

# CONSTITUTIVE EQUATIONS

## (LINEAR — HOOKE'S LAW)

For 3-D Linear Anisotropic Solid,  
elastic consts.

$$① \leftarrow \boxed{\sigma_{ij} = C_{ijkl} e_{kl}} \rightarrow \text{3D Linear Hooke's Law.}$$

Symm of  $\sigma_{ij}$  &  $e_{kl} \Rightarrow C_{ijkl} = C_{jikl} = C_{jilk} = C_{jikl}$

i.e., only 36 independent elastic constants

Alternatively, due to symm of  $\underline{\sigma}$  &  $\underline{e}$ , we can use extended notation, as follows:

$$(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) \equiv (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$$

$$(e_{11}, e_{22}, e_{33}, e_{23}, e_{13}, e_{12}) \equiv (e_1, e_2, e_3, e_4, e_5, e_6)$$

don't confuse with  $\dot{e}_i$  on p23 of STRAINS,  
i.e., relative disp. per unit length



Thus Hooke's Law for 3D Anisotropy is

$$\tau_i = C_{ij} e_j \rightarrow i, j = 1, \dots 6$$

②

so 36 indep  $C_{ij}$ 's.

Now consider strain energy per unit vol defined as,

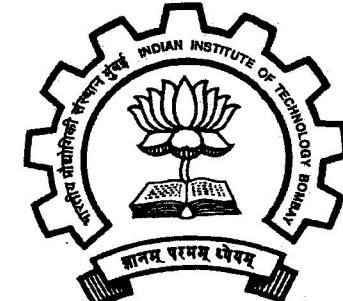
$$U = \frac{1}{2} \tau_i e_i = \frac{1}{2} C_{ij} e_j e_i = \frac{1}{2} C_{ji} e_i e_j$$

$$\Rightarrow C_{ij} = C_{ji} \Rightarrow 36 - 15 = \boxed{21 \text{ indep consts.}}$$

i.e.,  $C_{ijkl} = C_{kl, ij}$

Isotropic material: one whose mat. props are direction independent, i.e  $C_{ijkl}$  (or  $C_{ij}$ ) are invariant wrt coord axes / coord transform

Homogeneous:  $C_{ijkl}$  constant wrt spatial coords  $x_k$ . -Matrim

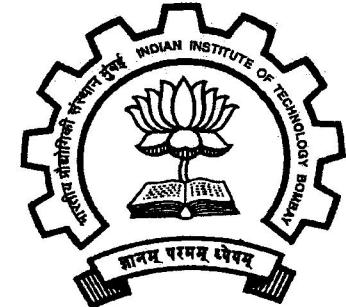


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$$C_{ijkl} = C'_{ijkl} \Rightarrow \text{isotropic}$$

$$\frac{\partial C_{ijkl}}{\partial x_k} = 0 \implies \text{homogeneous.}$$

and  $C_{ijkl}$  symmetries.



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Consider Isotropic Solid.  
 For transformation of rotation about x-axis,  
 $\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow C'_{1123} = a_{1i} a_{1j} a_{2k} a_{3l} C_{ijkl} = -C_{1132}$   
 $= -C_{1123} = C_{1123}$   
 $\Rightarrow C_{1123} = 0$  Isotropy

$$\text{Also } C'_{1213} = -C_{\frac{1312}{\text{Isotropy}}} = -C_{1213} \Rightarrow C_{1213} = 0$$

$$\left. \begin{aligned} C'_{1113} &= -C_{1112} = C_{1113} \\ C'_{1112} &= C_{1113} = C_{1112} \end{aligned} \right\} \Rightarrow C_{1112} = 0$$



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Thus if 2,3 appear once or 1 appears thrice then  $C_{ijkl} = 0$ . Doing  $90^\circ$  rot about  $x_2$  and  $x_3$  axes, similarly, we conclude that  $C_{ijkl} = 0$  for isotropic solid if any index appears odd times, ie.,

$C_{ijkl} = 0$  except when  $i=j, k=l$  or  $i=k, j=l$   
or  $i=l, j=k$ .

Now align axes along p-axes of  $\underline{\underline{e}}$ .

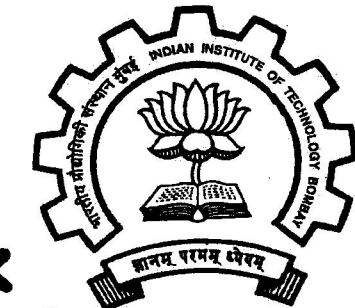
$\Rightarrow e_{ij} = 0$  for  $i \neq j$

$$\sigma_{ij} = C_{ijkl} e_{kl} = C_{i j 11} e_{11} + C_{i j 22} e_{22} + C_{i j 33} e_{33}$$

$\Rightarrow \tau_{ij} = 0$  for  $i \neq j$

$\Rightarrow$  P-axes of  $\underline{\underline{\sigma}}$   
 & e coincide  
 for Isotropic solid

$\because C_{ij11}, C_{ij22},$   
 $C_{ij33}$  are zero  
 for  $i \neq j$ , ie index  
 appearing odd times).



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Now consider  $90^\circ$  rotation about  $x_1$  axis  
 followed by reflection about  $x'_3$  axis, ie,

$$\underline{\underline{a}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{\text{90}^\circ \text{ rot abt } x_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

reflection abt  $x'_3$

$90^\circ$  rot  
abt  $x_1$

$$\Rightarrow C'_{1122} = C_{1133} = C_{1122}$$

$$C'_{2222} = C_{3333} = C_{2222}$$

Similarly by  $90^\circ$  rot about  $x_2$  followed  
by reflection abt  $x'_1$

& by  $90^\circ$  rot abt  $x_3$  followed by  
reflection abt  $x'_2$ , we get

$$C_{2211} = C_{2233}, C_{1111} = C_{3333}$$

$$\& C_{3311} = C_{3322}, C_{1111} = C_{2222}$$

$$\Rightarrow C_{1111} = C_{2222} = C_{3333} \rightarrow = (2\mu + \lambda)$$

$$C_{1122} = C_{1133} = C_{2233} \left. \begin{array}{l} \text{for Isotropic} \\ \text{solid material} \end{array} \right\} \rightarrow = \lambda$$

all other  $C_{ijkl} = 0$  Here  $\lambda, \mu$   
are iAME's const.

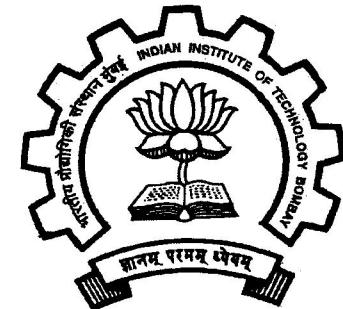


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$\Rightarrow$  only 2 indep consts for Isotropic Mat'l.  
With  $x_i$  as p-axes, we had,

$$\tau_{ij} = (2\mu + \lambda) e_{ij} + \lambda e_{kk} \delta_{ij} - \lambda e_{ij}$$

$$\boxed{\tau_{ij} = 2\mu e_{ij} + \lambda e_{\underbrace{kk}} \delta_{ij} \rightarrow 3a}$$



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$\rightarrow$  linear cubical dilatation

$\therefore$  this is a tensor eqn  $\Rightarrow$  it is invariant to coord transf, so it is valid in every (ie even in non-principal) system of coords.

$$\text{Contract } i=j \Rightarrow e_{RR} = \tau_{RR}/(3\lambda+2\mu)$$

Inverting,

-7-  $(3b)$

$$e_{ij} = \frac{1}{2\mu} \tau_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)} \tau_{kk} \delta_{ij}$$

This is Hooke's Law for linear Elastic Isotropic Material in terms of LAME's consts.

In terms of Engg consts  $E, \nu$ ,



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4a

$$e_{ij} = \frac{(1+\nu)}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{RR} \delta_{ij}$$

{By superposing normal stresses as in basic solid mech course.}

$$\text{Contract } i=j \Rightarrow \sigma_{RR} = \frac{E}{(1-2\nu)} e_{RR}$$

inverting  $\Rightarrow$

$$\sigma_{ij} = \frac{E}{1+\nu} \left[ e_{ij} + \frac{\nu}{1-2\nu} e_{RR} \delta_{ij} \right]$$

4b

Comparing,  $2G = \frac{E}{1+\nu} \rightarrow$  twice shear modulus

$$\lambda = \frac{EV}{(1+\nu)(1-2\nu)} \rightarrow \text{no physical meaning. for } \lambda$$

$$e_{kk} = \frac{1-2\nu}{E} \sigma_{kk} = 0 \text{ for } \nu = \frac{1}{2}$$

$p_m$  = mean pressure  
 $= \sigma_{kk}/3$

| Incompressible  
| solid  $\therefore D = e_{kk} = 0.$

$$= \frac{E}{3(1-2\nu)} e_{kk} = K e_{kk}$$

$\hookrightarrow$  Bulk Modulus.  $\rightarrow \infty$  for  
 incompressible solid



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