left(\frac{1}{12}\right)\left(\frac{1}{3}\right)\left(\frac{72}{27}\right)\right]^{(\star)}$$

$$+ a^3 \cdot 6\left[\frac{25}{108}(-1) - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{126}{216}\right) + \left(\frac{1}{36}\right)\left(\frac{1}{2}\right)\left(\frac{12}{9}\right) - \left(\frac{4}{108}\right)(2)\right]^{(\star\star)}$$

$$= -\frac{2}{3}a^3$$

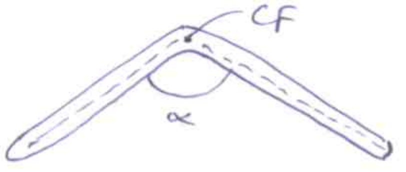
$$\Rightarrow\ x_{CF} = -\frac{2}{3}a. \blacktriangleleft$$

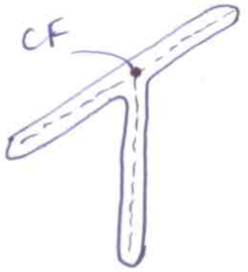
(see above).

$$y_{CF} = -\frac{I_x}{F}\int_C Q_y\,(x\,dy - y\,dx) + \frac{I_{xy}}{F}\int_C Q_x\,(x\,dy - y\,dx) = -\frac{I_x}{F}\overset{(*)}{\left[\quad\right]} + \frac{I_{xy}}{F}\overset{(**)}{\left[\quad\right]}$$

$$= a\left[-\frac{9}{8}(\star) - \frac{3}{2}(\star\star)\right] = a\left[-\frac{9}{8} \times \frac{144}{324} - \frac{3}{2}\left(-\frac{72}{324}\right)\right] = -\frac{a}{6} \blacktriangleleft$$

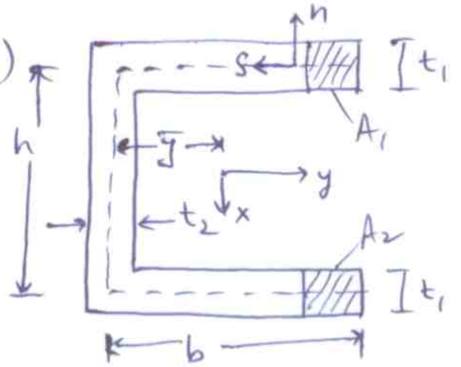
Thus CF lies at the corner of the angle, ie $(x_{CF}, y_{CF}) = \left(-\frac{2}{3}a, -\frac{a}{6}\right)$. This fact is obvious since the shear flows (along horz & vert legs) intersect at the corner, whereby the resultant forces also intersect at the corner, and the shear center is obviously located at such an intersection. More generally we have,





CF definition (alternative): Resultant shear flows, and hence shear forces, have zero moment about CF

(Ex)



$$\bar{y} = \frac{2(bt_1)(b/2)}{2bt_1 + ht_2} = \frac{b^2 t_1}{2bt_1 + ht_2}$$

$V_y = 0$ given. , $(t_1, t_2) \ll (h, b)$

Upper horizontal leg. $(ds\,dn = dx\,dy)$

$$Q_x = \int_y^{b-\bar{y}} \int_{-\frac{h}{2}+\frac{t_1}{2}}^{-h/2-t_1/2} y\,dx\,dy = -\frac{t_1}{2}\left[(b-\bar{y})^2 - y^2\right]$$

$$Q_y = \int_y^{b-\bar{y}} \int_{-\frac{h}{2}+\frac{t_1}{2}}^{-h/2-t_1/2} x\,dx\,dy = (b-\bar{y}-y)\frac{ht_1}{2}$$

Lower horizontal leg. $(ds\,dn = dx\,dy)$

$$Q_x = \int_y^{b-\bar{y}} \int_{\frac{h}{2}-\frac{t_1}{2}}^{h/2+t_1/2} y\,dx\,dy = \frac{t_1}{2}\left[(b-\bar{y})^2 - y^2\right]$$

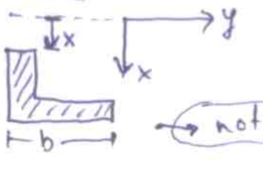
(expected that $[Q_x]_{\text{lower leg}}$ is negative of $(Q_x)_{\text{upper leg}}$ ∵ statical x-moments are same for A_1, A_2).

$$Q_y = \int_y^{b-\bar{y}} \int_{h/2-t_1/2}^{h/2+t_1/2} x\,dx\,dy = (b-\bar{y}-y)\frac{ht_1}{2}$$

(expected same as for upper leg ∵ y-statical moments of A_1, A_2 differ by negative sign).

Lower vertical leg. $ds\,dn = -dx\,dy$

$(x>0)$



→ noted as Case(a) later on

$$Q_x = -\int_x^{h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} y\,dy\,dx + [Q_x]_{\text{lower horz leg},\, y=-\bar{y}}$$

$\circledast$

$$\begin{cases} Q_x = \bar{y}\, t_2 \left(x - \frac{h}{2}\right) + \frac{t_1}{2}\left(b^2 - 2b\bar{y}\right) \;, & \boxed{x > 0} \text{ — relaxed later to } -\frac{h}{2} \le x \le \frac{h}{2} \\ Q_y = -\int_x^{h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} x\,dy\,dx + \left[Q_y\right]_{\text{Lower Horz leg, } y=-\bar{y}} & \\ \quad = \frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + b\frac{ht_1}{2} \;, & \underline{x > 0} \text{ (relaxed later to } -\frac{h}{2} \le x \le \frac{h}{2}) \end{cases}$$

uh = upper horz
lh = lower horz
uv = upper vert
lv = lower vert.

not required for solution. (It is for explanation/understanding only)

Upper vertical leg. ($x \le 0$)

By considering statical moments (shortcut) and hence noting that $[Q_x]_{lv} = -[Q_x]_{uv}$ and $[Q_y]_{lv} = [Q_y]_{lv}$, we can directly write

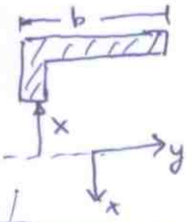
introduced ∵ for uv leg we have $x<0$, so the sign on odd power terms in x should remain same in expression for $[Q_x]_{lv}$ and $[Q_x]_{uv}$ so that their numerical values are -ve of each other

$$\left.\begin{aligned} Q_x &= -\bar{y}\, t_2 \left(\ominus x - \frac{h}{2}\right) - \frac{t_1}{2}\left(b^2 - 2b\bar{y}\right) \\ Q_y &= \frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + bh\frac{t_1}{2} \end{aligned}\right\} , \; \underline{x < 0} \text{ (relaxed later to } -\frac{h}{2} \le x \le \frac{h}{2})$$

or we get same result by first principles a. follows,

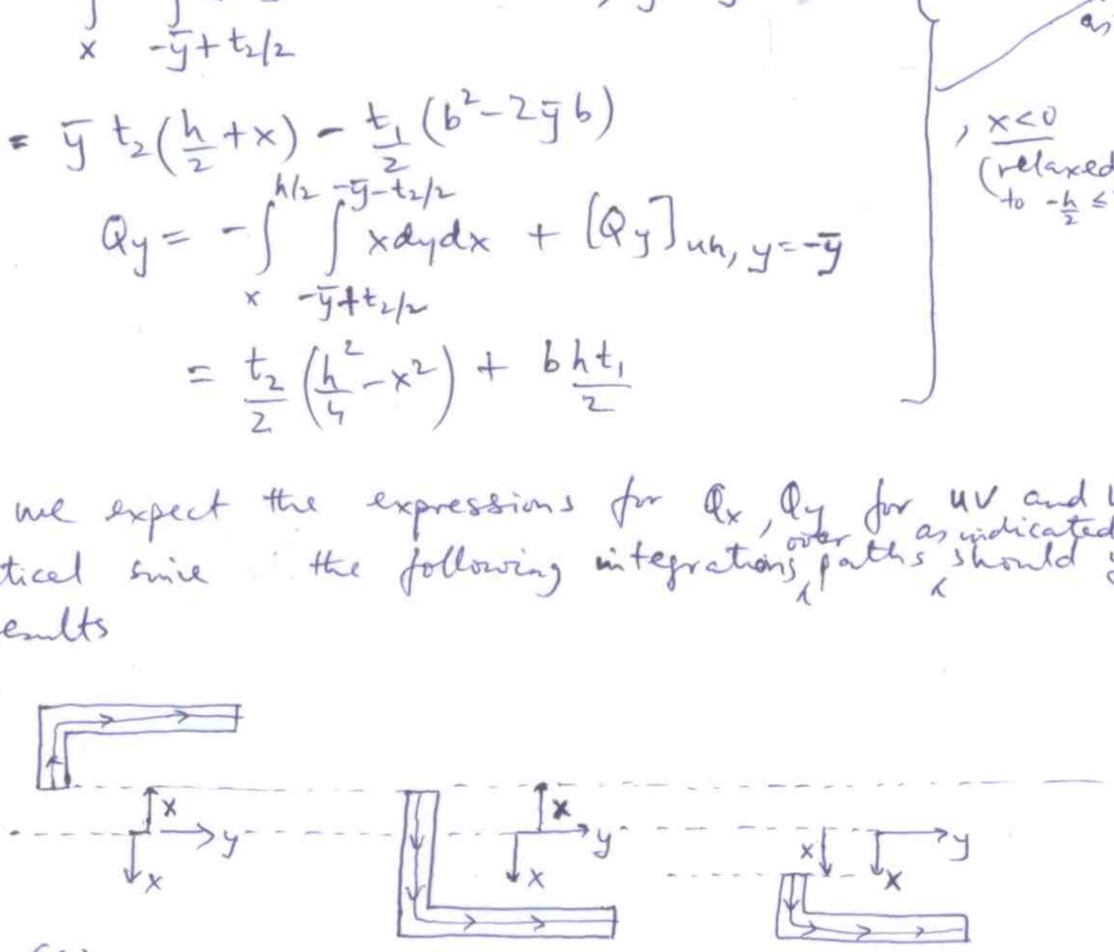
$$\left.\begin{aligned} Q_x &= -\int_x^{-h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} y\,dy\,dx + \left[Q_x\right]_{uh,\, y=-\bar{y}} \\ &= \bar{y}\, t_2\left(\frac{h}{2} + x\right) - \frac{t_1}{2}\left(b^2 - 2\bar{y}b\right) \\ Q_y &= -\int_x^{h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} x\,dy\,dx + \left[Q_y\right]_{uh,\, y=-\bar{y}} \\ &= \frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + b\frac{ht_1}{2} \end{aligned}\right\} , \; \underline{x<0} \text{ (relaxed later to } -\frac{h}{2} \le x \le \frac{h}{2})$$

(they match! as expected)



noted as case(c) later on

In fact we expect the expressions for Q_x, Q_y for uv and lv legs to be identical since the following integrations over paths as indicated should yield identical results



(a) (b) (c)

(a), (b) yield identical results (ie, numerical values for Q_x, Q_y, resp'ly)
(b), (c) yield identical expressions for Q_x, Q_y, respectively.
(a), (c) yield Q_x values differing by -ve sign.

Thus using any one of the expressions obtained from (a) or (b) or (c) for Q_x and using $Q_x[x] + Q_x[-x] = 0$ you get,

$$-\bar{y}\, t_2 \frac{h}{2} + \frac{t_1}{2} b^2 - \frac{t_1}{2} 2b\bar{y} = 0 \Rightarrow \bar{y} = \frac{b t_1^2}{2bt_1 + t_2 h}$$ which we had from before. Hence, <u>for entire vertical leg</u>, we could <u>directly</u> write from ⓧ on p. 12,

$$Q_x = \bar{y}\, t_2\, x, \qquad -\frac{h}{2} \le x \le \frac{h}{2}$$

$$Q_y = \frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + \frac{bht_1}{2}$$

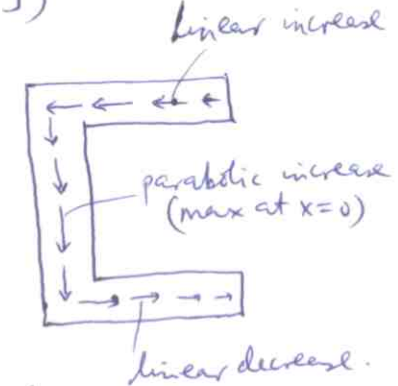
<u>Shear flows / stresses</u>

<u>Upper horizontal leg.</u>

$$\tau_{sz} = \frac{V_x Q_y}{I_y t} = \frac{V_x}{I_y t_1} \frac{ht_1}{2}(b - \bar{y} - y) = \frac{V_x h}{2 I_y}(b - \bar{y} - y)$$ ◀

<u>Vertical leg.</u>

$$\tau_{sz} = \frac{V_x Q_y}{I_y t} = \frac{V_x}{I_y t_2}\left[\frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + \frac{bht_1}{2}\right]$$ ◀



<u>Shear Center</u>

▶ $x_{CF} = 0$ from symmetry. Verify it as follows. $x_{cf} = \frac{1}{I_x}\int_C Q_x (x\,dy - y\,dx)$

$$x_{CF} = \frac{1}{I_x}\Bigg[\underbrace{\int_{C_{uh}} (Q_x)_{uh}\left(-\frac{h}{2}\right) dy + \int_{C_{Lh}} \left(\frac{h}{2}\right)(Q_x)_{Lh}\, dy}\; + \bar{y}\int_{C_v} (Q_x)_v\, dx\Bigg]$$

$$\left(C = \underset{\substack{\text{upper}\\\text{horz}}}{C_{uh}} + \underset{\text{vert}}{C_v} + \underset{\substack{\text{lower}\\\text{horz}}}{C_{Lh}}\right)$$

$\because (Q_x)_{uh} = -(Q_x)_{Lh}$, and $\int_{C_{uh}} = \int_{b-\bar{y}}^{-\bar{y}}$, and $\int_{C_{Lh}} = \int_{-\bar{y}}^{b-\bar{y}}$, the first two integrals cancel out.

The third integral is zero $\because$ it contains integrand that is odd power in x and limits are $\left(-\frac{h}{2}, \frac{h}{2}\right)$. Hence verified.

$$y_{CF} = -\frac{1}{I_y}\int_C Q_y (x\,dy - y\,dx) = -\frac{1}{I_y}\left[\int_{b-\bar{y}}^{-\bar{y}} (Q_y)_{uh}\left(-\frac{h}{2}\right) dy + \int_{-\bar{y}}^{b-\bar{y}} (Q_y)_{Lh}\left(\frac{h}{2}\right) dy + \int_{-h/2}^{h/2} (Q_y)_v\, \bar{y}\, dx\right]$$

Note, $(Q_y)_{uh} = (Q_y)_{Lh}$

$$\Rightarrow y_{CF} = -\frac{1}{I_y}\left[2\frac{h}{2}\int_{-\bar{y}}^{b-\bar{y}} (Q_y)_{uh} dy + \bar{y}\int_{-h/2}^{h/2} (Q_y)_v\, dx\right] = -\frac{1}{I_y}\left[\frac{h^2 t_1}{2}\int_{-\bar{y}}^{b-\bar{y}} (b - \bar{y} - y)\, dy + \bar{y}\int_{-h/2}^{h/2}\left[\frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + \frac{bht_1}{2}\right] dx\right]$$

$$= -\frac{1}{I_y}\left[\frac{h^2 t_1}{2}\left\{(b - \bar{y})(b) - \frac{1}{2}(b^2 - 2b\bar{y})\right\} + \bar{y}\left\{\frac{t_2}{2}\left(\frac{h^2}{4}\cdot h - \frac{h^3}{12}\right) + \frac{bh^2 t_1}{2}\right\}\right]$$

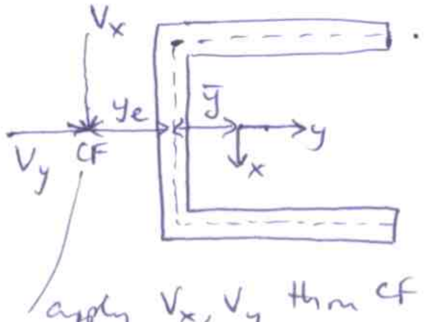
$$y_{CF} = -\frac{1}{I_y}\left[\frac{h^2t_1}{2}b^2 - \frac{h^2t_1}{2}b\bar{y} - \frac{h^2t_1}{2}\frac{b^2}{2} + \frac{h^2t_1}{2}b\bar{y} + \frac{h^3}{12}t_2\bar{y} + \frac{h^2t_1}{2}b\bar{y}\right]$$

$$= -\frac{1}{I_y}\left[\frac{h^2b^2t_1}{4} + \bar{y}\left(\frac{h^2t_1b}{2} + \frac{h^3t_2}{12}\right)\right]$$

$$I_y = bt_1\frac{h^2}{2} + \frac{h^3t_2}{12}$$

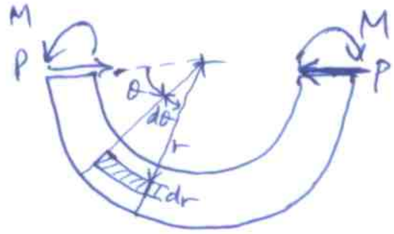
$$\Rightarrow y_{CF} = -\frac{1}{I_y}\frac{h^2b^2t_1}{4} \qquad \bar{y}$$

$$= -y_e - \bar{y}$$



apply V_x, V_y thru CF to have bending w/o twisting (ie $\alpha = 0$)

Curved beams. (Symmetric Sections)



→ Exact solution was done as plane stress problem in (r, θ) coordinates to get $\sigma_r, \sigma_\theta, \sigma_{r\theta}$. However, that
→ solution valid only for thin rectangular sections.

Now we discuss approximate (engineering) solution for non-rectangular but symmetric sections based on assumptions of:

(i) Plane sections remaining plane

(ii) σ_r, $\sigma_{r\theta}$ are negligible, and plane state of stress as before (in elasticity soln).

(iii) applied loads lie in plane of symmetry.

Refer Fig 1 showing the loading, cross section and forces on beam element.

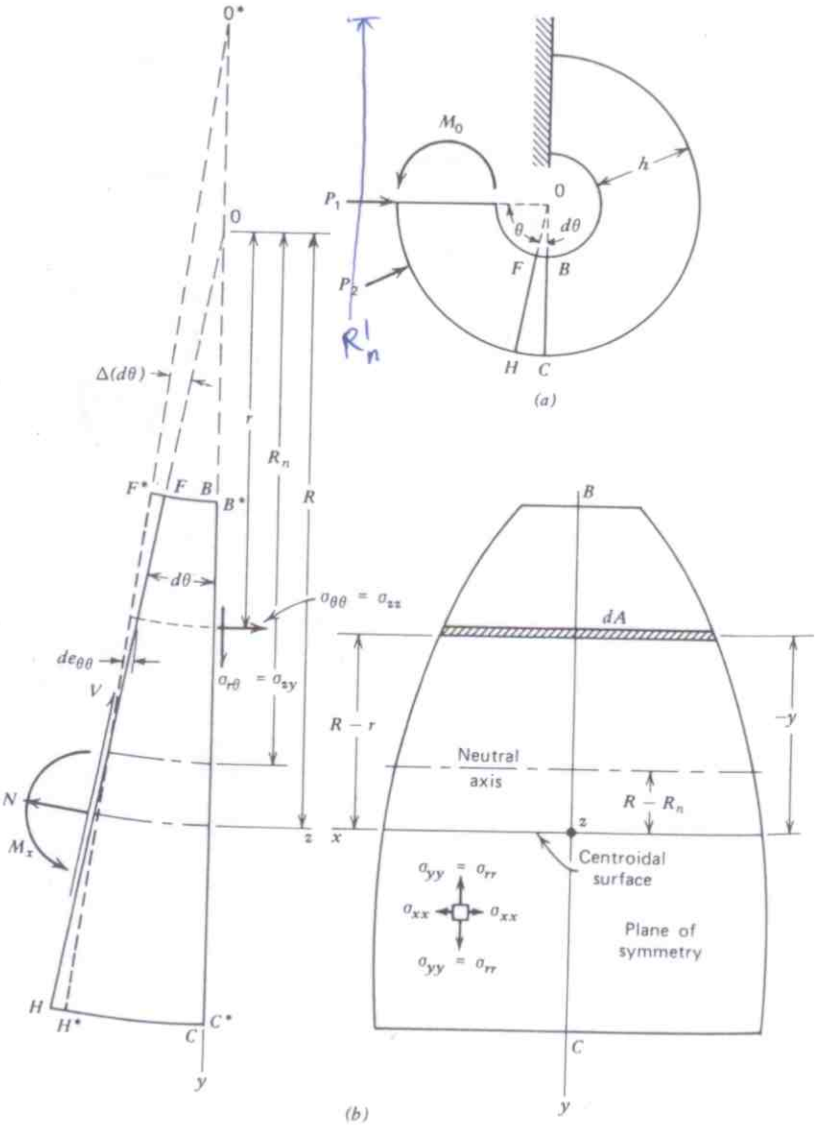


Fig 1

Kinematics: $F^*B^*C^*H^*$ is deformed element, FBCH is undeformed element. For simplicity we take B^*C^* to coincide with BC (and FH rotated by $\Delta(d\theta)$) (alternatively take FH and BC to rotate by $\frac{\Delta(d\theta)}{2}$ each in opposite sense so that curvature reduces, and then rotate deformed element thus obtained by $\frac{\Delta(d\theta)}{2}$ ccw to make B^*C^* coincide with BC — this way you get the same as Fig 1.).

R_n, R, r, denote position of NA, centroidal, nominal fiber from original (undeformed) center of curvature. Also, $de_{\theta\theta}$ is change in length of fiber.

$$\boxed{\varepsilon_{\theta\theta}} = \frac{de_{\theta\theta}}{r\,d\theta} = \frac{(R_n - r)\,\Delta(d\theta)}{r\,d\theta} \boxed{= \left(\frac{R_n}{r} - 1\right)\omega} \longrightarrow ①$$

$$\text{where, } \omega \triangleq \frac{\Delta(d\theta)}{d\theta} = \underbrace{\left(\frac{R_n}{R_n'} - 1\right)}_{\to ②} \quad \left(\to \text{from } R_n'(d\theta + \Delta(d\theta)) = R_n\,d\theta \text{ ie, neutral axis length unchanged}\right).$$

Here R_n' is position of NA in deformed beam (ie measured from 0^* to NA).

Note that although $de_{\theta\theta}$ varies linearly with r (as in straight beams) $\varepsilon_{\theta\theta}$ is nonlinear in r, due to undeformed lengths $(r\,d\theta)$ being function of depth (r) unlike in straight beams.

Hookes law: $\sigma_{rr} \simeq 0$, $\sigma_{xx} \simeq 0 \Rightarrow \sigma_{\theta\theta} = E\varepsilon_{\theta\theta} = \frac{R_n - r}{r} E\omega \longrightarrow ③$

Equilibrium:

$$\sum F_\theta : \quad N = \int_A \sigma_{\theta\theta}\, dA$$

$$\sum M_x : \quad M_x = \int_A \sigma_{\theta\theta}(R - r)\, dA$$

$$\Rightarrow N = E\omega \int_A \frac{R_n - r}{r}\, dA = R_n E\omega A_m - E\omega A \longrightarrow ④$$

$$M_x = E\omega \int_A \left(\frac{R_n - r}{r}\right)(R - r)\, dA = R_n E\omega (RA_m - A) \longrightarrow ⑤$$

$$\text{where } A_m \triangleq \int_A \frac{dA}{r} \longrightarrow ⑥$$

$$(4,5) \Rightarrow E\omega = \frac{A_m M_x}{A(RA_m - A)} - \frac{N}{A} \longrightarrow ⑦$$

$$(3),(5,7) \Rightarrow \sigma_{\theta\theta} = \frac{N}{A} + \frac{M_x(A - rA_m)}{Ar(RA_m - A)} \longrightarrow ⑧$$

$$(\sigma_{\theta\theta})_{r=R_n} = 0 \Rightarrow R_n = \frac{AM_x}{A_m M_x + N(A - RA_m)} \longrightarrow \text{location of NA} \longrightarrow$$

$$= \frac{A}{A_m} \quad (\text{for } N=0, \text{ pure bending}) \longrightarrow \Big\} ⑨$$

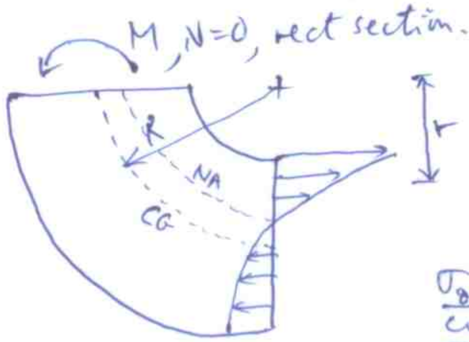
$$\underset{(N=0)\,(\text{use 9b})}{\text{pure bending}} \Rightarrow \sigma_{\theta\theta} = \frac{M_x(A - rA_m)}{Ar(RA_m - A)} = \frac{M_x(R_n - r)}{Ar(R - R_n)} = -\frac{M_x y}{Ae(R_n - y)} \Bigg\} \longrightarrow ⑩$$

ie $e = R - R_n$

$-y = R_n - r$

ie, y measured −ve from NA to center of curvature

The $\sigma_{\theta\theta}$ distribution is hyperbolic $\left(\frac{1}{r}\right)$ in form.



$\sigma_{\theta\theta}$ distribution for Rectangular sectioned curved beam.

Comparison with elasticity for Rectangular section case:

- For pure bending (**M**) and pure shear (P) loads, comparison of $(\sigma_{\theta\theta})^{\text{obt by (8)}}_{max}$ with the plane stress elasticity sol_n show that the pure bending results compare better than the pure shear loading ones. Further, as R/h increases, the comparison gets better for both types of loading. In fact for $R/h = 5$, even the straight-beam formula gives reasonably good results. (see table 8-2-1, Boresi + Sidebottom, p. 357). Thus for curved beams with $R/h > 5$ we use straight-beam flexure formula.

-

Table 8-2-1 ⟶ (Boresi)

Ratios of the Maximum Circumferential Stress in Rectangular Section Curved Beams as Computed by Elasticity Theory, by the Curved Beam Formula and by the Flexure Formula

	Pure Bending		*Shear Loading*	
$\frac{R}{h}$	$\frac{\sigma_{\theta\theta(CB)}}{\sigma_{\theta\theta(elast)}}$	$\frac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(elast)}}$	$\frac{\sigma_{\theta\theta(CB)}}{\sigma_{\theta\theta(elast)}}$	$\frac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(elast)}}$
0.65	1.046	0.439	0.855	0.407
0.75	1.012	0.526	0.898	0.511
1.0	0.997	0.654	0.946	0.653
1.5	0.996	0.774	0.977	0.776
2.0	0.997	0.831	0.987	0.834
3.0	0.999	0.888	0.994	0.890

In fact we can show that second term on RHS of ⑧ reduces to $-M_x y/I_x$ when $R/h \to \infty$, i.e., ⑧ approaches the straight beam formula. The details are:

$$RA_m = \int \frac{R\,dA}{r} = \int \frac{R}{R+y}\,dA = \int\left(1-\frac{y}{R+y}\right)dA = A - \int \frac{y}{R+y}\,dA$$

$$(RA_m - A) = -\int \frac{y}{R+y}\,dA \longrightarrow ⊛$$

$$\Rightarrow Ar\,(RA_m - A) = -A\,(R+y)\int \frac{y}{R+y}\,dA = -A\int \frac{Ry+y^2-y^2}{R+y}\,dA - Ay\int\frac{y}{R+y}\,dA$$

$$= -A\underbrace{\int y\,dA}_{=0} + A\int \frac{y^2}{R+y}\,dA - Ay\int\frac{y}{R+y}\,dA$$

$(\because y$ measured from centroid$)$

For $\frac{R}{h}\to\infty$,

$$\simeq \frac{A}{R}\int \frac{y^2}{1+\cancelto{\simeq 0}{y/R}}\,dA - \underbrace{A\frac{y}{R}}_{\simeq 0}\int\frac{y}{1+\cancelto{\simeq 0}{y/R}}\,dA \simeq \frac{A}{R}I_x$$

Thus, for $R/h\to\infty$, second term on RHS of ⑧ approximates to

$$\frac{M_x R}{AI_x}(A - rA_m) = \frac{M_x R}{AI_x}(A - RA_m - yA_m)$$

$$= \frac{M_x R}{AI_x}\left[\int\frac{y}{R+y}\,dA - y\int\frac{dA}{R+y}\right] = \frac{M_x R}{AI_x}\left[\underbrace{\frac{1}{R}\int\frac{y}{1+y/R}\,dA}_{\simeq 0} - \frac{y}{R}\underbrace{\int\frac{dA}{1+\frac{y}{R}}}_{\simeq A}\right]$$

$$\left.\frac{M_x\,(A-rA_m)}{Ar\,(RA_m-A)}\right|_{\frac{R}{h}\to\infty} \simeq -\frac{M_x y}{I_x}$$

Note: $\because RA_m \to A$ as $R/h\to\infty$ (see ⊛ above), we must the calculation of A_m quite accurately and retain sufficient significant digits.

Composite curved beams: They are made up of a combination of fundamental areas. So we have,

$$A = \sum_{i=1}^{n} A_i\,,\quad A_m = \sum_{i=1}^{n} A_{m_i}\,,\quad R = \frac{\sum_{i=1}^{n} R_i A_i}{\sum_{i=1}^{n} A_i}\,,$$

n = nos of fundamental areas comprising section. See Table 8-2-2 (Boresi) for A_i, A_{m_i}, R_i of regular sections.

Table 8-2.2

Analytical Expressions for A, R, and $A_m = \int \frac{dA}{r}$

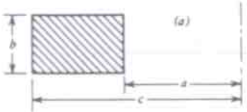
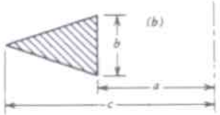
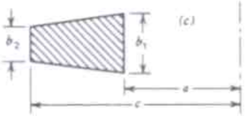
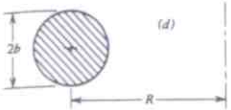
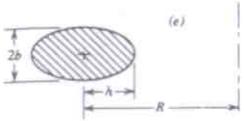
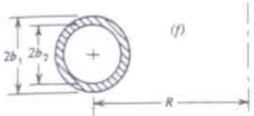
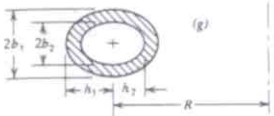
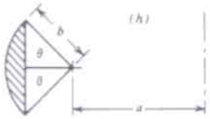
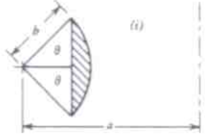
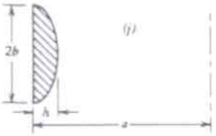
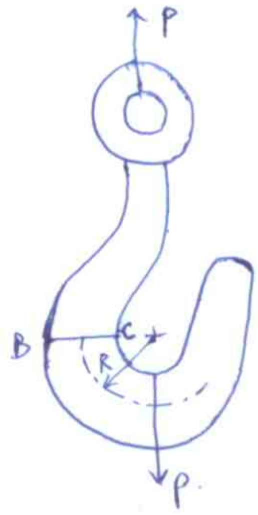
	$A = b(c-a)$; $A_m = b \ln \frac{c}{a}$	$R = \frac{a+c}{2}$
	$A = \frac{b}{2}(c-a)$; $A_m = \frac{bc}{c-a} \ln \frac{c}{a} - b$	$R = \frac{2a+c}{3}$
	$A = \frac{b_1+b_2}{2}(c-a)$; $A_m = \frac{b_1 c - b_2 a}{c-a} \ln \frac{c}{a} - b_1 + b_2$	$R = \frac{a(2b_1+b_2)+c(b_1+2b_2)}{3(b_1+b_2)}$

Table 8-2.2

Continued

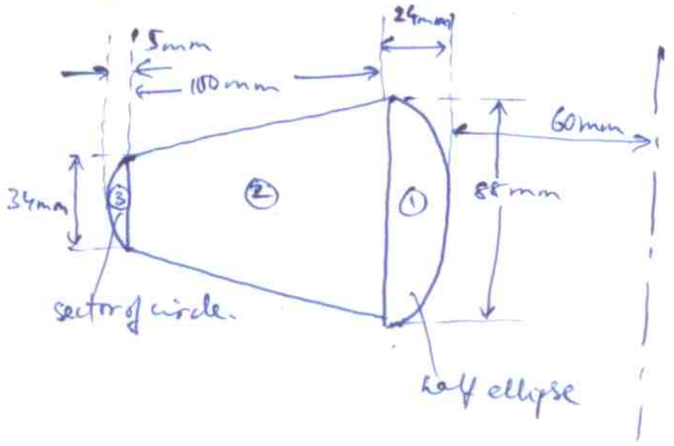
	$A = \pi b^2$ $A_m = 2\pi(R - \sqrt{R^2 - b^2})$
	$A = \pi bh$ $A_m = \frac{2\pi b}{h}(R - \sqrt{R^2 - h^2})$
	$A = \pi({b_1}^2 - {b_2}^2)$ $A_m = 2\pi\left(\sqrt{R^2 - {b_2}^2} - \sqrt{R^2 - {b_1}^2}\right)$
	$A = \pi(b_1 h_1 - b_2 h_2)$ $A_m = 2\pi\left(\frac{b_1 R}{h_1} - \frac{b_2 R}{h_2} - \frac{b_1}{h_1}\sqrt{R^2 - {h_1}^2} + \frac{b_2}{h_2}\sqrt{R^2 - {h_2}^2}\right)$
	$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; R = a + \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$. For $a > b$, $A_m = 2a\theta - 2b\sin\theta - \pi\sqrt{a^2 - b^2} + 2\sqrt{a^2 - b^2}\sin^{-1}\left[\frac{b + a\cos\theta}{a + b\cos\theta}\right]$ For $b > a$, $A_m = 2a\theta - 2b\sin\theta + 2\sqrt{b^2 - a^2}\ln\left[\frac{b + a\cos\theta + \sqrt{b^2 - a^2}\sin\theta}{a + b\cos\theta}\right]$
	$A = b^2\theta - \frac{b^2}{2}\sin 2\theta;\ R = a - \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$ $A_m = 2a\theta + 2b\sin\theta - \pi\sqrt{a^2 - b^2} - 2\sqrt{a^2 - b^2}\sin^{-1}\left[\frac{b - a\cos\theta}{a - b\cos\theta}\right]$
	$A = \frac{\pi bh}{2};\ R = a - \frac{4h}{3\pi}$ $A_m = 2b + \frac{\pi b}{h}(a - \sqrt{a^2 - h^2}) - \frac{2b}{h}\sqrt{a^2 - h^2}\sin^{-1}\left(\frac{h}{a}\right)$

(eg) Crane hook. (Boresi, Ex 8.2.2).



Given: Yield stress, $Y = 500\,MPa$, $SF = 2$

Find: Max P.

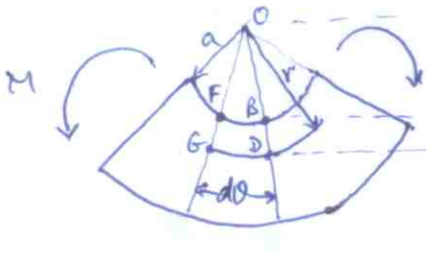


From Table 8-2-2 (Boresi), A_i, R_i, A_{m_i} for $i = 1, 2, 3$ can be obtained. Then obtain A, R, A_m for composite area as discussed on bot of p. 126.

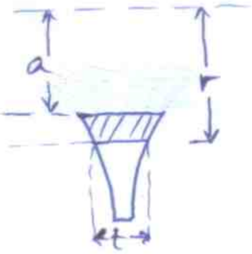
Since N and M_x are maximum for section BC, it is the critical section. Use eqn ⑧ to find $\sigma_{\theta\theta}$ at B and C and decide max load P based on the larger magnitude of $|(\sigma_{\theta\theta})_B|$ or $|(\sigma_{\theta\theta})_C|$, based on $SF = 2$. Use $N = P$, $M_x = PR$ in the eqn ⑧.

Radial Stresses in Curved Beams

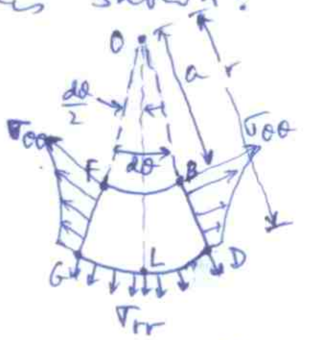
So far we assumed $\sigma_{rr} \simeq 0$. For sections with thin webs (H, T, I sections) the $(\sigma_{rr})_{max}$ may exceed $(\sigma_{\theta\theta})_{max}$. Consider the curved beam with section as shown:



Curved beam



Section



Stress distribution on slice FBGD ($\sigma_{\theta\theta}$, σ_{rr} shown).

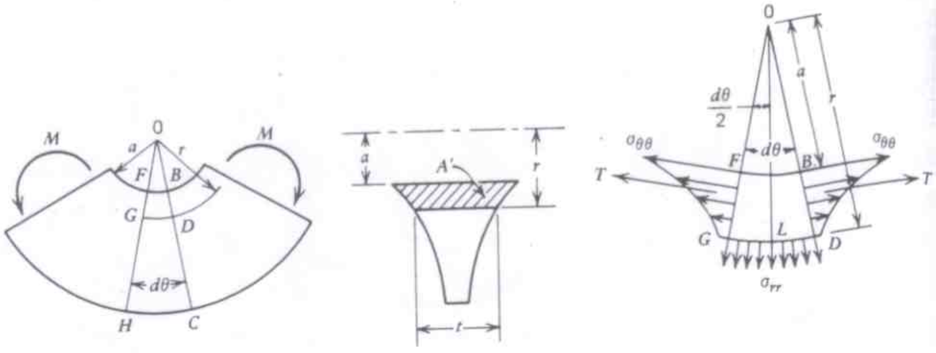
The radial stress, σ_{rr}, occurring at radius r is shown.

[illegible] (Boresi, Sec 8.2).

Use Table 8.1 (Boresi), A, B, Am for (r = [illegible]) can be [illegible] A ≠ [illegible] obtain A, B, Am for composite section [illegible] out of p 126

[illegible] V, M & M_x are maximum for section BC, [illegible] is the [illegible]. Use eq (?) to find $\sigma_{\theta\theta}$ at A and C [illegible] load P based on the larger magnitude of $|(\sigma_{\theta\theta})_A|$ or $|(\sigma_{\theta\theta})_C|$ based on SF = 2. Use [illegible] in eq (?).

Radial Stresses in Curved Beams



(a) (b) (c)

The radial stress [illegible]

Consider equilibrium of element BDGF. Resultant circumferential force on face BD or FG is,

$$T = \int_a^r \sigma_{\theta\theta}\, dA \longrightarrow \text{⑪}$$

Let r direction be OL. $\Sigma F_\theta = 0$ ensured since θ-component of T on faces BD & FG cancel out. $\Sigma F_r = 0$ requires,

$$\sigma_{rr}\, t r\, d\theta - 2T \sin(d\theta/2) = 0 \Rightarrow \boxed{\sigma_{rr} = \frac{T}{tr}} \longrightarrow \text{⑫}$$

From ⑧, ⑪,

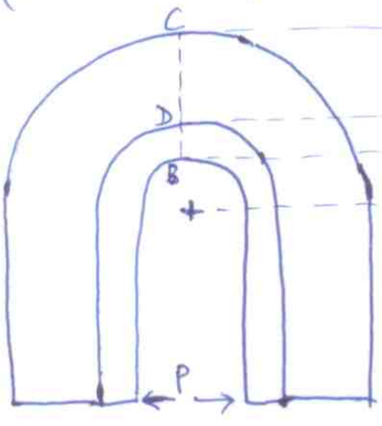
$$T = \frac{N}{A}\int_a^r dA + \frac{M_x}{(RA_m - A)}\int_a^r \frac{1}{r}\, dA - \frac{M_x A_m}{A(RA_m - A)}\int_a^r dA$$

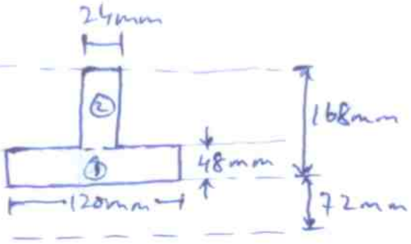
$$\boxed{T = \frac{A'}{A} N + \frac{AA_m' - A'A_m}{A(RA_m - A)} M_x} \longrightarrow \text{⑬}$$

where $\boxed{A_m' \triangleq \int_a^r \frac{dA}{r}, \quad A' \triangleq \int_a^r dA}$

- For only shear loading, in case of a rectangular ⊗-section beam, the above approximate solution provides a conservative estimate when compared to plane-stress elasticity solution. The differences remain ~~to~~ within 6% for $R/h > 1.0$ even if the axial force term (N) is neglected in ⑬.

(eg) (Boresi, Ex 8.3-1)





Given: P = 120kN.
Find: $(\sigma_{\theta\theta})_B$, $(\sigma_{rr})_D$.

$M = PR$, $N = P$.
$R_1 = 96$, $R_2 = 180$

Find A, R, A_m as discussed on p. 125 bottom, using Table 8.2.2 for A_m. Then use ⑧, ⑫, ⑬ to find respective stresses. For $(\sigma_{rr})_D$, A', A_m' based on area ① only.

Here the effect of stress concentration at the fillet joining web & flange is neglected. These need to be considered for brittle materials or even ductiles under cyclic loading.

Even the $\sigma_{\theta\theta}$ need to be corrected, especially in case of I or T sections where effect of σ_{rr} is non-negligible and it changes $\sigma_{\theta\theta}$. We use Bleich's correction factors (ref. Boresi).

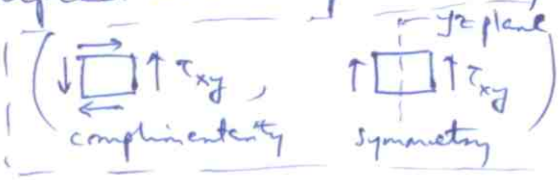
DEFLECTIONS IN CURVED BEAMS

We use Castigliano's theorem.

Strain energy:

$$U = \frac{1}{2}\int_V \sigma_{ij}\,\varepsilon_{ij}\,dV = \int_V U_0\,dV .$$

Symmetry of loading wrt yz plane, coupled with complimentarity of shear stresses, gives $\tau_{xy} = \tau_{xz} = 0$



Neglecting σ_{rr} $(\equiv \sigma_{yy})$ and σ_{xx}, ie,

retaining only $\sigma_{\theta\theta}$ $(\equiv \sigma_{zz})$ and $\sigma_{r\theta}$ $(\equiv \sigma_{yz})$ we have

$$U_0 = \frac{1}{2E}\sigma_{\theta\theta}^2 + \frac{1}{2G}\sigma_{r\theta}^2$$

$\sigma_{\theta\theta} \rightarrow$ contributes to U_N (axial due to N), U_M (bending) and U_{MN} (coupling).

$\sigma_{r\theta} \rightarrow$ contributes to U_s (shear).

$$U_s = \int \frac{kV^2}{2AG}\,dz = \int \frac{kV^2}{2AG}R\,d\theta$$

(based on straight beam approximation with V passing thru centroid).

$$U_N = \int \frac{N^2}{2AE}\,dz = \int \frac{N^2}{2AE}R\,d\theta$$

(based on straight beam approx with N thru centroid).

For U_M, U_{MN} we dont use straight beam approx.

For U_M: Let V and N be applied first and then moment be increased from zero to M_x. Since material is linearly elastic, the order of loading does not affect strain energy. Incremental bending strain energy is,

$$dU_M = \frac{1}{2} M_x (\Delta d\theta) = \frac{1}{2} M_x \omega \, d\theta$$

where $\Delta(d\theta)$ and ω arise due to M_x alone. Thus Using $N=0$ in ⑦ to obtain ω, we get,

(∵ $\Delta(d\theta)$, ω are due to M_x alone)

$$dU_M = \frac{A_m M_x^2}{2A(RA_m - A)E} d\theta$$

During application of M_x, the centroidal surface stretches an amount $d\bar{e}_{\theta\theta}$ and hence N does additional work given by,

$$dU_{MN} = N \, d\bar{e}_{\theta\theta} = N \bar{\varepsilon}_{\theta\theta} R \, d\theta \quad \text{(from ① with } r=R\text{)}.$$

Using $r=R$, ①, ⑤ for obtaining R_n ω, and ⑦ with $N=0$ to obtain ω, we have,

$$\bar{\varepsilon}_{\theta\theta} = \left(\frac{R_n}{R} - 1\right)\omega = \frac{M_x}{ER(RA_m - A)} - \frac{A_m M_x}{EA(RA_m - A)}$$

$$\Rightarrow dU_{MN} = \frac{N}{E}\left[\frac{M_x}{(RA_m - A)} - \frac{R A_m M_x}{A(RA_m - A)}\right] d\theta = -\frac{M_x N}{EA} d\theta$$

Thus, $$\boxed{U = \int \left(\frac{kV^2 R}{2AG} + \frac{N^2 R}{2AE} + \frac{A_m M_x^2}{2A(RA_m - A)E} - \frac{M_x N}{EA} \right) d\theta} \quad ⑭$$

- ⑭ based on assumptions that σ_{rr} negligible & plane sections remaining plane.
- If M_x and N are same sign then $U_{MN} < 0$. Usually U_{MN} is small and negative. If it is negative, neglect it to compensate somewhat for loss of strain energy due to $\sigma_{rr} \cong 0$ assumption.
- For shear-only loading $\delta_P = \delta_U = \frac{\partial U}{\partial P}$ and for moment-only loading (i.e. pure bending) $\delta_M = \delta_U = \frac{\partial U}{\partial M}$. These results are shown in Table 8-5-1 (Boresi) for a rectangular ⊗ section beam with $k=1.5$ and $\nu = 0.30$ used. Approximate results from Castigliano's theorem and plane-stress elasticity solution are compared.

Table 8-5.1

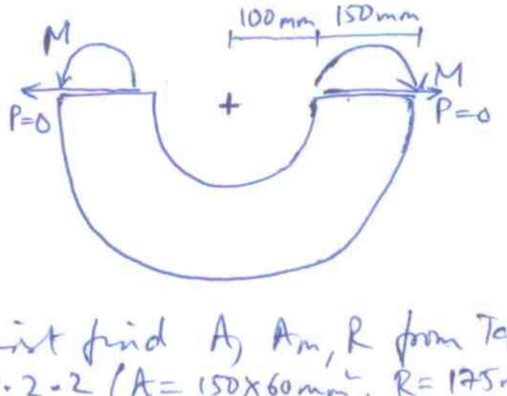
Ratios of Deflections in Rectangular Section Curved Beams as Computed by Elasticity Theory and by Approximate Strain Energy Solution

	Neglecting U_{MN}		*Including U_{MN}*	
	Pure Bending	*Shear Loading*	*Pure Bending*	*Shear Loading*
$\left(\frac{R}{h}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$
0.65	0.923	1.563	0.697	1.215
0.75	0.974	1.381	0.807	1.123
1.0	1.004	1.197	0.914	1.048
1.5	1.006	1.085	0.968	1.016
2.0	1.004	1.048	0.983	1.008
3.0	1.002	1.021	0.993	1.003
5.0	1.000	1.007	0.997	1.001

- Deflections in curved beams are much less influenced by curvature than are the $\sigma_{\theta\theta}$ stresses. Thus for $R/h > 2.0$ the $(U_M + U_{MN})$ terms can be replaced by straight beam version of U_M alone, ie by $\int \frac{M_x^2}{2EI_x} R\,d\theta$. The deflection results are conservative when $R/h = 2$ and the straight beam U_M is used, with straight beam approximation result being 7.7% higher than curved beam ones.

(eg)

(Boresi Ex 8-5-1)



Given: $M = 24$ kN.m applied, rectangular cross section with 60mm depth, $E = 72$ GPa

Find: (a) angle change between two horizontal faces where M applied

(b) Relative displ of centroids of horizontal face.

First find A, A_m, R from Table 8.2.2 ($A = 150 \times 60$ mm^2, $R = 175$ mm by observation).

(a) $\because V = N = 0$, only U_M term present, ie $U = U_M$. From ⑭ and Castiglianos theorem,

$$\theta = \frac{\partial U}{\partial M} = \int_0^{\pi} \frac{A_m M}{A(RA_m - A)E}\,d\theta = \frac{A_m M \pi}{A(RA_m - A)E}$$

(b) Vertical displ of centroids will be identical so the relative vertical displ is zero. For relative horizontal displ, apply fictitious horizontal force P thru centroids as shown and finally put $P = 0$.

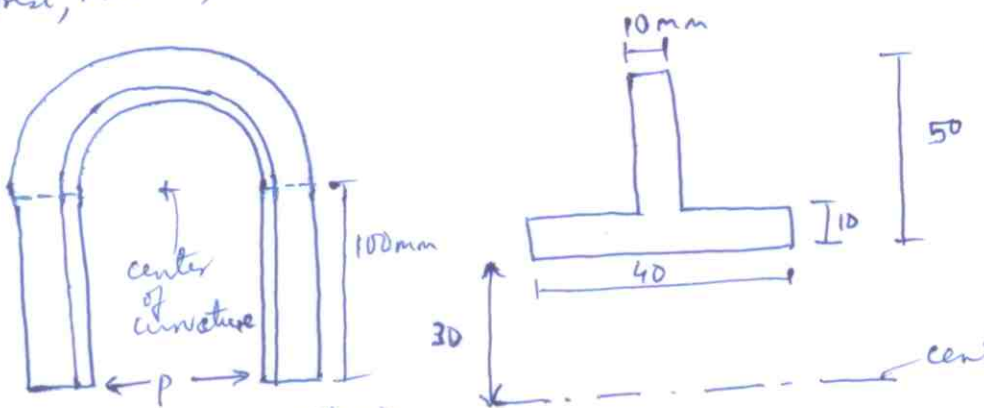
$$V = P\cos\theta, \quad N = P\sin\theta, \quad M_x = M + PR\sin\theta$$

From ⑭, and $\delta_P = \frac{\partial U}{\partial P}\Big|_{P=0}$,

$$\delta_P = \int_0^{\pi} \left\{ \frac{kP\cos^2\theta R}{AG} + \frac{P\sin^2\theta R}{AE} + \frac{A_m(MR\sin\theta + PR^2\sin^2\theta)}{A(RA_m - A)E} - \frac{M\sin\theta + 2PR\sin^2\theta}{EA} \right\}\Bigg|_{P=0} d\theta$$

$$= \int_0^{\pi} \left(\frac{A_m MR\sin\theta}{A(RA_m - A)E} - \underbrace{\frac{M\sin\theta}{EA}}_{\text{neglect since it deducts from displ (ie neglect } U_{Nm})} \right) d\theta$$

eg) (Boresi, Ex 8-5-2)



Given: $P = 11{\cdot}2\,kN$
$E = 200GPa$, $\nu = 0{\cdot}3$
Find: seperation of jaws.

Done without ~~using~~ considering Bleich's correction for T sections.

$V = P$, $N = 0$, $M_x = Pz$ in straight beam portion

$V = P\cos\theta$, $N = P\sin\theta$, $M_x = P(100) + PR\sin\theta$ for curved beam portion.

Use table 8-2-2 to get A, A_m, R. Note I_x is about centroidal section.

From ⑭,

$$\delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\Bigg[\underbrace{2\int_0^{100} \left\{ \frac{kP^2}{2AG} + \frac{P^2 z^2}{2EI_x} \right\}}_{\text{straight beam portion}} + \underbrace{\int_0^{\pi} \left\{ \frac{kP^2\cos^2\theta R}{2AG} + \frac{P^2\sin^2\theta R}{2AE} + \frac{A_m(100P + PR\sin\theta)^2}{2A(RA_m - A)E} - \frac{100P^2\sin\theta + P^2R\sin^2\theta}{EA} \right\} d\theta}_{\text{curved beam portion}} \Bigg]$$

$$= 2\int_0^{100} \left\{ \frac{kP}{AG} + \frac{Pz^2}{EI_x} \right\} dz$$

$$+ \int_0^{\pi} \left\{ \frac{kP\cos^2\theta R}{AG} + \frac{P\sin^2\theta R}{AE} + \frac{A_m P(100 + R\sin\theta)}{A(RA_m - A)E} \right\} d\theta$$

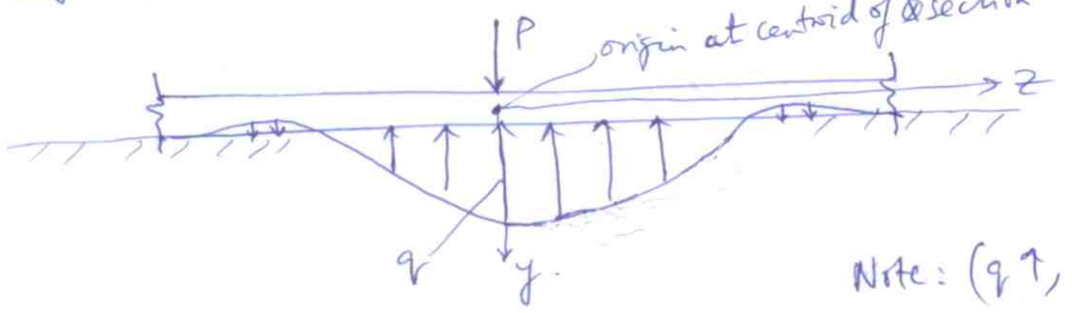
where the last term (ie U_{Nm}) in curved beam portion is neglected since it has negative contribution to δ_P.

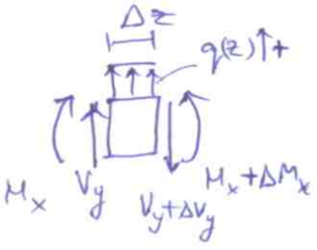
BEAMS ON ELASTIC FOUNDATIONS.

Assumptions:

1) Foundation stronger than beam so failure occurs first in beam (since that is often the case).
2) Linear elastic foundation. [Note that nonlinear foundations are generally of hardening type, ie, $f = k_1 x + k_2 x^3$, $(k_1, k_2) > 0$. Hardening nonlinearity reduces stresses/displacements when compared to linear case. So linear analysis is conservative] [Could also have viscoelastic foundation].
3) Beam attached to foundation (so foundation undergoes compression as well as tension).

(I) Infinite Beam with Point load





Note: (q ↑, w ↓) +ve

For Euler-Bernoulli beam,

$$\left.\begin{array}{l} \dfrac{dw}{dz} = \theta \ , \quad EI_x w'' = -M_x , \quad EI_x w''' = -V_y , \\ EI_x w^{IV} = -q(z) \end{array}\right\} \rightarrow ①.$$

For elastic foundation,

$$q = kw \ , \quad k = b k_0$$

where b is width of beam, k is $N.m^{-2}$, k_0 is foundation constant ie $N.m^{-3}$. Let $I_x \equiv I$, $V. \equiv V$, $M_x \equiv M$.

$$EIw^{IV} = -kw$$

$$w = e^{sz} \Rightarrow s^4 + 4\beta^4 = 0 \ , \quad \boxed{\beta^4 \triangleq \frac{k}{4EI}} \longrightarrow ②$$

$$(s^2 + 2i\beta^2)(s^2 - 2i\beta^2) = 0 \quad \Rightarrow s = \pm\sqrt{2}\beta\sqrt{\pm i} = \pm\sqrt{2}\beta e^{\frac{\pm i\pi}{4}}$$
$$= \pm\cancel{\sqrt{2}}\beta\frac{(1 \pm i)}{\cancel{\sqrt{2}}}$$

$$\Rightarrow w = \sum_{i=1}^{4} A_i e^{s_i z} = A_1 e^{\beta z} e^{i\beta z} + A_2 e^{\beta z} e^{-i\beta z} + A_3 e^{-\beta z} e^{+i\beta z} + A_4 e^{-\beta z} e^{-i\beta z}$$
$$= e^{\beta z}\left([A_1 + A_2]\cos\beta z + [A_1 - A_2]\, i \sin\beta z\right) + e^{-\beta z}\left([A_3 + A_4]\cos\beta z + i[A_3 - A_4]\sin\beta z\right)$$

$w = \text{real} \Rightarrow A_2 = \bar{A}_1,\ A_4 = \bar{A}_3$, let $A_1 + A_2 = C_1$, $i(A_1 - A_2) = C_2$, $(A_3 + A_4) = C_3$, $i(A_3 - A_4) = C_4$

$$\Rightarrow w = e^{\beta z}(C_1 \cos\beta z + C_2 \sin\beta z) + e^{-\beta z}(C_3 \cos\beta z + C_4 \sin\beta z)$$

$\because$ w = finite $\Rightarrow$ $C_1 = C_2 = 0$ for $z \geq 0$, $C_3 = C_4 = 0$ for $z \leq 0$.

$$\Rightarrow \boxed{\begin{array}{l} w = e^{-\beta z}(K_1 \cos\beta z + K_2 \sin\beta z) \quad \text{for } z \geq 0 \\ w[-z] = w[z] \quad (\text{from symmetry}) \end{array}} \rightarrow ③.$$

BC's: $w'\big|_{z=0} = 0$ (from symmetry) and $\frac{P}{2} = \int_0^\infty k w \, dz$ (from equilibrium)

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$-\cancel{\beta} K_1 + \cancel{\beta} K_2 = 0 \Rightarrow K_1 = K_2 = C \Longrightarrow \text{use } \int e^{-\beta z}(\sin\beta z + \cos\beta z)\,dz = -\frac{1}{\beta} e^{-\beta z}\cos\beta z$$

$$\Rightarrow C = \frac{P\beta}{2k}$$

$$\Rightarrow \boxed{\begin{array}{l} w = \frac{P\beta}{2k} \underbrace{e^{-\beta z}(\cos\beta z + \sin\beta z)}_{A_{\beta z}} = \frac{P\beta}{2k} A_{\beta z} \\ \theta = -\frac{P\beta^2}{k} \underbrace{e^{-\beta z}\sin\beta z}_{B_{\beta z}} = -\frac{P\beta^2}{k} B_{\beta z} \\ M = \frac{EI P \beta^3}{k} \underbrace{(\cos\beta z - \sin\beta z)\, e^{-\beta z}}_{C_{\beta z}} = \frac{P}{4\beta} C_{\beta z} \\ V = \frac{P}{4\beta}\beta(-2 \underbrace{e^{-\beta z}\cos\beta z}_{D_{\beta z}}) = -\frac{P}{2} D_{\beta z} \end{array}}$$

use ① ⇒ (θ)

→ see Table 9.2-1 Boresi for $A_{\beta z}$, $B_{\beta z}$, $C_{\beta z}$, $D_{\beta z}$

→ ④

→ $\boxed{\text{for } z \geq 0}$

→ see Fig 9.1.1, Boresi).

$$\text{i.e.,} \quad \boxed{\begin{array}{l} \int A_{\beta z}\,dz = -\frac{1}{\beta} D_{\beta z}, \quad \int B_{\beta z}\,dz = -\frac{1}{2\beta} A_{\beta z} \\ \int C_{\beta z}\,dz = \frac{1}{\beta} B_{\beta z}, \quad \int D_{\beta z}\,dz = -\frac{1}{2\beta} C_{\beta z} \end{array}} \rightarrow ⑤$$

Note symmetry of displ and BM and antisymmetry of slope and SF — these arise due to symmetry of problem and the sign convention adopted.

(see fig 9.1.1 Boresi).

$$\Rightarrow \boxed{\begin{array}{ll} w(-z) = w(z), & M(-z) = M(z) \\ V(-z) = -V(z), & \theta(-z) = -\theta(z) \end{array}} \rightarrow ⑥$$

140a

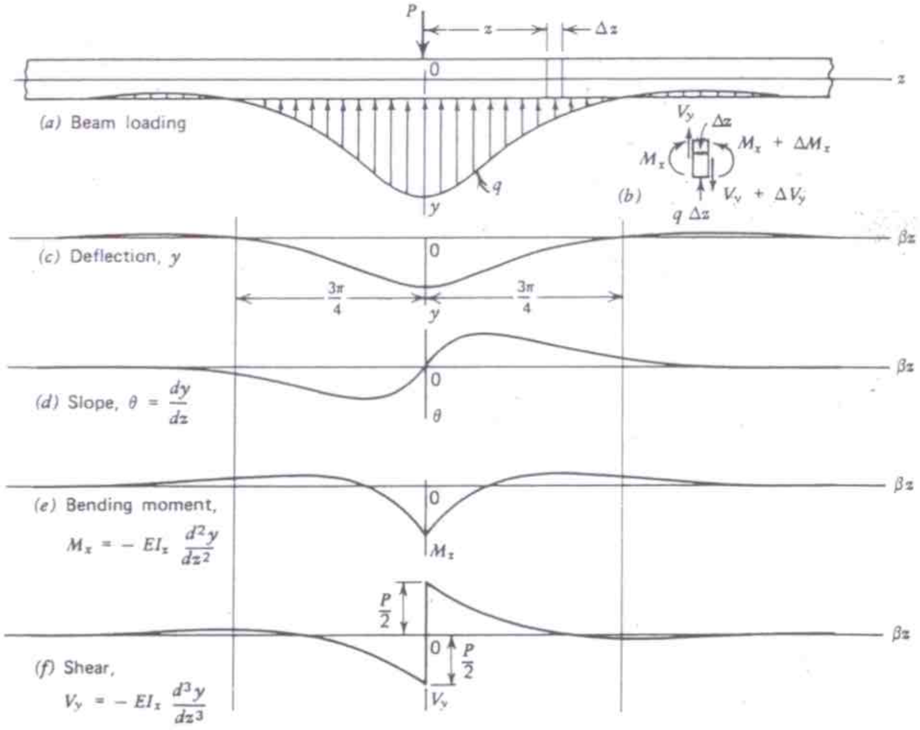


Fig. 9-1.1/Infinite beam on an elastic foundation and loaded at origin.

Table 9-2.1

βz	$A_{\beta z}$	$B_{\beta z}$	$C_{\beta z}$	$D_{\beta z}$
0	1	0	1	1
0.001	1.0000	0.0010	0.9980	0.9990
0.002	1.0000	0.0020	0.9960	0.9980
0.003	1.0000	0.0030	0.9940	0.9970
0.004	1.0000	0.0040	0.9920	0.9960
0.005	1.0000	0.0050	0.9900	0.9950
0.006	1.0000	0.0060	0.9880	0.9940
0.007	0.9999	0.0070	0.9861	0.9930
0.008	0.9999	0.0080	0.9841	0.9920
0.009	0.9999	0.0087	0.9821	0.9910
0.010	0.9999	0.0099	0.9801	0.9900
0.011	0.9999	0.0109	0.9781	0.9890
0.012	0.9999	0.0119	0.9761	0.9880
0.013	0.9998	0.0129	0.9742	0.9870
0.014	0.9998	0.0138	0.9722	0.9860
0.015	0.9998	0.0148	0.9702	0.9850
0.016	0.9997	0.0158	0.9683	0.9840
0.017	0.9997	0.0167	0.9663	0.9830
0.018	0.9997	0.0177	0.9643	0.9820
0.019	0.9996	0.0187	0.9624	0.9810
0.02	0.9996	0.0196	0.9604	0.9800
0.03	0.9991	0.0291	0.9409	0.9700
0.04	0.9984	0.0384	0.9216	0.9600
0.05	0.9976	0.0476	0.9025	0.9501
0.10	0.9906	0.0903	0.8100	0.9003
0.15	0.9796	0.1283	0.7224	0.8510
0.20	0.9651	0.1627	0.6398	0.8024
0.25	0.9472	0.1927	0.5619	0.7546
0.30	0.9267	0.2189	0.4888	0.7078
0.35	0.9036	0.2416	0.4204	0.6620
0.40	0.8784	0.2610	0.3564	0.6174
0.45	0.8515	0.2774	0.2968	0.5742
0.50	0.8231	0.2908	0.2414	0.5323
0.55	0.7934	0.3016	0.1902	0.4918
0.60	0.7628	0.3099	0.1430	0.4529
0.65	0.7315	0.3160	0.0996	0.4156
0.70	0.6997	0.3199	0.0599	0.3798
0.75	0.6676	0.3220	0.0237	0.3456
$\frac{1}{4}\pi$	0.6448	0.3224	0	0.3224
0.80	0.6353	0.3223	−0.0093	0.3131
0.85	0.6032	0.3212	−0.0391	0.2821
0.90	0.5712	0.3185	−0.0658	0.2527
0.95	0.5396	0.3146	−0.0896	0.2250
1.00	0.5083	0.3096	−0.1109	0.1987

Table 9-2.1

Continued

1.05	0.4778	0.3036	−0.1294	0.1742
1.10	0.4476	0.2967	−0.1458	0.1509
1.15	0.4183	0.2890	−0.1597	0.1293
1.20	0.3898	0.2807	−0.1716	0.1091
1.25	0.3623	0.2719	−0.1815	0.0904
1.30	0.3355	0.2626	−0.1897	0.0729
1.35	0.3098	0.2530	−0.1962	0.0568
1.40	0.2849	0.2430	−0.2011	0.0419
1.45	0.2611	0.2329	−0.2045	0.0283
1.50	0.2384	0.2226	−0.2068	0.0158
1.55	0.2166	0.2122	−0.2078	0.0044
$\frac{1}{2}\pi$	0.2079	0.2079	−0.2079	0
1.60	0.1960	0.2018	−0.2077	−0.0059
1.65	0.1763	0.1915	−0.2067	−0.0152
1.70	0.1576	0.1812	−0.2046	−0.0236
1.75	0.1400	0.1720	−0.2020	−0.0310
1.80	0.1234	0.1610	−0.1985	−0.0376
1.85	0.1078	0.1512	−0.1945	−0.0434
1.90	0.0932	0.1415	−0.1899	−0.0484
1.95	0.0795	0.1322	−0.1849	−0.0527
2.00	0.0667	0.1230	−0.1793	−0.0563
2.05	0.0549	0.1143	−0.1737	−0.0594
2.10	0.0438	0.1057	−0.1676	−0.0619
2.15	0.0337	0.0975	−0.1613	−0.0638
2.20	0.0244	0.0895	−0.1547	−0.0652
2.25	0.0157	0.0820	−0.1482	−0.0663
2.30	0.0080	0.0748	−0.1416	−0.0668
2.35	0.0008	0.0679	−0.1349	−0.0671
$\frac{3}{4}\pi$	0	0.0671	−0.1342	−0.0671
2.40	−0.0056	0.0613	−0.1282	−0.0669
2.45	−0.0114	0.0550	−0.1215	−0.0665
2.50	−0.0166	0.0492	−0.1149	−0.0658
2.55	−0.0213	0.0435	−0.1083	−0.0648
2.60	−0.0254	0.0383	−0.1020	−0.0637
2.65	−0.0289	0.0334	−0.0956	−0.0623
2.70	−0.0320	0.0287	−0.0895	−0.0608
2.75	−0.0347	0.0244	−0.0835	−0.0591
2.80	−0.0369	0.0204	−0.0777	−0.0573
2.85	−0.0388	0.0167	−0.0721	−0.0554
2.90	−0.0403	0.0132	−0.0666	−0.0534
2.95	−0.0415	0.0100	−0.0614	−0.0514
3.00	−0.0422	0.0071	−0.0563	−0.0493
3.05	−0.0427	0.0043	−0.0515	−0.0472
3.10	−0.0431	0.0019	−0.0469	−0.0450
π	−0.0432	0	−0.0432	−0.0432

Table 9-2.1

Continued

3.15	−0.0432	−0.0004	−0.0424	−0.0428
3.20	−0.0431	−0.0024	−0.0383	−0.0407
3.25	−0.0427	−0.0042	−0.0343	−0.0385
3.30	−0.0422	−0.0058	−0.0306	−0.0365
3.35	−0.0417	−0.0073	−0.0271	−0.0344
3.40	−0.0408	−0.0085	−0.0238	−0.0323
3.45	−0.0399	−0.0097	−0.0206	−0.0303
3.50	−0.0388	−0.0106	−0.0177	−0.0283
3.55	−0.0378	−0.0114	−0.0149	−0.0264
3.60	−0.0366	−0.0121	−0.0124	−0.0245
3.65	−0.0354	−0.0126	−0.0101	−0.0227
3.70	−0.0341	−0.0131	−0.0079	−0.0210
3.75	−0.0327	−0.0134	−0.0059	−0.0193
3.80	−0.0314	−0.0137	−0.0040	−0.0177
3.85	−0.0300	−0.0139	−0.0023	−0.0162
3.90	−0.0286	−0.0140	−0.0008	−0.0147
$\frac{5}{4}\pi$	−0.0278	−0.0140	0	−0.0139
3.95	−0.0272	−0.0139	0.0005	−0.0133
4.00	−0.0258	−0.0139	0.0019	−0.0120
4.50	−0.0132	−0.0108	0.0085	−0.0023
$\frac{3}{2}\pi$	−0.0090	−0.0090	0.0090	0
5.00	−0.0046	−0.0065	0.0084	0.0019
$\frac{7}{4}\pi$	0	−0.0029	0.0058	0.0029
5.50	0.0000	−0.0029	0.0058	0.0029
6.00	0.0017	−0.0007	0.0031	0.0024
2π	0.0019	0	0.0019	0.0019
6.50	0.0018	0.0003	0.0012	0.0018
7.00	0.0013	0.0006	0.0001	0.0007
$\frac{9}{4}\pi$	0.0012	0.0006	0	0.0006
7.50	0.0007	0.0005	−0.0003	0.0002
$\frac{5}{2}\pi$	0.0004	0.0004	−0.0004	0

$A_{\beta z}$, $B_{\beta z}$, $C_{\beta z}$, $D_{\beta z}$ tabulated in Table (9.2.1) — Boresi & Sidebottom.

From this table, note that

(1) • $\left(A_{\beta z}, B_{\beta z}, C_{\beta z}, D_{\beta z}\right) \to 0$ as $\beta z \to \infty$.

(2) • $\left|A_{\beta z}, C_{\beta z}, D_{\beta z}\right|_{max} = 1$ at $\beta z = 0$.

(3) • $\left|B_{\beta z}\right|_{max} = 0.3224$ at $\beta z = \pi/4$.

(4) • $A_{\beta z} = 0$ at $\beta z = \frac{3\pi}{4}$

#Thus ∞ beam theory can be used for long finite beam (length L) (due to (1) above). Further, due to (4) above, as a guideline we use ∞ theory for finite beam if distance from load to nearer end is $\geq 3\pi/4\beta$ (in which case $L \geq 3\pi/2\beta$).

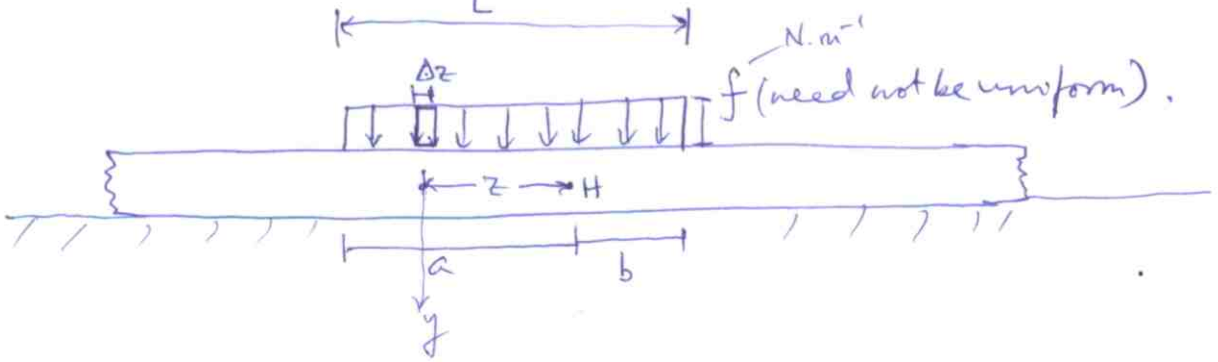
Although the error of this approx is nonconservative (i.e, (∞ theory applied to finite beam) yields smaller w, BM, etc than if we used the more exact finite theory) the error is very small.

#Since $A_{\beta z}$ and hence w are +ve for $\beta z < \frac{3\pi}{4}$ (i.e (4) above), a beam having $L \leq 3\pi/2\beta$ need not be attached to foundation & still the ∞ theory herein will give exact solution.

#For a infinite beam supported on discrete springs, equally spaced (ℓ mts apart), having identical stiffness (K Nm^{-1}), we smear the springs (over ℓ, i.e $\ell/2$ to left & right), i-e, use $\boxed{k = K/\ell}$. The solution is quite accurate when $\boxed{\ell \leq \pi/\beta}$.

#For finite beam (actual length L) on discrete springs (stiffness K, spacing ℓ) use effective length $\boxed{L'' = m\ell}$, where m = nos of equally spaced springs (i-e, extend beam length from L to L'' so that each spring is smeared over $\ell/2$ to its left and right). Then use ∞ beam approx method as above. This is fairly accurate if $\boxed{\beta L'' \geq 3\pi/2}$.

(II) Infinite beam subject to distributed (f) load segment.



We concentrate on portion under f, since max values of w, BM, SF usually occur under f region.

For $\Delta P = f\Delta z$,

$$\Delta w_H = \frac{f\Delta z\,\beta}{2k}\,\overbrace{e^{-\beta z}(\cos\beta z + \sin\beta z)}^{A_{\beta z}}$$

$$\Rightarrow\quad w_H = \int_0^a \frac{f\beta}{2k} A_{\beta z}\,dz + \int_0^b \frac{f\beta}{2k} A_{\beta z}\,dz = \frac{f\beta}{2k}\left(\frac{-1}{\beta}\right)\left(D_{\beta z}\Big|_0^a + D_{\beta z}\Big|_0^b\right)$$

$$\boxed{w_H = \frac{f}{2k}\left(2 - D_{\beta a} - D_{\beta b}\right)} \longrightarrow \text{(7a)}$$

$$④,⑤ \rightarrow\quad \theta = -\int_0^a \frac{f\beta^2}{k} B_{\beta z}\,dz \oplus \int_0^b \frac{f\beta^2}{k} B_{\beta z}\,dz = \frac{f\beta}{2k}\left(A_{\beta z}\Big|_0^a - A_{\beta z}\Big|_0^b\right)$$

$(\because\ \theta(-z) = -\theta(z))$.
for region 0–b.

$$\boxed{\theta = \frac{f\beta}{2k}\left(A_{\beta a} - A_{\beta b}\right)} \longrightarrow \text{(7b)}$$

$$M = \frac{f}{4\beta}\left(\int_0^a C_{\beta z}\,dz + \int_0^b C_{\beta z}\,dz\right) = \frac{f}{4\beta^2}\left(B_{\beta z}\Big|_0^a + B_{\beta z}\Big|_0^b\right)$$

$$\boxed{M = \frac{f}{4\beta^2}\left(B_{\beta a} + B_{\beta b}\right)} \longrightarrow \text{(7c)}$$

$(\because\ V(-z) = -V(z))$

$$V = -\frac{f}{2}\left[\int_0^a D_{\beta z} \ominus \int_0^b D_{\beta z}\right] = \frac{f}{4\beta}\left(C_{\beta z}\Big|_0^a - C_{\beta z}\Big|_0^b\right)$$

$$\boxed{V = \frac{f}{4\beta}\left(C_{\beta a} - C_{\beta b}\right)} \longrightarrow \text{(7d)}$$

\# Max deflection occurs at center of segment L' (can see it from physical symmetry or from Table 9.2.1).

\# Max BM.:

(i) $\beta L' \leq \pi$: From Table 9-2-1, see that max BM occurs at center of L' segment.

(ii) $\beta L' \to \infty$: $\beta L' \to \infty \Rightarrow (\beta a, \beta b) \to \infty$ except for points 'H' located near ends of L' segment.

$\Rightarrow$ (from (7)) $(\theta, M, V) \to 0$ and $w \to \frac{f}{k}$ except near ends of L' segment.

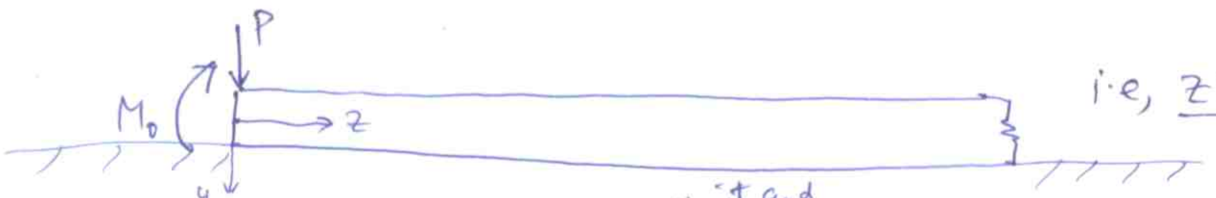
Then, Table 9.2.1 shows that max BM occurs ~~when~~ ie at pt. H for which either $\left\{\beta a = \frac{\pi}{4}, \text{ ie } \beta b \to \infty \; (\because \beta L' \to \infty)\right\}$ or $\left\{\beta b = \frac{\pi}{4}, \text{ ie } \beta a \to \infty \; (\because \beta L' \to \infty)\right\}$

(iii) $\beta L' > \pi$ (ie intermediate values of β) :

In this case max BM may lie outside load segment L'. However, sufficient accuracy can still be obtained by assuming max BM location as in (ii) above (ie at $\frac{\pi}{4\beta}$ from either end).

\# Note: Eqs (7a, c) valid only for points under distributed load (see Boresi Prob 9-3-3 p. 419).

(III) Semi-infinite beam with end loads



i.e, $z \geq 0$

Note that ③ is still valid. Use it and Apply bc's,

$$EI\, w''\Big|_{z=0} = -M_0 \quad , \quad EI\, w'''\Big|_{z=0} = -V = P$$

and get,

$$K_2 = \frac{2\beta^2 M_0}{k} \; , \quad K_1 = \frac{2\beta P}{k} - K_2$$

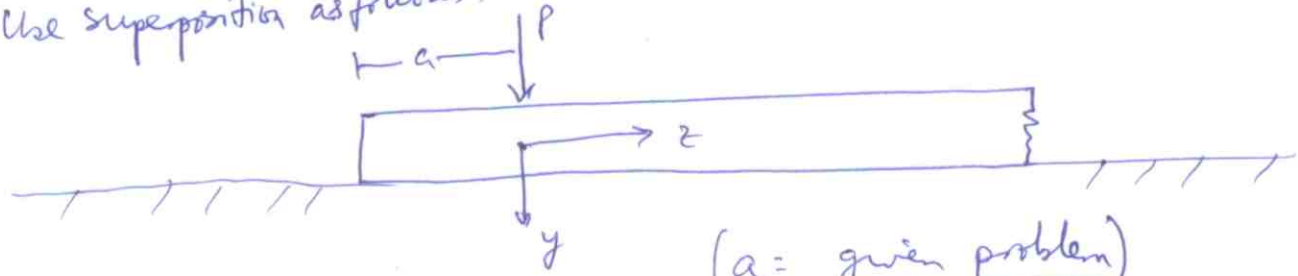
$$\Rightarrow \quad w = \frac{2\beta e^{-\beta z}}{k}\left[P\cos\beta z - \beta M_0\left(\cos\beta z - \sin\beta z\right)\right]$$

$$\boxed{w = \frac{2P\beta}{k} D_{\beta z} - \frac{2\beta^2 M_0}{k} C_{\beta z}} \longrightarrow (8a)$$

$$①, ⑤, (8a) \rightarrow \boxed{\begin{aligned} \theta &= -\frac{2P\beta^2}{k} A_{\beta z} + \frac{4\beta^3 M_0}{k} D_{\beta z} \\ M &= -\frac{P}{\beta} B_{\beta z} + M_0 A_{\beta z} \\ V &= -P C_{\beta z} - 2M_0 \beta B_{\beta z} \end{aligned}} \begin{aligned} &\longrightarrow (8b) \\ &\longrightarrow (8c) \\ &\longrightarrow (8d) \end{aligned} \quad \left.\right\} \; \underline{z \geq 0}$$

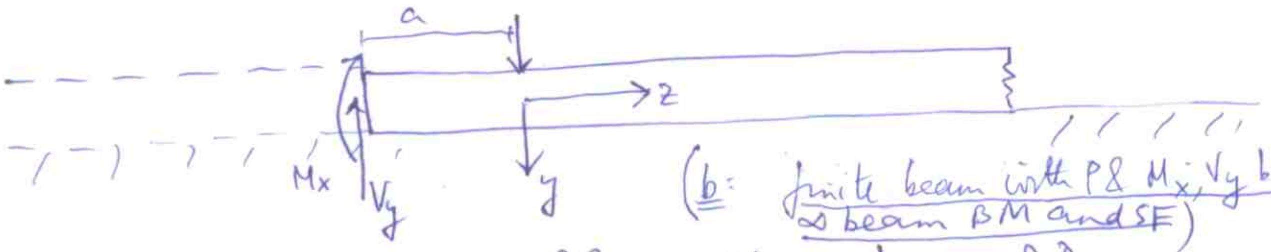
(IV) Semi-infinite beam with concentrated load near its end.

Use superposition as follows:



(a: given problem)

||



(b: finite beam with P & M_x, V_y based on ∞ beam BM and SF)

$$M_x = M_x\big|_{z=-a} = \frac{PC_{\beta a}}{4\beta}, \quad V_y = V_y\big|_{z=-a} = \frac{PD_{\beta a}}{2}$$

as obtained from ∞ beam (eqns 4(c,d))

+

(c: finite beam with end loads M, Q as shown)

$$⑨ \longleftarrow \boxed{M = \frac{PC_{\beta a}}{4\beta}, \quad Q = \frac{PD_{\beta a}}{2}} \rightarrow \text{chosen to negate end loads in Probl (b)}$$

Thus $\underset{\text{Sol}}{\text{Prob}\,(\underline{a})} = \underset{\text{Sol}}{\text{Prob}\,(\underline{b})} + \underset{\text{Sol}}{\text{Prob}\,(\underline{c})}$

Since origin is dist 'a' to the right of end of beam, use $z \rightarrow (a+z)$ in eqns (8) when superposing. Using (4a), 4(c) and shifted version of 8(a) and 8(c) (with $M_0 \equiv -M$, $P \equiv Q$) and ⑨, and adding them, respectively,

$$\boxed{\begin{aligned} W &= \frac{P\beta}{2k}\left[A_{\beta z} + 2D_{\beta a}D_{\beta(z+a)} + C_{\beta a}C_{\beta(z+a)}\right] \\ M &= \frac{P}{4\beta}\left[C_{\beta z} - 2D_{\beta a}B_{\beta(z+a)} - C_{\beta a}A_{\beta(z+a)}\right] \end{aligned}} \longrightarrow \text{⑩}$$

(Ex) (example 9.2-1 Boresi) — Infinite beam, multiple point loads

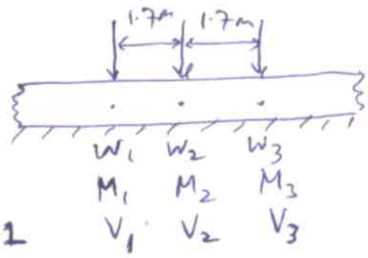
Given: Railroad with steel rails ($E = 200$ GPa) having depth 184mm, and distance from top of rail to centroid is 99.1mm, and $I = 36.9 * 10^6$ mm^4. Also $k = 14$ N/mm^2

Find: (a) Max deflection, max BM, max bending stress for case of single wheel with $P = 170$ kN.

(b) Do (a) if locomotive has three equally spaced wheels, spacing $= 1.70$m, with $P = 170$ kN for each wheel

(a) For single wheel, max defl, max BM occur at $\beta z = 0$, ie., under the wheel ($\because A_{\beta z}$ and $C_{\beta z}$ are max value of unity at $\beta z = 0$). Now β is found by ②, w_{max}, M_{max} by 4(a,c) with $A_{\beta z} = C_{\beta z} = 1$. From given section, y_{max} occurs at top ($= 99.1$mm) which gives max bending stress. Thus $w_{max} = \frac{P\beta}{2k}$, $M_{max} = \frac{P}{4\beta}$, $\sigma_{max} = \frac{M_{max}\,y_{max}}{I}$

(b) Need to use superposition. In general max values depend on inter-load spacing and value of β. For large inter-load spacing and/or large β value (representing steep exponential decay, it is evident that the max values will be under the loads (or near about that). Table (9.2-1) reveals that to be the case here, but this is not generally true. In general we just use superposition to find max values. Thus for the present case, let $z_0=0$, $z_1 = 1.7\times10^3$ mm, $z_2 = 3.4\times10^3$ mm. Then,



$$W_1 = W_3 = \frac{P\beta}{2k}\left(A_{\beta z_0} + A_{\beta z_1} + A_{\beta z_2}\right)$$

$$M_1 = M_3 = \frac{P}{4\beta}\left(C_{\beta z_0} + C_{\beta z_1} + C_{\beta z_2}\right)$$

$$V_1 = -\frac{P}{2}\left(\pm D_{\beta z_0} - D_{\beta z_1} - D_{\beta z_2}\right),$$ + for just to right of wheel 1, − for just to left of wheel 1

$$V_3 = -\frac{P}{2}\left(\mp D_{\beta z_0} + D_{\beta z_1} + D_{\beta z_2}\right),$$ − for just to left of wheel 3, + for just to right of wheel 3

consistent with the physical intuition that the shear forces have a numerically antisymmetric distribution about middle wheel, as per sign convention, i.e., a physically symmetric distribution about middle wheel.

$$W_2 = \frac{P\beta}{2k}\left(2A_{\beta z_1} + A_{\beta z_0}\right)$$

$$M_2 = \frac{P}{4\beta}\left(C_{\beta z_0} + 2C_{\beta z_1}\right)$$

$$V_2 = -\frac{P}{2}\left(\pm D_{\beta z_0} + \cancel{D_{\beta z_1}} - \cancel{D_{\beta z_1}}\right) = -\frac{P}{2}\left(\pm D_{\beta z_0}\right),$$ + for just to right of wheel 2, − " " "left" " "

The shear force analysis is not required.

Choose $_{max} = \max(W_1, W_2)$

$M_{max} = \max(M_1, M_2)$

Then $\sigma_{max} = \dfrac{M_{max}\, y_{max}}{I}$

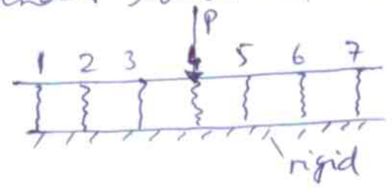
(Ex) (9.2.2 example, Boresi) – Finite beam on discrete springs.

Given: I Beam with depth = 100 mm, $I_x = 2.45\times10^6$ mm^4, E = 72 GPa, L = 6.8 m, supported on 7 springs, K = 110 N/mm, spacing = 1.10 m (l"), load P = 12 kN at center.

Find: load carried by each spring, deflection under the load, max bending moment and associated bending stress, based on approximation using smearing concept.

First check that $l \le \pi/\beta$ so that smearing approx is good and that $L''(= ml = 7l) \geq 3\pi/2\beta$ so that the infinite beam formula gives a good approx. These conditions are satisfied in the present case so we can proceed.

M_{max}, $W_{max} = W_4$ occur at $\beta z = 0$ (ie $A_{\beta z} = C_{\beta z} = 1 = max$), ie under load at center spring position. Also $\sigma_{max} = \frac{M_{max}(50)}{I}$.



→ Easy to find springforces – first find deflections of w_5, w_6, w_7 and by K.

The spring forces in the leftmost & right most spring are in considerable error from this smearing type solution as compared to the exact solution based on Castigliano's theorem using discrete springs. However, other spring forces, deflections, and BM compare very well (error ≃ 7% when compared to exact solution).

(<u>Ex</u>) (Boresi Ex 9.4-1)

Given: I beam, $E = 200 GPa$, depth $= 102$ mm, width $= 68$ mm, $I = 2.53 * 10^6 mm^4$, $L = 4m$, $k_0 = 0.350 \ N/mm^3$, concentrated downward load $P = 30 kN$ at end.

Find: Max defl and max bending stress, and their location.

$k = 0.350 * 68 \ N/mm^2$

Here $L (= 4000) > \frac{3\pi}{2\beta}$ so ∞ beam theory can be used.

$W = \frac{2P\beta}{k} D_{\beta z} \Rightarrow W_{max}$ at $\beta z = 0$, $D_{\beta z}\big|_0 = 1$, $z = 0$.

$M_x = -\frac{P}{\beta} B_{\beta z} \Rightarrow M_{max}$ at $\beta z = \pi/4$, $B_{\beta z}\big|_{\pi/4} = 0.3224$

$\sigma_{max} = M_{max}(51)/I$.

(Ex)

VISCOELASTICITY – Linear Theory.

Recall: $\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\bar{\sigma}}_{ij}$, $\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(A_{ij}, T)$ = conservative stress tensor, $A_{ij} \rightarrow$ generic strain tensor.
$\bar{\bar{\sigma}}_{ij}$ = dissipative stress tensor

For viscoelasticity, we do not have such a kinetic equation of state to relate the conservative part of the stress tensor to the strain tensor. Instead, we relate the stress tensor to the strain tensor thru a differential eqn as follows:

$$\cdots \tilde{\tilde{B}}_{ijkr}\,\ddot{\sigma}_{kr} + \tilde{B}_{ijkr}\,\dot{\sigma}_{kr} + \sigma_{ij} = \overset{(=0 \text{ when no prestress is assumed})}{\cancel{C_{ij}}} + C_{ijkr}\,\ell_{kr} + \tilde{C}_{ijkr}\,\dot{e}_{kr} + \cdots$$

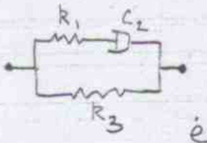
Linear Viscoelasticity – Assumptions.

(i) Coeffs $\bar{\bar{B}}_{ijkr}$, $\bar{C}_{ijkr}$, etc. are not functions of stress/strain tensors or their time rates.

(ii) Small displ. gradients assumed (i.e, linear strain tensors ℓ_{kr}, e_{kr}, considered)

NOTE: In viscoelasticity, we have dissipation of energy due to which the process is irreversible, in general.

Three Parameter Solid. – 1-D modelling



$\sigma_1 = k_1 e_1$, $\sigma_2 = c_2 \dot{e}_2$, $\sigma_1 = \sigma_2$
$\sigma_3 = k_3 e_3$, $e = e_3 = e_1 + e_2$, $\sigma = \sigma_1 + \sigma_3$ } 7 eqns, 8 unknowns, so we can eliminate and get one eqn relating the two unknowns σ, e.

$$\dot{e} = \dot{e}_1 + \dot{e}_2 = \frac{\dot{\sigma}_1}{k_1} + \frac{\sigma_2}{c_2} = \frac{\dot{\sigma}_1}{k_1} + \frac{\sigma_1}{c_2} = \frac{\dot{\sigma} - \dot{\sigma}_3}{k_1} + \frac{\sigma - \sigma_3}{c_2} = \frac{\dot{\sigma} - k_3\dot{e}_3}{k_1} + \frac{\sigma - k_3 e_3}{c_2}$$

$$\dot{e} = \frac{\dot{\sigma}}{k_1} - \frac{k_3}{k_1}\dot{e} + \frac{\sigma}{c_2} - \frac{k_3}{c_2}e \Rightarrow \boxed{\frac{\dot{\sigma}}{k_1} + \frac{\sigma}{c_2} = \frac{k_3}{c_2}e + \left(1 + \frac{k_3}{k_1}\right)\dot{e}}$$

Elements of Laplace Transform.

Definition of L.T. $F(s) \triangleq \int_0^\infty f(t)\,e^{-st}\,dt$ provided the integral exists (i.e, it is finite).

This means that $f(t)$ must satisfy the condt. $\int^\infty |f(t)e^{-\sigma t}|\,dt < \infty$ where σ is real & finite, & $s = \sigma + i\omega$. It also implies that $f(t)$ is piecewise continuous.

Note: For a causal system wherein response (output) does not precede excitation (input), the field functions (response, excitation, etc), which we will call $f(t)$, are assumed to satisfy $f(t) = 0$ for $t < 0$.

Inverse L.T. $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,e^{st}\,ds$ where c is a real constant that is greater than the real parts of all the singularities of $F(s)$. Thus we need to determine all the singularities of $F(s)$ and then evaluate the above line integral in the s-plane. Generally this definition is not used.

eorems on L.T.

(1) $\mathcal{L}[k f(t)] = k F(s)$, where k is a complx const.

(2) $\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$.

(3) $\mathcal{L}\left[\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\int_0^{t_1} f(\tau)\,d\tau\,dt_1\,dt_2\cdots dt_{n-1}\right] = \frac{F(s)}{s^n}$

(4) Shift in time : $\mathcal{L}[f(t-T)\,H(t-T)] = e^{-Ts} F(s)$, where $H(t-T)$ is shifted step f^n.

(5) Initial value theorem : $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$ if the limit exists

(6) Final value theorem : If $sF(s)$ is analytic on the imag axis (ie, $s=i\omega$) and the right half complx plane (ie, $s=\sigma+i\omega$, $\sigma>0$) then,

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ → valid only if $sF(s)$ has no poles with $\sigma \geq 0$.

(7) Shift of axis : $\mathcal{L}[e^{\pm\alpha t} f(t)] = F(s\mp\alpha)$

(8) Convolution Theorem : Let $f_1(t)=0$, $f_2(t)=0$, for $t<0$. Then,

$$\mathcal{L}\left[\int_0^t f_1(\tau)\, f_2(t-\tau)\,d\tau\right] = \underbrace{\mathcal{L}\left[\int_0^\infty f_1(\tau)\, f_2(t-\tau)\,d\tau\right]}_{\mathcal{L}[f_1 * f_2]} = F_1(s)\,F_2(s)$$

Also, we have the dual relation, $\mathcal{L}[f_1(t)\,f_2(t)] = F_1(s) * F_2(s)$.

Some commonly used L.T.'s.

(i) Unit impulse function $[\delta(t)]$.

consider $\delta(t-a)$, $a>0$.

$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$, $(a>0)$, $(s>0)$

Then for $a=0$, we get $\mathcal{L}\{\delta(t)\} = 1$. We see that strictly speaking, L.T. should be defined as $\int_{0^-}^\infty e^{-st} f(t)\,dt$ and then we can directly get this result.

(ii) $\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,dt = \int_a^\infty e^{-s(t-a)} H(t-a)\,e^{-as}\,dt = e^{-as}\int_0^\infty e^{-s\tau} H(\tau)\,d\tau = e^{-as}\left[\frac{-1}{s}e^{-s\tau}\right]_0^\infty$

$= \frac{e^{-as}}{s}$, $s>0$ (can also do by Theorem #4 above by putting $f(t-a) = H(t-a)$ and noting that $\mathcal{L}[H(t-a)\,H(t-a)] = \mathcal{L}[H(t-a)]$ ∵ $(H(t-a))^n = H(t-a)$)

(iii) $\mathcal{L}\{e^{-\alpha t}\} = \int_0^\infty e^{-\alpha t} e^{-st}\,dt = \left[-\frac{e^{-(s+\alpha)t}}{s+\alpha}\right]_0^\infty = \frac{1}{s+\alpha}$, $s+\alpha>0$

Coming back to 3-parameter solid, rewriting the differential constitutive law, we get,

$\dot{\sigma} + \frac{\sigma}{T_R} = E_D\left\{\dot{e} + \frac{e}{T_C}\right\}$ where, $T_R^{-1} = k_1/C_2$, $E_D = k_1 + k_3$, $T_C^{-1} = \frac{k_1 k_3}{C_2(k_1+k_3)}$ (see reverse for reconciliation with YCF notation)

Taking the L.T. for zero IC's (ie, $\sigma(0)=e(0)=0$), we get,

$\sigma(s) = E(s)\,e(s)$, $E(s) = E_D\left(\frac{s+T_C^{-1}}{s+T_R^{-1}}\right)$ = transform modulus.

us in general (ie, taking L.T. of differential const. law for a general viscoelastic material) we get (assuming zero IC's),

$$\sigma(s) = E(s)\,e(s) \Rightarrow \sigma(t) = \int_0^t E(t-\tau)\,e(\tau)\,d\tau \quad \text{(from convolution integral).}$$

↳ shows that σ(t) depends on history of e — i.e., hereditary effect

Creep function

Consider 3-parameter solid with $\sigma(t) = \sigma_0 H(t)$; $H(t)$ = step f^n $= 0,\ t<0$; $= \sigma_0,\ t>0$

Then, for zero I.C.'s, (ie, $\sigma(0) = e(0) = 0$)

$$e(s) = \frac{\sigma_0}{E_D}\frac{1}{s}\left(\frac{s+T_R^{-1}}{s+T_C^{-1}}\right) = \frac{\sigma_0}{E_D}\left[\frac{1}{s+T_C^{-1}} + \frac{T_R^{-1}}{T_C^{-1}}\left(\frac{1}{s} - \frac{1}{s+T_C^{-1}}\right)\right]$$

$$\Rightarrow e(t) = \frac{\sigma_0}{E_D}\left[\frac{T_R^{-1}}{T_C^{-1}}H(t) + \left(1-\frac{T_R^{-1}}{T_C^{-1}}\right)e^{-t/T_C}\right] = \frac{\sigma_0}{E_D}\left[\left(\frac{T_R^{-1}}{T_C^{-1}}+1-1\right)H(t) + \left(1-\frac{T_R^{-1}}{T_C^{-1}}\right)e^{-t/T_C}\,\overset{\uparrow}{H(t)}\right]$$

$H(t)$ introduced ∵ we assume system to be causal i.e, output $e(t)$ cannot precede input $\sigma(t)$.

$$\underset{\text{(creep } f^n \text{ in YCF)}}{c(t)} = e(t) = \frac{\sigma_0}{E_D}\left[1 + \underbrace{\left(\frac{T_C}{T_R}-1\right)\left(1-e^{-t/T_C}\right)}_{}\right]H(t)$$

($\psi(t)$ → creep f^n in S.C. Hunter, defined: $\psi(0) = 0$)

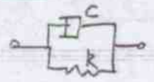
valid for t>0 For t=0 use IC's

$$\therefore \boxed{e(t) = \frac{\sigma_0}{E_D}\left(1+\psi(t)\right)H(t)}$$

→ response to unit step σ(t). The first term in ( ), i.e, 1, represents the elastic component and second term in ( ), i.e, ψ(t), represents the visco-el comp.

NOTE: This is a general form for unit step response for any viscoelastic material. For the case when the $\dot{\sigma}$ term does not appear in the differential form of the constitutive law, we can argue that $e(t) = \frac{\sigma_0}{E_D K}\left(1+K\psi(t)\right)$ where $K\to\infty$, so that $e(0^+) = 0$. (see details below).

Voigt element



$\sigma_S = ke_S$, $\sigma_D = c\dot{e}_D$, $e = e_S = e_D$, $\sigma = \sigma_S + \sigma_D$

$\dot{\sigma} = ke + c\dot{e} = c(\dot{e} + \frac{k}{c}e)$

not ψ but e(t) 16/11/97
→ $\psi(t) = 0$ elastic solid
→ $\psi(\infty) \to 0$ visco solid
→ $\psi(\infty) \to \infty$ visco liquid

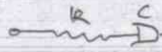
For $\sigma(t) = \sigma_0 H(t)$ & zero IC's we get, $e(s) = \frac{\sigma_0}{c}\frac{1}{s}\left(\frac{k}{c}+s\right)^{-1} = \frac{\sigma_0}{k}\left(\frac{1}{s} - \frac{1}{s+k/c}\right)$

$$e(t) = \underset{\text{(of YCF)}}{c(t)} = \frac{\sigma_0}{k}\underbrace{\left(1-e^{-k/ct}\right)}_{\psi(t)/K}H(t) = \frac{\sigma_0}{kK}\left(1+\psi(t)\right) \text{ where } K \text{ is large (ie, } K\to\infty).$$

So we see that here the elastic comp vanishes ∵ K is large.

Also, $e(0^+) = 0$

Maxwell element



$\sigma = \sigma_S = \sigma_D$; $e = e_S + e_D$; $\sigma_S = ke_S$, $\sigma_D = c\dot{e}_D$.

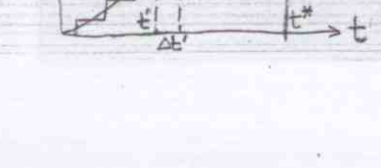
$\therefore \dot{e} = \dot{e}_S + \dot{e}_D = \frac{\dot{\sigma}}{k} + \frac{\sigma}{c} \Rightarrow \dot{\sigma} + \frac{k}{c}\sigma = k\dot{e}$

For $\sigma(t) = \sigma_0 H(t)$ & zero IC's, $e(s) = \frac{1}{k}\frac{1}{s}\frac{\sigma_0}{s}\left(s + k/c\right) \Rightarrow e(t) = \underset{\text{of YCF}}{c(t)} = \frac{\sigma_0}{k}\left(H(t) + \frac{k}{c}t\right) \overset{\text{(due to causality)}}{=} \frac{\sigma_0}{k}\left(1+\underbrace{\frac{k}{c}t}_{\psi(t)}\right)H(t)$

Thus, $e(0^+) = \sigma_0/k$

Returning to 3-p model, $e(0^+) = \sigma_0/E_D = \sigma_0/(k_1+k_3)$.

General Response for viscoelastic material — Boltzmann's constitutive Law



We sum the responses ($\Delta e(t^*)$) due to step inputs Δσ applied at t', where $t' = 0 \to t^*$

$$\therefore \Delta e(t^*) = \frac{\Delta\sigma|_{t'}}{E_D}\left(1+\psi[t^*-t']\right)H(t^*-t')$$

or $\Delta\sigma\Big|_{t'} = \frac{d\sigma(t)}{dt}dt\Big|_{t=t'} = \frac{d\sigma(t')}{dt'}dt'$

$\therefore e(t^*) = \lim_{\Delta\sigma\to 0} \frac{1}{E_D}\sum \Delta\sigma|_{t'}\left(1+\psi[t^*-t']\right)H(t^*-t') = \frac{1}{E_D}\int_0^{t^*}\frac{d\sigma(t')}{dt'}\left(1+\psi[t^*-t']\right)dt'$, $\underline{t^*>0}$.

Dropping the * notation and considering zero IC's ($\sigma(0)=0$), we get,

$$\boxed{e(t)=\frac{1}{E_D}\Big[\underbrace{\sigma(t)}_{\text{instantaneous, elastic part}} + \underbrace{\int_0^t \frac{d\sigma(t')}{dt'}\psi[t-t']dt'}_{\text{hereditary part}}\Big] \rightarrow \text{Boltzmann's constitutive Law} \quad t>0}$$

Taking the L.T. of the above we get, for zero IC's, $e(s) = \sigma(s)\underline{\left(1+s\psi(s)\right)}/E_D$

Thus we get $\boxed{E(s) = E_D\left(1+s\psi(s)\right)^{-1}}$ → relationship between transform modulus and creep function ψ.

NOTE: In YCF he gets $e(t) = \int_0^t c(t-t')\frac{d\sigma(t')}{dt'}dt'$. This is equivalent to the above since in YCF, $c(t-t') = \frac{(1+\psi(t-t'))}{E_D}$.

In both YCF & Hunter, $e(0)=0$. Note that IC's for various visco models, ie, Maxwell, Voight, & 3-p solid given in YCF are based on physical considerations. They are not unique to the respective models since any IC's can be considered for any visco model. Thus we should not use the step response (ie creep fn) to evaluate $e(0)$ since the step response is valid (strictly speaking) only for $t>0$. For $t=0$ we should just use the given IC's. That is to say we should separate out the homogenous sol. & particular sol. Thus,

$e(t) = e_H(t) + e_p(t)$, $t>0$, where

$e_H(t)$ is due to IC's ($\sigma(0)$) alone for which $\mathcal{L}\{\dot\sigma(t)\} = s\overset{=0}{\cancel{\sigma(s)}} - \sigma(0)$, and

$e_p(t)$ is due to forcing f^n ($\sigma(t) = \sigma_0 H(t)$) alone for which $\mathcal{L}\{\dot\sigma(t)\} = s\underset{\sigma_0/s}{\cancel{\sigma(s)}} - \underset{0}{\cancel{\sigma(0)}}$.

Also, $e(0) = e_p(0)$.

+done detail

<u>Relaxation function</u>

Input is $e(t) = e_0 H(t)$. Output is $\sigma(t)$

In general we can write, $\overset{\text{(YCF notation for relaxation } f^n)}{k(t)} = \sigma(t) = E_D e_0(1-\phi(t))$. for any visco model. This is analogous to the general expression for creep f^n $c(t)$.

(rel. f^n of Hunter)

Maxwell : $k(t) = \sigma(t) = e_0 k e^{-k/c\, t}H(t)$

Voight : $k(t) = \sigma(t) = e_0(c\delta(t) + \mu)H(t)$

3-p solid : $k(t) = \sigma(t) = E_D e_0\left[1-(1-T_R/T_c)(1-e^{-t/T_R})\right]$