CONSTITUTIVE EQUATIONS

(LINEAR — HOOKE'S LAW)

For 3-D Linear Anisotropic Solid,

$$(1) \leftarrow \boxed{\sigma_{ij} = \underbrace{C_{ijkl}}_{\text{elastic consts.}} e_{kl}} \rightarrow \text{3D Linear Hooke's Law.}$$

Symm of σ_{ij} & $e_{kl} \Rightarrow C_{ijkl} = C_{jikl} = C_{jilk} = C_{jikl}$

i.e., only 36 independent elastic constants

Alternatively, due to symm of $\underline{\underline{\sigma}}$ & $\underline{\underline{e}}$, we can use extended notation, as follows:

$$(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) \equiv (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$$

$$(e_{11}, e_{22}, e_{33}, e_{23}, e_{13}, e_{12}) \equiv (e_1, e_2, e_3, e_4, e_5, e_6)$$

dont confuse with e_i on p23 of STRAINS, ie, relative displ per unit length

Thus Hookes Law for 3D Anisotropy is

$\boxed{\sigma_i = C_{ij} e_j}$ → $i, j = 1, \dots 6$ (2)

So 36 indep C_{ij}'s.

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Now consider strain energy per unit vol defined as,

$$U = \frac{1}{2}\sigma_i e_i = \frac{1}{2} C_{ij} e_j e_i = \frac{1}{2} C_{ji} e_i e_j$$

$$\Rightarrow C_{ij} = C_{ji} \Rightarrow 36 - 15 = \boxed{21 \text{ indep consts.}}$$

ie, $C_{ijkl} = C_{kl,ij}$

Isotropic material: one whose mat. props are direction independent, ie C_{ijkl} (or C_{ij}) are invariant wrt coord axes/coord transformation

Homogenous: C_{ijkl} constant wrt spatial coords x_k.

$C_{ijkl} = C'_{ijkl} \Rightarrow$ isotropic

$\dfrac{\partial C_{ijkl}}{\partial x_k} = 0 \Rightarrow$ homogenous.

and C_{ijkl} symmetries.

Consider Isotropic Solid.

For transformation of 90° rotation about x_1 axis,

$$\underline{\underline{a}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow C'_{1123} = a_{1i} a_{1j} a_{2k} a_{3l} C_{ijkl} = -C_{1132}$$

$$= -C_{1123} \underset{\text{Isotropy}}{=} C_{1123}$$

$$\Rightarrow C_{1123} = 0$$

Also $C'_{1213} = -C_{1312} = -C_{1213} \underset{\text{Isotropy}}{=} C_{1213} \Rightarrow C_{1213} = 0$

$$\left.\begin{aligned} C'_{1113} &= -C_{1112} = C_{1113} \\ C'_{1112} &= C_{1113} = C_{1112} \end{aligned}\right\} \Rightarrow \begin{aligned} C_{1112} &= 0 \\ C_{1113} &= 0 \end{aligned}$$

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Thus if 2,3 appear once or 1 appears thrice then $C_{ijkl} = 0$. Doing 90° rot about x_2 and x_3 axes, similarly, we conclude that $C_{ijkl} = 0$ for isotropic solid if any index appears odd times, ie.,

$$C_{ijkl} = 0 \text{ except when } i=j, k=l \text{ or } i=k, j=l \text{ or } i=l, j=k.$$

Now align axes along p-axes of $\underline{\underline{e}}$.

$\Rightarrow e_{ij} = 0$ for $i \neq j$

$$\sigma_{ij} = C_{ijkl} e_{kl} = C_{ij11} e_{11} + C_{ij22} e_{22} + C_{ij33} e_{33}$$

$\Rightarrow \sigma_{ij} = 0$ for $i \neq j$ $\left(\because C_{ij11}, C_{ij22}, C_{ij33} \text{ are zero for } i \neq j \text{, ie index appearing odd times}\right)$.

$\Rightarrow$ $\boxed{\text{p-axes of } \underline{\underline{\sigma}} \text{ \& } \underline{\underline{e}} \text{ coincide for Isotropic solid}}$

Now consider 90° rotation about x_1 axis followed by reflection about x_3' axis, ie,

$$\underline{\underline{a}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{reflection abt } x_3'} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{90^\circ \text{ rot abt } x_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow C'_{1122} = C_{1133} = C_{1122}$

$C'_{2222} = C_{3333} = C_{2222}$

Similarly by 90° rot about x_2 followed by reflection abt x_1'

& by 90° rot abt x_3 followed by reflection abt x_2', we get

$C_{2211} = C_{2233}$, $C_{1111} = C_{3333}$

& $C_{3311} = C_{3322}$, $C_{1111} = C_{2222}$

$$\Rightarrow \left.\begin{array}{l} C_{1111} = C_{2222} = C_{3333} \rightarrow = (2\mu + \lambda) \\ C_{1122} = C_{1133} = C_{2233} \rightarrow = \lambda \\ \text{all other } C_{ijkl} = 0 \end{array}\right\}$$

for Isotropic ~~solid~~ material

Here λ, μ are LAME's Consts.

$\Rightarrow$ only 2 indep consts for Isotropic mat'l.

With x_i as p-axes, we had,

$$\sigma_{ij} = (2\mu+\lambda)e_{ij} + \lambda e_{kk}\delta_{ij} - \lambda e_{ij}$$

$$\boxed{\sigma_{ij} = 2\mu e_{ij} + \lambda \underbrace{e_{kk}}_{D}\delta_{ij}} \rightarrow (3a)$$

$D \rightarrow$ linear cubical dilatation.

$\because$ this is a tensor eqn $\Rightarrow$ it is invariant to coord transf, so it is valid in every (ie even in non-principal) system of coords.

Contract $i \equiv j \Rightarrow e_{kk} = \sigma_{kk}/(3\lambda+2\mu)$

Inverting,

$$\boxed{e_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij}} \quad (3b)$$

This is Hooke's Law for Linear Elastic Isotropic Material in terms of LAME's Consts.

In terms of Engg consts E, ν,

$$\boxed{e_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}} \quad (4a)$$

{By superposing normal stresses as in basic solid mech course.}

Contract $i \equiv j \Rightarrow \sigma_{kk} = \frac{E}{(1-2\nu)} e_{kk}$

inverting $\Rightarrow$ $$\boxed{\sigma_{ij} = \frac{E}{1+\nu}\left[e_{ij} + \frac{\nu}{1-2\nu} e_{kk}\delta_{ij}\right]} \quad (4b)$$

Comparing, $2\mu = \frac{E}{1+\nu}$ → twice shear modulus

$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ → no physical meaning for λ.

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$$e_{kk} = \frac{1-2\nu}{E}\sigma_{kk} = 0 \text{ for } \nu = 1/2$$

Incompressible solid $\therefore D = e_{kk} = 0.$

p_m = mean pressure

$$= \sigma_{kk}/3$$

$$= \frac{E}{3(1-2\nu)} e_{kk} = K e_{kk}$$

K → Bulk Modulus. → ∞ for incompressible solid

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