Compatibility Equations.

(LINEAR)

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \rightarrow \text{STRAIN-DISPL EQNS}$$

IIT Bombay

- If e_{ij} determined (or given) first, then we need to solve u_i (3) by integrating 6 SD eqns, ie, 3 unknowns (u_i) from 6 eqns.
- so to get unique u_i, the e_{ij} must satify compatibility equations.
- Another way of looking at this is

that u_i should be single-valued and unique. This means that a particle cannot occupy two points after deformation (ie no voids/tearing), and two particles occupying distinct positions before deformation cannot coalesce after deformation.

- So e_{ij} are not independent of each other and cannot be chosen arbitrarily.
- Consider solid composed of small cubic elements. The e_{ij} should be such

IIT Bombay

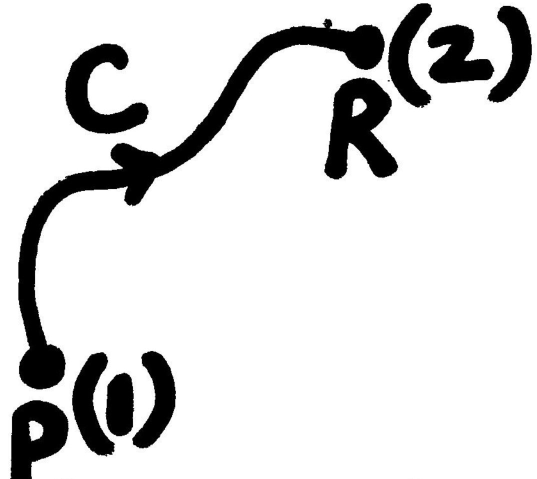
that all deformed cubes should fit together after deformation without any holes or overlaps.

Derivation: based on single valuedness of displ's.

Consider two points denoted (1) & (2) in the solid. Then,

$$u_i(2) = u_i(1) + \int_C du_i$$

$$= u_i(1) + \int_C (e_{ij} + \omega_{ij})\, dx_j$$



C is any path from (1) to (2) in the simply connected domain

$$u_i(2) = u_i(1) + \int_C e_{i\cancel{j}\,k}\, dx_{\cancel{j}\,k}$$

$$+ \omega_{ij}(2)\, x_j(2) - \omega_{ij}(1)\, x_j(1)$$

$$- \int_C x_j \underbrace{\omega_{ij,k}\, dx_k}_{= d\omega_{ij}} \qquad \text{(int. by parts)}$$

Now, $\omega_{ij,k} = \frac{1}{2}\left(u_{i,jk} - u_{j,ik} + u_{k,ij} - u_{k,ij}\right)$

$$= e_{ik,j} - e_{jk,i}$$

$$\Rightarrow u_i(2) = u_i(1) + \omega_{ij}(2)\, x_j(2) - \omega_{ij}(1)\, x_j(1) + \int_C F_{ik}\, dx_k$$

Where, $F_{ik} = e_{ik} - x_j\left(e_{ik,j} - e_{jk,i}\right)$

For $u_i(2)$ to be singlevalued, irrespective of path C from (1) to (2), we must have

$$F_{ik}\,dx_k = \text{perfect differential} = dG_i = G_{i,k}\,dx_k$$

$$\Rightarrow F_{ik} = G_{i,k} \Rightarrow F_{ik,l} = F_{il,k}$$

$$\Rightarrow \cancel{e_{ik,l}} - x_j\left(e_{ik,jl} - e_{jk,il}\right) - \left(\cancel{e_{ik,l}} - \cancel{e_{lk,i}}\right)$$

$$= \cancel{e_{il,k}} - x_j\left(e_{il,kj} - e_{jl,ik}\right) - \left(\cancel{e_{il,k}} - \cancel{e_{kl,i}}\right)$$

$x_j \neq 0$ in general $\Rightarrow e_{ik,jl} - e_{jk,il} = e_{il,kj} - e_{jl,ik}$

IIT Bombay

Even if $x_j = 0$ (ie at origin) we can always do a shift of axis to include origin $\therefore$ $\partial/\partial x_j$ will not be affected by shift of axis.

Rearranging indices, $j \rightleftarrows k$, then $i \rightleftarrows j$,

$$\boxed{e_{ij,kl} + e_{kl,ij} = e_{ik,jl} + e_{jl,ik}} \quad (1)$$

$\rightarrow$ Saint Venant Strain Compatibility Eqns

$3^4 = 81$ Eqns, of which only following 6 are independent:-

	I	II	III	IV	V	VI eqn.
i	1	3	2	1	2	3
j	1	3	2	1	2	3
k	2	1	3	2	3	1
l	2	1	3	3	1	2

In full form, in physical notation,

$$e_{xx,yy} + e_{yy,xx} = 2e_{xy,xy}$$

$$e_{xx,yz} + e_{yz,xx} = e_{xy,xz} + e_{xz,xy}$$

(and other four, $x \to y$, $y \to z$, $z - x$)

Now we derive Compat Eqns in terms of stress, for Isotropic linear solid.

Constitutive Law $\rightarrow$ $e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{mm}$

Subst CL in Compat eqns,

$$(1+\nu)\left[\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik}\right] =$$

$$\nu\left[\delta_{ij}\sigma_{mm,kl} + \delta_{kl}\sigma_{mm,ij} - \delta_{ik}\sigma_{mm,jl} - \delta_{jl}\sigma_{mm,ik}\right]$$

Now you can use same table for the 6 indep eqns. However easier & compact way is as follows:

Perform contraction $k \equiv l$,

$$(1+\nu)\left[\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik}\right] =$$

$$(\nu)\left[\delta_{ij}\sigma_{mm,kk} + \underbrace{\cancel{\delta_{kk}}}_{3}\sigma_{mm,ij} - \delta_{ik}\sigma_{mm,jk} - \delta_{jk}\sigma_{mm,ik}\right]$$

Subst Equil eqn $\rightarrow \sigma_{ij,j} + f_i = 0$ in above, (f_i → b.f per vol.)

$$(1+\nu)\left[\sigma_{ij,kk} + \sigma_{kk,ij} + f_{i,j} + f_{j,i}\right] = \nu\left[\delta_{ij}\sigma_{mm,kk} + \sigma_{mm,ij}\right]$$

6 indep eqns.

Do contraction $i \equiv j$, solve $\sigma_{mm,kk} = -\dfrac{(1+\nu)}{(1-\nu)} f_{i,i}$ subst

Thus,

$$\underbrace{\sigma_{ij,kk}}_{\nabla^2 \sigma_{ij}} + \frac{1}{1+\nu}\underbrace{\sigma_{kk,ij}}_{I_{1,ij}} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu}\delta_{ij}\overbrace{f_{k,k}}^{\nabla \cdot \underset{\sim}{f}} = 0$$

→ BELTRAMI–MICHELL STRESS COMPATIBILITY EQNS.

(2)