PLANE PROBLEMS — 2D ELASTICITY

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- Plane Stress / Plane Strain.
- Airy Stress function
- Compatibility Eqn for plane stress / plane strain.

PLANE STRESS

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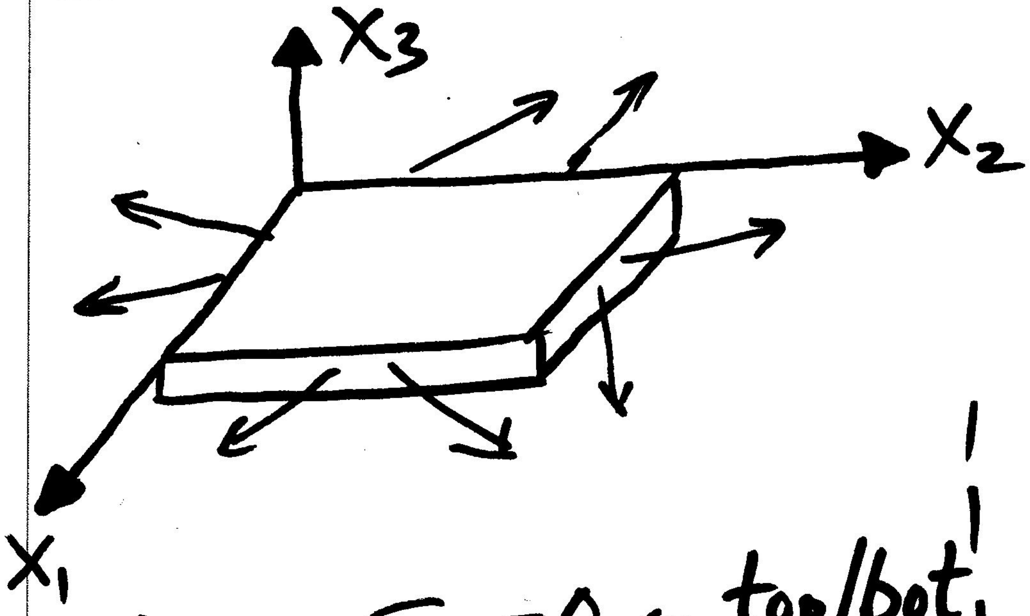

- Thin plate
- Loading in $x_1 - x_2$ plane along thin edges.
- No loading on top/bottom faces
- Loading indep. of x_3 coord. ie uniform in x_3 direction.
- $f_3 = 0$ (no b.f in x_3-dir).

BC's $\Rightarrow \sigma_{i3} = 0$ on top/bot face.

Thinness $\Rightarrow \sigma_{i3} = 0$ thruout & $\partial / \partial x_3 = 0$

Equil: $\sigma_{11,1} + \sigma_{12,2} + f_1 = 0 \rightarrow$ (A)

$\sigma_{21,1} + \sigma_{22,2} + f_2 = 0 \rightarrow$ (B)

3^{rd} eqn $\rightarrow 0 = 0$ (identity)

Airy Stress function $\phi(x_1, x_2)$

Let $f_1(x_1, x_2) = \psi_{,1}$; $f_2(x_1, x_2) = \psi_{,2}$

$\psi(x_1, x_2) \rightarrow$ Body force Potential

(ie for conservative b.f. $\underline{f} = \underline{\nabla}\psi$).

Let $\sigma_{11}(x_1, x_2) = \phi_{,22} - \psi$; $\sigma_{22}(x_1, x_2) = \phi_{,11} - \psi$;

$\sigma_{12} = -\phi_{,12}$

$\phi(x_1, x_2) \rightarrow$ Airy Stress $f^{\underline{n}}$

Equil: $\sigma_{11,1} + \sigma_{12,2} + f_1 = \phi_{,221} - \psi_{,1} - \phi_{,122} + \psi_{,1} = 0$

$\sigma_{21,1} + \sigma_{22,2} + f_2 = -\phi_{,121} + \phi_{,112} - \psi_{,2} + \psi_{,2} = 0$

So Equil is identically satisfied (i.s.) if we recast problem in terms of ϕ, ψ.

Thus instead of solving for σ_{11}, σ_{22}, σ_{12}, we solve for a single $\phi(x_1, x_2)$. How ??
By satisfying compatibility.

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B.M. Compat Eqns:

$i=1,\ j=1$: $\nabla^2\sigma_{11} + \frac{1}{1+\nu}(\sigma_{11,11}+\sigma_{22,11}) + 2f_{1,1} + \frac{\nu}{1-\nu}(f_{1,1}+f_{2,2}) = 0 \rightarrow$ (I)

$i=2,\ j=2$: $\nabla^2\sigma_{22} + \frac{1}{1+\nu}(\sigma_{11,22}+\sigma_{22,22}) + 2f_{2,2} + \frac{\nu}{1-\nu}(f_{1,1}+f_{2,2}) = 0 \rightarrow$ (II)

$i=1,\ j=2$: $\nabla^2\sigma_{12} + \frac{1}{1+\nu}(\sigma_{11,12}+\sigma_{22,12}) + f_{1,2} + f_{2,1} = 0 \rightarrow$ (III)

$i=3,\ j=3$: $\frac{\nu}{1-\nu}(f_{1,1}+f_{2,2}) = 0 \rightarrow$ (IV)

$\left.\begin{matrix} i=1,\ j=3 \\ i=2,\ j=3 \end{matrix}\right\} \rightarrow 0=0 \rightarrow$ (V, VI)

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From equil, $\frac{\partial(A)}{\partial x_2} + \frac{\partial(B)}{\partial x_1} = 0 = \nabla^2 \sigma_{12} + (\sigma_{11}+\sigma_{22})_{,12} + f_{1,2} + f_{2,1}$

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$\Rightarrow$ IIIrd Compat violated if $\nu \neq 0$

Also IVth Compat violated unless $\nu = 0$ or f_1, f_2 const.

due to $\partial/\partial x_3 = 0$ assumption

So we cannot satisfy III, IV, compat. (i)

Compat (I)+(II) $\Rightarrow$ $\nabla^2(\sigma_{11}+\sigma_{22}) = -\frac{2(1+\nu)}{(2+\nu)(1-\nu)}(f_{1,1}+f_{2,2})$

Now Consider Strain Compat (St Venants) eqns.

$e_{11,22} + e_{22,11} = 2e_{12,12} \rightarrow$ 1 ; $e_{33,11} = 0 \rightarrow$ 2

$e_{33,22} = 0 \rightarrow$ 3 ; $0=0 \rightarrow$ 4,5 ; $e_{33,12} = 0 \rightarrow$ 6

Numbering as per table on p.6 of "Compatibility Eqn"

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CL $\rightarrow e_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\sigma_{22}$; $e_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}\sigma_{11}$

$$e_{33} = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22}), \quad e_{12} = \frac{1+\nu}{E}\sigma_{12}, \quad e_{13}=e_{23}=0$$

CL in (1) $\rightarrow \cancel{\frac{1}{E}}\left\{(\sigma_{11}-\nu\sigma_{22})_{,22} + (\sigma_{22}-\nu\sigma_{11})_{,11}\right\} = \frac{2(1+\nu)}{\cancel{E}}\sigma_{12,12} \rightarrow (*)$

Equil $\rightarrow \frac{\partial(A)}{\partial x_1} + \frac{\partial(B)}{\partial x_2} = 0 = \sigma_{11,11}+\sigma_{22,22}+2\sigma_{12,12}+f_{1,1}+f_{2,2} = 0 \rightarrow (**)$

Eliminate σ_{12} from (*), (**) $\rightarrow \boxed{\nabla^2(\sigma_{11}+\sigma_{22}) = -(1+\nu)\,\underset{\sim}{\nabla}\cdot\underset{\sim}{f}} \rightarrow$ (ii)

CL in (2) $\rightarrow -\frac{\nu}{E}(\sigma_{11}+\sigma_{22})_{,11} = 0$

in (3) $\rightarrow -\frac{\nu}{E}(\sigma_{11}+\sigma_{22})_{,22} = 0$

in (6) $\rightarrow -\frac{\nu}{E}(\sigma_{11}+\sigma_{22})_{,12} = 0$

$\Rightarrow (\sigma_{11}+\sigma_{22}) = ax_1+bx_2+c$

This places a severe restriction on σ_{11}, σ_{22}, e_{33} which is unacceptable.

Hence (2), (3), (6) cannot be satisfied. (again due to $\frac{\partial}{\partial x_3}=0$)

RHS of (i) & (ii) differ by factor, ie $-\frac{2(1+\nu)}{(2+\nu)(1-\nu)}$ in (i) & $-(1+\nu)$ in (ii). Why??

Strain Compat (St. Venant) & Stress Compat (BM) do not have one-to-one correspondance amongst the Six eqns. Since some eqns in each set are non-satisfiable, we cannot expect the remaining eqn (ie (i) or (ii)) which we intend to satisfy to be identical.

Do we use (i) or (ii)??

(i) requires ignoring III, V which in turn imply $\nu = 0 \rightarrow$ this is a severe restriction on the solution.

(ii) requires $(\sigma_{11} + \sigma_{22})$ to be linear $f^{\underline{n}}$ of x_1, x_2. This is less severe restriction

So we choose Strain Compat (St. Venant) version ie (ii).

Subst ϕ, ψ in Strain Compat, Eq(ii)

$$\nabla^2(\nabla^2\phi - 2\psi) = -(1+\nu)\underline{\nabla}.\underline{\nabla}\psi = -(1+\nu)\nabla^2\psi$$

$$\boxed{\nabla^4\phi = (1-\nu)\nabla^2\psi}$$

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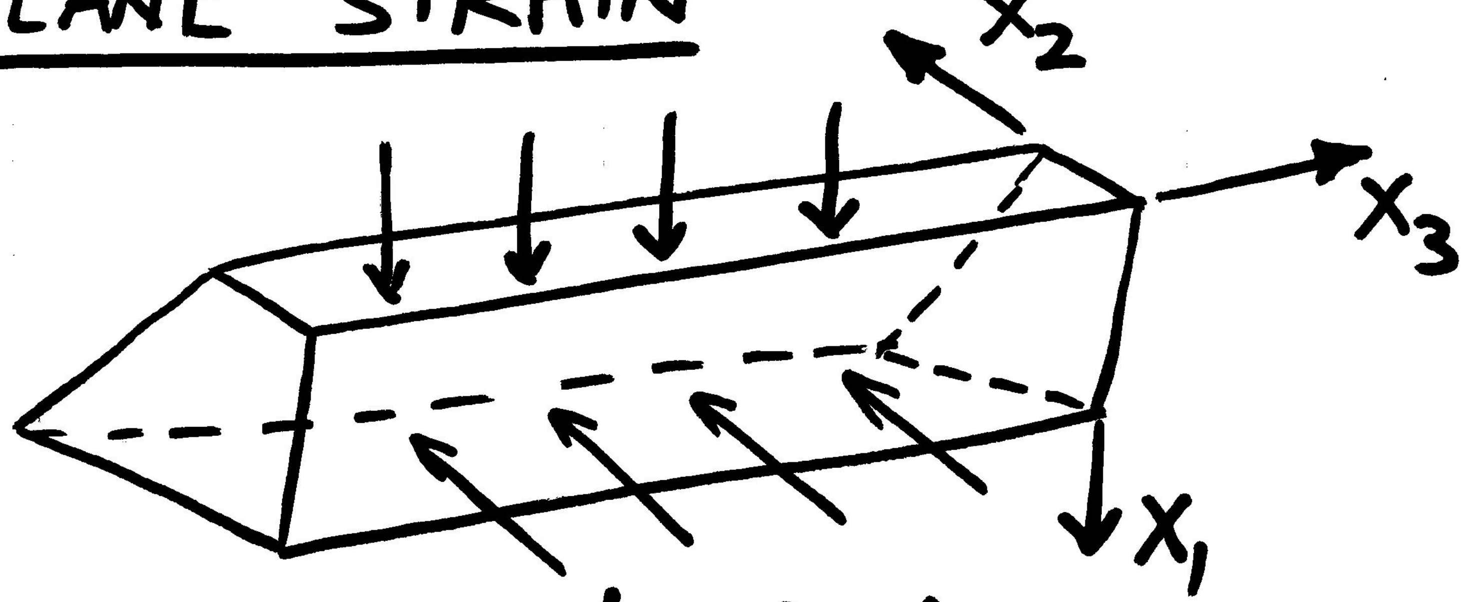
PLANE STRAIN



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- Long prismatic body.
- Load $\perp^{ar}$ to longitudinal axis, ie load has no x_3 component.
- Load does not vary in long. dir. (ie x_3 dir).
- Assume ends $x_3 = 0, L$ restrained from long. displ., ie $u_3 = 0$ at ends. Symmetry $\Rightarrow u_3 = 0$ thruout ($\div$ half, 1/4, ...). So $u_3(x_1, x_2, x_3) = 0$

• $\because \dfrac{\partial(\text{load})}{\partial x_3} = 0 \rightarrow u_1, u_2, \underline{\underline{\sigma}}$ indep of x_3

ie $\dfrac{\partial(\,)}{\partial x_3} = 0$

Example → dam, retaining wall, pressure vessel

$\because u_3 = 0, \dfrac{\partial}{\partial x_3} = 0 \rightarrow e_{i3} = 0 \rightarrow$ Plane strain.

Get same equil eqns as on p.2.

<u>STRAIN COMPAT EQS</u>: (St. Venant)

Eqs (2)–(6) i.s (0=0). {ref. numbering scheme, p.6 of "Compat Eqn" notes}

Eq(1) $e_{11,22} + e_{22,11} = 2e_{12,12} \rightarrow (*)$

$C.L. \rightarrow e_{33} = 0 \rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \rightarrow e_{11} = \dfrac{1-\nu^2}{E}\left(\sigma_{11} - \dfrac{\nu}{1-\nu}\sigma_{22}\right)$

$e_{22} = \dfrac{1-\nu^2}{E}\left(\sigma_{22} - \dfrac{\nu}{1-\nu}\sigma_{11}\right)$

Subst CL in $(*)$

Subst CL in (*),

$$\frac{1-\nu^2}{E}\left[\left(\sigma_{11}-\frac{\nu}{1-\nu}\sigma_{22}\right)_{,22}+\left(\sigma_{22}-\frac{\nu}{1-\nu}\sigma_{11}\right)_{,11}\right]=2\frac{(1+\nu)}{E}\sigma_{12,12}$$

Equil $\rightarrow \frac{\partial(A)}{\partial x_1}+\frac{\partial(B)}{\partial x_2} \rightarrow -2\sigma_{12,12}=\sigma_{11,11}+\sigma_{22,22}+f_{1,1}+f_{2,2}$

Eliminate $\sigma_{12,12}$ from compat above,

$$\nabla^2(\sigma_{11}+\sigma_{22}) = -\frac{1}{1-\nu}\underline{\nabla}\cdot\underline{f}$$

[$\because$ all compat eqns satisfied, we expect Stress Compat (BM) eqns to yield identical result.]

Subst Airy stress fn & Body force potential,

$$\nabla^2(\nabla^2\phi-2\psi) = -\frac{1}{1-\nu}\underline{\nabla}\cdot\underline{\nabla}\psi = -\frac{1}{1-\nu}\nabla^2\psi$$

$$\boxed{\nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2\psi}$$

Hooke's Law:

Plane Stress:

$$\sigma_{11} = \frac{E}{1+\nu}\left(e_{11} + \frac{\nu}{1-2\nu}\left[e_{11}+e_{22}+e_{33}\right]\right)$$

$$\boxed{e_{33} = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22})\ ;\ e_{11} = \frac{\sigma_{11}-\nu\sigma_{22}}{E},\ e_{22} = \frac{\sigma_{22}-\nu\sigma_{11}}{E}}$$

$$\Rightarrow e_{33} = -\frac{\nu}{E}\frac{E}{1-\nu}(e_{11}+e_{22}) \Rightarrow \sigma_{11} = \frac{E}{1+\nu}\left(e_{11} + \frac{\nu}{1-\nu}(e_{11}+e_{22})\right)$$

$$\sigma_{11} = \frac{E}{1-\nu^2}(e_{11}+\nu e_{22})$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(e_{22}+\nu e_{11})$$

$$\sigma_{12} = \frac{E}{1+\nu}e_{12}$$

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Plane strain:

$$\sigma_{11} = \frac{E}{1+\nu}\left[e_{11} + \frac{\nu}{1-2\nu}(e_{11}+e_{22})\right]$$

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{11} + \nu e_{22}\right]$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{22} + \nu e_{11}\right]$$

$$\sigma_{12} = \frac{E}{1+\nu}e_{12} \quad ; \quad \sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu(e_{11}+e_{22})\right]$$

$$\sigma_{33} = \nu(\sigma_{11}+\sigma_{22})$$

$$e_{11} = \frac{1}{E}\left[(1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22}\right], \quad e_{22} = \frac{1}{E}\left[(1-\nu^2)\sigma_{22} - \nu(1+\nu)\sigma_{11}\right]$$

Plane stress $\rightarrow$ Plane strain, $E \rightarrow \frac{E_1}{1-\nu_1^2}$, $\nu \rightarrow \frac{\nu_1}{1-\nu_1}$, then drop "1"

POLYNOMIAL SOLUTIONS of 2D Problems in Cartesian Coords

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$$\nabla^4 \phi = R \nabla^2 \psi$$ (R depends on ν, diff for plane-stress/strain).

If b.f. due to gravity,

$$\psi = px + qy,$$

$$\Rightarrow \boxed{\nabla^4 \phi = 0} \rightarrow \text{Biharmonic eqn}$$

choose $\phi = \sum_{N=1}^{P} \phi_N$; $\phi_N = \sum_{i=0}^{N} A_{Ni}\, x^{N-i} y^i$

to satisfy Biharmonic.

$$\nabla^4 \phi_N = \sum_{i=0}^{N-4} B_{Ni}\, x^{(N-4-i)} y^i = 0$$

$\Rightarrow B_{Ni} = 0,\ i = 0 \ldots N-4 \rightarrow$ constraint eqns. must be satisfied for $N > 3$

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$N=1$: $\phi_1 = a_1 x + b_1 y \rightarrow \nabla^4 \phi_1 \equiv 0$

$\sigma_{xx} = \sigma_{yy} = -\cancel{\psi}$, $\sigma_{xy} = 0$

0 if b.f = 0

$N=2$: $\phi_2 = a_2 x^2 + b_2 xy + c_2 y^2 \rightarrow \nabla^4 \phi_2 \equiv 0$

$\sigma_{xx} = 2c_2$, $\sigma_{yy} = 2a_2$, $\sigma_{xy} = -b_2 \rightarrow$ <u>const σ</u>

if b.f = 0.

$N=3$: $\phi_3 = a_3 x^3 + b_3 x^2 y + c_3 xy^2 + d_3 y^3$

$\nabla^4 \phi_3 \equiv 0$, $\sigma_{xx} = 2(c_3 x + 3d_3 y)$; $\sigma_{yy} = 2(b_3 y + 3a_3 x)$

$\sigma_{xy} = -2(b_3 x + c_3 y) \longrightarrow$ <u>Linear variation</u>.

If $a_3 = b_3 = c_3 = 0$ we get solⁿ. for pure bending

$N=4$: $\phi_4 = a_4 x^4 + b_4 x^3 y + c_4 x^2 y^2 + d_4 xy^3 + e_4 y^4$

$\nabla^4 \phi = 0 = \cancel{24}\,3\, a_4 + \cancel{8}\, c_4 + \cancel{24}\,3\, e_4 = 0 \rightarrow$ constraint.

$\sigma_{xx} = 2(c_4 x^2 + 3d_4 xy + 6e_4 y^2)$; $\sigma_{yy} = 2(c_4 y^2 + 3b_4 xy + 6a_4 x^2)$

$$\sigma_{xy} = -3b_4 x^2 - 4c_4 xy - 3d_4 y^2$$

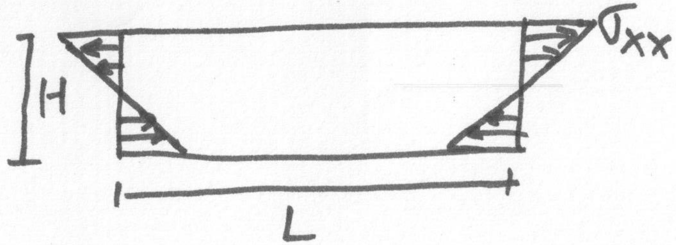
$$N=5: \ \phi_5 = a_5 x^5 + b_5 x^4 y + c_5 x^3 y^2 + d_5 x^2 y^3 + e_5 x y^4 + f_5 y^5$$

$$\left.\begin{aligned} \nabla^4\phi = 0 = \ & x\,\underbrace{(5a_5 + e_5 + c_5)}_{=0}\cdot 24 \\ & + y\,\underbrace{(5f_5 + b_5 + d_5)}_{=0}\cdot 24 \end{aligned}\right\} \text{constraints}$$

$\underline{\underline{\sigma}} \rightarrow$ cubic.

NOTE: For linear in (x,y) $\overbrace{\text{body forces}}^{(\psi,\text{ i.e constant})}$, $\nabla^4\phi = 0$ for plane stress & plane strain. So σ_{xx}, σ_{yy}, σ_{xy} same in both cases. However $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ in plane strain, and $e_{zz} = \frac{-\nu}{1-\nu}(e_{xx} + e_{yy})$ in plane stress. Recall also that $E \rightarrow E_1/1-\nu_1^2$, $\nu \rightarrow \nu_1/1-\nu_1$ in CL for plane stress $\rightarrow$ plane strain. So strains not same in both cases $\Rightarrow$ hence displ's not same

In ϕ_3, if $a_3 = b_3 = c_3 = 0$, $\sigma_{xx} = 6d_3 y$, $\sigma_{yy} = \sigma_{xy} = 0$
ie solution for pure bending w/o restrictions on dimensions L, H, or assumption of plane sections remaining plane.



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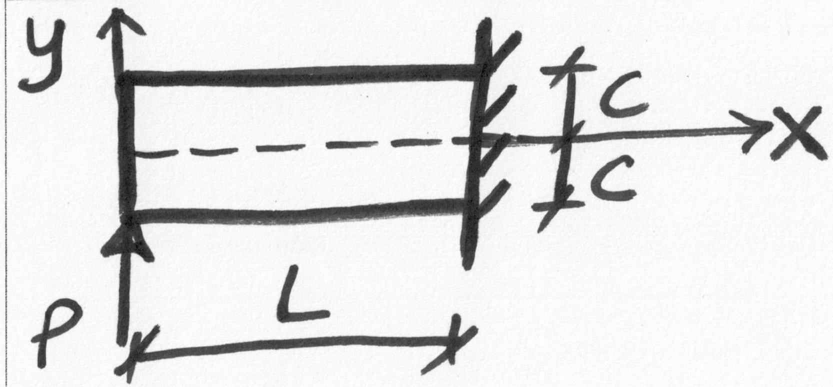
Ex1 Tip loaded cantilever.



From basic solid mech. →

$$\sigma_{xx} = \frac{My}{I}, \quad M \propto x \Rightarrow \sigma_{xx} \propto xy$$

$$\Rightarrow \phi = O(4) = \phi_2 + \phi_3 + \phi_4$$

BC's $\sigma_{xx}\big|_{x=0} = 0 \Rightarrow$
(equate coeffs of y^0, y^1, y^2 to zero)

$$c_2 = 0$$
$$d_3 = 0$$
$$e_4 = 0 = -a_4 - \frac{c_4}{3}$$

$\sigma_{yy}\big|_{y=\pm c} = 0 \Rightarrow a_2 \pm b_3 c + c_4 c^2 = 0$

(coeffs of x^0, x^1, x^2) $\quad a_3 \pm b_4 c = 0$

$a_4 = 0$

$\sigma_{xy}\big|_{y=\pm c} = 0 \Rightarrow -b_2 \mp 2c_3 c - 3d_4 c^2 = 0 \rightarrow \circledast$

(coeffs x^0, x^1, x^2) $\quad b_3 \pm 2c_4 c = 0$

$b_4 = 0$

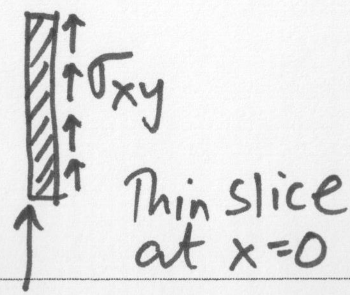
$\sigma_{xy}\big|_{x=0} = 0 \Rightarrow b_2 = c_3 = d_4 = 0 \leftrightarrow$ relax this.

(coeffs y^0, y^1, y^2)

soln $\rightarrow$ all coeffs $a_2 = \ldots = e_4 = 0$, ie $\phi = 0$

So relax strong bc $\sigma_{xy}\big|_{x=0} = 0$ in favor of

$$\int_{-c}^{c} \sigma_{xy}\big|_{x=0} dy + P = 0.$$



$\rightarrow$ NOTE: This is the weak bc, obvious $\because$ $\sigma_{xy}\big|_{x=0}$ must add up to P.

$$\int_{-c}^{c} \sigma_{xy}\Big|_{x=0} dy = -P \Rightarrow 2\left(-b_2 c - d_4 c^3\right) = -P \rightarrow (**)$$

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Solve $(*)\ (**) \rightarrow b_2 = \dfrac{3P}{4c}, \quad d_4 = -\dfrac{P}{4c^3}$

other coeffs $= 0$.

$$\Rightarrow \sigma_{xx} = -\frac{3}{2}\frac{P}{c^3} xy = -\frac{P}{I} xy \ ; \quad \sigma_{yy} = 0$$

$$\sigma_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right) = -\frac{P}{I}\frac{1}{2}\left(c^2 - y^2\right)$$

use $\sigma_{xx} = \phi_{,yy}$ etc.

This sol$^{\underline{n}}$ is exact if P applied not as point load but through a parabolically varying $\sigma_{xy}\big|_{x=0}$.

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Strains & Displacements:

CL & Str-displ relations →

$$e_{xx} = u_{,x} = \frac{\sigma_{xx}}{E} = -\frac{P}{EI}xy$$

$$e_{yy} = v_{,y} = -\frac{\nu}{E}\sigma_{xx} = \frac{\nu P}{EI}xy$$

$$e_{xy} = \frac{1}{2}(u_{,x} + v_{,y}) = \frac{1+\nu}{E}\sigma_{xy} = -\frac{P}{4IG}(c^2 - y^2)$$

$$\left\{ G = \frac{E}{2(1+\nu)} \right\}$$

Note: above strains have same functional dependency on (x, y) as in Tutorial problm

Integrate →

$$u = -\frac{P}{2EI}x^2y + f(y)$$

$$v = \frac{\nu P}{2EI}x\,y^2 + g(x)$$

subst in e_{xy} →

$$\left.\begin{aligned} g' - \frac{P}{2EI}x^2 &= k_1 \\ f' + \frac{\nu P}{2EI}y^2 - \frac{P}{2IG}y^2 &= k_2 \end{aligned}\right\}$$

where, $k_1 + k_2 = -\frac{Pc^2}{2IG}$ (★)

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Integrate for f, g, $\rightarrow$

$$u = -\frac{P}{2EI}x^2y - \frac{\nu P}{6EI}y^3 + \frac{P}{6IG}y^3 + k_2 y + k_3$$

$$v = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 + k_1 x + k_4$$

Solve $k_1, \dots, k_4$ by setting RBM to zero.

R.B. Translation zero $\rightarrow$ $u = v = 0$ at $y=0,\ x=L$

$$\Rightarrow k_3 = 0 \ (**),\quad k_4 = -\frac{PL^3}{6EI} - k_1 L \rightarrow (***)$$

R.B Rot. zero: Horizontal line element at $x=L,\ y=0$ has no rot.

(i) $\Rightarrow v_{,x}\Big|_{\substack{x=L\\y=0}} = 0 \rightarrow k_1 = -\frac{PL^2}{2EI} \rightarrow (****) \rightarrow$ solve $\forall\ k_1 \cdots k_4$

↳ same as $du_2\Big|_{\substack{x=L\\y=0}} = u_{2,j}\, dx_j = 0$ for $dx_j = (1,0,0)$

gives $u_{2,1} = v_{,x} = 0$ at $x=L,\ y=0$.

(ii) Vertical line element at $x=L$, $y=0$ has no rot.

$$\Rightarrow u_{,y}\Big|_{\substack{x=L\\y=0}} = 0 \rightarrow R_2 = \frac{PL^2}{2EI} \rightarrow (\star\star\star\star) \rightarrow \text{solve } R_1, \dots R_4$$

↳ same as $du_1\Big|_{\substack{x=L\\y=0}} = u_{1,j}\,dx_j = 0$ for $dx_j = (0,1,0)$

↓

gives $u_{1,2} = u_{,y} = 0$ at $x=L$, $y=0$.

(iii) Rotation component $\omega_3 = -\omega_{12} = 0$ at $x=L$, $y=0$

i.e., $(v_{,x} - u_{,y})\Big|_{\substack{x=L\\y=0}} = 0 \longrightarrow (\star\star\star\star) \longrightarrow$ solve $R_1, \dots, R_4$

The three conditions [(i), (ii), (iii)] for zero R.B. rotation yield different results for displacements.

For condition (i), elastic curve, ie $v|_{y=0}$, ie displ of neutral line is $v|_{y=0} = \frac{1}{EI}\left(\frac{Px^3}{6} - \frac{PL^2}{2}x + \frac{PL^3}{3}\right) \longrightarrow$ same result as Euler Bernoulli beam theory of basic SM course

For conditions (ii) & (iii) you get extra terms, ie,

$$\frac{Pc^2}{2IG}(L-x) \longrightarrow \text{for condt (ii)}$$

$$\frac{Pc^2}{4IG}(L-x) \longrightarrow \text{for condt (iii)}.$$

Now elementary "Bernoulli-Euler" beam theory of CE221 assumes $\gamma_{xy} = 0$, ie $G \to \infty$. This is also implied for $\frac{c}{L}$ large, ie slender beam (not thick beam). So put $G \to \infty$ or $c \to 0$ in condt (ii) or (iii) & you get condt (i) result.

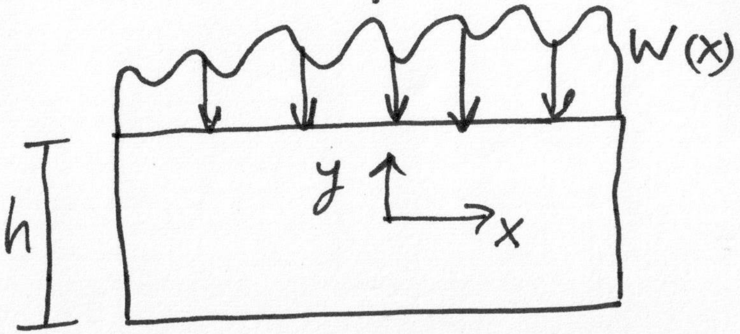
NOTE: You must always satisfy strong BC's on long boundaries $y = \pm c$. On short boundaries you can try to satisfy strong BC's, else satisfy some/all of these as weak BC's (ie in integral sense).

Q: What about satisfying following conditions at fixed end $x = L$.

$$\int_{-c}^{c} \sigma_{xx}\, dy = 0 \quad ; \quad \int_{-c}^{c} \sigma_{xy}\, dy = -P \quad ; \quad \int_{-c}^{c} \sigma_{xx}\, y\, dy = -PL$$

A: Since $\nabla^4 \phi = 0$ includes equilibrium eqns, equil is satisfied pointwise. Above conditions are a statement of lumpsum equil (ie of whole beam). Now pointwise equil implies lumpsum equil. So above condts are automatically (identically) satisfied, and they cannot be of any use to obtain any constants.

Order of polynomial.



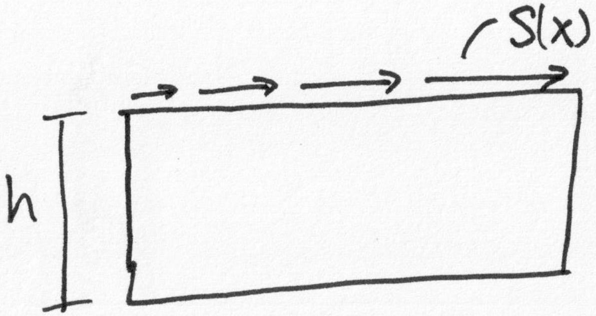
Normal traction $W(x) \propto x^n$

$\Rightarrow M(x) = \int (W\,dx)\,x \propto x^{n+2}$

$\sigma_{xx} \propto x^{n+2}\,y$

$\because \sigma_{xx} = \phi_{,yy} \Rightarrow \phi$ contains term $x^{n+2}y^3$

$\Rightarrow \phi = O(n+5)$ polynomial.



Shear traction $S(x) \propto x^m$

$\Rightarrow M(x) = \int (S\,dx)\,\frac{h}{2} \propto x^{m+1}$; ie $\sigma_{xx} \propto x^{m+1}y$

So ϕ contains $x^{m+1}y^3$; ie $\phi = O(m+4)$

$\Rightarrow O(\phi) = \max\,(n+5,\ m+4)$ where n, m are order of normal & shear tractions, respectively.

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Symmetry (to reduce $O(\phi)$)

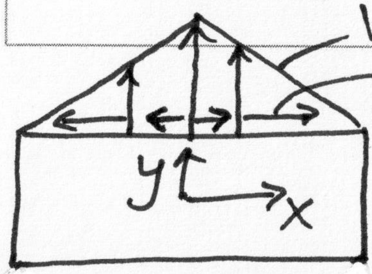
If beam (including support conditions) symmetric about y-axis, then:

(i) If load symm abt y-axis, i.e.,

$w(x) = w(-x)$; $S(x) = -S(-x)$

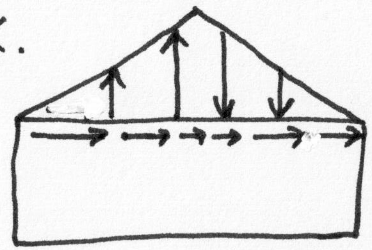
then $\sigma_{xx}(x) = \sigma_{xx}(-x)$; $\sigma_{xy}(x) = -\sigma_{xy}(-x)$

i.e, ϕ = even in x.



(i)
SX

(ii) If load antisymm abt y-axis, ϕ = odd in x.



(ii)
ASX

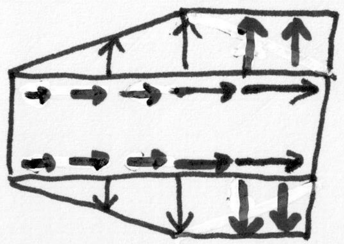
Similarly if beam symm abt x-axis, then:

(iii) Similarly, if load is symm abt x-axis, i.e,

$w(y) = w(-y)$; $S(y) = -S(-y)$

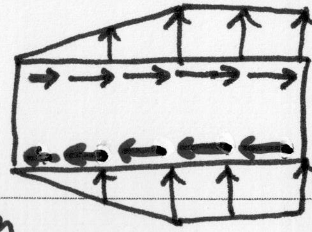
then $\sigma_{yy}(y) = \sigma_{yy}(-y)$; $\sigma_{xy}(y) = -\sigma_{xy}(-y)$

ie ϕ = even in y



(iii)
SY

(iv) If load antisymm abt x-axis, ϕ = odd in y



(iv)
ASY

NOTE: Load should "appear" symm or antisymm, or use sign convention for normal/shear traction

Algorithm for polynomial solutions

(i) Obtain $O(\phi) = \max(n+5, m+4)$. Then use symmetry/antisymmetry to reduce nos of terms in ϕ.

(ii) Substitute ϕ in biharmonic eqn to get constraint equations.

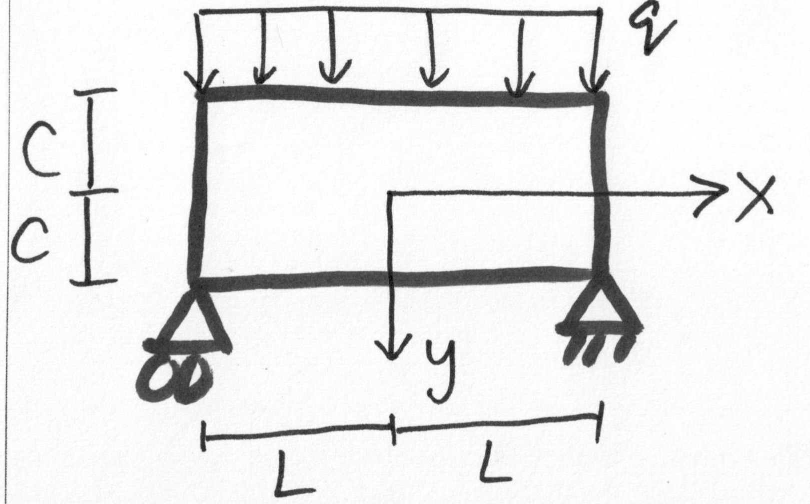
(iii) Substitute ϕ to get $\sigma_{xx} = \phi_{,yy}$, $\sigma_{yy} = \phi_{,xx}$, $\sigma_{xy} = -\phi_{,xy}$

(iv) Apply BC's $\rightarrow$ strong for long boundaries ($y=$ const) and if required weak for short boundaries ($x=$ const).

NOTE: The number of equations obtained after applying BC's may exceed nos of ~~constants~~ coefficients in ϕ that you need to solve for. But solution of these consts will be unique, ie not all the BC equations will be independent.

(v) Solve constraint eqns & BC eqns for coefficients of ϕ.

Ex2 Simply supported uniformly loaded beam.



$M \propto x^2 \Rightarrow \sigma_{xx} \propto x^2 y$

$\Rightarrow \phi = O(5)$

or directly, $n=0$, $m=?$

$O(\phi) = \max(0+5, -) = 5$

(i) $\phi = \phi_2 + \phi_3 + \phi_4 + \phi_5 \longrightarrow$ 18 terms $\longrightarrow$ Strong BC's $\left\{ \sigma_{xy}\Big|_{y=\pm c} = 0 \right.$;

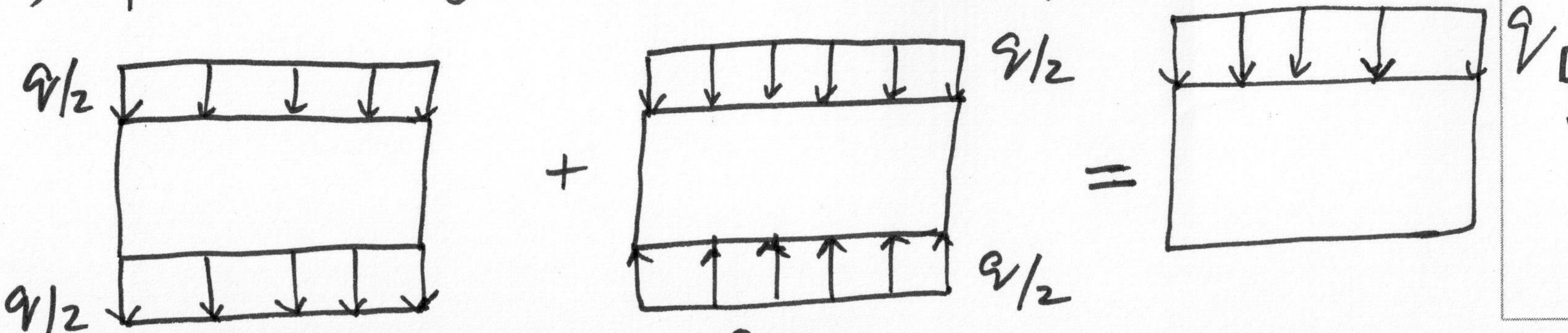
$\left. \sigma_{yy}\Big|_{y=-c} = -q \; ; \; \sigma_{yy}\Big|_{y=c} = 0 \right\}$; Weak BC's $\left\{ \int_{-c}^{c} \sigma_{xx}\Big|_{x=\pm L} dy = 0 \right.$;

$\left. \int_{-c}^{c} (\sigma_{xx}\, dy)\, x \Big|_{x=\pm L} = 0 \; ; \; \int_{-c}^{c} \sigma_{xy}\Big|_{x=\pm L} dy = \mp qL \right\}$.

Too cumbersome.

(ii) Load symmetric in x $\Rightarrow \phi$ = even in $x \longrightarrow$ 10 terms $\longrightarrow$ above BC's $\longrightarrow$ still too cumbersome.

(iii) Split as symmetric & antisymm problems.



A-Symm	Symm
$\Downarrow$	$\Downarrow$
S-x, AS-y	S-x, S-y $\longrightarrow$ but for this we directly see soln is $\sigma_{yy} = q/2$ all other stresses zero
$\phi_{AS} = x^{even} y^{odd}$	$\phi_S = x^{even} y^{even}$
$\downarrow$	so $\phi_S = \dfrac{q}{4} x^2 \longleftarrow$ This satisfies equil, compat & BC's.
$\phi_{AS} = 5^{th}$ order	

$$\phi = \underbrace{b_3 x^2 y + d_3 y^3 + b_5 x^4 y + d_5 x^2 y^3 + f_5 y^5}_{\phi_{AS}} + \underbrace{\frac{q}{4} x^2}_{\phi_S}$$

Constraint eqn $\longrightarrow$ $b_5 + d_5 + 5 f_5 = 0$

When applying bc's you can treat A-Symm problem seperately (ie only ϕ_{AS}) or both problems combined (ie ϕ). We will do former.

$$\sigma_{yy}\Big|_{y=\pm c} = \pm\frac{q}{2} \Rightarrow \pm 2b_3 c \pm 2d_5 c^3 = \pm\frac{q}{2} \rightarrow (\star)$$

$$\pm 12 b_5 c = 0$$

$$\sigma_{xy}\Big|_{y=\pm c} = 0 \Rightarrow -2b_3 - 6d_5 c^2 = 0$$

$$4b_5 = 0$$

$$d_5 = -\frac{q}{8c^3}$$

$$b_3 = \frac{3}{8}\frac{q}{c}$$

$$\int_{-c}^{c} \sigma_{xx}\Big|_{x=\pm L} dy = 0 \Rightarrow 0 = 0 \quad (\text{i.s}).$$

$$\int_{-c}^{c} \sigma_{xx}\Big|_{x=\pm L} y\,dy = 0 \Rightarrow 4d_3 c^3 + 4d_5 L^2 c^3 + 8f_5 c^5 = 0$$

$$\int_{-c}^{c} \sigma_{xy}\Big|_{x=\pm L} dy = \mp qL \Rightarrow \pm 4b_3 Lc \pm 8b_5 L^3 c \pm 4d_5 Lc^3 = \pm qL$$

same as $\star$ above

$\therefore b_5 = 0.$

$b_3 = \frac{3}{8}\frac{q}{c}$; $d_5 = -\frac{q}{8c^3}$; $b_5 = 0$; $f_5 = \frac{q}{40c^3}$

$d_3 = \frac{q}{8c^3}\left(L^2 - \frac{2}{5}c^2\right)$

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$$\phi = \frac{q}{40c^3}\left(15c^2x^2y + 5L^2y^3 - 2c^2y^3 - 5x^2y^3 + y^5 + 10c^3x^2\right)$$

So only one weak BC gives useful info, other two i.s.

$$\sigma_{xx} = \phi_{,yy} = \frac{q}{2I}\left[(L^2 - x^2)y + \underline{\underline{\left(\frac{2}{3}y^2 - \frac{2}{5}c^2y\right)}}\right] \qquad \left(I = \frac{2}{3}c^3\right)$$

$$\sigma_{yy} = \phi_{,xx} = -\frac{q}{2I}\left[\underline{\underline{\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3}}\right]$$

$$\sigma_{xy} = -\phi_{,xy} = -\frac{q}{2I}\left[c^2 - y^2\right]x$$

Non-underlined terms $\rightarrow$ Euler Bernoulli theory

underlined terms $\rightarrow$ correction to E-B-T, corrections vanish for $\frac{c}{L} \ll 1$ (ie $c \rightarrow 0$, $y \rightarrow 0$)

Deflections obtained in usual manner by integrating SD relations (HW4, P1).

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BC's: $v\big|_{\substack{y=-c\\x=\pm L}} = 0$ prevents RB translation $-y$ & RB rotation $-xy$ plane

$u\big|_{\substack{y=-c\\x=-L}} = 0$ prevents RB transl $-x$.

$$\text{Result} \rightarrow \quad v\big|_{y=0} = \frac{5}{24}\frac{qL^4}{EI}\Bigg[\underbrace{1-\frac{6}{5}\left(\frac{x}{L}\right)^2+\frac{1}{5}\left(\frac{x}{L}\right)^4}_{\text{Term I}} + \underbrace{\frac{12}{5}\left(\frac{c}{L}\right)^2\left\{\left(\frac{4}{5}+\frac{\nu}{2}\right)\left(1-\left(\frac{x}{L}\right)^2\right) + \underbrace{\left(\frac{1}{64}-\frac{\nu}{480}\right)\left(\frac{h}{L}\right)^2}_{\text{Term II a}}\right\}}_{\text{Term II}}\Bigg]$$

Term I $\rightarrow$ Classical E-BT

Term II $\rightarrow$ Correction due to shear effect in deformation, useful when $\frac{c}{L} \cancel{\ll} 1$

Alternatively, if supports at $y=0$, then apply above BC's at $y=0$ instead of $y=-c$. Term II a vanishes (Timoshenko, Shames & Dym).

Alternatively, $u, v, (u_{,y}-v_{,x})$ all zero at $x=-L$, $y=-c$. Gives diff result.

SEMI-INVERSE METHOD.

STEP-I Based on loading & BC's, assume functional form for some/all of stresses/displ's.

STEP-II Satisfy governing equations to obtain explicit form of solution; then satisfy BC's to get constants.

For stress function approach, assume fun form of stresses, integrate to obtain **fun form** of ϕ, then make ϕ satisfy biharmonic eqn to get explicit ϕ, then satisfy BC's to get solution.

STEP-I $\because \left.\sigma_{yy}\right|_{y=\pm c} = 0,\ -q$, ie indep of x, assume $\sigma_{yy} = f(y)$

$$\Rightarrow \phi_{,xx} = f(y) \longrightarrow \phi = \frac{x^2}{2} f(y) + x\, g(y) + h(y)$$

STEP-II

$$\nabla^4\phi = \frac{x^2}{2} f^{IV} + x g^{IV} + h^{IV} + 2f'' = 0$$

$$\Rightarrow f^{IV} = g^{IV} = h^{IV} + 2f'' = 0$$

$$f = Ay^3 + By^2 + Cy + D$$

$$g = Ey^3 + Fy^2 + Gy \qquad \text{(const in } g \text{ dropped as it wont affect stresses)}$$

$$h^{IV} = -2f'' = -12Ay - 4B \Rightarrow h = \frac{A}{10}y^5 - \frac{B}{6}y^4 + Hy^3 + Ky^2$$

$$\Rightarrow \phi = \frac{x^2}{2}\left(Ay^3 + By^2 + Cy + D\right) + x\left(Ey^3 + Fy^2 + Gy\right) - \frac{A}{10}y^5 - \frac{B}{6}y^4 + Hy^3 + Ky^2$$

Stresses: $\sigma_{xx} = \phi_{,yy} = \frac{x^2}{2}(6Ay + 2B) + x(6Ey + 2F) - 2Ay^3 - 2By^2 + 6Hy + 2K.$

$$\sigma_{yy} = \phi_{,xx} = Ay^3 + By^2 + Cy + D$$

$$\sigma_{xy} = -\phi_{,xy} = -x\left(3Ay^2 + 2By + C\right) - \left(3Ey^2 + 2Fy + G\right)$$

Apply BC's (strong & weak). Some weak BC's will be redundant (i.e, I.S.), ie

$$\int_{-c}^{c} \sigma_{xy}\Big|_{x=\pm L} dy = \mp qL$$ in this case. Also you can use symmetry in 'x' to reduce work $\rightarrow$ get $(E, F, G) = 0$

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FOURIER SERIES METHOD.

Polynomial solutions possible only when loading (tractions) can be written as power series.
When not so, e.g. concentrated / discontinuous load, we use Fourier series.

Consider,

$\phi = \sin(\alpha x) f(y)$ or $\phi = \cos(\alpha x) f(y)$ (when loading on top/bot faces).

$\nabla^4 \phi = 0 \Rightarrow \left(f^{IV} - 2\alpha^2 f'' + \alpha^4 f\right) = 0$ ($\because \sin(\alpha x), \cos(\alpha x)$ nonzero).

$$f = A\cosh\alpha y + B\sinh\alpha y + C y\cosh\alpha y + D y\sinh\alpha y$$

$$\phi = \sum_{n=1}^{\infty} \underbrace{\left(A_n \cosh\alpha_n y + D_n y \sinh\alpha_n y\right)}_{P_n} \cos\alpha_n x \longrightarrow f_1, g_1 \text{ problem} \rightarrow S_x S_y, A_x A_y$$

$$+ \sum \underbrace{\left(A_n' \cosh\alpha_n y + D_n' y \sinh\alpha_n y\right)}_{P_n} \sin\alpha_n x \longrightarrow f_2, g_2 \rightarrow A_x S_y, S_x A_y$$

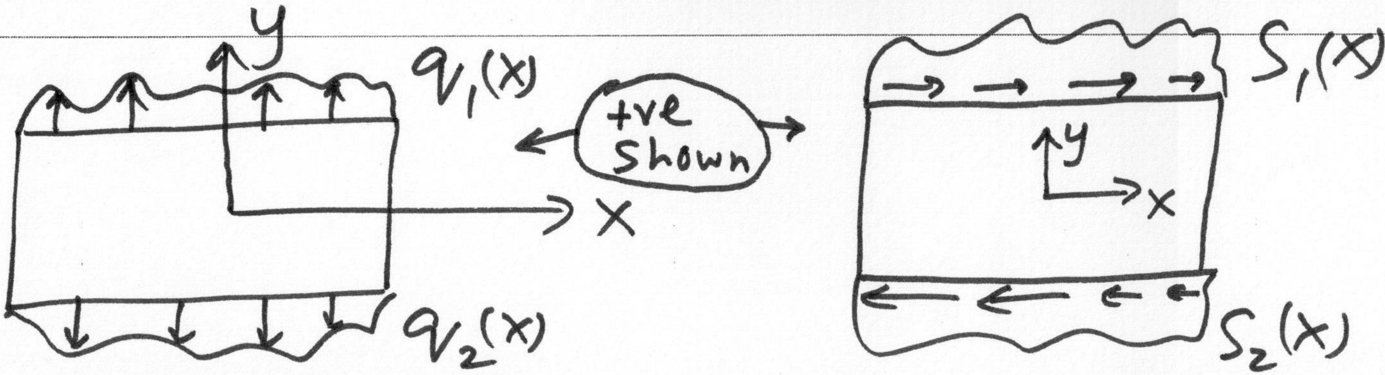
$$+ \sum \underbrace{\left(B_n \sinh\alpha_n y + C_n y \cosh\alpha_n y\right)}_{Q_n} \cos\alpha_n x \longrightarrow f_3, g_3 \rightarrow S_x A_y, A_x S_y$$

$$+ \sum \underbrace{\left(B_n' \sinh\alpha_n y + C_n' y \cosh\alpha_n y\right)}_{Q_n} \sin\alpha_n x \longrightarrow f_4, g_4 \rightarrow A_x A_y, S_x S_y$$

$$
\begin{array}{ll}
S_xS_y \leftarrow & \sigma_{xx} = \sum \big(A_n\alpha_n^2\cosh\alpha_n y + 2D_n\alpha_n\cosh\alpha_n y + D_n\alpha_n^2 y\sinh\alpha_n y\big)\underbrace{\quad}_{R_n}\cos\alpha_n x \\
A_xS_y \leftarrow & \quad + \sum \big(\qquad\qquad\big)\underbrace{\quad}_{R_n}\sin\alpha_n x \\
S_xA_y \leftarrow & \quad + \sum \big(B_n\alpha_n^2\sinh\alpha_n y + 2C_n\alpha_n\sinh\alpha_n y + C_n\alpha_n^2 y\cosh\alpha_n y\big)\underbrace{\quad}_{S_n}\cos\alpha_n x \\
A_xA_y \leftarrow & \quad + \sum \big(\qquad\qquad\big)\underbrace{\quad}_{S_n}\sin\alpha_n x
\end{array}
$$

$$
\begin{array}{lll}
\sigma_{yy} = & -\sum P_n\alpha_n^2\cos\alpha_n x & \longrightarrow S_xS_y \\
& -\sum P_n\alpha_n^2\sin\alpha_n x & \longrightarrow A_xS_y \\
& -\sum Q_n\alpha_n^2\cos\alpha_n x & \longrightarrow S_xA_y \\
& -\sum Q_n\alpha_n^2\sin\alpha_n x & \longrightarrow A_xA_y
\end{array}
$$

$$
\begin{array}{lll}
\sigma_{xy} = & +\sum \big(A_n\alpha_n^2\sinh\alpha_n y + D_n\alpha_n\sinh\alpha_n y + D_n\alpha_n^2 y\cosh\alpha_n y\big)\underbrace{\quad}_{T_n}\sin\alpha_n x & \rightarrow A_xA_y \\
& -\sum \big(\qquad\qquad\big)\underbrace{\quad}_{T_n}\cos\alpha_n x & \rightarrow S_xA_y \\
& +\sum \big(B_n\alpha_n^2\cosh\alpha_n y + C_n\alpha_n\cosh\alpha_n y + C_n\alpha_n^2 y\sinh\alpha_n y\big)\underbrace{\quad}_{U_n}\sin\alpha_n x & \rightarrow A_xS_y \\
& -\sum \big(\qquad\qquad\big)\underbrace{\quad}_{U_n}\cos\alpha_n x & \rightarrow S_xS_y
\end{array}
$$



Define,

$$
\begin{array}{lllll}
f_1(x) = +q_1(x) & +\ q_1(-x) & +\ q_2(x) & +\ q_2(-x) & \longrightarrow S_x, S_y \\
f_2(x) = + & - & + & - & \longrightarrow A_x, S_y \\
f_3(x) = + & + & - & - & \longrightarrow S_x, A_y \\
f_4(x) = + & - & - & + & \longrightarrow A_x, A_y
\end{array}
$$

So, eg f_3 is S_x

$$
\begin{array}{lllll}
g_1(x) = +S_1(x) & -\ S_1(-x) & -\ S_2(x) & +\ S_2(-x) & \longrightarrow A_x, A_y \\
g_2(x) = + & + & - & - & \longrightarrow S_x, A_y \\
g_3(x) = + & - & + & - & \longrightarrow A_x, S_y \\
g_4(x) = + & + & + & + & \longrightarrow S_x, S_y
\end{array}
$$

so, eg., g_2 is S_x

So we decompose into the above 4 sub-problems, ie,
(f_1, g_1), (f_2, g_2), (f_3, g_3), (f_4, g_4)

$$\Rightarrow q_1 = (f_1 + f_2 + f_3 + f_4)/4$$
$$q_2 = (f_1 + f_2 - f_3 - f_4)/4$$
$$s_1 = (g_1 + g_2 + g_3 + g_4)/4$$
$$s_2 = (-g_1 - g_2 + g_3 + g_4)/4$$

So, eg., f_3 is Ay

So, eg, g_2 is Ay

Here its possible to satisfy one of the short-edge BC's in strong form. Say,

$$\sigma_{xx} = 0 \text{ at } x = \pm L \Longrightarrow \alpha_n = (2n-1)\frac{\pi}{2L} \text{ for } f_1, f_3 \qquad n = 1, 2, \ldots$$
$$= \frac{n\pi}{L} \text{ for } f_2, f_4$$

Strong BC's:

$$\sigma_{yy}\Big|_{y=c} = q_1 = \tfrac{1}{4} f_1 = \tfrac{1}{4} f_2 = \tfrac{1}{4} f_3 = \tfrac{1}{4} f_4$$

Note: $\sigma_{yy}\Big|_{y=-c} = q_2 = \tfrac{1}{4} f_1 = \tfrac{1}{4} f_2 = -\tfrac{1}{4} f_3 = -\tfrac{1}{4} f_4$

gives no new info.

$\because f_1, f_2$ are Sy & f_3, f_4 are Ay

$$\Rightarrow \frac{1}{4} f_1 = -\sum P_n\big|_c \alpha_n^2 \cos\alpha_n x \longrightarrow f_1 \text{ problm}$$

$$\frac{1}{4} f_2 = -\sum P_n\big|_c \alpha_n^2 \sin\alpha_n x \longrightarrow f_2$$

$$\frac{1}{4} f_3 = -\sum Q_n\big|_c \alpha_n^2 \cos\alpha_n x \longrightarrow f_3$$

$$\frac{1}{4} f_4 = -\sum Q_n\big|_c \alpha_n^2 \sin\alpha_n x \longrightarrow f_4$$

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$\Rightarrow$ for

Ⓐ

$$\left\{\begin{array}{l}
\underset{\text{prblm}}{f_1} \rightarrow \frac{1}{4}\int_{-L}^{L} f_1 \cos\alpha_n x\, dx = -P_n\big|_c \alpha_n^2 L = -\alpha_n^2 L\left(A_n \cosh\alpha_n c + D_n c \sinh\alpha_n c\right) \\
f_2 \rightarrow \frac{1}{4}\int_{-L}^{L} f_2 \sin\alpha_n x\, dx = -P_n\big|_c \alpha_n^2 L = -\alpha_n^2 L\left(\quad\right) \\
f_3 \rightarrow \frac{1}{4}\int_{-L}^{L} f_3 \cos\alpha_n x\, dx = -Q_n\big|_c \alpha_n^2 L = -\alpha_n^2 L\left(B_n \sinh\alpha_n c + C_n c \cosh\alpha_n c\right) \\
f_4 \rightarrow \frac{1}{4}\int_{-L}^{L} f_4 \sin\alpha_n x\, dx = -Q_n\big|_c \alpha_n^2 L = -\alpha_n^2 L\left(\quad\right)
\end{array}\right.$$

Shear BC $\rightarrow$ $\sigma_{xy}\big|_{y=c} = S_1 = \frac{1}{4} g_1$

$= \frac{1}{4} g_2$

$= \frac{1}{4} g_3$

$= \frac{1}{4} g_4$

Note: $\sigma_{xy}\big|_{y=-c} = S_2 = -\frac{1}{4} g_1$

$= -\frac{1}{4} g_2$

$= \frac{1}{4} g_3$

$= \frac{1}{4} g_4$

gives no new info

$\therefore g_1, g_2$ are Ay

& g_3, g_4 are Sy

$$\Rightarrow \frac{1}{4} g_1 = \sum T_n\big|_c \sin\alpha_n x$$

$$\frac{1}{4} g_2 = -\sum T_n\big|_c \cos\alpha_n x$$

$$\frac{1}{4} g_3 = \sum U_n\big|_c \sin\alpha_n x$$

$$\frac{1}{4} g_4 = -\sum U_n\big|_c \cos\alpha_n x$$

$\Rightarrow$ for g_1 prblm

B

$$g_1 \rightarrow \frac{1}{4}\int_{-L}^{L} g_1 \sin\alpha_n x\,dx = T_n\big|_c L = L\left[A_n\alpha_n^2 \sinh\alpha_n c + D_n\alpha_n \sinh\alpha_n c + D_n\alpha_n^2 c\cosh\alpha_n c\right]$$

$$g_2 \rightarrow -\frac{1}{4}\int_{-L}^{L} g_2 \cos\alpha_n x\,dx = T_n\big|_c L = L\left[\ \ \right]$$

$$g_3 \rightarrow \frac{1}{4}\int_{-L}^{L} g_3 \sin\alpha_n x\,dx = U_n\big|_c L = L\left[B_n\alpha_n^2 \cosh\alpha_n c + C_n\alpha_n \cosh\alpha_n c + C_n\alpha_n^2 c\sinh\alpha_n c\right]$$

$$g_4 \rightarrow -\frac{1}{4}\int_{-L}^{L} g_4 \cos\alpha_n x\,dx = U_n\big|_c L = L\left[\ \ \right]$$

Solve $[A_1, B_1]$, $[A_2, B_2]$, $[A_3, B_3]$, $[A_4, B_4]$ for (A_n, D_n) and (B_n, C_n)

We get,

$$D_n = \frac{\sinh \alpha_n c \displaystyle\int_{-L}^{L} \frac{1}{4} \left\{ \begin{matrix} f_1 \cos \alpha_n x \, dx \\ f_2 \sin \alpha_n x \, dx \end{matrix} \right\} \pm \cosh \alpha_n c \int_{-L}^{L} \frac{1}{4} \left\{ \begin{matrix} g_1 \sin \alpha_n x \, dx \\ g_2 \cos \alpha_n x \, dx \end{matrix} \right\}}{L\left(\alpha_n^2 c + \alpha_n \sinh \alpha_n c \cosh \alpha_n c\right)}$$

$$A_n = \frac{-\frac{1}{4} \displaystyle\int_{-L}^{L} \left\{ \begin{matrix} f_1 \cos \alpha_n x \, dx \\ f_2 \sin \alpha_n x \, dx \end{matrix} \right\} - \alpha_n^2 L D_n c \sinh \alpha_n c}{\alpha_n^2 L \cosh \alpha_n c}$$

$\rightarrow (f_1, g_1), (f_2, g_2)$

$$C_n = \frac{\cosh \alpha_n c \displaystyle\int_{-L}^{L} \frac{1}{4} \left\{ \begin{matrix} f_3 \cos \alpha_n x \, dx \\ f_4 \sin \alpha_n x \, dx \end{matrix} \right\} \pm \sinh \alpha_n c \int_{-L}^{L} \frac{1}{4} \left\{ \begin{matrix} g_3 \sin \alpha_n x \, dx \\ g_4 \cos \alpha_n x \, dx \end{matrix} \right\}}{L\left(-\alpha_n^2 c + \alpha_n \sinh \alpha_n c \cosh \alpha_n c\right)}$$

$$B_n = \frac{-\frac{1}{4} \displaystyle\int_{-L}^{L} \left\{ \begin{matrix} f_3 \cos \alpha_n x \, dx \\ f_4 \sin \alpha_n x \, dx \end{matrix} \right\} - \alpha_n^2 L C_n c \cosh \alpha_n c}{\alpha_n^2 L \sinh \alpha_n c}$$

$\rightarrow (f_3, g_3), (f_4, g_4)$

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Procedure:

(i) From give loads q_1, q_2, S_1, S_2, obtain $f_1, \cdots f_4$, $g_1, \cdots, g_4$, as given on p 38.

(ii) Obtain $A_n, \cdots, D_n$ (p. 42)

(iii) Obtain stresses (p. 37).

All above expressions are easily programmable.

If loads are periodic in x, you will need finite nos of terms in the series soln (ie n= finite).

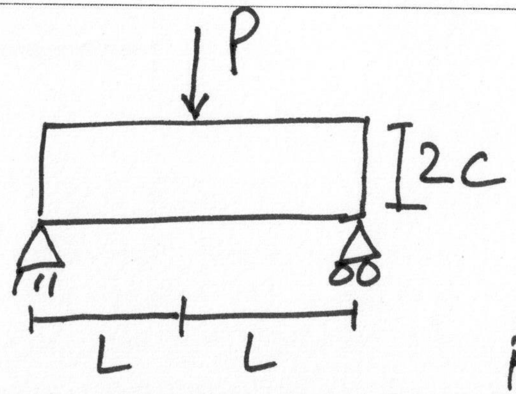
If loads are not periodic, truncate to n-term series solution, where n is obtained by convergence, ie

$$\left|(\text{soln})_{n+1} - (\text{soln})_n\right| / \left|(\text{soln})_n\right| < \varepsilon. \text{ (a small number)}$$

NOTE: Strong BC on short edge satisfied for σ_{xx}

Weak BC on short edge for σ_{xy} will be automatically satisfied — dont need to check it.

(Ex)



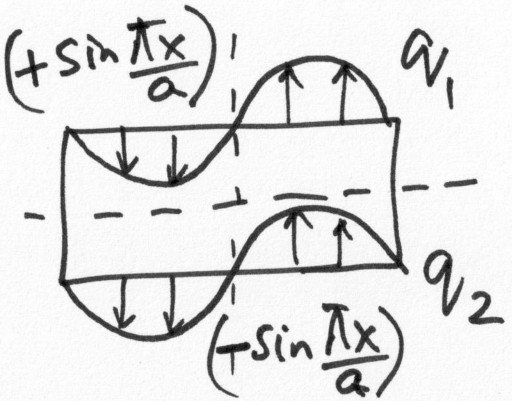
$q_1(x) = P\delta(x-0)$; $q_2 = s_1 = s_2 = 0$

$f_1 = f_3 = 2P\,\delta(x-0)$

$f_2 = f_4 = g_1 = g_2 = g_3 = g_4 = 0.$

Proceed to find $A_n, \ldots, D_n$ & soln for stresses.

Ex



(q_1, q_2) are $A_x\, A_y \longrightarrow$ So its f_4 problem.

$f_1 = f_2 = f_3 = 0$, $\quad f_4 = 4\sin\frac{\pi x}{a} \longrightarrow$ proceed for soln.

2D PROBLEMS IN POLAR COORDINATES

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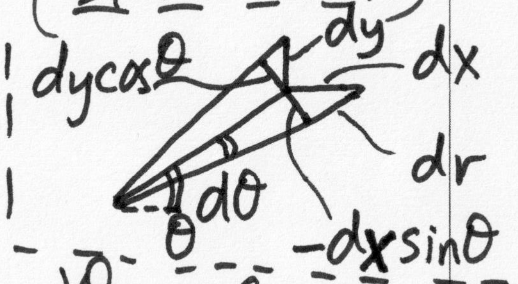
Transformation of equilibrium equations:

$$\boxed{\begin{aligned} &\sigma_{r,r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\sigma_{r\theta,\theta}}{r} + B_r = 0 \\ &\sigma_{r\theta,r} + \frac{2\sigma_{r\theta}}{r} + \frac{\sigma_{\theta,\theta}}{r} + B_\theta = 0 \end{aligned}}$$

$\longrightarrow$ r - eqn

$\longrightarrow$ θ - eqn.

Transformation of derivatives & Compatibility (Biharmonic) equation:



$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \quad ; \quad x = r c\theta, \quad y = r s\theta$$

$$r^2 = x^2 + y^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = c\theta \; ; \quad \frac{\partial r}{\partial y} = \frac{y}{r} = s\theta \; ; \quad \frac{\partial \theta}{\partial x} = -\frac{s\theta}{r} = \frac{-y}{r^2} \; ; \quad \frac{\partial \theta}{\partial y} = \frac{c\theta}{r} = \frac{x}{r^2}$$

$$\frac{\partial}{\partial x} = c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x^2} = \left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}\right)\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}\right) = c^2\theta\frac{\partial^2}{\partial r^2} + s^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) - s\theta c\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$\boxed{-\frac{s\theta c\theta}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\right) + \frac{s\theta c\theta}{r^2}\frac{\partial}{\partial \theta}} = -s\theta c\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

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$$\Rightarrow \frac{\partial^2}{\partial x^2} = c^2\theta \frac{\partial^2}{\partial r^2} + s^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) - 2s\theta c\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

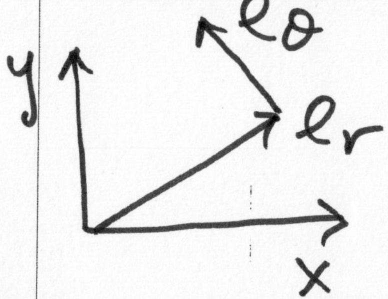
For $\frac{\partial}{\partial y}, \frac{\partial^2}{\partial y^2}$ do $\left\{\begin{array}{l} s\theta \rightarrow -c\theta \\ c\theta \rightarrow s\theta \end{array}\right\}$ in $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$

$$\frac{\partial^2}{\partial x \partial y} = -s\theta c\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial r^2}\right) + (c^2\theta - s^2\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

Thus, $\nabla^2_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

$$\nabla^4_{xy} = \nabla^2_{xy}\nabla^2_{xy} \Rightarrow \boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0} \longrightarrow \text{COMPATIBILITY EQN.}$$

Tranformation of stresses & stress function relations.



$$\left.\begin{array}{l} \sigma_r = \sigma_{xx}c^2\theta + \sigma_{yy}s^2\theta + 2\sigma_{xy}s\theta c\theta \\ \sigma_\theta = \sigma_{xx}s^2\theta + \sigma_{yy}c^2\theta - 2\sigma_{xy}s\theta c\theta \\ \sigma_{r\theta} = -(\sigma_{xx} - \sigma_{yy})s\theta c\theta + \sigma_{xy}(c^2\theta - s^2\theta) \end{array}\right\} \longrightarrow \star$$

Substitute $\sigma_{xx} = \phi_{,yy}$, $\sigma_{yy} = \phi_{,xx}$, $\sigma_{xy} = -\phi_{,xy}$ in transformation relation of stresses (★), and transform x,y derivatives on ϕ to r,θ derivatives (bot. p. 45, top p. 46), get,

$$\boxed{\begin{aligned} \sigma_r &= \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} \\ \sigma_\theta &= \phi_{,rr} \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\phi_{,\theta}\right) \end{aligned}}$$

Transformation of strain displ relations.

Displacements $(u, v) \rightarrow$ cartesian, $(u_r, u_\theta) \rightarrow$ polar.

$$u = c\theta\, u_r - s\theta\, u_\theta \quad ; \quad v = s\theta\, u_r + c\theta\, u_\theta$$

$$e_{xx} = u_{,x} = c^2\theta\, e_r + s^2\theta\, e_\theta - 2e_{r\theta}\, s\theta c\theta = c^2\theta\, u_{r,r} - c\theta s\theta\, u_{\theta,r} - \frac{s\theta}{r}\left(-s\theta\, u_r - c\theta\, u_\theta + c\theta\, u_{r,\theta} - s\theta\, u_{\theta,\theta}\right)$$

use transf of strains

use transf of displs & derivatives.

Similarly for e_{yy}, e_{xy}. Then solve 3 eqns for e_r, e_θ, $e_{r\theta}$ in terms of u_r, u_θ and their derivatives.

Result,

$$\boxed{\begin{aligned} e_r &= u_{r,r} \\ e_\theta &= \frac{u_r}{r} + \frac{1}{r} u_{\theta,\theta} \\ e_{r\theta} &= \frac{1}{2}\left(u_{\theta,r} + \frac{1}{r} u_{r,\theta} - \frac{u_\theta}{r}\right) \end{aligned}}$$

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Transformation of Constitutive Relations.

These remain same, except $e_{xx} \to e_r$, $\sigma_{xx} \to \sigma_r$, $e_{yy} \to e_\theta$, $\sigma_{yy} \to \sigma_\theta$, $e_{xy} \to e_{r\theta}$, $\sigma_{xy} \to \sigma_{r\theta}$

$$\boxed{\begin{aligned} e_r &= \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \\ e_\theta &= \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \\ e_{r\theta} &= \frac{1+\nu}{E}\sigma_{r\theta} \\ e_z &= -\frac{\nu}{E}(\sigma_r + \sigma_\theta) \end{aligned}} \rightarrow \text{PLANE STRESS}$$

$$\boxed{\begin{aligned} e_r &= \frac{1-\nu^2}{E}\left[\sigma_r - \frac{\nu}{1-\nu}\sigma_\theta\right] \\ e_\theta &= \frac{1-\nu^2}{E}\left[\sigma_\theta - \frac{\nu}{1-\nu}\sigma_r\right] \\ e_{r\theta} &= \frac{1+\nu}{E}\sigma_{r\theta} \end{aligned}} \rightarrow \text{PLANE STRAIN.}$$

i.e $E \to E'/(1-\nu'^2)$, $\nu \to \nu'/(1-\nu')$ in plane stress.

AXISYMMETRIC PROBLEMS.

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Loading, & hence stresses/strains, are invariant about an axis (i.e., independent of θ). Hence $\phi = \phi(r)$ (except the case $\phi = c\theta$, which we will treat seperately).
Note that this $\not\Rightarrow$ that displacements are indep of θ.

$$\nabla^4 \phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 = \phi^{IV} + \frac{2}{r}\phi^{III} - \frac{1}{r^2}\phi^{II} + \frac{1}{r^3}\phi' = 0$$

put $r = e^t$ to convert to ODE with constant coeffs.

details only

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = e^{-t}\frac{d\phi}{dt} \;;\quad \frac{d^2\phi}{dr^2} = e^{-2t}\left[-\frac{d\phi}{dt} + \frac{d^2\phi}{dt^2}\right]$$

$$\frac{d^3\phi}{dr^3} = e^{-3t}\left[2\frac{d\phi}{dt} - 3\frac{d^2\phi}{dt^2} + \frac{d^3\phi}{dt^3}\right];\quad \frac{d^4\phi}{dr^4} = e^{-4t}\left[-6\frac{d\phi}{dt} + 11\frac{d^2\phi}{dt^2} - 6\frac{d^3\phi}{dt^3} + \frac{d^4\phi}{dt^4}\right]$$

$$\Rightarrow \nabla^4\phi = \frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

$$\phi = e^{\lambda t} \rightarrow \lambda^4 - 4\lambda^3 + 4\lambda^2 = 0 \rightarrow \lambda = 0, 0, 2, 2$$

$$\phi = A^* + B^* t + C^* e^{2t} + D^* t e^{2t}$$

$$\phi = A \ln r + B r^2 \ln r + C r^2 + D$$

$$\Rightarrow \boxed{\begin{aligned} \sigma_r &= \frac{1}{r}\phi_{,r} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C \\ \sigma_\theta &= \phi_{,rr} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \\ \sigma_{r\theta} &= -\left(\frac{1}{r}\phi_{,\theta}\right)_{,r} = 0 \end{aligned}}$$

If $r=0$ is part of domain $\Rightarrow A = B = 0$, $\sigma_r = \sigma_\theta = 2C$ for finite stresses.

Displacements :

$$\varepsilon_r = u_{r,r} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\Rightarrow u_r = \frac{1}{E}\left[-\frac{A}{r}(1+\nu) + 2B(1-\nu) r \ln r - B(1+\nu) r + 2C(1-\nu) r\right] + f(\theta)$$

$$u_{\theta,\theta} = r e_\theta - u_r = r\frac{1}{E}(\sigma_\theta - \nu\sigma_r) - u_r$$

$$= \frac{1}{E}4Br - f(\theta)$$

$$\Rightarrow u_\theta = \frac{4}{E}Br\theta - \int f(\theta)\,d\theta + g(r)$$

Const of intgr absorbed in $g(r)$

$$e_{r\theta} = 0 \Rightarrow u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r} = \cancel{\frac{4B\theta}{E}} + g' + \frac{1}{r}f' - \cancel{\frac{4B\theta}{E}} + \frac{1}{r}\int f\,d\theta - \frac{g}{r} = 0$$

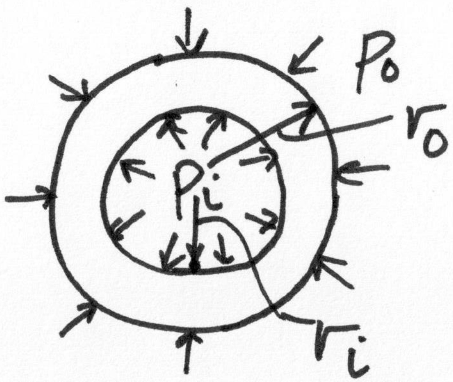
$$\Rightarrow g - rg' = \underbrace{Z}_{\text{(Const)}} = f' + \int f\,d\theta$$

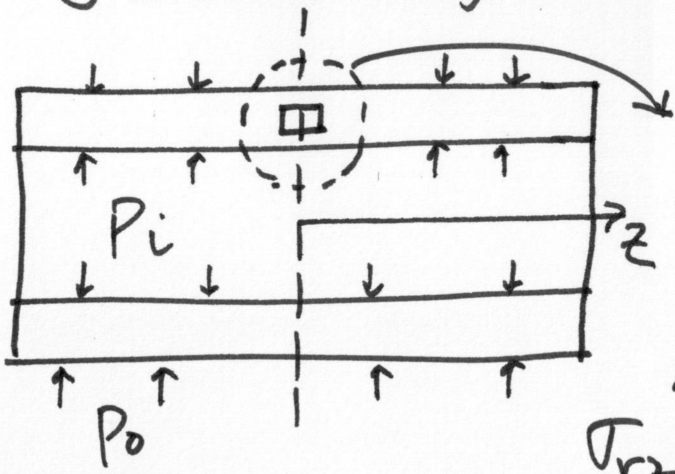
$$f'' + f = 0 \rightarrow f = H\sin\theta + K\cos\theta \rightarrow Z = Hc\theta - Ks\theta - Hc\theta + Ks\theta = 0$$

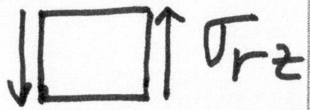
$$rg' = g \rightarrow g = Gr$$

$$\boxed{\begin{aligned} \Rightarrow u_r &= \frac{1}{E}\left[-(1+\nu)\frac{A}{r} + 2(1-\nu)Br\ln r - B(1+\nu)r + 2C(1-\nu)r\right] + H\sin\theta + K\cos\theta \\ u_\theta &= \frac{4}{E}Br\theta + H\cos\theta - K\sin\theta + Gr \end{aligned}}$$

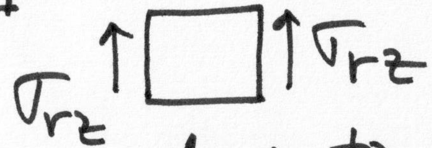
(I) Thick Walled Cylinder subject to pressure







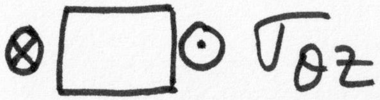
+ve convention



due to symmetry in z-dir.

Symmetry in z-direction

$$\Rightarrow \sigma_{rz} = \sigma_{\theta z} = 0$$

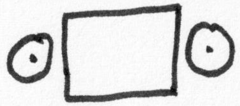


+ve convention

Also

+ve conv.

due to axisymm (ie symm in θ-dir)

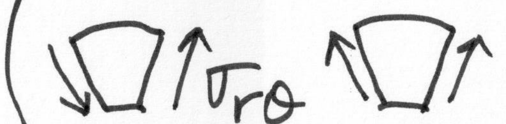


due to symmetry in z-dir

$$\Rightarrow \sigma_{\theta z} = 0.$$

Case A: Ends un-capped (ie open) and unrestrained $\Rightarrow \sigma_{zz} = 0$

Thus its plane stress problm ($\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = 0$)

Axisymmetry $\Rightarrow$ $\sigma_{r\theta} = 0$ ()

Axisymm $\Rightarrow$ $u_\theta = 0 \Rightarrow B = 0$ (also due to single-valuedness)

$$\Rightarrow Gr = \underbrace{K \sin\theta - H\cos\theta}_{} = \text{const} = 0$$

$$G = 0$$

$$u_r \neq u_r[\theta] \Rightarrow H\sin\theta + K\cos\theta = 0 \longrightarrow \Rightarrow K = H = 0$$

Stress BC's: $\sigma_r\big|_{r=r_o} = -p_o$; $\sigma_r\big|_{r=r_i} = -p_i$ $\rightarrow$ gives A, C.

$$\Rightarrow \boxed{\sigma_{r/\theta} = \pm \underbrace{\left[\frac{r_i^2 r_o^2 (p_o - p_i)}{r_o^2 - r_i^2}\right]}_{A} \frac{1}{r^2} + \underbrace{\left[\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}\right]}_{C}}$$

- $\sigma_{r/\theta}$ independent of E, ν
- $e_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\cancelto{0}{u_{\theta,\theta}} \neq 0$ despite $u_\theta = 0$ $\because u_r \neq 0$.
- $e_z = -\dfrac{\nu}{E}\underbrace{(\sigma_r + \sigma_\theta)}_{\text{Const}} = \text{Const} \Rightarrow$ sections $\perp^{ar}$ to z-axis remain plane.

- If $p_0 = 0 \rightarrow \sigma_r < 0$ (compr), $\sigma_\theta > 0$ (tensile)

$$(\sigma_\theta)_{max} = (\sigma_\theta)_{r=r_i} = p_i \frac{r_i^2 + r_o^2}{r_o^2 - r_i^2} > p_i$$

$$(\sigma_r)_{min} = (\sigma_r)_{r=r_i} = -p_i$$

So max Shear stress at $r = r_i$ $= \dfrac{\sigma_\theta - \sigma_r}{2}$

- Thin walled $(p_0 = 0)$ $\rightarrow r_o - r_i = t$, $r_o + r_i = 2r \cong 2r_o \cong 2r_i$

$$\underbrace{\sigma_\theta \cong \frac{p_i r}{t}, \quad \sigma_r \cong 0}_{\text{same result from CE221 for closed t.w. cylinder, where additionally } \sigma_z = \frac{pr}{2t}}$$

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Case B: Ends uncapped but restrained

i.e. $e_{zz} = 0$.

$\sigma_{\theta z} = \sigma_{rz} = e_{\theta z} = e_{rz} = 0$ as before.

i.e., PLANE STRAIN.

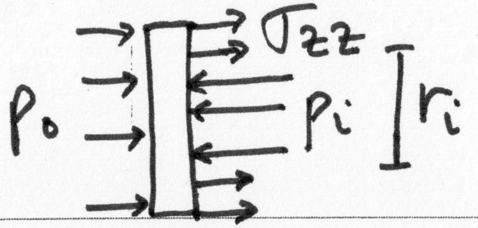
So results as before for stresses σ_r, σ_θ

$$\sigma_{zz} = \nu(\sigma_r + \sigma_\theta) = 2\nu \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} = \text{constant thru } r.$$

For displ's u_r, u_θ, put $\nu \longrightarrow \frac{\nu}{1-\nu}$, $E \rightarrow \frac{E}{1-\nu^2}$ in previous expressions.

Case C: Ends capped, unrestrained

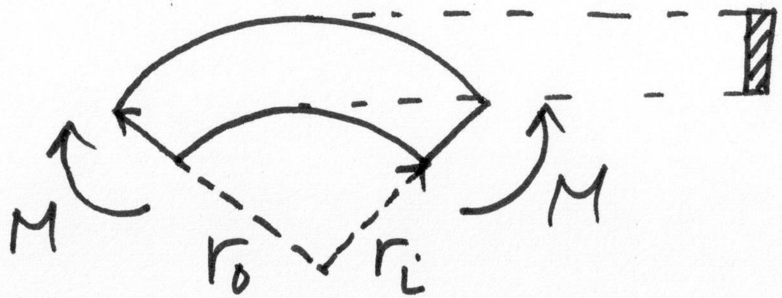
If caps are rigid disks, we expect $\sigma_{zz} = \text{const}$ thru r at sections away from ends (St. Venant), as in Case B (but value of σ_{zz} changes).



$$\Rightarrow \text{Equil} \longrightarrow \sigma_{zz} = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$$

$\sigma_r, \sigma_\theta \rightarrow$ as before.

(II) Pure Bending of Curved Beam.



Beam in shape of circular arc with narrow rectangular section, loaded by end couples (in-plane).

$\Rightarrow M = \text{const} \neq M(\theta)$

i.e. axisymmetric prblm.

BC's

$$\sigma_r\Big|_{\substack{r=r_0\\ r=r_i}} = 0 \longrightarrow \frac{A}{r_i^2} + B(1+2\ln r_i) + 2C = 0.$$

$$\frac{A}{r_0^2} + B(1+2\ln r_0) + 2C = 0$$

$\sigma_{r\theta} = 0$ on entire bndry $\rightarrow$ i.S.

$$\int_{r_i}^{r_0} \sigma_\theta \, dr = 0 = \int_{r_i}^{r_0} \frac{\partial^2 \phi}{\partial r^2} dr = \frac{\partial \phi}{\partial r}\Big|_{r_i}^{r_0} = r\sigma_r\Big|_{r_i}^{r_0} = 0 \text{ (due to } \sigma_r\big|_{\substack{r=r_0\\ r=r_i}} \text{ BC)}$$

$= \underline{\underline{\text{i.S.}}}$ if this BC satisfied.

$$\int_{r_i}^{r_o} \sigma_\theta r\,dr = -M = \int \phi_{,rr} r\,dr = \underbrace{\cancel{\phi_{,r}} r\Big|_{r_i}^{r_o}}_{=\sigma_r = 0 \text{ at } r_o,\, r_i} - \phi\Big|_{r_i}^{r_o}$$

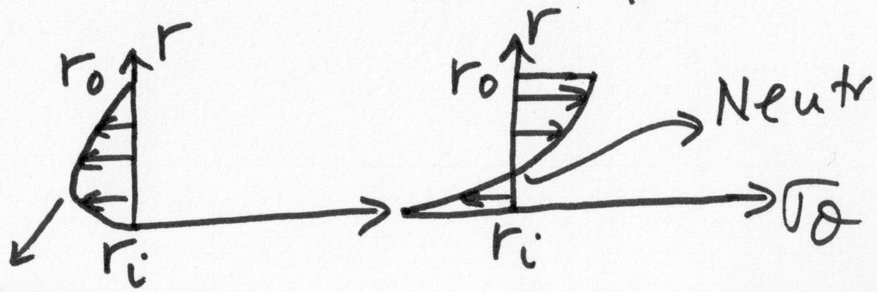
$$= \phi(r_i) - \phi(r_o)$$

$$\Rightarrow M = A \ln\frac{r_o}{r_i} + B\left(r_o^2 \ln r_o - r_i^2 \ln r_i\right) + C\left(r_o^2 - r_i^2\right)$$

Solve for A, B, C, result is,

$$\sigma_r = -\frac{4M}{N}\left[\frac{r_o^2 r_i^2}{r^2}\ln\frac{r_o}{r_i} + r_o^2 \ln\frac{r}{r_o} + r_i^2 \ln\frac{r_i}{r}\right]$$

$$\sigma_\theta = -\frac{4M}{N}\left[-\frac{r_o^2 r_i^2}{r^2}\ln\frac{r_o}{r_i} + r_o^2 \ln\frac{r}{r_o} + r_i^2 \ln\frac{r_i}{r} + r_o^2 - r_i^2\right]$$

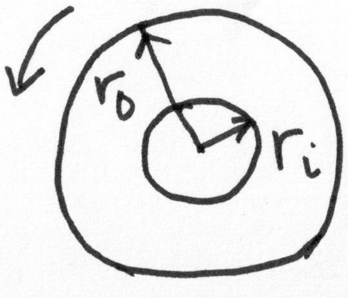


where,

$$N = \left(r_o^2 - r_i^2\right)^2 - 4 r_o^2 r_i^2 \left(\ln\frac{r_o}{r_i}\right)^2$$

Solution compares well with Solid Mech soln which assumes $\sigma_r = 0$ (ref. Timoshenko & Goodier p.74 table; Popov p.362-363 for Sol Mech soln).

III Rotating Disk. (Displ. formulation).



Thin rotating disk.

$\hookrightarrow \sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = 0.$

Axisymmetry $\Rightarrow \frac{\partial}{\partial\theta} = 0.$

Here $\nabla^4\phi \neq 0$ ($\because$ B.F. nonzero). So we use displacement formulation using equil eqns.

B.F.'s $\rightarrow$ $B_r = \rho\omega^2 r$; $B_\theta = \rho r\alpha$

r equil eqn $\rightarrow$ $\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0.$

S.D. eqns $\rightarrow$ $e_r = u_{r,r}$; $e_\theta = u_r/r$

C.L eqns $\rightarrow$ $\sigma_r = \frac{E}{1-\nu^2}(e_r + \nu e_\theta) = \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right)$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right)$$

subst in equil $\rightarrow$ $r^2 u_r'' + r u_r' - u_r = -\frac{(1-\nu^2)}{E}\rho\omega^2 r^3$

$(\,{}'\,) = \frac{d}{dr}$

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$$u_r = (u_r)_h + \cancel{(u_r)_p} \rightarrow \text{proportional to } r^3$$

put $r = e^t$

$$\dot{(\,)} = \frac{d}{dt}$$

$$\ddot{u}_r - u_r = -\left(\frac{1-\nu^2}{E}\right)\rho\omega^2 e^{3t}$$

$$u_r = \underbrace{C_1 e^t + C_2 e^{-t}}_{(u_r)_h} \underbrace{- \left(\frac{1-\nu^2}{E}\right)\frac{\rho\omega^2}{8} e^{3t}}_{(u_r)_p}$$

$$= C_1 r + \frac{C_2}{r} - \left(\frac{1-\nu^2}{E}\right)\frac{\rho\omega^2}{8} r^3$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2} - (3+\nu)\frac{(1-\nu^2)}{8E}\rho\omega^2 r^2\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2} - (1+3\nu)\frac{(1-\nu^2)}{8E}\rho\omega^2 r^2\right].$$

Solid Disk: $r_i = 0$; finite stresses at $r = 0 \longrightarrow C_2 = 0$

$$(\sigma_r)_{r=r_0} = 0 \xrightarrow{\text{get}} C_1 \rightarrow \sigma_r = \left(\frac{3+8}{\nu}\right)\rho\omega^2(r_0^2 - r^2)$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho\omega^2 r_0^2 - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2$$

- σ_r, σ_θ max at $r = r_i = 0$, $(\sigma_r)_{max} = (\sigma_\theta)_{max} = \frac{3+\nu}{8}\rho\omega^2 r_o^2$

<u>Disk with hole</u> $\quad r_i \neq 0$

$$BC's \rightarrow \left.\sigma_r\right|_{\substack{r=r_i \\ r=r_o}} = 0 \rightarrow \text{get } C_1, C_2$$

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left[r_o^2 + r_i^2 - \frac{r_o^2 r_i^2}{r^2} - r^2\right]$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left[r_o^2 + r_i^2 + \frac{r_o^2 r_i^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2\right]$$

Observations.

$$\frac{d\sigma_r}{dr} = 0 \rightarrow (\sigma_r)_{max} = \frac{3+\nu}{8}\rho\omega^2 (r_o - r_i)^2, \text{ at } r = \sqrt{r_o r_i}$$

$$(\sigma_\theta)_{max} = \frac{3+\nu}{4}\rho\omega^2\left[r_o^2 + \frac{1-\nu}{3+\nu} r_i^2\right], \text{ at } r = r_i$$

$$(\sigma_\theta)_{max} > (\sigma_r)_{max} \qquad \because r_o^2 > (r_o - r_i)^2$$

$$r_i \rightarrow 0 \Rightarrow (\sigma_\theta)_{max} = \frac{3+\nu}{4}\rho\omega^2 r_o^2 = 2 * (\sigma_\theta)_{max}, \text{ disk w/o hole}$$

i.e., stress concentration

What about θ-equil eqn ?.?

$\underline{\theta\text{-equil}}$ eqn:

$$\underbrace{\sigma_{r\theta,r} + \frac{2\sigma_{r\theta}}{r}}_{=0 \text{ for } (\sigma_{r\theta})_h} + \cancelto{0}{\frac{\sigma_{\theta,\theta}}{r}} + \cancelto{\rho r\alpha}{B_\theta} = 0$$

$$\Longrightarrow -\int \frac{2\,dr}{r} = \int \frac{d\sigma_{r\theta}}{\sigma_{r\theta}} \Rightarrow (\sigma_{r\theta})_h = C/r^2$$

$$(\sigma_{r\theta})_p = Kr^2 = -\frac{\rho\alpha}{4} r^2$$

$$\sigma_{r\theta} = \frac{C}{r^2} - \frac{\rho\alpha}{4} r^2$$

BC: $\sigma_{r\theta}\big|_{r=r_0} = 0 \longrightarrow$ find $C \longrightarrow \sigma_{r\theta} = \frac{\rho\alpha}{4}\left[\frac{r_0^4}{r^2} - r^2\right]$

for ω = const, $\alpha = 0$, $\sigma_{r\theta} = 0$.

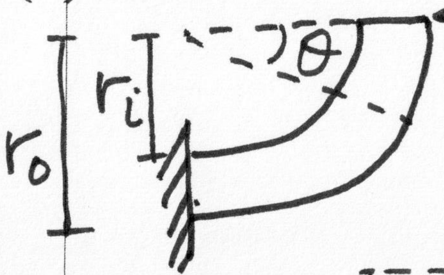
for $\alpha \neq 0$, torque T is applied thru stress distribution $\sigma_{r\theta}\big|_{r=r_i}$

$$\Rightarrow T = \sigma_{r\theta}\big|_{r=r_i} \cdot 2\pi r_i \cdot \cancel{1} = \frac{\rho\alpha}{4} \cdot 2\pi\left[\frac{r_0^4}{r_i} - r_i^3\right]$$

unit thickness

NON-AXISYMMETRIC PROBLEMS.

(I) END LOADED CURVED BEAM



Circular arc with narrow rectangular section.
Force P in radial direction.

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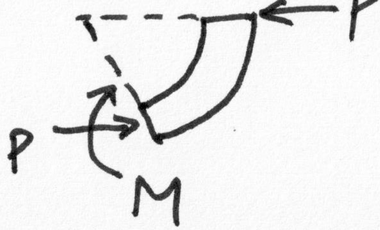


$\Rightarrow M \propto \sin\theta$

$\sigma_\theta \propto \sin\theta$

$\phi_{,rr} \propto \sin\theta \longrightarrow$ so try $\phi = f(r)\sin\theta$.

$$\nabla^4\phi = 0 \Rightarrow \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2}\right)\underbrace{\sin\theta}_{\neq 0} = 0$$

$$\Rightarrow f^{IV} - \frac{2}{r}f^{III} - \frac{3}{r^2}f^{II} + \frac{3}{r^3}f^{I} - \frac{3}{r^4}f = 0$$

$$\xrightarrow{r=e^t} \ddddot{f} - 4\dddot{f} + 2\ddot{f} + 4\dot{f} - 3f = 0 \qquad (\cdot) = \frac{d}{dt} \quad \xRightarrow{f=e^{st}} s^4 - 4s^3 + 2s^2 + 4s - 3 = 0$$

$$s = 3, -1, +1, +1$$

$$f = Ae^{3t} + Be^{-t} + Ce^{t} + Dte^{t}$$
$$= Ar^3 + \frac{B}{r} + Cr + Dr\ln r$$

$$\sigma_r = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2} = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$$

$$\sigma_\theta = \phi_{,rr} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\phi_{,\theta}\right) = -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta$$

<u>BC's</u>: $\sigma_r\Big|_{\substack{r=r_0\\ r=r_i}} = \sigma_{r\theta}\Big|_{\substack{r=r_0\\ r=r_i}} = 0 \longrightarrow$ yields only 2 indep eqns

$$\left.\begin{aligned} 2Ar_i - \frac{2B}{r_i^3} + \frac{D}{r_i} &= 0 \\ 2Ar_0 - \frac{2B}{r_0^3} + \frac{D}{r_0} &= 0 \end{aligned}\right]$$

$$P = \int_{r_0}^{r_i} \sigma_{r\theta}\Big|_{\theta=0} dr = -\int_{r_i}^{r_0} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\Big|_{\theta=0} dr = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r_0}^{r_i}\Big|_{\theta=0}$$

$$\Rightarrow P = -A\left(r_0^2 - r_i^2\right) + B\left(\frac{r_0^2 - r_i^2}{r_i^2 r_0^2}\right) - D\ln\frac{r_0}{r_i}$$

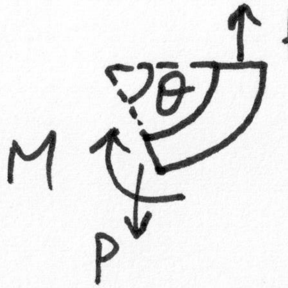
Solve for A, B, D, $A = \frac{P}{2N}$, $B = -\frac{P r_i^2 r_0^2}{2N}$, $D = -\frac{P}{N}\left(r_i^2 + r_0^2\right)$,

$$N = r_i^2 - r_0^2 + \left(r_i^2 + r_0^2\right)\ln\left(\frac{r_0}{r_i}\right)$$

Note: $\sigma_\theta\big|_{\theta=0} = 0$ is <u>i.s.</u>

$\bar{r} = \frac{r_o + r_i}{2}$

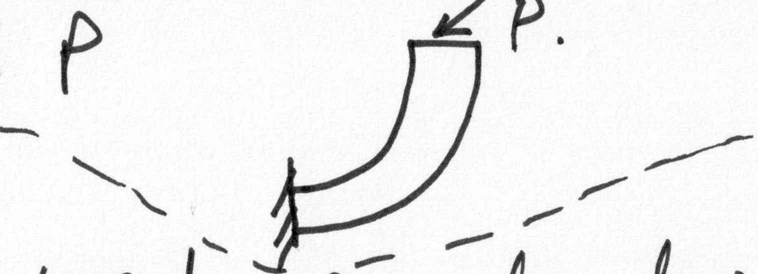
• For vertical load P, $M = \underbrace{P\bar{r}} - P\bar{r}\cos\theta$



use solution due to pure bending (axisymm)

use $\phi = f(r)\cos\theta$ and proceed similarly.

• Then combine solutions due to horz & vert loads to get soln for inclined P



• These solutions can also be used when loading on curved faces are non-zero, ie $(\sigma_r, \sigma_{r\theta}) \propto \sin\theta$ or $\cos\theta$

(II) Plate with circular hole.

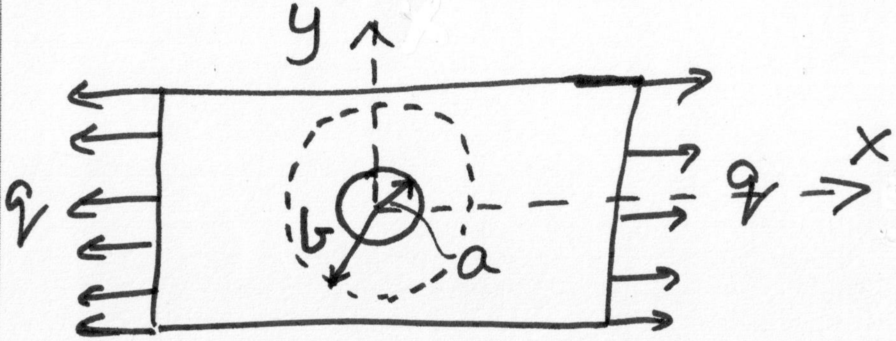


Plate uniformly loaded in x-dir. Fictitious boundary at $r=b$, $b \gg a$, a = hole radius.

$\because b \gg a$, $\sigma_{xx}\big|_{r=b} = q$; $\sigma_{yy}\big|_{r=b} = 0$; $\sigma_{xy}\big|_{r=b} = 0 \rightarrow$ approx. far field stresses

$$\text{Transformation} \Rightarrow \sigma_r\big|_{r=b} = \frac{q}{2}(1+\cos 2\theta) \; ; \; \sigma_\theta\big|_{r=b} = \frac{q}{2}(1-\cos 2\theta)$$

$$\sigma_{r\theta}\big|_{r=b} = -\frac{q}{2}\sin 2\theta$$

So an equivalent problem is that of annular disk ($r_i = a$, $r_o = b$) loaded with $\sigma_r\big|_{r=b}$ and $\sigma_{r\theta}\big|_{r=b}$ at outer boundary $r = r_o = b$. We solve this problem.

Part (A) soln : Due to Axisymmetric loading

$$\sigma_r\Big|_{r=b} = \frac{q}{2}$$

Put $p_o = -\frac{q}{2}$, $p_i = 0$ in soln. of Thick-walled cylinder with open (uncapped, unrestrained) ends. Use $b \gg a$, get,

$$\sigma_r = \frac{q}{2}\left(1 - \frac{a^2}{r^2}\right) \; ; \; \sigma_\theta = \frac{q}{2}\left(1 + \frac{a^2}{r^2}\right) \; ; \; \sigma_{r\theta} = 0$$

Part (A) soln

Part (B) soln : $\sigma_r\Big|_{r=b} = \frac{q}{2}\cos 2\theta$, $\sigma_{r\theta}\Big|_{r=b} = -\frac{q}{2}\sin 2\theta$ applied

Semi-inverse method $\rightarrow$ try $\phi = f(r)\cos 2\theta$ $\rightarrow$ gives $\sigma_r \propto \cos 2\theta$, $\sigma_{r\theta} \propto \sin 2\theta$

$$\nabla^4\phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right)\cos 2\theta = 0$$

$$\Rightarrow \left[f^{IV} + \frac{2}{r}f^{III} - \frac{9}{r^2}f^{II} + \frac{9}{r^3}f^{I}\right] = 0 \qquad \left\{(I) = \frac{d}{dr}\right\} \text{etc.}$$

$$r = e^t \Rightarrow \ddddot{f} - 4\dddot{f} - 4\ddot{f} + 16\dot{f} = 0 \rightarrow s^4 - 4s^3 - 4s^2 + 16s = 0, \; s = 0, 2, -2, 4$$

$$f(r) = Ar^4 + Br^2 + C + \frac{D}{r^2}$$

$$\sigma_r = -\cos 2\theta \left(2B + \frac{4C}{r^2} + \frac{6D}{r^4}\right) \; ; \; \sigma_\theta = \cos 2\theta \left(12Ar^2 + 2B + \frac{6D}{r^4}\right)$$

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$$\sigma_{r\theta} = \sin 2\theta \left(6Ar^2 + 2B - \frac{2C}{r^2} - \frac{6D}{r^4}\right)$$

BC's:

$$\sigma_r\big|_{r=a} = 0 \rightarrow 2B + \frac{4C}{a^2} + \frac{6D}{a^4} = 0 \longrightarrow \text{(i)}$$

$$\sigma_{r\theta}\big|_{r=a} = 0 \rightarrow 6Aa^2 + 2B - \frac{2C}{a^2} - \frac{6D}{a^4} = 0 \rightarrow \text{(ii)}$$

$$\sigma_r\big|_{r=b} = \frac{q}{2}\cos 2\theta \rightarrow 2B + \frac{4C}{b^2} + \frac{6D}{b^4} = -\frac{q}{2} \rightarrow \text{(iii)}$$

$$\sigma_{r\theta}\big|_{r=b} = -\frac{q}{2}\sin 2\theta \rightarrow 6Ab^2 + 2B - \frac{2C}{b^2} - \frac{6D}{b^4} = -\frac{q}{2} \longrightarrow \text{(iv)}$$

Solve using $\underbrace{\frac{a}{b}}_{\varepsilon} \rightarrow 0$, $B = -\frac{q}{4}$ (from (iii)) $\rightarrow 2Ba^4 + 4C\cancel{\varepsilon^2}^{\,0}a^2 + 6D\cancel{\varepsilon^4}^{\,0} = -\frac{q}{2}a^4$

$A = 0$ (from (iv)) $\rightarrow 6Ab^2a^4 + 2\underbrace{\cancel{B}}_{-\frac{q}{4}}a^4 - 2C\underbrace{\cancel{\varepsilon^2}}_{0}a^2 - 6D\cancel{\varepsilon^4}^{\,0} = -\frac{q}{2}a^4$

$C = \frac{qa^2}{2}$, $D = -q\frac{a^4}{4}$ (from (i), (ii)).

Combining Part (A) & Part (B) soln,

$$\sigma_r = \frac{q}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{q}{2}\cos 2\theta\left(1-\frac{a^2}{r^2}\right)\left(1-\frac{3a^2}{r^2}\right)$$

$$\sigma_\theta = \frac{q}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{q}{2}\cos 2\theta\left(1+\frac{3a^4}{r^4}\right)$$

$$\sigma_{r\theta} = -\frac{q}{2}\sin 2\theta\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)$$

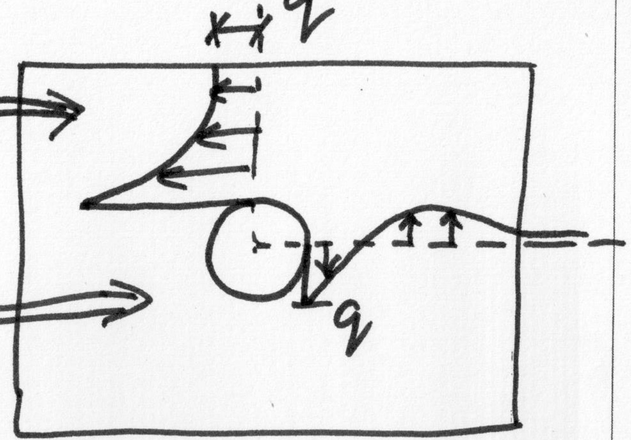
At $r=a$, $\sigma_r = 0$, $\sigma_\theta = q(1-2\cos 2\theta)$

$= 3q$, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\longrightarrow$ Stress concentration.

$= -q$, $\theta = 0, \pi$

At $\theta = \frac{\pi}{2}$, $\sigma_\theta = q\left(1+\frac{1}{2}\frac{a^2}{r^2}+\frac{3}{2}\frac{a^4}{r^4}\right)$

At $\theta = 0$, $\sigma_\theta = -\frac{q}{2}\frac{a^2}{r^2}\left(3\frac{a^2}{r^2}-1\right)$



If q acts in y-direction, use this solution with $\theta \to \theta - \frac{\pi}{2}$.

An <u>approximate</u> way to find stresses in a plate with multiple holes is as follows:

(i) Find stresses in plate assuming no holes. — ie far field stresses.

(ii) Find principal stresses at the location of each hole, say q_1, q_2 are p-stresses at a particular hole.

(iii) Use q_1 in solution derived & q_2 with $\theta \to \theta - \frac{\pi}{2}$ in soln derived & add results, to get stresses around (ie in vicinity of hole).

(iv) If holes are close to each other then the solutions in (iii) above will need to be superposed.

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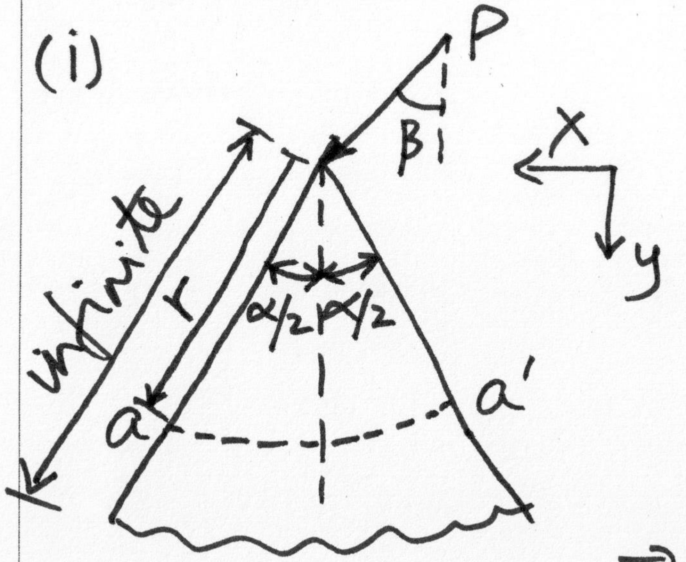
Wedges.

(i)



Wedge angle $= \alpha$.

P = concentrated load per unit thickness, ie N/m.

Stresses depend on $P, \alpha, \beta, r, \theta$.

Stresses $\equiv \frac{N}{m^2} \equiv \frac{P}{r}\lambda$, λ = dimensionless function of α, β, θ

$\Rightarrow$ From stress – ϕ relation,

$\boxed{\phi = r f(\theta)} \longrightarrow f(\theta)$ contains $P, \lambda(\alpha, \beta, \theta)$

$$\nabla^4 \phi = 0 = \frac{1}{r^3}\left[\frac{d^4 f}{d\theta^4} + 2\frac{d^2 f}{d\theta^2} + f\right] = 0 \xrightarrow{f = e^{s\theta}} (s^2+1)^2 = 0,\ s = \pm i, \pm i$$

$$\Rightarrow f = A\cos\theta + B\sin\theta + \theta(C\cos\theta + D\sin\theta)$$

$$\Rightarrow \sigma_r = \frac{2}{r}(D\cos\theta - C\sin\theta) \quad ; \quad \sigma_\theta = 0 \quad ; \quad \sigma_{r\theta} = 0$$

BC's :- $\sigma_\theta \Big|_{\theta = \pm\frac{\alpha}{2}} = 0$; $\sigma_{r\theta}\Big|_{\theta = \pm\frac{\alpha}{2}} = 0 \longrightarrow$ i.s.

Use lumpsum equil to find C, D, by taking section at aa' at radial coord r.

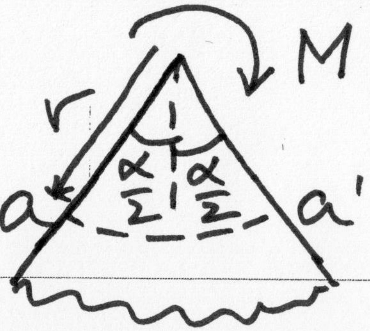
$$\int_{-\alpha/2}^{\alpha/2} \sigma_r \, r \, d\theta \cos\theta + P\cos\beta = 0 \qquad \{\Sigma F_y = 0\}$$

$$\int_{-\alpha/2}^{\alpha/2} \sigma_r \, r \, d\theta \sin\theta + P\sin\beta = 0 \qquad \{\Sigma F_x = 0\}$$

$$\left.\begin{aligned} D(\sin\alpha + \alpha) + P\cos\beta &= 0 \\ C(\sin\alpha - \alpha) + P\sin\beta &= 0 \end{aligned}\right\} \text{Solve for } C, D.$$

$$\Rightarrow \boxed{\sigma_r = -\frac{2P}{r}\left[\frac{\cos\beta\cos\theta}{\alpha + \sin\alpha} + \frac{\sin\beta}{\alpha - \sin\alpha}\sin\theta\right] ; \ \sigma_\theta = \sigma_{r\theta} = 0}$$

(ii)



M = concentrated torque per unit thk, $\frac{Nm}{m}$.

Stresses depend on M, α, r, θ

Stresses $\equiv \frac{N}{m^2} = \frac{M}{r^2}\lambda$, λ = dimensionless $f^{\underline{n}}$ of α, θ.

From stress $-\ \phi$ rel. $\longrightarrow\ \phi = f(\theta)$

$$\nabla^4 \phi = 0 \longrightarrow \frac{1}{r^4}\left(\frac{\partial^4 f}{\partial \theta^4} + 4\frac{\partial^2 f}{\partial \theta^2}\right) = 0 \longrightarrow s = 0, 0, \pm 2i$$

$$\phi = f = A\cos 2\theta + B\sin 2\theta + C\theta + D$$

$$\sigma_r = -\frac{4}{r^2}(A\cos 2\theta + B\sin 2\theta) \quad ; \quad \sigma_\theta = 0 ;$$

$$\sigma_{r\theta} = \frac{1}{r^2}(-2A\sin 2\theta + 2B\cos 2\theta + C)$$

BC's: $\sigma_\theta \Big|_{\theta = \pm\frac{\alpha}{2}} = 0 \longrightarrow \underline{\underline{\text{i.s.}}}$

$$\sigma_{r\theta}\Big|_{\theta = \pm\frac{\alpha}{2}} = 0 \implies A = 0 \ \ \& \ \ C = -2B\cos\alpha$$

Use lump sum equilibrium to get another relation between B & C.

$$\{\Sigma M = 0\} \rightarrow \int_{-\alpha/2}^{\alpha/2} \sigma_{r\theta}\, r^2\, d\theta + M = 0 \implies (2B\sin\alpha + C\alpha)\frac{\cancel{r^2}}{\cancel{r^2}} + M = 0$$

$$\sigma_r = \frac{2M \sin 2\theta}{(\sin\alpha - \alpha\cos\alpha)r^2} \quad ; \quad \sigma_\theta = 0$$

$$\sigma_{r\theta} = \frac{M(\cos 2\theta - \cos\alpha)}{(\alpha\cos\alpha - \sin\alpha)r^2}$$

Note: You can verify that lumpsum $\sum F_x = 0$, $\sum F_y = 0$, are i.s.

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FINITE DIFFERENCE METHOD FOR PLANE PROBLEMS

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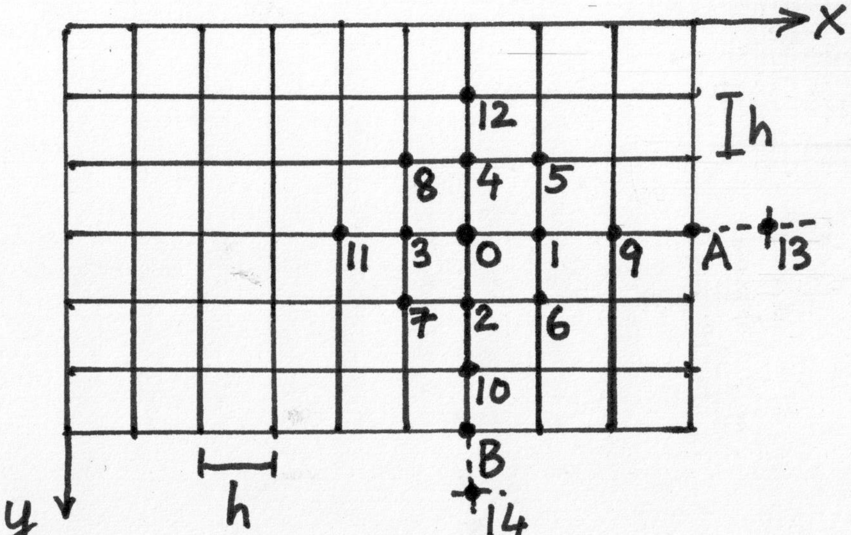


$$f_1 = f_0 + h\left(\frac{\partial f}{\partial x}\right)_0 + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + \cdots \rightarrow (1)$$

$$f_3 = f_0 - h\left(\frac{\partial f}{\partial x}\right)_0 + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + \cdots \rightarrow (2)$$

$$(1),(2) \rightarrow \left(\frac{\partial f}{\partial x}\right)_0 = \frac{f_1 - f_3}{2h} \rightarrow (3) \quad ; \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_0 = \frac{f_1 + f_3 - 2f_0}{h^2} \rightarrow (4)$$

Similarly in y-dir $\rightarrow \left(\frac{\partial f}{\partial y}\right)_0 = \frac{f_2 - f_4}{2h} \rightarrow (3a) \quad ; \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_0 = \frac{f_2 + f_4 - 2f_0}{h^2} \rightarrow (4a)$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]_0 = \frac{\left(\frac{\partial f}{\partial x}\right)_2 - \left(\frac{\partial f}{\partial x}\right)_4}{2h} = \frac{\frac{f_6 - f_7}{2h} - \frac{f_5 - f_8}{2h}}{2h} = \frac{f_6 + f_8 - f_5 - f_7}{4h^2} \rightarrow (4b)$$

Similarly,

$$\left(\frac{\partial^4 f}{\partial x^4}\right)_0 = \frac{1}{h^4}\left[6f_0 - 4(f_1+f_3) + (f_9+f_{11})\right] \longrightarrow (5a)$$

$$\left(\frac{\partial^4 f}{\partial^2 x\, \partial^2 y}\right)_0 = \frac{1}{h^4}\left[4f_0 - 2(f_1+f_2+f_3+f_4) + (f_5+f_6+f_7+f_8)\right] \longrightarrow (5c)$$

$$\left(\frac{\partial^4 f}{\partial y^4}\right)_0 = \frac{1}{h^4}\left[6f_0 - 4(f_2+f_4) + (f_{10}+f_{12})\right] \longrightarrow (5b)$$

[3 – 5 are central difference formulae. More accurate than forward/backward diff. formulae (done next).]

$$f_9 = f_0 + 2h\left(\frac{\partial f}{\partial x}\right)_0 + 2h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + \cdots \longrightarrow (1a)$$

1, 1a $\rightarrow$ $\left(\frac{\partial f}{\partial x}\right)_0 = \frac{-3f_0 + 4f_1 - f_9}{2h} \longrightarrow (6)$; $\left(\frac{\partial^2 f}{\partial x^2}\right)_0 = \frac{f_0 - 2f_1 + f_9}{h^2} \longrightarrow (7)$

Similarly, $\left(\frac{\partial f}{\partial y}\right)_0 = \frac{-3f_0 + 4f_2 - f_{10}}{2h} \longrightarrow (6a)$; $\left(\frac{\partial^2 f}{\partial y^2}\right)_0 = \frac{f_0 - 2f_2 + f_{10}}{h^2} \longrightarrow (7a)$

Similarly, using $f_{11} = f_0 - 2h\left(\frac{\partial f}{\partial x}\right)_0 + 2h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + \dots \rightarrow$ (2a)

(2), (2a) $\rightarrow \left(\frac{\partial f}{\partial x}\right)_0 = \frac{3f_0 - 4f_3 + f_{11}}{2h} \rightarrow$ (8) ;

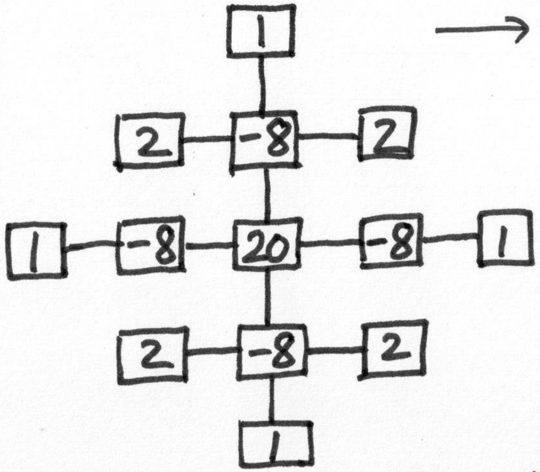
$$\left(\frac{\partial^2 f}{\partial x^2}\right)_0 = \frac{f_0 - 2f_3 + f_{11}}{h^2} \rightarrow (9);$$

Similarly, $\left(\frac{\partial f}{\partial y}\right)_0 = \frac{3f_0 - 4f_4 + f_{12}}{2h} \rightarrow$ (8a) ; $\left(\frac{\partial^2 f}{\partial y^2}\right)_0 = \frac{f_0 - 2f_4 + f_{12}}{h^2} \rightarrow$ (9a)

[(6), (7) are Forward difference formulae
(8), (9) are Backward difference formulae].

Use Forward/Backward diff formulae only for end-point (boundary pt) derivatives $\because$ they are less accurate.

Thus, $\nabla^4\phi = 0 = 20\phi_0 - 8(\phi_1 + \phi_2 + \phi_3 + \phi_4) + 2(\phi_5 + \phi_6 + \phi_7 + \phi_8)$
$+ (\phi_9 + \phi_{10} + \phi_{11} + \phi_{12}) = 0 \longrightarrow$ (10)



$\longrightarrow$ Eqn (10) in pictorial form.

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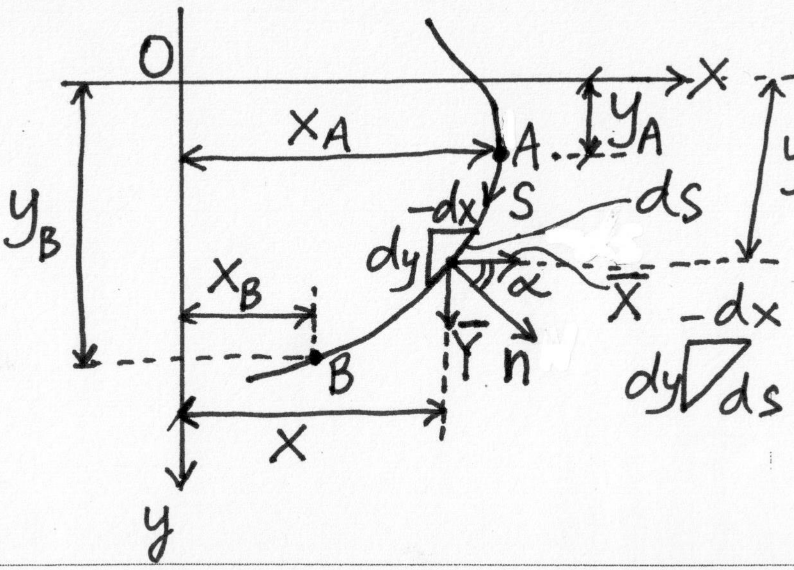
Can write (10) for every interior point. However if point lies at distance 'h' from boundary, then fictitious nodes lying outside the domain boundary get involved (eg nodes 13, 14). So we do the following.



$$n = \{l, m\}^T$$

$$l = \cos\alpha = \frac{dy}{ds} \quad ; \quad m = \sin\alpha = -\frac{dx}{ds}$$

Stress vector $\underline{t} = \bar{X}\,\underline{i} + \bar{Y}\,\underline{j} = \underline{\underline{\sigma_s}}\;\underline{n}$

$$\bar{X} = l(\sigma_x)_s + m(\sigma_{xy})_s$$

$$\bar{Y} = l(\sigma_{xy})_s + m(\sigma_y)_s$$

$\underline{\underline{\sigma_s}}$ is stress vector on boundary S.

$$\left.\begin{aligned} \overline{X} &= \frac{dy}{ds}\left(\frac{\partial^2\phi}{\partial y^2}\right)_s + \frac{dx}{ds}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_s = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)_s \\ \overline{Y} &= -\frac{dy}{ds}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_s - \frac{dx}{ds}\left(\frac{\partial^2\phi}{\partial x^2}\right)_s = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)_s \end{aligned}\right\} \longrightarrow (11)$$

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Integrate wrt s, from A to B,

$$\left(\frac{\partial\phi}{\partial y}\right)_B = \underbrace{\left(\frac{\partial\phi}{\partial y}\right)_A}_{=0\text{ (see later).}} + \int_A^B \overline{X}\,ds \quad ; \quad \left(\frac{\partial\phi}{\partial x}\right)_B = \underbrace{\left(\frac{\partial\phi}{\partial x}\right)_A}_{=0\text{ (see later)}} - \int_A^B \overline{Y}\,ds \longrightarrow (12)$$

Also, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = \frac{\partial\phi}{\partial x}\frac{dx}{ds}ds + \frac{\partial\phi}{\partial y}\frac{dy}{ds}ds$

Integrate $\rightarrow \phi_B = \phi_A + \left(x\frac{\partial\phi}{\partial x}\right)_A^B - \int_A^B x\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)ds + \left(y\frac{\partial\phi}{\partial y}\right)_A^B - \int_A^B y\frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)ds$

Use (11) & (12) $\Rightarrow \phi_B = \phi_A + \underbrace{(x_B - x_A)\left(\frac{\partial\phi}{\partial x}\right)_A + (y_B - y_A)\left(\frac{\partial\phi}{\partial y}\right)_A}_{=0\text{ (see later)}} + \int_A^B (y_B - y)\overline{X}\,ds + \int_A^B (x - x_B)\overline{Y}\,ds \rightarrow (13)$

Now we know that (constant + linear) terms in stress fn. dont affect stresses. So we can consider stress fn. $(\phi + a + bx + cy)$ & solve for a, b, c from the 3 conditions $\phi_A = \left(\frac{\partial\phi}{\partial x}\right)_A = \left(\frac{\partial\phi}{\partial y}\right)_A = 0$ without stresses being affected.

So w/o loss of generality take $\phi_A = \left(\frac{\partial\phi}{\partial x}\right)_A = \left(\frac{\partial\phi}{\partial y}\right)_A = 0$ (analogous to choosing a datum in potential energy).

This works only for simply-connected domains.

The integrals in (12), (13) have following physical interpretation:

- Integral in (12a), (12b) represent sum of applied surface forces between pts A & B in x & y directions, respectively.
- Integrals in (13) represent sum of moments (+ve CW) due to surface forces applied between A & B, moments being taken about B.

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Value of ϕ at fictitious (exterior) nodes are **given** by Central Difference formulae (3), (3a)

i.e,

$$\phi_{13} = \phi_9 + 2h\left(\frac{\partial\phi}{\partial x}\right)_A \; ; \; \phi_{14} = \phi_{10} + 2h\left(\frac{\partial\phi}{\partial y}\right)_B \longrightarrow (14)$$

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Summary:

Step (1) Choose datum A on boundary, set $\phi_A = \left(\frac{\partial\phi}{\partial x}\right)_A = \left(\frac{\partial\phi}{\partial y}\right)_A = 0$. Calculate ϕ_B, $\left(\frac{\partial\phi}{\partial x}\right)_B$, $\left(\frac{\partial\phi}{\partial y}\right)_B$ for all boundary points using (12), (13) and physical interpretation of integrals appearing in (12), (13).

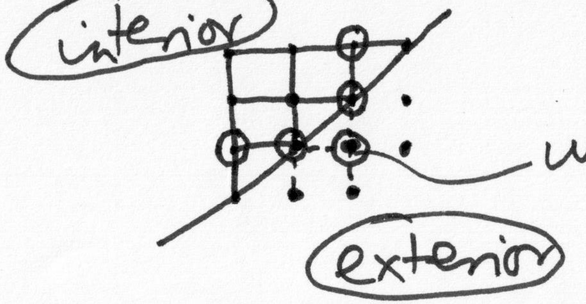
Step (2) Express ϕ at fictitious (exterior) nodes in terms of ϕ at interior nodes using (14)

Step (3) Formulate $\nabla^4\phi = 0$ at every interior node using (10)

Step (4) Solve equations obtained in <u>Step (2)</u> & <u>Step (3)</u> above for ϕ at interior & fictitious (exterior) nodes.

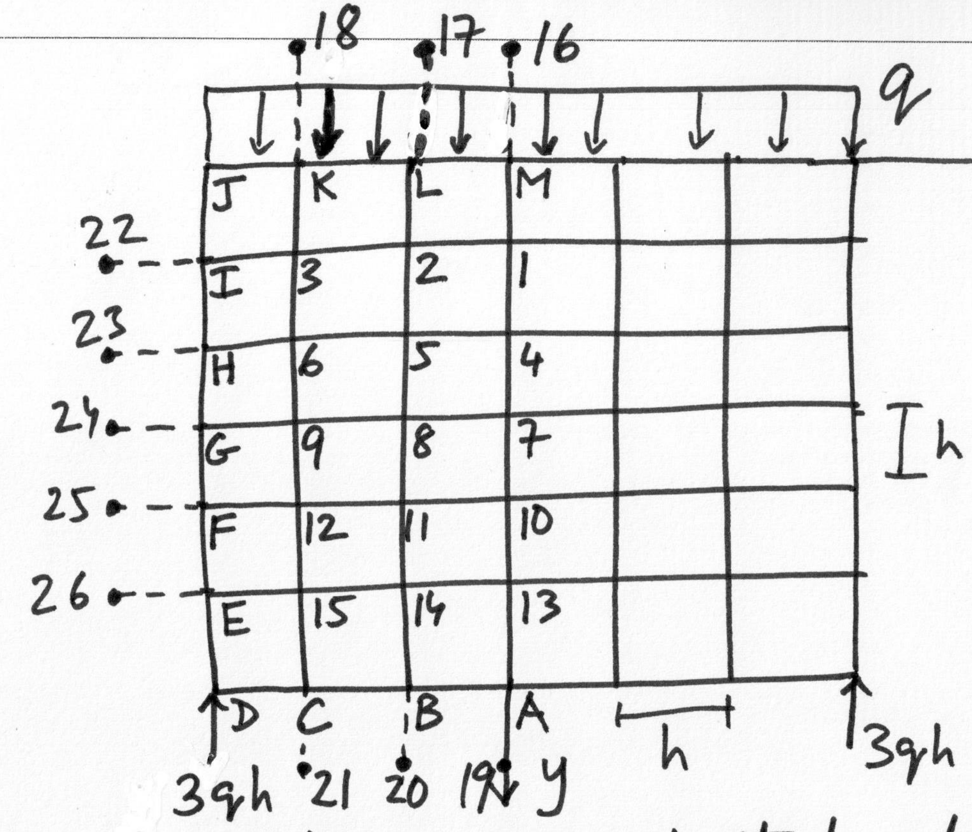
Step (5) Find stress components using nodal ϕ's in eqns (4), (4a), (4b).

This procedure works if domain has horizontal & vertical boundaries. For inclined boundaries we will have to resort to forward/backward difference version of $\nabla^4\phi = 0$ (eqn 10) for interior nodes lying at distance 'h' from boundary; and/or we will have to obtain ϕ at fictitious nodes by averaging from x & y derivatives (ie averaging of eqn 14).



use averaged value of ϕ obtained from x, y derivatives involving circled nodes.

Ex



Square beam size $6h \times 6h$ (ie deep beam) with udl $= q$.

Use symmetry, so analyze left half only.

Assume reactions concentrated at corners. This wont affect stresses much away from corners.

Step I Take datum point as A., ie $\phi_A = \left(\frac{\partial \phi}{\partial x}\right)_A = \left(\frac{\partial \phi}{\partial y}\right)_A = 0$

Node	A	B,C	D	E,F,G,H,I	J	K	L	M
ϕ	0	0	0	0	0	$2.5qh^2$	$4qh^2$	$4.5qh^2$
$\partial\phi/\partial x$	0	—	—	$3qh$	—	—	—	—
$\partial\phi/\partial y$	0	0	—	—	—	0	0	0

— means not required in Eq 14 So not computed.

$$\left(\frac{\partial\phi}{\partial x}\right)_E = -\sum F_x \text{ surface loads between } A \,\&\, E = 0$$

$$(\phi)_E = (\phi)_A + \sum \text{C.W. moments due to surface loads from } A \text{ to } E\text{, taken about } E.$$

$$(\phi)_L = \underbrace{\cancel{(\phi)_A}}_{0} + (3qh)(2h) - (2hq)(h) = 4qh^2$$

<u>Step II</u> Express ϕ at fictitious nodes (16–26) in terms of internal nodes, using Eq (14)

$$\phi_{16} = \phi_1\,;\quad \phi_{17} = \phi_2\,;\quad \phi_{18} = \phi_3\,;\quad \phi_{19} = \phi_{13}\,;\quad \phi_{20} = \phi_{14}\,;\quad \phi_{21} = \phi_{15}$$

$$\phi_{22} = \phi_3 - 2h\left(\frac{\partial\phi}{\partial x}\right)_I = \phi_3 - 6qh^2\,;\quad \phi_{23} = \phi_6 - 6qh^2\,;\quad \phi_{24} = \phi_9 - 6qh^2$$

$$\phi_{25} = \phi_{12} - 6qh^2\,;\quad \phi_{26} = \phi_{15} - 6qh^2$$

<u>Step III</u> Formulate $\nabla^4\phi = 0$ for internal nodes (1–15), using Eq (10)

$$20\phi_1 - 8\Big(2\phi_2 + \phi_4 + \underbrace{\phi_M}_{4.5qh^2}\Big) + 2\Big(2\phi_5 + 2\underbrace{\phi_L}_{4qh^2}\Big) + \Big(2\phi_3 + \phi_7 + \underbrace{\phi_{16}}_{\phi_1}\Big) = 0$$

$$\Rightarrow 21\phi_1 - 16\phi_2 + 2\phi_3 - 8\phi_4 + 4\phi_5 + \phi_7 - 20qh^2 = 0.$$