STRAIN ANALYSIS.

(Deformation analysis).

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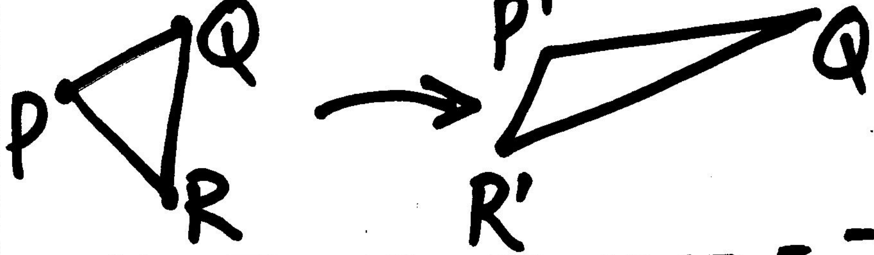
- Kinematics & Nonlinear Strain Tensor.
- Physical interpretation of Strain Tensor Components.
- Change in angle between two arbitrary line elements.
- Strain Transformations.
- Principal Strains & axes.
- Linear Strain Theory.
- Analogies with stress
- Infinitesimal rotation, relative displ, p-strain (Linear).
- Linear Cubical Dilatation.



Total motion = STRAINING + RB Motion

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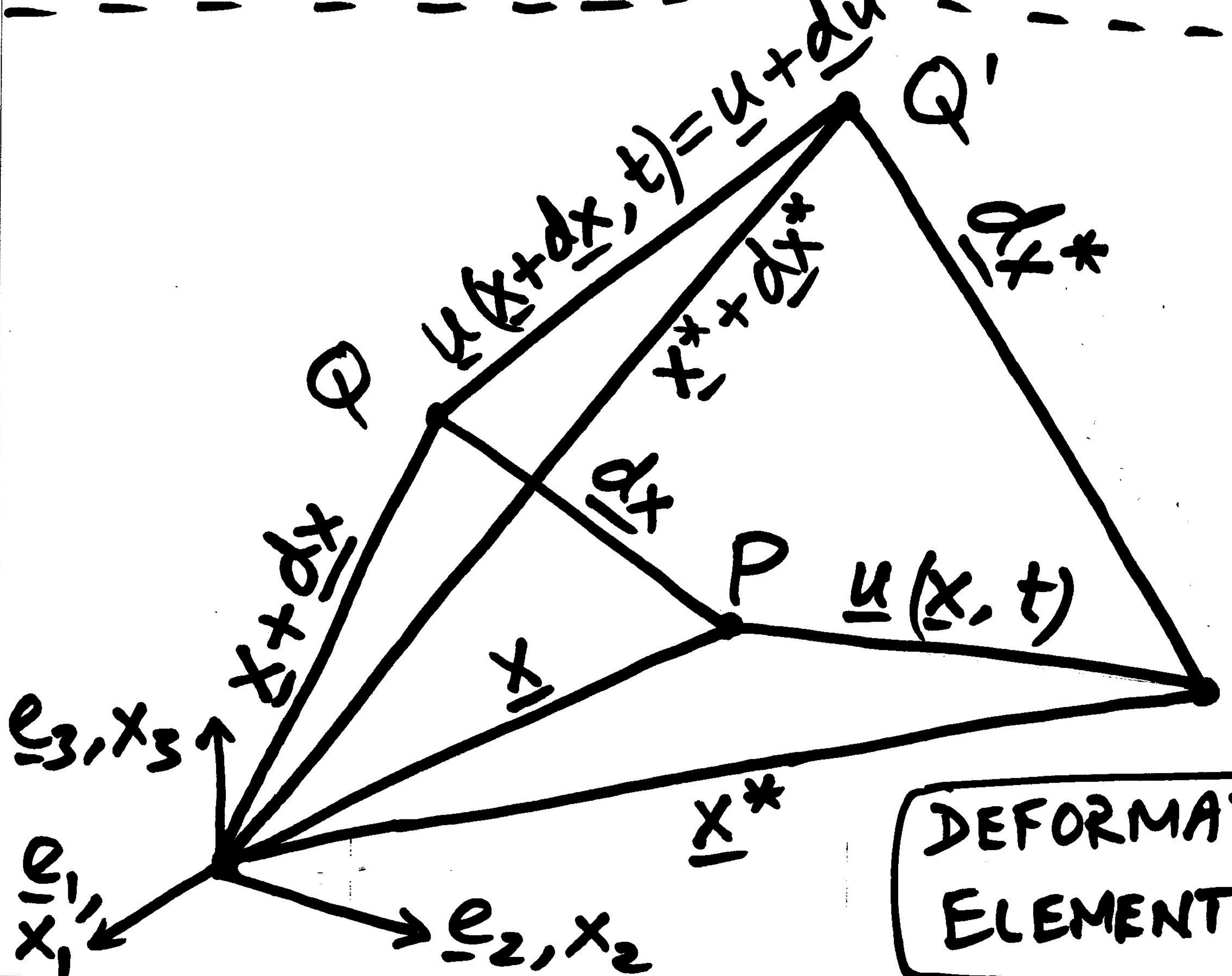

$\underline{X} \equiv \underline{r}$ = p.v. (before def)
$= (x_1, x_2, x_3)$
$= x_i$

$\underline{u}$ = displacement
$= (u_1, u_2, u_3)$
$= u_i$

DEFORMATION OF LINE ELEMENT PQ AT AN INSTANT.

$X^* =$ P.V. after deformation

$$\underset{\sim}{x}^*[\underset{\sim}{x}, \underset{\sim}{u}, t] = \underset{\sim}{x} + u[\underset{\sim}{x}, t]$$

$$= x_i^*[x_1, x_2, x_3] = x_i + u_i[x_1, x_2, x_3, t]$$

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STRETCH RATIO $\triangleq (P'Q')^2 - (PQ)^2 = (dx^*)^2 - (dx)^2$

$$= dx_i^* dx_i^* - dx_i dx_i$$

$$= x^*_{i,j} x^*_{i,k} dx_j dx_k - dx_i dx_i$$

$$x^*_{i,j} = \frac{\partial x_i^*}{\partial x_j}$$

$$= (\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) dx_j dx_k - dx_i dx_i$$

$$x_{i,j} = \delta_{ij}$$

STRETCH $= \underbrace{(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j})}_{\to 2e_{ij}} dx_i dx_j$

$$= 2e_{ij} dx_i dx_j$$

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where we used simplifications

$$\delta_{ij}\,\delta_{ik}\,dx_j\,dx_k = dx_i\,dx_i$$

$$\delta_{ij}\,u_{i,k}\,dx_j\,dx_k = u_{j,\underset{i}{\cancel{k}}}\,dx_j\,dx_{\underset{i}{\cancel{k}}}$$

$$\delta_{ik}\,u_{i,j}\,dx_j\,dx_k = u_{i,j}\,dx_j\,dx_i$$

$$u_{\underset{k}{\cancel{i}},j}\,u_{\underset{k}{\cancel{i}},\underset{i}{\cancel{k}}}\,dx_j\,dx_{\underset{i}{\cancel{k}}} = u_{k,i}\,u_{k,j}\,dx_i\,dx_j$$

Alternatively can do $dx_i^* = dx_i + du_i$
$= dx_i + u_{i,j}\,dx_j$
and get same result.

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$$M = \text{magnification factor}$$

$$= \frac{1}{2}\,\frac{(P'Q')^2-(PQ)^2}{(PQ)^2} = \frac{1}{2}\,\frac{(dx^*)^2-(dx)^2}{(dx)^2}$$

$$= e_{ij}\,n_i\,n_j \qquad \text{where } \frac{dx_i}{dx} = n_i \text{, } \frac{dx_j}{dx} = n_j$$

unit normal along PQ

$$= \frac{1}{2}\left[\left(\frac{dx^*}{dx}\right)^2 - 1\right]$$

$$= \frac{1}{2}\left[(1+\varepsilon_E)^2 - 1\right]$$

$$= \varepsilon_E + \frac{\varepsilon_E^2}{2}$$

Here ε_E is engg extensional strain (physics!)

Summary

$$\boxed{M = \frac{1}{2}\,\frac{(dx^*)^2-(dx)^2}{(dx)^2} = e_{ij}\,n_i\,n_j = \varepsilon_E + \frac{\varepsilon_E^2}{2}}$$

where,

$$\varepsilon_E = \frac{dx^*-dx}{dx}, \qquad e_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}\right]$$

STRAIN-DISPLACEMENT RELATION.

(i) e_{ij} → strain tensor.
→ measure of straining

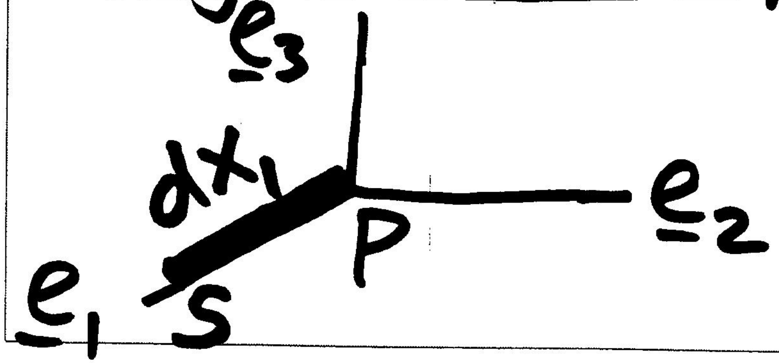
$$dx^* \neq dx \Rightarrow e_{ij} \neq 0$$

$= e_{ji}$ (symmetric by definition).

(ii) R.B motion does not affect M

(iii) M analogous to N (normal stress).
(compare with $N = \sigma_{ij} n_i n_j$)

Physical interpretation of e_{ij} (nonlinear).



Normal strains (diagonal comp's)

Consider element PS along x_1 axis.

$$\varepsilon_E = \frac{dx^* - dx}{dx} = \frac{(dx^*)^2 - (dx)^2}{(dx)^2(\varepsilon_E + 2)}$$

$$= \frac{2e_{ij}n_in_j}{(\varepsilon_E+2)} = \frac{2e_{11}}{(\varepsilon_{E1}+2)}$$

← used $\underline{n} = (1,0,0)$

i.e $n_1 = 1, n_2 = n_3 = 0$

$\varepsilon_E \to \varepsilon_{E1}$

$\because$ element in 1-dir.

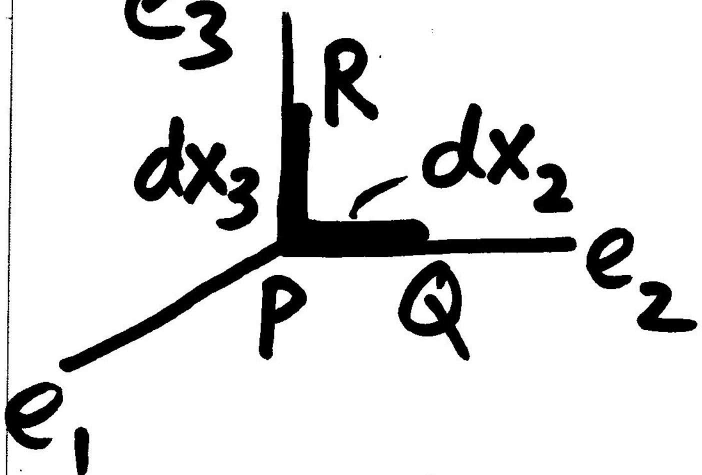
$$\Rightarrow \boxed{\varepsilon_{E\underline{1}} = \sqrt{1+2e_{11}} - 1} \rightarrow (2)$$

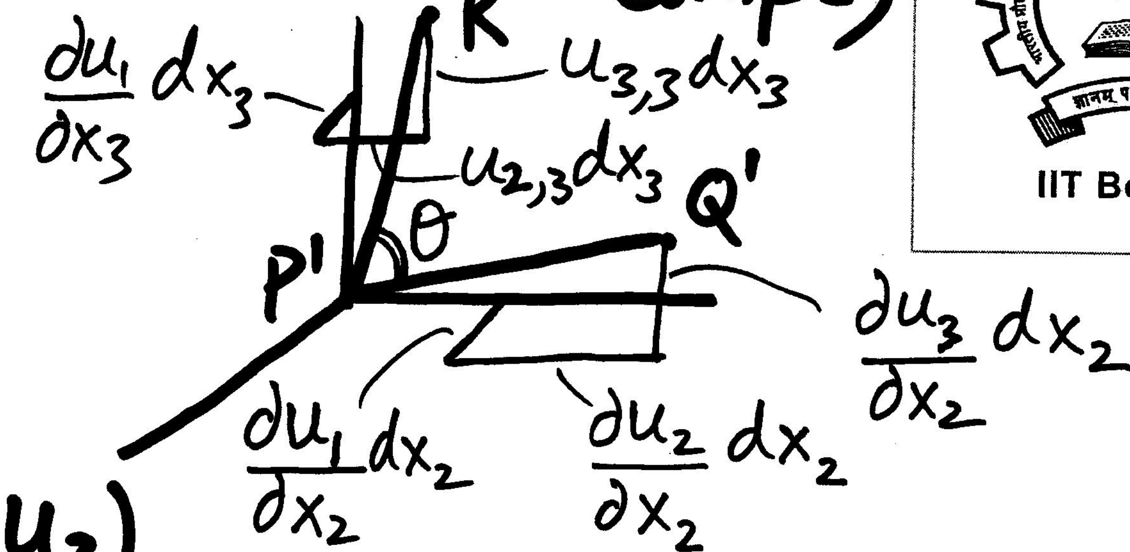
$$M_1 = \varepsilon_{E1} + \frac{\varepsilon_{E1}^2}{2} = e_{11}$$

similarly for e_{22}, ε_{E2}

e_{33}, ε_{E3}

Shear strains (off-diagonal comp's)





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$$\underline{u} = \underline{u}_P = (u_1, u_2, u_3)$$

$$\underline{P'Q'} \cdot \underline{P'R'} = (P'Q')(P'R')\cos\theta = (\varepsilon_{E2}+1)_P dx_2 (\varepsilon_{E3}+1)_P dx_3 \cdot \cos\theta$$

$$= \left(\sqrt{1+2e_{22}}\sqrt{1+2e_{33}}\right)_P dx_2 \, dx_3 \cos\theta \longrightarrow (a)$$

Also

$$\underline{P'Q'} \cdot \underline{P'R'} = \underline{dx}^*\big|_{PQ} \cdot \underline{dx}^*\big|_{PR} = dx_i^*\big|_{PQ} \, dx_i^*\big|_{PR}$$

$$= \left(\frac{\partial x_i^*}{\partial x_j}\right)_P dx_j\big|_{PQ} \left(\frac{\partial x_i^*}{\partial x_k}\right)_P dx_k\big|_{PR}$$

use $dx_j\big|_{PQ} = (0, dx_2, 0)$, $dx_k\big|_{PR} = (0, 0, dx_3)$

$$\underline{P'Q'}\cdot\underline{P'R'} = \left(\frac{\partial x_i^*}{\partial x_2}\,\frac{\partial x_i^*}{\partial x_3}\right)_P dx_2\,dx_3$$

Recall, p. 3,

$$x^*_{i,j}\,x^*_{i,k}\,dx_j\,dx_k - dx_{\cancel{i}\,j}\,\overbrace{dx_{\cancel{i}\,j}}^{=dx_k\delta_{jk}} = 2e_{\cancel{i}\,k\,j}\,dx_{\cancel{i}\,k}\,dx_j$$

$$\left(x^*_{i,j}\,x^*_{i,k} - \delta_{jk}\right)dx_j\,dx_k = 2e_{jk}\,dx_j\,dx_k$$

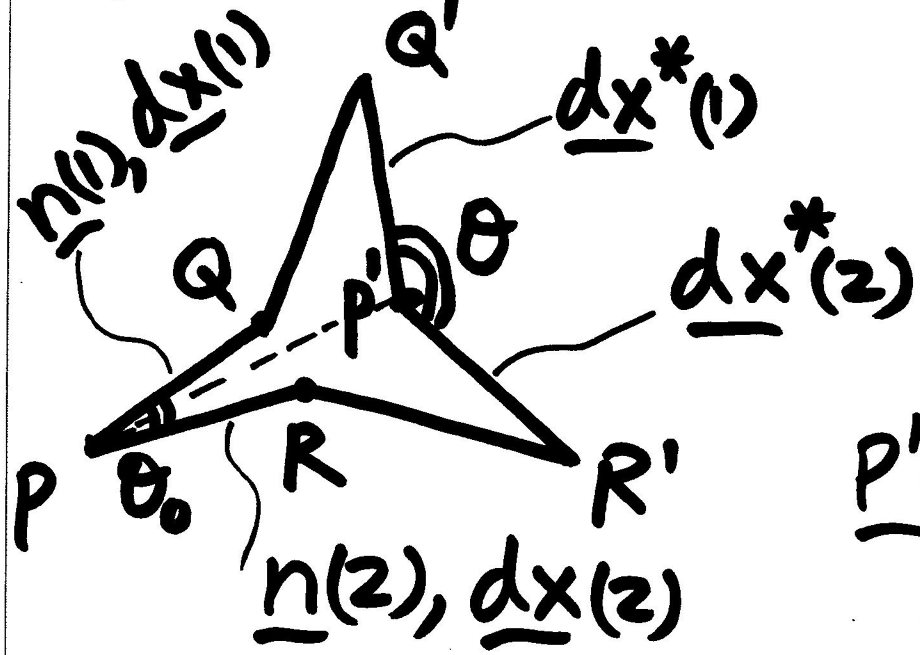
$$\underline{P'Q'}\cdot\underline{P'R'} = \left(x^*_{i,2}\,x^*_{i,3}\right)_P dx_2\,dx_3 = 2\left(e_{23}\right)_P dx_2\,dx_3 \rightarrow (b)$$

(a), (b) $\Rightarrow$

$$\boxed{\begin{aligned} 2e_{23} &= \sqrt{1+2e_{22}}\,\sqrt{1+2e_{33}}\,\cos\theta \\ &= (1+\varepsilon_{E2})(1+\varepsilon_{E3})\cos\theta \end{aligned}} \rightarrow (3)$$

similarly for e_{13}, e_{12}

Angle between two line elements after deformation — Generalization of shear strain interpretation.



Consider two line elements (1) → PQ, (2) → PR before & after deformation.

$$\underline{P'Q'}\cdot\underline{P'R'} = \underline{dx}^*(1)\cdot\underline{dx}^*(2)$$

$$= dx^*_i(1)\,dx^*_i(2)$$

$$= \left(dx_i + u_{i,j}\,dx_j\right)_{(1)}\left(dx_i + u_{i,k}\,dx_k\right)_{(2)}$$

$$= dx_i(1)\,dx_i(2) + dx_i(1)\,dx_j(2)\left[u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}\right]_P \rightarrow (A)$$

Also $\underline{P'Q'} \cdot \underline{P'R'} = \sqrt{(dx^*(1))^2}\sqrt{(dx^*(2))^2}\cos\theta$

see ① p.4

$$= \sqrt{dx_i(1)\,dx_i(1) + 2e_{ij}\,dx_i(1)\,dx_j(1)}$$
$$* \sqrt{dx_p(2)\,dx_p(2) + 2e_{pq}\,dx_p(2)\,dx_q(2)}$$
$$* \cos\theta \rightarrow \text{(B)}$$

÷ (A) & (B) by $dx(1)\,dx(2)$, note that $n_i^{(1)} = \dfrac{dx_i(1)}{dx(1)}$

$n_j(2) = \dfrac{dx_j(2)}{dx(2)}$, $n_i(1)n_i(1) = n_j(2)n_j(2) = 1$,

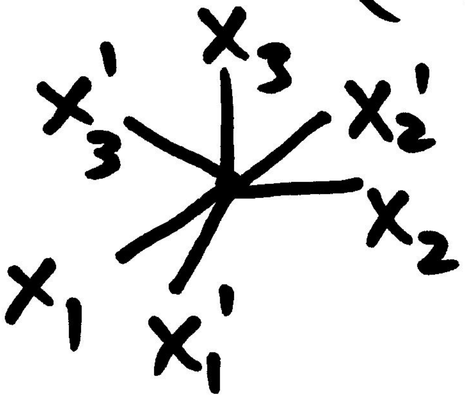
$n_i(1)\,n_i(2) = \cos\theta_0$, you get

all strain comps at P

$$\boxed{\cos\theta = \frac{\cos\theta_0 + 2n_i(1)\,n_j(2)\,e_{ij}}{\sqrt{1+2n_r(1)\,n_s(1)\,e_{rs}}\,\sqrt{1+2n_p(2)\,n_q(2)\,e_{pq}}}} \rightarrow ④$$

Strain Transformations.

(ie, transf of coords.)



Consider element along x_1'.

(1) p4 $M = e_{ij} n_i n_j = e_{ij} a_{1i} a_{1j}$ $\quad n_i = a_{1i}$

$\because \underline{n} = (a_{11}, a_{12}, a_{13})$

Also $M = e'_{ij} n'_i n'_j = e'_{11}$

$\because \underline{n}' = (1, 0, 0)$

$$\Rightarrow \begin{array}{c} e'_{11} \\ 22 \\ 33 \end{array} = e_{ij} \begin{array}{c} a_{1i} \\ 2 \\ 3 \end{array} \begin{array}{c} a_{1j} \\ 2 \\ 3 \end{array} \rightarrow \text{(i)}$$

(3) p.9 gives θ between elements along x_2', x_3' axes after deformation as

$$\cos\theta = \frac{2e'_{23}}{\sqrt{1+2e'_{22}}\ \sqrt{1+2e'_{33}}} \rightarrow \text{(ii)}$$

This is in terms of strains referred to x_i' system. We can get $\cos\theta$ in terms of strains referred to x_i system by using (4) p. 11. Here, $\theta_0 = 90^\circ$, line (1) is x_2' axis so $n_i(1) = a_{2i} = (a_{21}, a_{22}, a_{23})$, line (2) is x_3' axis, so $n_j(2) = a_{3j} = (a_{31}, a_{32}, a_{33})$.

$$\Rightarrow \cos\theta = \frac{\underset{0}{\cancel{\cos 90}} + 2 a_{2i} a_{3j} e_{ij}}{\sqrt{1 + 2\underbrace{a_{2r} a_{2s} e_{rs}}_{e'_{22}}}\ \sqrt{1 + 2\underbrace{a_{3p} a_{3q} e_{pq}}_{e'_{33}}}} \rightarrow \text{(iii)}$$

$\boxed{\text{see (i) p. 12}} \rightarrow$

Compare (ii), (iii), p. 12, 13,

$$e'_{23} = a_{2i} a_{3j} e_{ij} \rightarrow \text{(iv)}$$

$$\begin{array}{ccc} 12 & 1 & 2 \\ 13 & 1 & 3 \end{array}$$

Combine transf. law for normal & shear strains (ie (i), (iv), p 12, 14)

$$\boxed{e'_{ij} = a_{ir} a_{js} e_{rs}} \rightarrow (5)$$

<u>STRAIN TRANSF LAW</u>

<u>Same as STRESS TRANSF LAW.</u>

So e_{ij} is 2^{nd} order tensor.

So e_{ij} analogous (mathematically) to σ_{ij}

Both are symmetric.

Principal Strains & Directions.

Eigenvalue problem (due to σ_{ij} e_{ij} analogy) is

$$⑥ \leftarrow (e_{ij} - \lambda \delta_{ij}) n_j = 0 = (\underline{\underline{e}} - \lambda \underline{\underline{I}}) \underline{n}$$

This gives p-strains $\lambda = M(1), M(2), M(3)$ (analogy $N(1), N(2), N(3)$) and p-directions $n(1), n(2), n(3)$, ie a coord system x_1', x_2', x_3' such that e_{ij} gets diagonlized, ie $e'_{ij} = 0$ for $i \neq j$.

Analogies: M (N) stationary at p-axes of strain (stress). Also p-axes orthogonal

From (1) p. 4, $M = \varepsilon_E + \frac{\varepsilon_E^2}{2}$

$\Rightarrow dM = d\varepsilon_E(1+\varepsilon_E)$

Now ε_E = physical quantity, ie engg extensional strain

$$= \frac{dx^* - dx}{dx} = \frac{dx^*}{dx} - 1$$

so $(1+\varepsilon_E) > 0 \because dx^*, dx$ are lengths (+ve).

So, M stationary $\Rightarrow dM = 0 = d\varepsilon_E \Rightarrow \varepsilon_E$ stationary

Thus M(1), M(2), M(3) stationary values of magnification factor correspond to $\varepsilon'_{E1}, \varepsilon'_{E2}, \varepsilon'_{E3}$, ie stationary values of engg ext str which act along x'_1, x'_2, x'_3 ie $\underline{n}(1), n(2), n(3)$ axes

Thus CE (Characteristic Eqn) is

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0$$

$J_1 = e_{ii}$, $J_2 = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - e_{12}^2 - e_{23}^2 - e_{13}^2$

$J_3 = \det|\underline{\underline{e}}|$

$J_1, J_2, J_3 \rightarrow$ strain invariants (analogous to I_1, I_2, I_3)

$\lambda(1) = M(1)$, etc, (analogous to $N(1)$, etc).

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Linear Theory — Small displacement gradient theory.

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$\boxed{u_{i,j} \ll 1} \Rightarrow \boxed{e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})}$

(1A)

$\Rightarrow du_i = u_{i,j}\, dx_j = \text{small} \Rightarrow dx^* \simeq dx$

$\Rightarrow M = \frac{1}{2}\frac{(dx^*)^2 - (dx)^2}{(dx)^2} \cong \frac{(dx^* - dx)\, 2dx}{2(dx)^2}$

$= \varepsilon_E$ (can also get directly from (1) p.4, putting $\varepsilon_E \ll 1$).

$= e_{ij} n_i n_j$

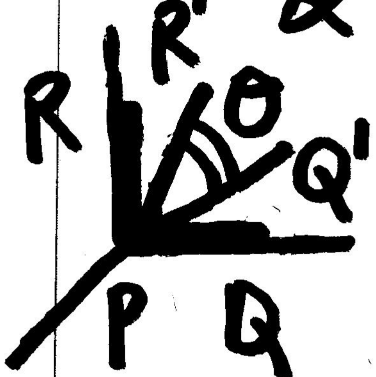
Physical interpretation of e_{ij} (Linear)

$u_{i,j} \ll 1 \Rightarrow e_{ij} \ll 1$

(2) p.7 $\Rightarrow \varepsilon_{E1} = e_{11}$

(can also get from $\varepsilon_E = e_{ij} n_i n_j$ valid for linear theory with $\underline{n} = \{1, 0, 0\}$

(3) p.9 $\Rightarrow e_{23} = \frac{1}{2}\cos\theta = \frac{\sin(\overbrace{\pi/2 - \theta}^{=\alpha})}{2} = \frac{\alpha}{2}$



α = change in angle between PQ & PR originally along x_2, x_3 axes, resp'ly = small $\because u_{i,j} \ll 1$ ie. $e_{ij} \ll 1$

e_{23} = tensorial shear strain

Note: $e_{23} = \frac{\alpha}{2} = \frac{\gamma_{23}}{2}$

γ_{23} = engg shear strain.

Note:

$$\gamma_{ij} = 2e_{ij}, \quad i \neq j$$

$e_{ij} \rightarrow$ Tensorial strains. They transform as a tensor for all i, j.

$\gamma_{ij} \rightarrow$ Engg shear strains

$\gamma_{ij} = e_{ij}, \ i \neq j \rightarrow$ one transf law.

$\gamma_{ij} = 2e_{ij}, \ i \neq j \rightarrow$ another transf law.

Analogies with $\underline{\sigma}$

(i) $\hat{e}_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}\, e_{mm} \rightarrow (7)$

deviatoric = strain − spherical.

(ii) Pure shear state of strain − iff $e_{mm} = 0$

Physically it means that at point p there exist three mutually perpendicular directions $\underline{n}(1), \underline{n}(2), \underline{n}(3)$ along which no engineering extensional strain occurs, ie e_{E1}, e_{E2}, e_{E3}, respty are zero.

Infinitesimal Rotation, Relative Displacement, P-strains — LINEAR

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$$du_i = u_{i,j}\,dx_j = \frac{1}{2}\Big[\underbrace{(u_{i,j}+u_{j,i})}_{e_{ij}\text{ for Linear}} + \underbrace{(u_{i,j}-u_{j,i})}_{\omega_{ij} = -\omega_{ji}}\Big]dx_j$$

$$= [e_{ij} + \omega_{ij}]\,dx_j$$

RB motion only $\Rightarrow e_{ij} = 0 \Rightarrow du_i = \omega_{ij}\,dx_j$

$$du_i = \underbrace{\begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}}_{\text{Linear Rot Tensor.}} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \underline{\omega} \times \underline{dx}$$

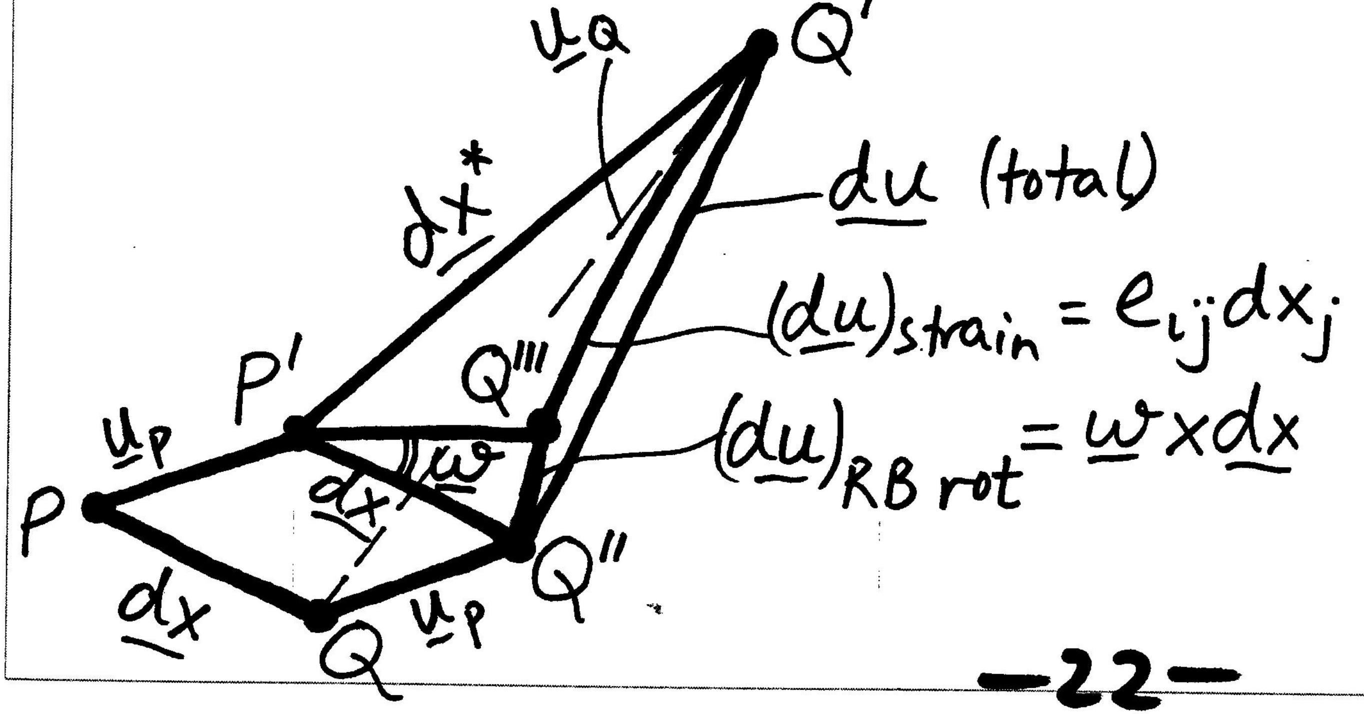
where

$\underline{\omega} = \{\omega_{32}, \omega_{13}, \omega_{21}\}^T$

Linear Rot. vector

ie, due to skew symmetry of $\underline{\underline{w}} \equiv w_{ij}$ it can be written as $\underline{w} \equiv w_i$ $\because$ only three independent components.

So $\underline{w} \times \underline{dx}$ = Rot. comp. of. $\underline{du}$



So relative motion $\underline{du}$ = RB motion + Straining.

Another explanation of P-strains: (LINEAR)

Consider straining only.

$$e_i \triangleq \frac{du_i}{dx} = \frac{e_{ij}\,dx_j}{dx} = e_{ij} n_j \longrightarrow (8)$$

= relative displ per unit length of element due to straining only.

In general e_i neither along PQ nor P'Q', it is along Q'''Q'.

We seek orientation of (element) PQ, ie n_i, such that the element remains perpendicular (after deformation) to the plane of particles that it was originally (before def.) perpendicular to. Since RB motion is excluded, this means that e_i and n_i are in same direction,

$$e_i = \gamma n_i \Rightarrow e_{ij} n_j = \gamma \delta_{ij} n_j$$

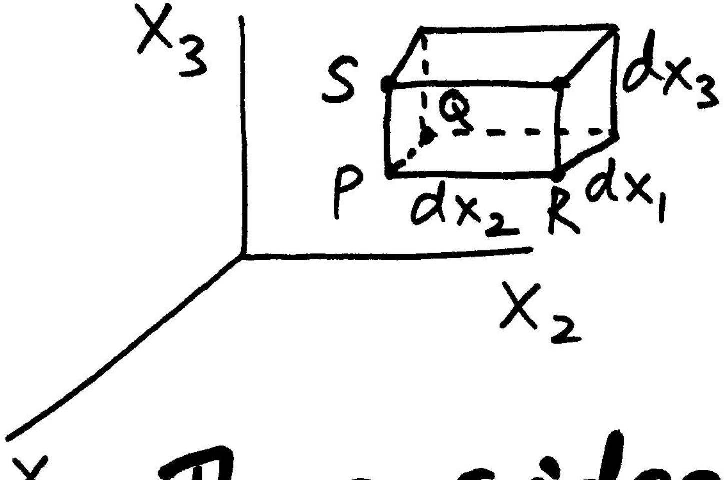
$$(e_{ij} - \gamma \delta_{ij}) n_j = 0$$

Note that this does not mean that direction PQ remains same after deformation, since RB motion will be present in general. All it means is that dir. PQ remains

Same when RB motion filtered out. So with RB motion included we would have $P'Q'''$ and $Q'''Q'$ and $P'Q'$ having same direction if PQ is a principal element (ie directed along p-axis).

Note that e_i & t_i are analogous.

Linear Cubical Dilatation (Volumetric Strain).



Consider rectangular parallelopiped with sides oriented along p-coord axes x_1, x_2, x_3.

Thus sides remain perpendicular after deformation since they lie along p-axes.

$$dV^* = dx_1^* \, dx_2^* \, dx_3^* = (1+\varepsilon_{E1})(1+\varepsilon_{E2})(1+\varepsilon_{E3}) * dx_1 \, dx_2 \, dx_3$$

$$D \triangleq \frac{dV^* - dV}{dV} = \varepsilon_{E1} + \varepsilon_{E2} + \varepsilon_{E3}$$

$$D = \varepsilon_{E1} + \varepsilon_{E2} + \varepsilon_{E3} \rightarrow \text{engg strains}$$

$$= e_{11} + e_{22} + e_{33} \rightarrow \text{linear normal strains}$$

$$= \gamma(1) + \gamma(2) + \gamma(3) \rightarrow \text{p-strains}$$

9

e_{kk}

Here we have linearized by neglecting products of engg strains.

NOTE: $\hat{D}$ = cubical dilatation for Deviatoric strain tensor $= 0 \because \hat{e}_{ii} = 0$

So Deviatoric part of strain tensor causes no volumetric strain, it causes only distortion strain (shear). Similarly spherical part causes only vol. strain and no distortion.