Lecture 1

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STRESS ANALYSIS

- Surface force, stress/traction vector
- Stress tensor
- Body forces & body moments
- Relating stress vector with stress tensor
- Symmetry of stress tensor
- Normal & Shearing stresses.

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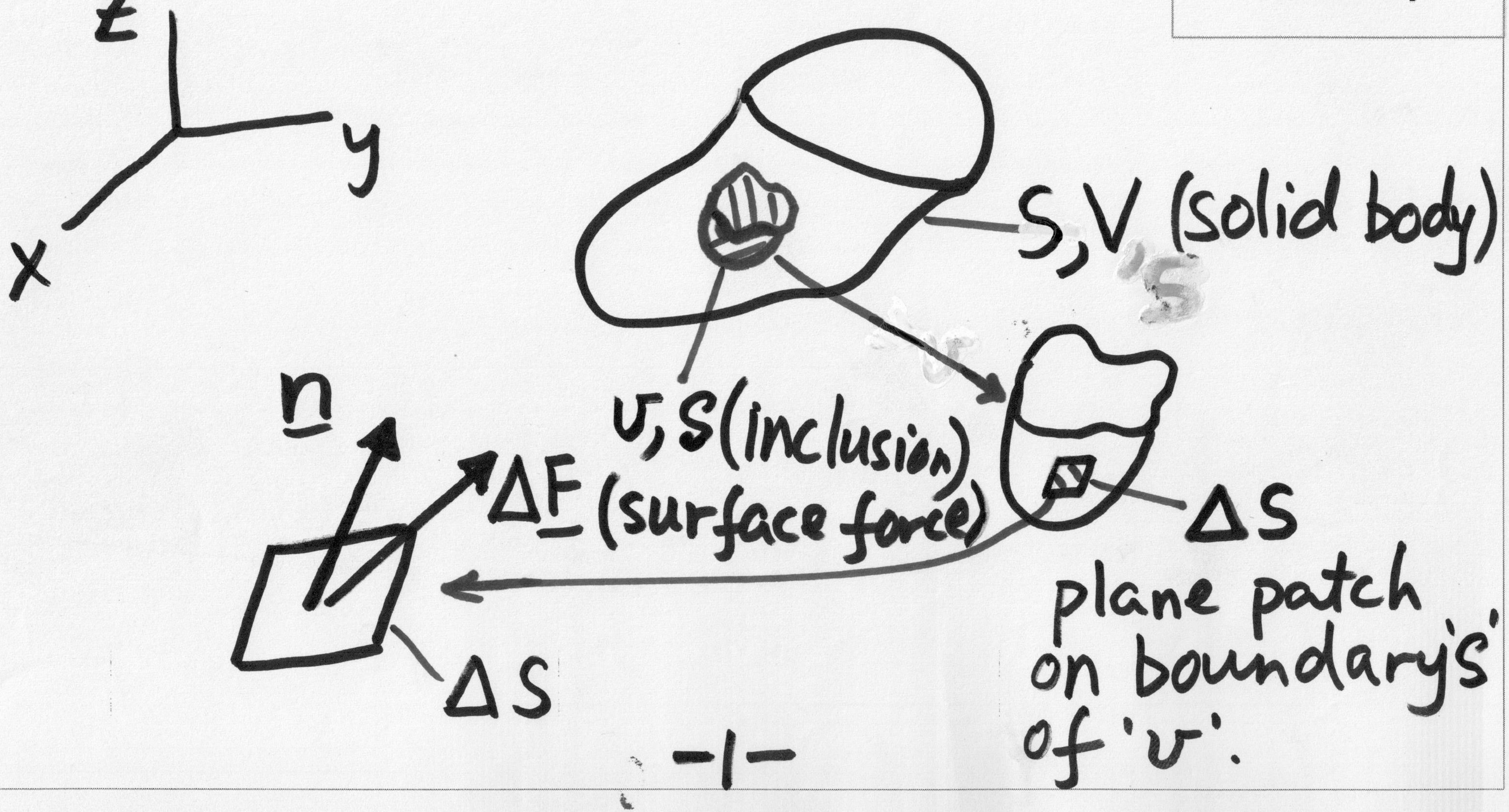
Surface force, Stress/Traction vector:



$$\boxed{\underline{t} = \lim_{\Delta S \to 0} \frac{\Delta \underline{F}}{\Delta S}}$$

↳ definition of stress/traction vector

Stress vector depends on :

1) Location of ΔS, i.e., $P(x, y, z)$

2) Orientation of ΔS, i.e, $\underline{n}$

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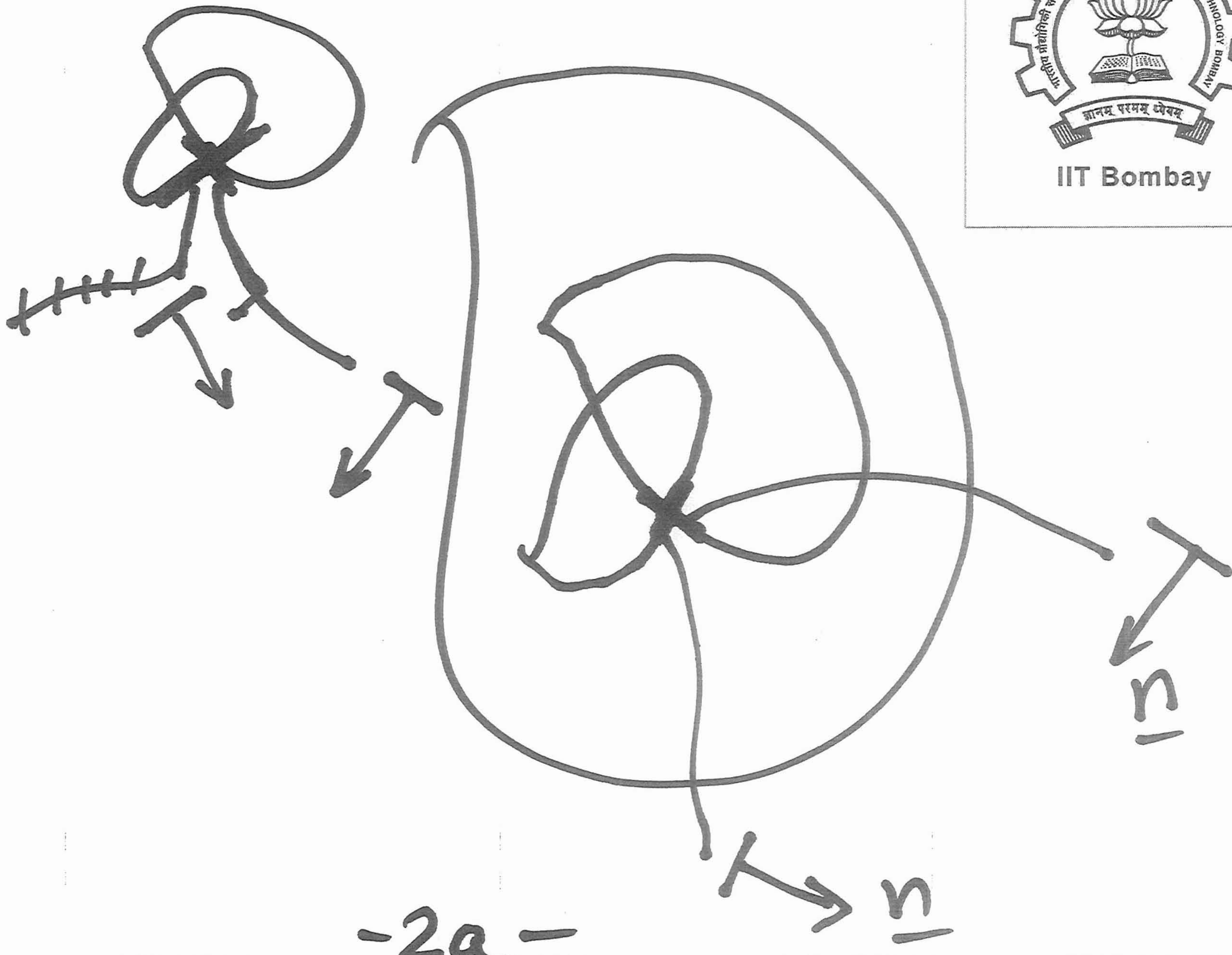
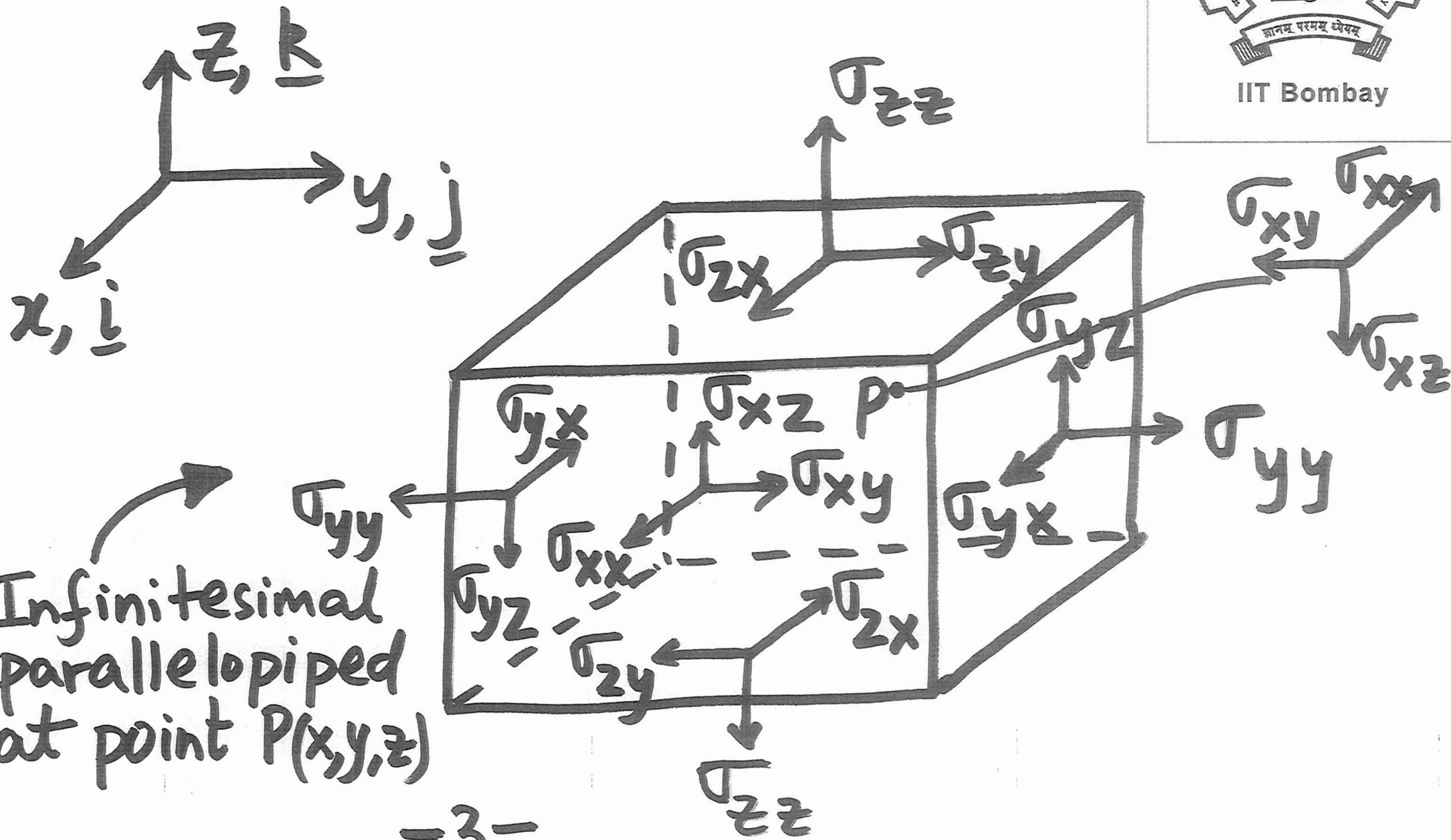


-2a-

Stress Tensor :



Infinitesimal parallelopiped at point P(x,y,z)

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$$\left.\begin{aligned}
\underline{t}_{(\underline{j})} &= \sigma_{yx}\,\underline{i} + \sigma_{yy}\,\underline{j} + \sigma_{yz}\,\underline{k} \\
\underline{t}_{(\underline{k})} &= \sigma_{zx}\,\underline{i} + \sigma_{zy}\,\underline{j} + \sigma_{zz}\,\underline{k} \\
\underline{t}_{(\underline{i})} &= \sigma_{xx}\,\underline{i} + \sigma_{xy}\,\underline{j} + \sigma_{xz}\,\underline{k}
\end{aligned}\right\} \rightarrow \star$$

$$\begin{Bmatrix} \underline{t}_{(\underline{i})} \\ \underline{t}_{(\underline{j})} \\ \underline{t}_{(\underline{k})} \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{Bmatrix} = \underline{\underline{\sigma}} \begin{Bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{Bmatrix}$$

$\hookrightarrow$ Stress Tensor $\underline{\underline{\sigma}}$ at $P(x, y, z)$

$\underline{\underline{\sigma}}$ depends on:

1) Coordinates (x, y, z) of P
2) Loads, displacements applied on body.

$\underline{\underline{\sigma}}$ does not depend on $\underline{n}$, ie orientation of plane at P.

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Body forces, Body moments

- Surface force $\Delta \underline{F}$ due to direct contact across ΔS
- Body forces, body moments due to action at a distance (eg. gravitational, magnetic field).
- Bf, Bm, proportional to mass/vol.

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$$\underline{f}_b = \lim_{\Delta m \to 0} \frac{\Delta \underline{F}_b}{\Delta m} = \lim_{\Delta V \to 0} \frac{1}{\rho} \frac{\Delta \underline{F}_b}{\Delta V}$$

$$= \frac{1}{\rho} \tilde{\underline{f}}_b$$

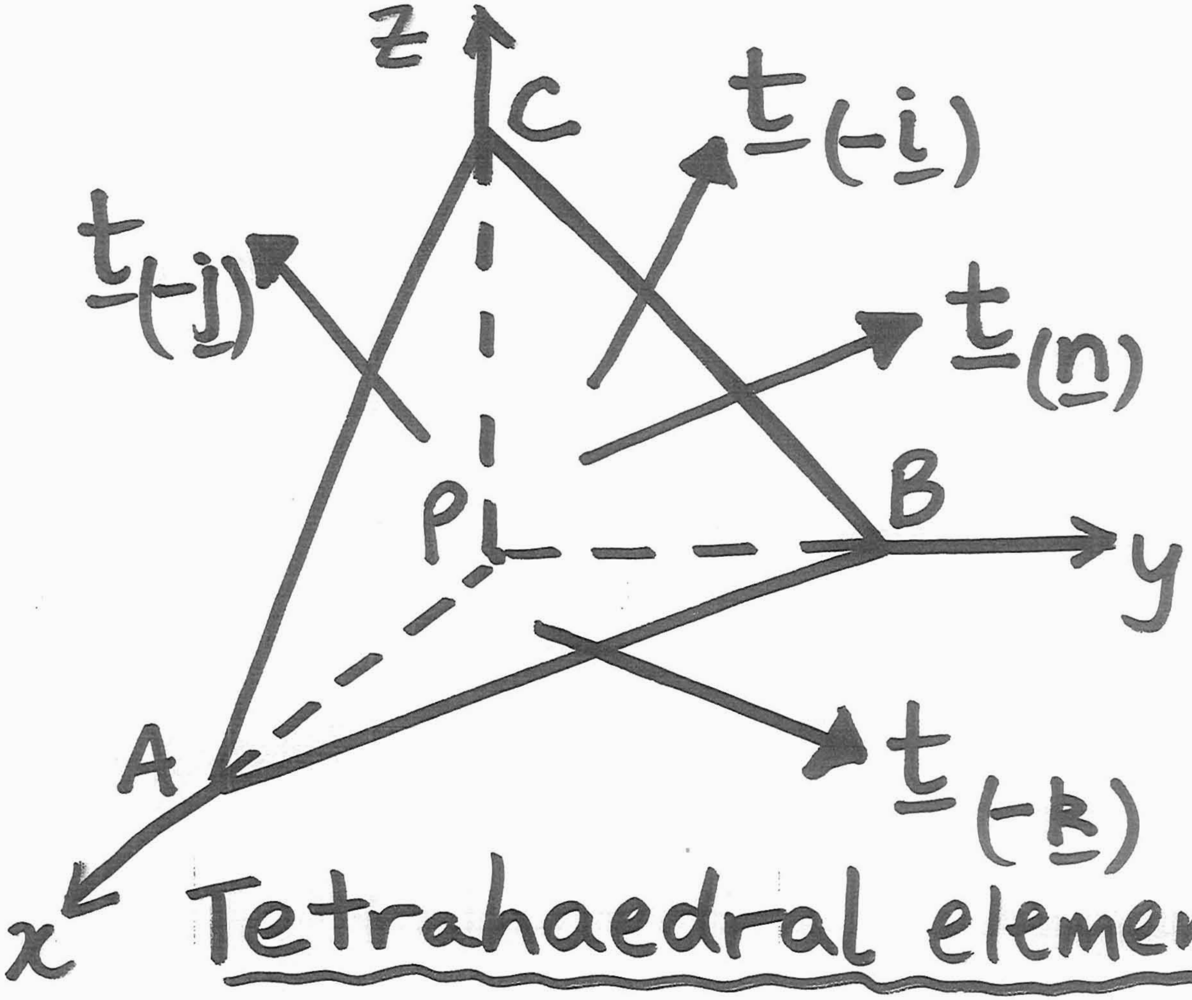
$$\underline{m}_b = \lim_{\Delta m \to 0} \frac{\Delta \underline{M}_b}{\Delta m} = \lim_{\Delta V \to 0} \frac{1}{\rho} \frac{\Delta \underline{M}_b}{\Delta V} = \frac{1}{\rho} \tilde{\underline{m}}_b$$

$\rho(x, y, z) \rightarrow$ density

$\underline{f}_b(x, y, z)$, $\tilde{\underline{f}}_b(x, y, z) \rightarrow$ BF per unit mass, vol

$\underline{m}_b(x, y, z)$, $\tilde{\underline{m}}_b(x, y, z) \rightarrow$ BM " " " "

Relating $\underline{t}$ with $\underline{\underline{\sigma}}$, $\underline{n}$ at P – Cauchy relation:



- $-\underline{i}$, $-\underline{j}$, $-\underline{k}$, $\underline{n}$, are normals to planes PBC, PAC, PAB, ABC, resply.

Tetrahaedral element at P.

• $\underline{t}_{(-\underline{i})}, \underline{t}_{(-\underline{j})}, \underline{t}_{(-\underline{k})}, \underline{t}_{(\underline{n})}$ are AVERAGE stress vectors on planes PBC, PAB, PAC, ABC, resply.

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Equilibrium of forces $\Rightarrow$

$$\underline{t}_{(-\underline{i})} \Delta PBC + \underline{t}_{(-\underline{j})} \Delta PAC + \underline{t}_{(-\underline{k})} \Delta PAB + \underline{t}_{(\underline{n})} \Delta ABC + \underline{\tilde{f}}_b * \frac{1}{3} h \Delta ABC = \underline{0}$$

vol of tetra

$\div$ by ΔABC

Note $\dfrac{\Delta PBC}{\Delta ABC} = n_x$,

$$\frac{\Delta PAC}{\Delta ABC} = n_y \,, \quad \frac{\Delta PAB}{\Delta ABC} = n_z$$

where,

$$\underline{n} = n_x \underline{i} + n_y \underline{j} + n_z \underline{k}$$

$$\Rightarrow \underline{t}_{(-\underline{i})} n_x + \underline{t}_{(-\underline{j})} n_y + \underline{t}_{(-\underline{k})} n_z$$

$$+ \underline{t}_{(\underline{n})} + \underline{\tilde{f}}_b \cdot \frac{1}{3} h = 0$$

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Let tetra shrink to P such that $\underline{n}$ unchanged.

$\Rightarrow h \to 0$, all $\underline{t}$'s are exact (not AVERAGE) at P.

Use $\underline{t}_{(-\underline{i})}$, $\underline{t}_{(-\underline{j})}$, $\underline{t}_{(-\underline{k})}$ in terms of components of $\underline{\underline{\sigma}}$ (see p.4, ★)

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$$\Rightarrow \boxed{\begin{aligned}\underline{t}_{(n)} = &(\sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z)\underline{i} \\ &+ (\sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z)\underline{j} \\ &+ (\sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z)\underline{k}\end{aligned}}$$

→ Cauchy relation, $\underline{t}$ in terms of $\underline{\underline{\sigma}}$, $\underline{n}$, at P.

let $\underline{i} \to \underline{e}_1$, $\underline{j} \to \underline{e}_2$, $\underline{k} \to \underline{e}_3$, $x \to 1$, $y \to 2$, $z \to 3$

$$\boxed{\underline{t}_{(n)} = \sum \sum \sigma_{ij} n_i \underline{e}_j} \to (1)$$

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Cauchy relation

$$\boxed{\begin{aligned} &\underline{t}_{(\underline{n})} = \sum_{j=1}^{3}\sum_{i=1}^{3} \sigma_{ij}\, n_i\, \underline{e}_j \\ &\text{or} \\ &\underline{t} = \underline{\underline{\sigma}}\,\underline{n} \quad \text{ie } \{t\} = [\sigma]\{n\} \end{aligned}} \rightarrow (1)$$

where $\{t\} = \{t_1 \; t_2 \; t_3\}^T = \{t_x \; t_y \; t_z\}^T$

$[\sigma]$ = stress tensor (P. 4)

$\{n\} = \{n_1 \; n_2 \; n_3\}^T = \{n_x \; n_y \; n_z\}^T$

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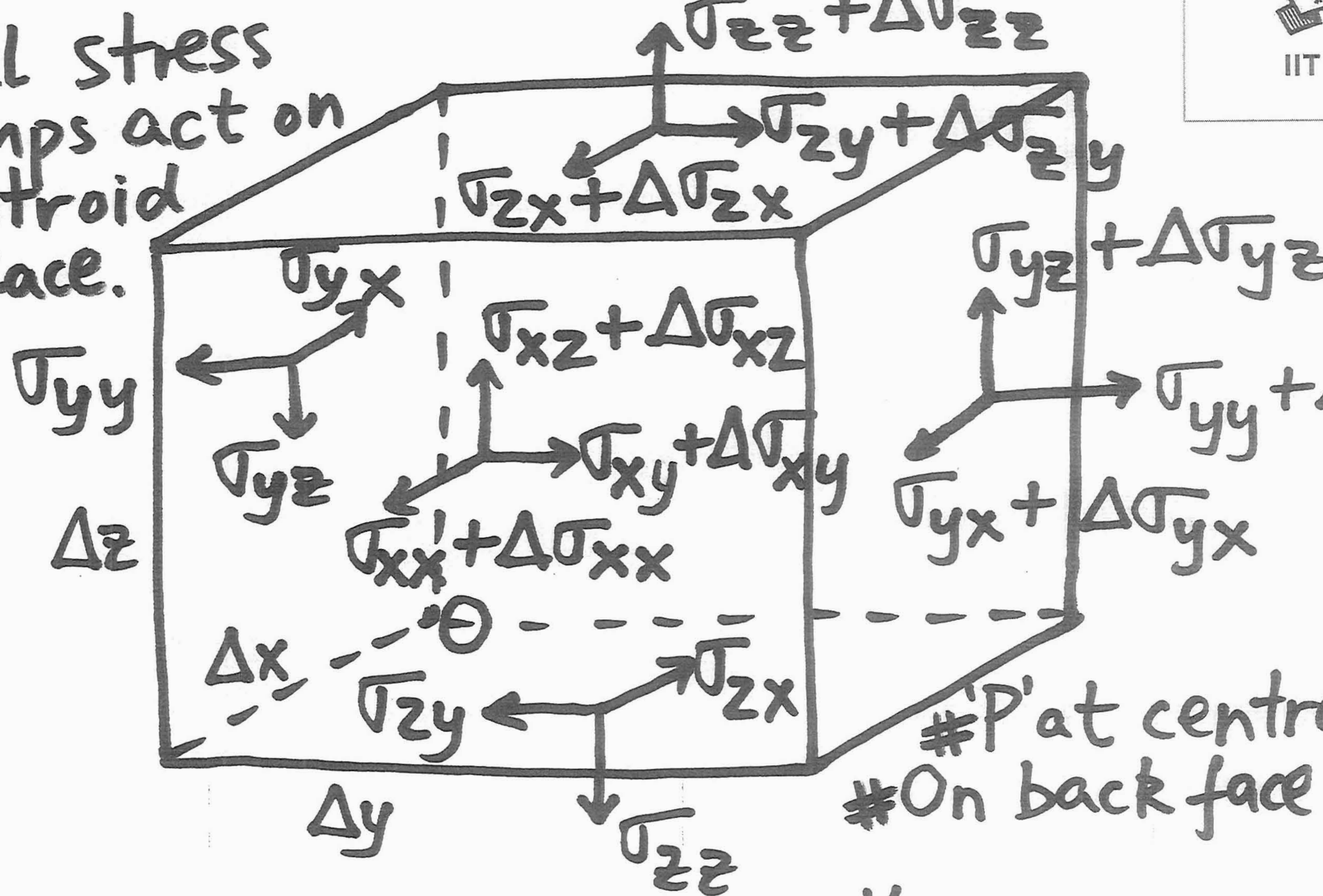
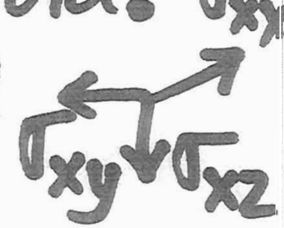
Symmetry of $\underline{\underline{\sigma}}$

All stress comps act on centroid of face.



'P' at centroid.

On back face



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Moment equilibrium:

$\sum \underline{M} = \underline{0}$ about P.

$= M_x \underline{i} + M_y \underline{j} + M_z \underline{k}$

M_x, M_y, M_z are moments about x, y, z, axes, resply, thru P.

$$M_x = \left(\sigma_{yz} + \frac{\partial \sigma_{yz}}{\partial y}\Delta y + \sigma_{yz}\right)\Delta x \Delta z \Delta y/2$$
$$-\left(\sigma_{zy} + \frac{\partial \sigma_{zy}}{\partial z}\Delta z + \sigma_{zy}\right)\Delta x \Delta y \Delta z/2$$
$$+\left(\tilde{m}_{bx} + \tilde{f}_{by}\,\alpha \Delta z + \tilde{f}_{bz}\,\beta \Delta y\right)\Delta x \Delta y \Delta z$$
$$= 0$$

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$\div \Delta x \Delta y \Delta z$

Neglect higher order terms

$(\tau_{yz,y}\Delta y; \tau_{zy,z}\Delta z; \bar{f}_{by} \propto \Delta z;$

(eqvt to shrink to P) $\bar{f}_{bz} \beta \Delta y)$

Assume no body moments, ie $\bar{m}_{bx} = 0$

$\Rightarrow \boxed{\tau_{yz} = \tau_{zy}}$

Similarly, $\boxed{\tau_{xy} = \tau_{yx}}$ → from $M_z = 0$

$\boxed{\tau_{xz} = \tau_{zx}}$ → from $M_y = 0$

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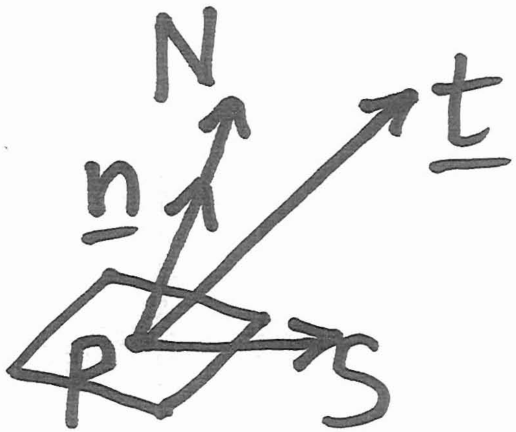
Normal (N) and Shear (S) stresses on plane ($\underline{n}$)



$$\underline{n} = n_x \underline{i} + n_y \underline{j} + n_z \underline{k}$$

$$\underline{t} = t_x \underline{i} + t_y \underline{j} + t_z k$$

(p.12)

$$\boxed{\begin{aligned} N &= \underline{t} \cdot \underline{n} \\ &= \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + \sigma_{zz} n_z^2 \\ &\quad + 2\sigma_{xy} n_x n_y + 2\sigma_{xz} n_x n_z + 2\sigma_{yz} n_y n_z \end{aligned}} \quad \rightarrow (2a)$$

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In index (ie $x \to 1$, $y \to 2$, $z \to 3$) notation

$$\boxed{N} = \underbrace{\left(\sum_{j,i=1}^{3} \sum \sigma_{ij} n_i \underline{e}_j \right)}_{\underline{t}} \cdot \underbrace{\left(\sum_{k=1}^{3} n_k \underline{e}_k \right)}_{\underline{n}} = \boxed{\sum_{i,j=1}^{3} \sum \sigma_{ij} n_i n_j} \to 2b$$

$\underline{t} = [\sigma]\{n\} \to$ column vector

$$\Rightarrow \boxed{N = \{n\}^T [\sigma] \{n\}} \to \text{Matrix-vector notation.}$$

$\to$ 2c

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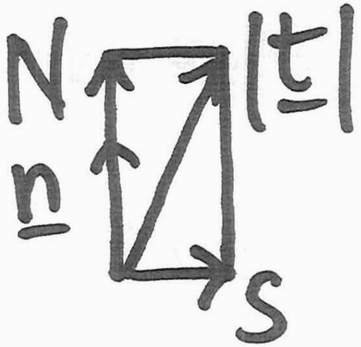

$$\boxed{S^2 = \left|\underline{t}_{(\underline{n})}\right|^2 - N^2}$$

($\underline{n}$ shown for clarity, can omit.).

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Ex 1 At $P(1,3,2)$

$$\underline{\underline{\sigma}} = \begin{bmatrix} x^2 & z & x \\ & xy & y \\ \text{symm} & & x^2 z \end{bmatrix}$$

On plane $\underline{n} = \frac{1}{\sqrt{3}}\underline{i} - \frac{1}{\sqrt{6}}\underline{j} + \frac{1}{\sqrt{2}}\underline{k}$

find $\underline{t}$, N, S. $\rightarrow$ use 1, 2, 3 (p13, 17–19)

$$\underline{t} = \{t\} = [\sigma]\{n\} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ -1/\sqrt{6} \\ 1/\sqrt{2} \end{Bmatrix} = \begin{Bmatrix} 0.47 \\ 2.05 \\ 0.77 \end{Bmatrix}$$

$\underline{t} = 0.47\underline{i} + 2.05\underline{j} + 0.77\underline{k}$

For N, can use 2a or 2b or 2c (p17, 18)

Using 2b,

$$N = 1.\frac{1}{3} + 2.2.\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{6}}\right) + 2.1.\frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}} + 3.\frac{1}{6} + 2.3\left(-\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{2}{2}$$

$= -0.025$ (ie. acts along inward normal to plane)

or simply do $\underline{t}.\underline{n} = \{t\}^T\{n\}$

$S = \pm\sqrt{|\underline{t}|^2 - N^2} = 2.2396$ directed along $(\underline{n} \times \underline{e}_{(\underline{t})}) \times \underline{n}$

Note $|\underline{t}| \simeq S \because N \simeq 0$

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Lecture 3

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STRESS ANALYSIS

- Transformation of $\underline{\underline{\sigma}}$
- Transf. of vector
- Introduction to 3D Cartesian Tensors.

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Transformation of Stress Tensor $\underline{\underline{\sigma}}$

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Given $\underline{\underline{\sigma}}$ at P in (x, y, z) system

Find $\underline{\underline{\sigma}}'$ at P, ie stress tensor in (x', y', z') system.

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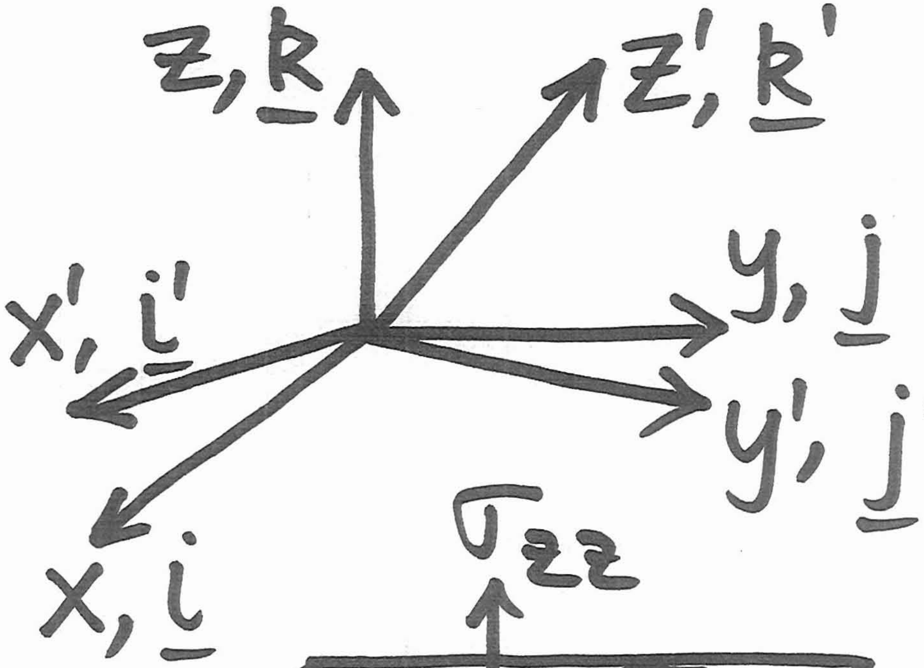

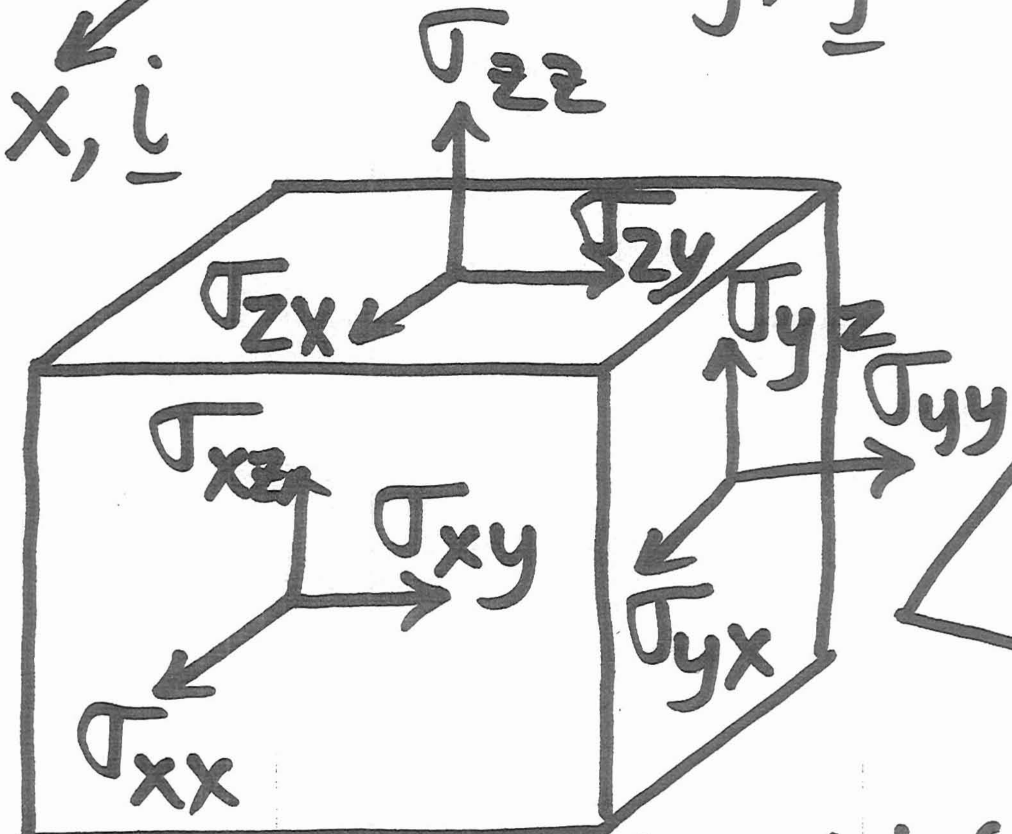

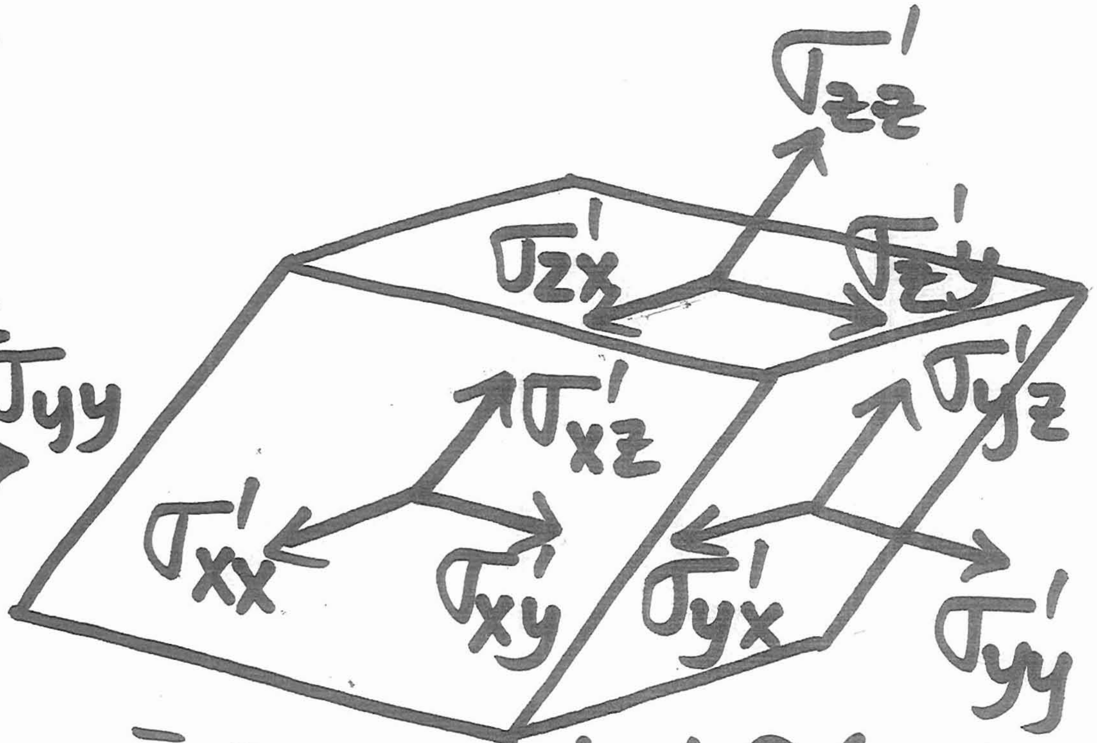

$\underline{\underline{\sigma}}$ at P (referred to (x, y, z)) $\underline{\underline{\sigma}}'$ at P (ref to (x', y', z'))

Transformation matrix ($\underline{\underline{a}}$) between (x, y, z) & (x', y', z) systems — matrix of direction cosines:

	$x, \underline{i}$	$y, \underline{j}$	$z, \underline{k}$
$x', \underline{i}'$	$l_1 \equiv a_{11}$	$m_1 \equiv a_{12}$	$n_1 \equiv a_{13}$
$y', \underline{j}'$	$l_2 \equiv a_{21}$	$m_2 \equiv a_{22}$	$n_2 \equiv a_{23}$
$z', \underline{k}'$	$l_3 \equiv a_{31}$	$m_3 \equiv a_{32}$	$n_3 \equiv a_{33}$

a_{ij} is cosine(angle between $x'_i/\underline{e}'_i$ & $x_j/\underline{e}_j$)

$x' \equiv x'_1$, $y' \equiv x'_2$, $z' \equiv x'_3$, $\underline{i}' \equiv \underline{e}'_1$, $\underline{j}' \equiv \underline{e}'_2$, $\underline{k}' \equiv \underline{e}'_3$

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Need $\underline{\underline{\sigma}}'$ in terms of $\underline{\underline{\sigma}}$ & transformation matrix.

Diagonal (Normal) Components.

σ_{xx}' = Normal stress on plane with $\underline{i}'$ as normal.

$$\underline{i}' = l_1 \underline{i} + m_1 \underline{j} + n_1 \underline{k} = a_{11} \underline{i} + a_{12} \underline{j} + a_{13} \underline{k}$$

$$\Rightarrow \boxed{\sigma_{xx}' = \sigma_{xx} l_1^2 + \sigma_{yy} m_1^2 + \sigma_{zz} n_1^2 + 2\sigma_{xy} l_1 m_1 + 2\sigma_{yz} m_1 n_1 + 2\sigma_{xz} l_1 n_1} \rightarrow (1a)$$

(ref (2a)/(2b) (L2, P.17, 18))

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In index notation,

$$\boxed{\sigma_{11}' = \sigma_{11} a_{11}^2 + \sigma_{22} a_{12}^2 + \sigma_{33} a_{13}^2 + 2\sigma_{12} a_{11} a_{12} + 2\sigma_{23} a_{12} a_{13} + 2\sigma_{13} a_{11} a_{13}} \rightarrow \text{1b}$$

Similarly σ_{yy}' = Normal stress on $\underline{j}'$ plane

σ_{zz}' = " " " $\underline{k}'$ "

σ_{yy}' = 1a with $l_1 \to l_2$, $m_1 \to m_2$, $n_1 \to n_2$

σ_{22}' = 1b with $1 \to 2$ in first subscript of a_{ij}

σ_{zz}' = 1a, $l_1 \to l_3$, $m_1 \to m_3$, $n_1 \to n_3$; σ_{33}' = 1b, 1st subscr of a_{ij}, $1 \to 3$.

Off-diagonal (Shear) components

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Ref. Fig P.2.

$\underline{t}_{(\underline{j}')}$ = stress vector on $\underline{j}'$ plane.

$$\sigma'_{yz} = \underline{t}_{(\underline{j}')} \cdot \underline{k}' = \underline{t}_{(\underline{k}')} \cdot \underline{j}'$$

From transf. matrix

$$\underline{j}' = l_2 \underline{i} + m_2 \underline{j} + n_2 \underline{k} \; ; \; \underline{k}' = l_3 \underline{i} + m_3 \underline{j} + n_3 \underline{k}$$

$$\underline{t}_{(\underline{j}')} = (\sigma_{xx} l_2 + \sigma_{xy} m_2 + \sigma_{xz} n_2)\underline{i} + (\sigma_{xy} l_2 + \sigma_{yy} m_2 + \sigma_{yz} n_2)\underline{j} + (\sigma_{xz} l_2 + \sigma_{yz} m_2 + \sigma_{zz} n_2)\underline{k}$$

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$$\Rightarrow \boxed{\sigma'_{yz} = \sigma_{xx} l_2 l_3 + \sigma_{yy} m_2 m_3 + \sigma_{zz} n_2 n_3 + \sigma_{xy}(m_2 l_3 + m_3 l_2) + \sigma_{xz}(n_2 l_3 + n_3 l_2) + \sigma_{yz}(n_2 m_3 + n_3 m_2)} \rightarrow (2a)$$

index not...

$$\boxed{\sigma'_{23} = \sigma_{11} a_{21} a_{31} + \sigma_{22} a_{22} a_{32} + \sigma_{33} a_{23} a_{33} + \sigma_{13}(a_{23} a_{31} + a_{33} a_{21}) + \sigma_{23}(a_{23} a_{32} + a_{33} a_{22}) + \sigma_{12}(a_{22} a_{31} + a_{32} a_{21})} \rightarrow (2b)$$

Similarly, $\sigma_{xz}' = \underline{t}_{(\underline{i}')} \cdot \underline{k}' = \underline{t}_{(\underline{k}')} \cdot \underline{i}'$

$$\sigma_{xy}' = \underline{t}_{(\underline{i}')} \cdot \underline{j}' = \underline{t}_{(\underline{j}')} \cdot \underline{i}'$$

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$\sigma_{xz}' =$ 2a with $l_2 \to l_1$; $m_2 \to m_1$; $n_2 \to n_1$

$\sigma_{13}' =$ 2b with $a_{21} \to a_{11}$; $a_{22} \to a_{12}$; $a_{23} \to a_{13}$

$\sigma_{xy}' =$ 2a with $l_3 \to l_1$; $m_3 \to m_1$; $n_3 \to n_1$

$\sigma_{12}' =$ 2b with $a_{31} \to a_{11}$; $a_{32} \to a_{12}$; $a_{33} \to a_{13}$

$\underline{\underline{\sigma}}'$ in index notation (1b, 2b etc) in compact form:

$$\boxed{\sigma'_{ij} = \sum_{r=1}^{3}\sum_{s=1}^{3} a_{ir}\, a_{js}\, \sigma_{rs}} \longrightarrow 3a$$

$\hookrightarrow$ Transf Law to get $\underline{\underline{\sigma}}'$ from $\underline{\underline{\sigma}}$, given $\underline{\underline{a}}$

$$= \sum\sum_{r,s=1}^{3} (a_{ir}\,\sigma_{rs})(a_{sj})^T = \underbrace{(\underline{\underline{a}}\,\underline{\underline{\sigma}})}(\underline{\underline{a}}^T),\ a^T_{sj}$$

$$\boxed{\underline{\underline{\sigma}}' = \underline{\underline{a}}\,\underline{\underline{\sigma}}\,\underline{\underline{a}}^T} \longrightarrow 3b$$

$$\underline{\underline{P}} = P_{is}\, \cancel{b}_{sj}$$

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Note: $\boxed{\underline{\underline{a}}^{-1} = \underline{\underline{a}}^{T}}$, ie $\boxed{\underline{\underline{a}}\,\underline{\underline{a}}^{T} = \underline{\underline{I}}}$

Proof: $\underline{i}' = a_{11}\,\underline{i} + a_{12}\,\underline{j} + a_{13}\,\underline{k}$

ie 1st row of $\underline{\underline{a}}$ is $\underline{i}'$, 2nd is $\underline{j}'$, 3rd is $\underline{k}'$

$$\Rightarrow \underline{\underline{a}}\,\underline{\underline{a}}^{T} = \begin{bmatrix} \underline{i}' \rightarrow \\ \underline{j}' \rightarrow \\ \underline{k}' \rightarrow \end{bmatrix} \begin{bmatrix} \underline{i}' & \underline{j}' & \underline{k}' \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \underline{\underline{I}}$$

$\because \underline{i}' \cdot \underline{i}' = 1, \; \underline{i}' \cdot \underline{j}' = 0$, etc

$$\Rightarrow \underline{\underline{a}}^{T} \; (3b) \; \underline{\underline{a}} \Rightarrow \boxed{\underline{\underline{\sigma}} = \underline{\underline{a}}^{T}\,\underline{\underline{\sigma}}'\,\underline{\underline{a}}} \rightarrow (3c)$$

↳ Inverse transf.

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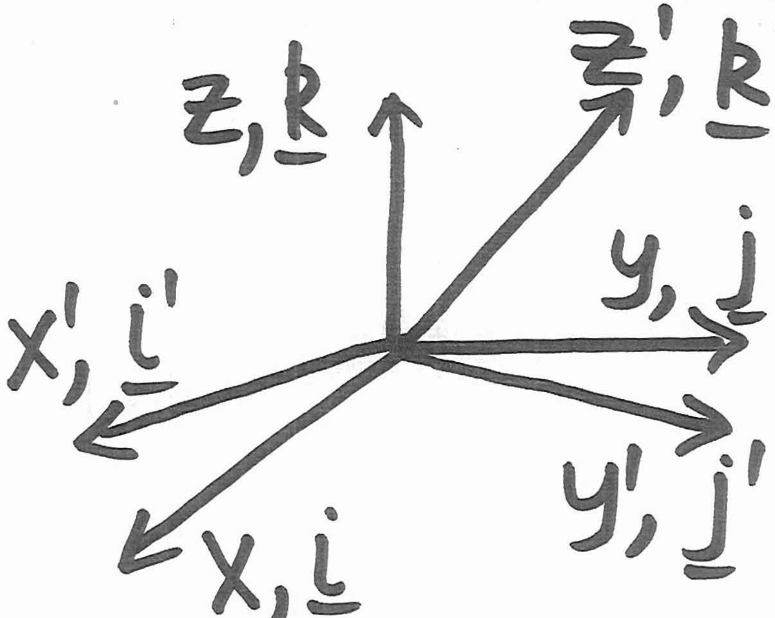
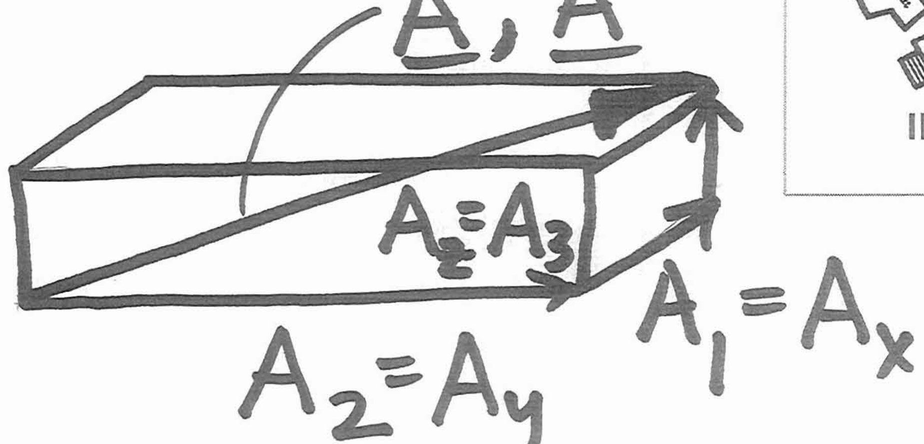
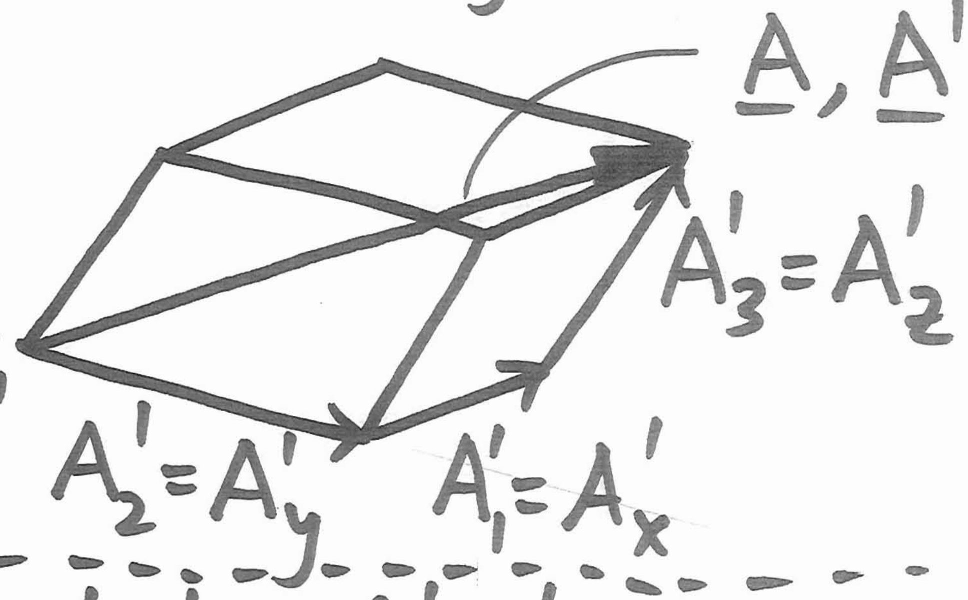
Transf. of vector $\underline{A}$





$\underline{A} = \underline{A}'$



Given: $\underline{A} = A_1\underline{e}_1 + A_2\underline{e}_2 + A_3\underline{e}_3$

Find: $\underline{A}' = A_1'\underline{e}_1' \quad A_2'\underline{e}_2' + A_3'\underline{e}_3'$

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Note $\underline{e}_1 \equiv \underline{i} \quad \underline{e}_2 \equiv \underline{j} \quad \underline{e}_3 \equiv \underline{k}$

$\underline{e}'_1 = \underline{i} \quad \underline{e}_2 \equiv \underline{j}' \quad \underline{e}_3 = \underline{k}'$

$$\underline{A}' = \underline{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k}$$

Subst $\underline{i}, \underline{j}, \underline{k}$ in terms of $\underline{i}' \; \underline{j}' \; \underline{k}'$ from $\underline{\underline{a}}$, ie $\underline{i} = a_{11} \underline{i}' + a_{21} \underline{j}' + a_{31} \underline{k}'$, etc

$$\begin{aligned} \Rightarrow \underline{A}' = & (a_{11} A_1 + a_{12} A_2 + a_{13} A_3) \underline{i}' \\ & + (a_{21} A_1 + a_{22} A_2 + a_{23} A_3) \underline{j}' \\ & + (a_{31} A_1 + a_{32} A_2 + a_{33} A_3) \underline{k}' \\ \equiv \; & A'_1 \underline{i}' + A'_2 \underline{j}' + A'_3 \underline{k}' \end{aligned}$$

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$$\Rightarrow \boxed{A'_i = \sum_{r=1}^{3} a_{ir} A_r} \rightarrow 4a$$

$$\begin{Bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{Bmatrix} = \boxed{\underline{A}' = \underline{\underline{a}}\, \underline{A}} = \underline{\underline{a}} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$$

$\rightarrow$ 4b

Introduction to 3D Cartesian Tensors

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1) Range convention: Unrepeated index, or subscript, in a term, can take values 1, 2, or 3. Its called range/free index.

Summation convention: (due to Einstein) Index/subscript appearing twice (and only twice), in a term, implies summation over that index for index ranging from 1 to 3.

L3

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(eg) $\sigma_{ij}\sigma_{rj}$ is a term in an Eqn

i, r, range indices, can take values 1, 2, or 3. So above represents $3\times 3 = 9$ terms in the 9 respective Eqns (ie the i, r^{th} eqn), ie

$\sigma_{1j}\sigma_{1j}$, $\sigma_{1j}\sigma_{2j}$, $\sigma_{1j}\sigma_{3j}$, $\sigma_{2j}\sigma_{1j}$, $\sigma_{2j}\sigma_{2j}$, $\sigma_{2j}\sigma_{3j}$, $\sigma_{3j}\sigma_{1j}$, $\sigma_{3j}\sigma_{2j}$, $\sigma_{3j}\sigma_{3j}$, appearing in the $i=1, r=1$ eqn, $i=1, r=2$ eqn, 13, 21, 22, 23, 31, 32, 33, eqn, respectively.

Also, from $\sum$ convention,

$$\sigma_{ij}\sigma_{rj} = \sum_{j=1}^{3} \sigma_{ij}\sigma_{rj}$$

So each of the 9 terms represent (contain) a sum of 3 terms, ie, for example,

$\sigma_{2j}\sigma_{3j}$ appearing in 2-3 eqn is

$$\hookrightarrow = \sigma_{21}\sigma_{31} + \sigma_{22}\sigma_{32} + \sigma_{23}\sigma_{33}$$

or $\sigma_{ij}\sigma_{rj} = \sigma_{i1}\sigma_{r1} + \sigma_{i2}\sigma_{r2} + \sigma_{i3}\sigma_{r3}$

$\sigma_{ij}\sigma_{rj} = \sigma_{ik}\sigma_{rk} = \sigma_{is}\sigma_{rs}$, i.e.,

L3

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Σ indices are dummy indices.

However, range indices must be same in each term of an equation, ie.,

$$\sigma_{ijtt} + p_{ijnn} = q_{ijmm} + b_{ijss}, \quad \checkmark$$

$$\sigma_{ijtt} + p_{rqtt} = q_{lmtt} + b_{rstt} \quad \times$$

eqn not homogenous in range indices, but it should be so.

Thus,

$$\boxed{A'_i = a_{ir} A_r}$$

→ (4a), p13, Σ dropped

→ Transf of Vector.

$$\boxed{\sigma'_{ij} = a_{ir} a_{js} \sigma_{rs}}$$

→ (3a), p. 9, Σ dropped

→ Transf Of $\underline{\underline{\sigma}}$

$$\boxed{A'_{\underbrace{ijpq}\ldots\ldots} = a_{ib} a_{jc} a_{pd} a_{qe} \ldots\ldots A_{\underbrace{bcde}\ldots}}$$

'n' range indices

→ (5)

→ Transf Law for n^{th} order tensor.

'n' range indices.

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Thus: vector is 1st order tensor
$\underline{\underline{\sigma}}$ is 2nd order tensor
scalar is 0th order tensor

Also, $\boxed{t_j = \sigma_{ij} n_i}$ → ①, p.13, L2, Σ omit
→ Cauchy relation.

eg $j=2$, $t_2 = t_y = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$

$\boxed{N = \sigma_{ij} n_i n_j}$ → (2b), p18, L2, Σ omit
→ Normal stress.

$$\boxed{S^2 = \underline{t} \cdot \underline{t} - N^2 = t_i t_i - N^2 = \sigma_{ji} n_j \sigma_{ki} n_k - (\sigma_{ij} n_i n_j)^2}$$

$$\underbrace{\sigma_{ij}\, n_j}_{t_i} \qquad \underbrace{\sigma_{\cancel{j}i}{}_{k}\, n_{\cancel{j}k}}_{t_i}$$

$$\underbrace{\sigma_{ji}\, n_j}_{t_i} \qquad \underbrace{\sigma_{\cancel{j}i}{}_{k}\, n_{\cancel{j}k}}_{t_i}$$

19a

L3

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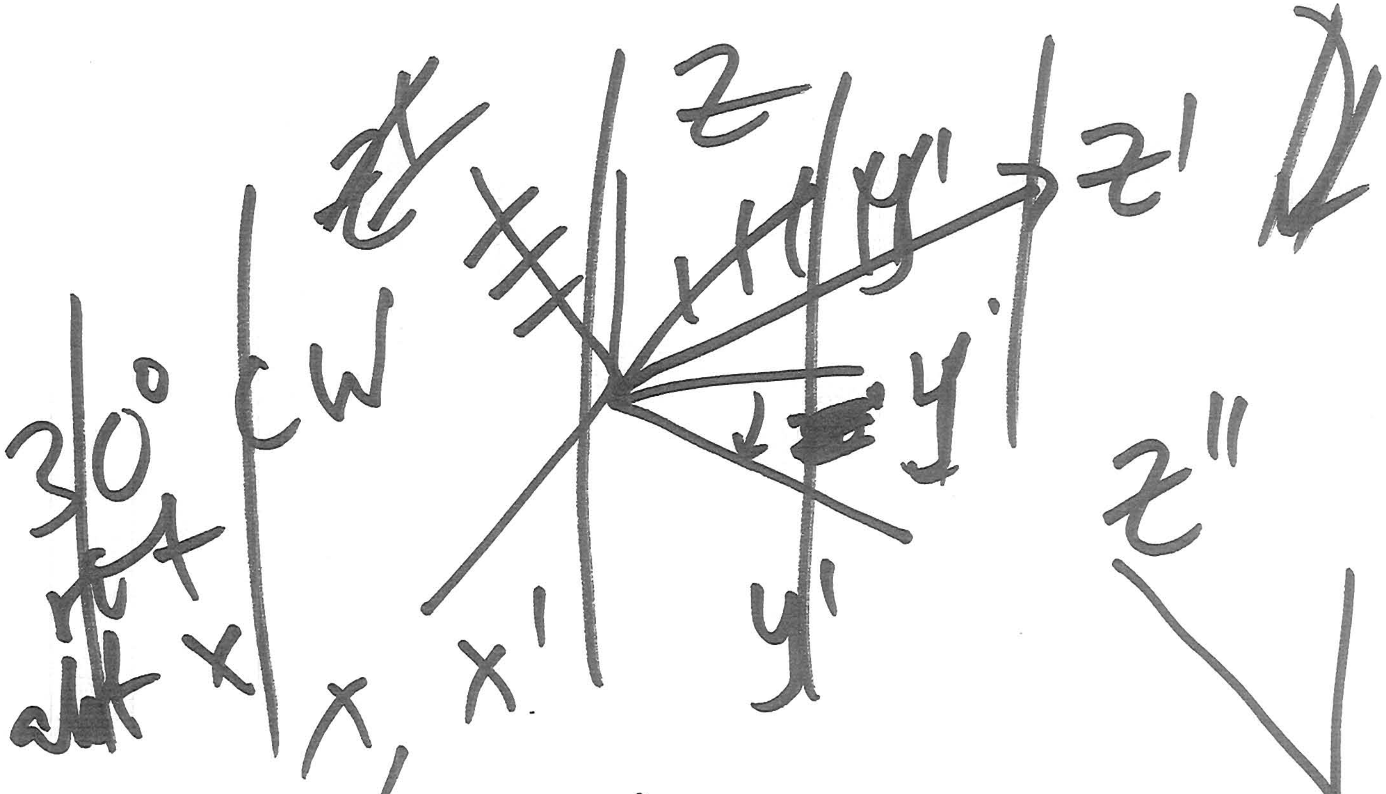
Ex 2 Given $\underline{\underline{\sigma}} = \begin{bmatrix} 10 & 20 & 30 \\ 20 & 5 & 25 \\ 30 & 25 & 15 \end{bmatrix}$ MPa at P

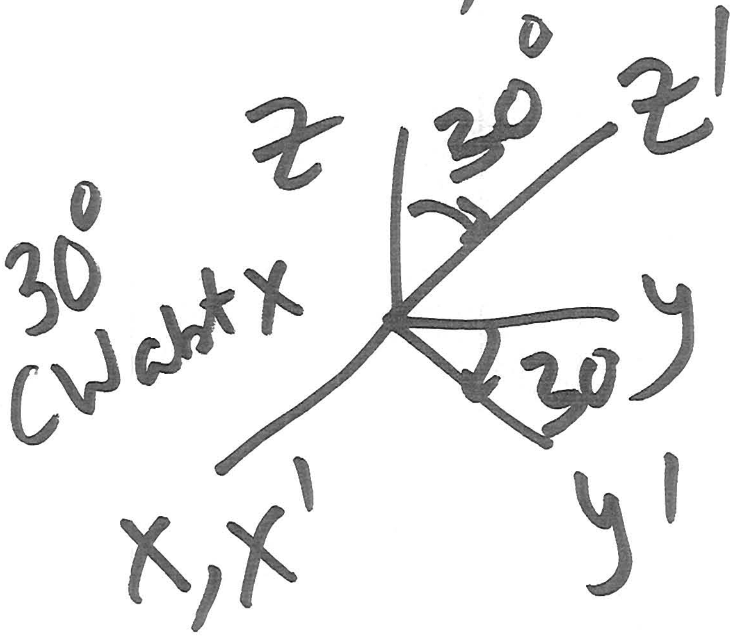
(x, y, z) rotated 30° CW about x (ie, $-\underline{i}$ rot)

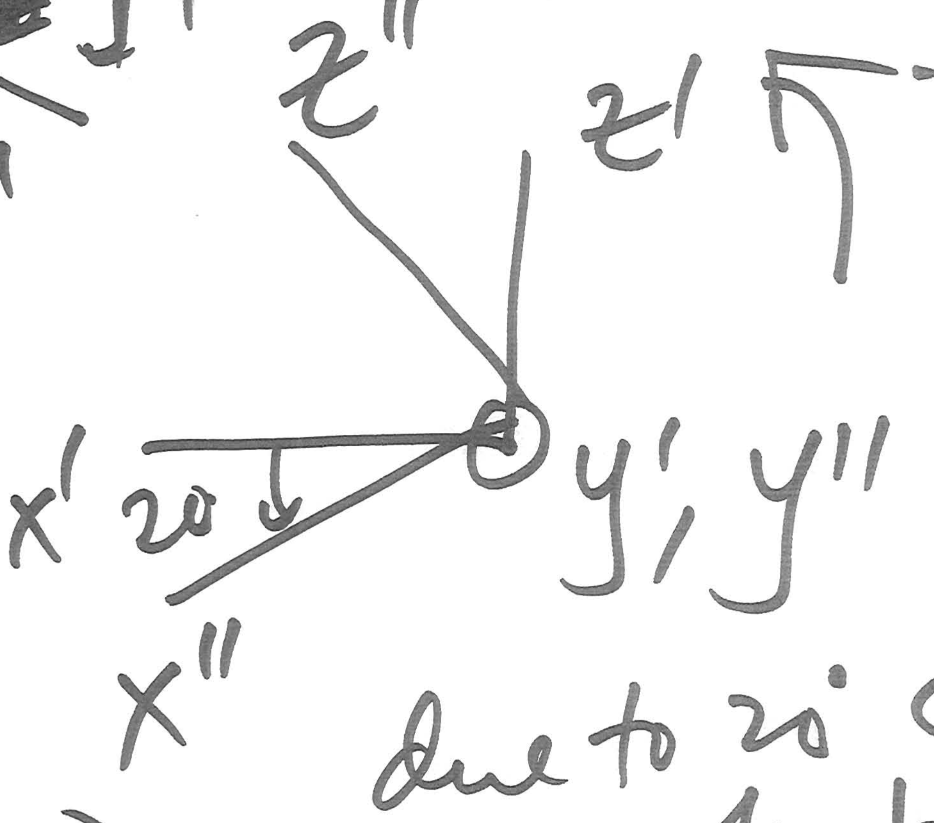
Then (x', y', z') rotated 20° CCW abt y' (ie, $+\underline{j}'$ rot).

Find $\underline{\underline{\sigma}}''$.

$$\begin{Bmatrix} \underline{i}' \\ \underline{j}' \\ \underline{k}' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} \begin{Bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{Bmatrix} = \underline{\underline{a}}_I \begin{Bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{Bmatrix}$$







due to 20° CCW rot abt y'

20a

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$$\begin{Bmatrix} \underline{i}'' \\ \underline{j}'' \\ \underline{k}'' \end{Bmatrix} = \begin{bmatrix} \cos 20 & 0 & -\sin 20 \\ 0 & 1 & 0 \\ \sin 20 & 0 & \cos 20 \end{bmatrix} \begin{Bmatrix} \underline{i}' \\ \underline{j}' \\ \underline{k}' \end{Bmatrix} = \underline{\underline{a}}_{II} \begin{Bmatrix} \underline{i}' \\ \underline{j}' \\ \underline{k}' \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \underline{i}'' \\ \underline{j}'' \\ \underline{k}'' \end{Bmatrix} = \underbrace{\underline{\underline{a}}_{II}\,\underline{\underline{a}}_{I}}_{\underline{\underline{a}}} \begin{Bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{Bmatrix} = \begin{bmatrix} c20 & -s20s30 & -s20c30 \\ 0 & c30^\circ & -s30 \\ s20 & c20s30 & c20c30 \end{bmatrix}$$

$$\underline{\underline{\sigma}}'' = \underline{\underline{a}}\,\underline{\underline{\sigma}}\,\underline{\underline{a}}^T = \begin{bmatrix} 10.96 & 33.70 & 14.71 \\ 33.70 & 22.38 & 20.78 \\ 14.71 & 20.78 & -3.34 \end{bmatrix}$$

LECTURE 4

STRESS ANALYSIS

- Intro to 3D Tensors (contd)
- Principal Stresses & Axes.

2) Kronecker delta

$$\delta_{ij} = 1, \text{ if } i = j$$
$$\quad = 0, \text{ if } i \neq j$$

$\Rightarrow \delta_{ij} \equiv \underline{\underline{I}}$ (identity matrix).

Recall $\underline{\underline{a}}\,\underline{\underline{a}}^T = \underline{\underline{I}} \Rightarrow a_{ik}\cancel{b_{kj}} = \delta_{ij}$

a_{jk}

$\Rightarrow \boxed{a_{ik}a_{jk} = \delta_{ij}} \rightarrow$ index notation form of $\underline{\underline{a}}\,\underline{\underline{a}}^T = \underline{\underline{I}}$

(eg) $A_v \delta_{uv}$: for $u=1$, $A_1\delta_{11} + A_2\cancelto{0}{\delta_{12}} + A_3\cancelto{0}{\delta_{13}} = A_1$

for $u=2$, $A_v \delta_{uv} = A_2$

$u=3$, $\quad = A_3$

$$\Rightarrow A_v \delta_{uv} = A_u$$

So, $\sigma_{ij} \delta_{jk} = \sigma_{ik}$; $\quad \sigma_{ij} \delta_{ij} = \sigma_{jj}$

$$= \sigma_{11} + \sigma_{22} + \sigma_{33}$$

3) Addition of tensors.

$$\underbrace{A'_{ijk} + B'_{ijk}}_{C'_{ijk}} = a_{ir} a_{js} a_{kt} (\underbrace{A_{rst} + B_{rst}}_{C_{rst}})$$

$$C'_{ijk} = a_{ir} a_{js} a_{kt} C_{rst}$$

So adding like tensors (ie with same range indices) is allowed & it yields a tensor.

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4) Tensor Multiplication.

a) Inner (dot) product of vectors

$$\underline{A} \cdot \underline{B} = A_i B_i \quad \boxed{\left(\begin{array}{l} \text{or } A_i \underline{e}_i \cdot B_j \underline{e}_j,\ \underline{e}_i \cdot \underline{e}_j = \delta_{ij} \\ \Rightarrow A_i B_j \delta_{ij} = A_i B_i \end{array}\right)}$$

b) Cross product of vectors

$$\underline{A} \times \underline{B} = \underline{C} \longrightarrow C_i = \epsilon_{ijk} A_j B_k$$

or

$$\underline{A} \times \underline{B} = A_j \underline{e}_j \times B_k \underline{e}_k$$

$$\underline{e}_j \times \underline{e}_k = \epsilon_{ijk} \underline{e}_i$$

$$\underline{C} = \underline{A} \times \underline{B} \Rightarrow \underbrace{C_i}\, \underline{e}_i = \underbrace{\epsilon_{ijk} A_j B_k}\, \underline{e}_i$$

$$=$$

where permutation symbol

$\epsilon_{ijk} = 0$, repeated indices $(i=j,\ i=k,\ j=k)$ → eg $\epsilon_{121} = 0$

$= 1$, ijk form $1 \to 2 \to 3 \to 1$ → eg. $\epsilon_{312} = 1$

$= -1$, ijk form $1 \to 3 \to 2 \to 1$ → eg $\epsilon_{132} = -1$

eg, $i=2$,

$$C_2 = \epsilon_{213} A_1 B_3 + \epsilon_{231} A_3 B_1$$

$$= -A_1 B_3 + A_3 B_1 \quad \text{(as expected)}$$

ie $C_y = -A_x B_z + A_z B_x$

c) Outer product

eg $A_{ijk} B_{rs}$

d) Inner product

Outer product followed by contraction (i.e., identifying two indices)

eg $\underbrace{A_{ijk} B_{rs}}_{\text{outer product}} \xrightarrow[i=s]{\text{contr.}} A_{ijk} B_{ri}$

Note : contraction lowers order of tensor by 2.

$$A'_{ijk} B'_{r\cancel{s}\,i} = a_{ib}\, a_{jc}\, a_{kd}\, a_{re}\, a_{\cancel{s}f\,i}\, A_{bcde} B_{ef}$$

$$\delta_{bf}$$

$$= a_{jc}\, a_{kd}\, a_{re}\, A_{fca} B_{ef} \quad \text{(QED)}$$

e) **Differentiation of tensors:**

It increases order by 1.

$$\frac{\partial A'_{uv\ldots}}{\partial x'_w} = \frac{\partial (a_{ui}\, a_{vj} \cdots A_{ij\ldots})}{\partial x_k} \frac{\partial x_k}{\partial x'_w}$$

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Now, $x_k = a_{wk} x'_w$ (inv. transf of p.v.).

$$\Rightarrow \frac{\partial x_k}{\partial x'_w} = a_{wk},$$

$$\Rightarrow \frac{\partial A'_{uv\cdots}}{\partial x'_w} = a_{ui} a_{vj} a_{wk} \cdots \frac{\partial A_{ij\cdots}}{\partial x_k} \quad \text{(QED)}$$

f) <u>Gradient of scalar $\phi(x_1, x_2, x_3)$</u>

$$\underline{\nabla}\phi = \left(\frac{\partial(\ )}{\partial x_i}\underline{e}_i\right)\phi = \frac{\partial \phi}{\partial x_i}\underline{e}_i = \phi_{,i}\,\underline{e}_i$$

$$\equiv \phi_{,i}$$

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If $\phi(x_1, x_2, x_3) = \text{constant}$ defines surface S, and $\underset{\sim}{dx}$ is a differential tangent vector on S,

$$d\phi = \underset{\sim}{\nabla}\phi \cdot \underset{\sim}{dx} = \phi_{,i}\, dx_i = 0 \quad (\text{on } S)$$

$\Rightarrow \underset{\sim}{\nabla\phi}$ is normal to S

$= \phi_{,i} \longrightarrow \phi_{,i}/|\phi_{,i}| =$ unit normal

9) <u>Divergence of vector $\underset{\sim}{A} = A_i \underset{\sim}{e_i}$</u>

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{A} = \nabla_i A_i = A_{i,i} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

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or,

$$\underline{\nabla} \cdot \underline{A} = \nabla_i \underline{e}_i \cdot A_j \underline{e}_j = \nabla_i A_i$$

h) <u>Curl of vector $\underline{A}$</u>

$$\underline{C} = \underline{\nabla} \times \underline{A} = \epsilon_{ijk} \nabla_j A_k \underline{e}_i$$

$$= \underbrace{\epsilon_{ijk} A_{k,j}}_{= C_i} \underline{e}_i = C_i \underline{e}_i$$

eg $i=2$, $C_2 = \epsilon_{231} A_{1,3} + \epsilon_{213} A_{3,1}$

$$= A_{1,3} - A_{3,1}$$

Principal Stresses, Principal Axes

Given : $\underline{\underline{\sigma}}$ at P in (x, y, z) system

Find : Planes $\underline{n}$ on which $\underline{t}$ is wholly normal, ie $\underline{t}$ & $\underline{n}$ have same direction.

Will be seen later that this is equivalent to finding planes $\underline{n}$ on which N is an extremum (ref Shames, Engg Mech).

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$\underline{t} = \lambda \underline{n}$ // ($\underline{t}$ 'wholly' along $\underline{n}$)

$t_i = \lambda n_i = \lambda n_j \delta_{ij}$

$\underline{t} = \underline{\underline{\sigma}}\, \underline{n}$ // $t_i = \sigma_{ij} n_j$ (Cauchy rel)

$\Rightarrow (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\underline{n} = \underline{0}$ // $(\sigma_{ij} - \lambda \delta_{ij}) n_j = 0_i$

(1)

Eigenvalue prob for $\underline{\underline{\sigma}}$

3 Linear homogenous eqns for 4 unknowns n_1, n_2, n_3, λ

Non-trivial soln, $\underline{n} \neq \underline{0}$ iff $\det|\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}| = 0$

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$$\det\left|\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}\right| = \boxed{\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0} \rightarrow (2)$$

where, $I_1 = \text{trace}(\underline{\underline{\sigma}}) = \sigma_{ii}$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2$$

$$I_3 = \det\left|\underline{\underline{\sigma}}\right|$$

Solve (2) for $\lambda(1)$, $\lambda(2)$, $\lambda(3)$. $\rightarrow$ evalues

Subst evalue $\lambda(i)$ in (1), solve $\underline{n}(i) \rightarrow$ evector as follows:

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$$\left.\begin{aligned}(\sigma_{11}-\lambda(i))n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 &= 0\\ \sigma_{12}n_1 + (\sigma_{22}-\lambda(i))n_2 + \sigma_{23}n_3 &= 0\\ \sigma_{13}n_1 + \sigma_{23}n_2 + (\sigma_{33}-\lambda(i))n_3 &= 0\end{aligned}\right\} \rightarrow 3(a,b,c)$$

At most two of 3(a, b, c) independent.

Use those two, &

$$n_1^2 + n_2^2 + n_3^2 = 1 \rightarrow 3d \rightarrow$$ ie, make $\underline{n}$ unit vector

& solve for $(n_1, n_2, n_3) \equiv \underline{n}(i) \rightarrow$ evector corresponding to $\lambda(i)$

$\lambda(i) \rightarrow$ Principal Stresses (evalues)

$\underline{n}(i) \rightarrow$ Principal axes (evectors)

ie, normal to planes

on which $\underline{t} = \lambda \underline{n}$

P-stresses are real:

Assume $\lambda = a + ib$, complex

$\because \underline{\underline{\sigma}}$ real, polynomial has real coeffs.

$\Rightarrow \bar{\lambda}$ also a root (evalue).

$\Rightarrow (\lambda, \underline{n})$ & $(\bar{\lambda}, \bar{\underline{n}})$ are e-solutions satisfying evp

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ie, $\underline{\underline{\sigma}}\,\underline{n} = \lambda\,\underline{n} \longrightarrow (a)$

$\underline{\underline{\sigma}}\,\bar{\underline{n}} = \bar{\lambda}\,\bar{\underline{n}} \longrightarrow (b)$

$\bar{\underline{n}}^T * (a) - \underline{n}^T * (b)$ gives

$$\underbrace{\bar{\underline{n}}^T\,\underline{\underline{\sigma}}\,\underline{n}} - \underbrace{\underline{n}^T\,\underline{\underline{\sigma}}\,\bar{\underline{n}}} = \lambda\,\underbrace{\bar{\underline{n}}^T\underline{n}} - \bar{\lambda}\,\underbrace{\underline{n}^T\bar{\underline{n}}}$$

all scalars, so equal to their transpose.

$$\Rightarrow \underline{n}^T(\underline{\underline{\sigma}}^T - \underline{\underline{\sigma}})\,\bar{\underline{n}} = (\lambda - \bar{\lambda})\,\underline{n}^T\bar{\underline{n}}$$

$\underline{\underline{0}}$ $\because$ $\underline{\underline{\sigma}}$ symm $\Rightarrow \lambda = \bar{\lambda} =$ real (QED)

for $\underline{n}, \bar{\underline{n}}, \neq \underline{0}$

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P-axes are orthogonal.

$$\left.\begin{array}{l}\underline{\underline{\sigma}}\,\underline{n}(i) = \lambda(i)\,\underline{n}(i) \rightarrow (c)\\[4pt] \underline{\underline{\sigma}}\,\underline{n}(j) = \lambda(j)\,\underline{n}(j) \rightarrow (d)\end{array}\right\}$$ evp satisfied by i^{th}, j^{th} eigensolution.

$\underline{n}^T(j) * (c) - \underline{n}^T(i) * (d)$ gives

(note all triple vector matrix products & vector [dot] products are scalars, and $\underline{\underline{\sigma}}$ symm, as before)

$$0 = \left(\lambda(i) - \lambda(j)\right)\underline{n}^T(i)\,\underline{n}(j)$$

$\because \lambda(i) \neq \lambda(j)$, in general,

$\underline{n}^T(i)\,\underline{n}(j) = 0$ (QED)

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L4

For $\underline{\underline{\sigma}}$ real, symm, even if $\lambda(i)=\lambda(j)$ (repeated roots) we can find evectors $\underline{n}(i)$, $\underline{n}(j)$ & orthogonalize them if reqd.

Thus if two p-stresses are same, then p-axis corresponding to third p-stress is uniquely determinable & other two p-axes are arbitrary

but orthogonal to 3rd p-axis, & orthogonalizable wrt each other. Thus for third [two repeated] p-stresses two [one] of 3 (a, b, c) are independent.

If all p-stresses same, then any three indep axes are p-axes. This corresponds to case when only one of 3 [a, b, c] are indep. However, p-axes still orthogonalizable.

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L4

L4

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<u>Ex1</u> Given

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 & -cx_2 \\ 0 & 0 & cx_1 \\ -cx_2 & cx_1 & 0 \end{bmatrix} \text{ at } P \equiv (1,2,4)$$

Find p-stresses & axes.

$I_1 = 0,\ I_2 = -c^2 \cdot 1^2 - c^2 \cdot 2^2 = -5c^2,\ I_3 = 0$

$\lambda^3 - 5c^2\lambda = 0,\quad \lambda = 0, \pm c\sqrt{5}$ (p-stresses)

$\underline{\lambda = 0}$:

$$\left.\begin{aligned} -2cn_3 &= 0 \\ cn_3 &= 0 \\ -2cn_1 + cn_2 &= 0 \end{aligned}\right\} \Rightarrow \underline{n}(1) = \frac{\pm 1}{\sqrt{5}}\underline{e}_1 \pm \frac{2}{\sqrt{5}}\underline{e}_2$$

after normalizing

$\lambda(1) = 0$

L4

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$\underline{\lambda = c\sqrt{5}}$: $-c\sqrt{5}\,n_1 - 2c\,n_3 = 0$ (a)

$-c\sqrt{5}\,n_2 + c\,n_3 = 0$ (b)

$-2c\,n_1 + c\,n_2 - c\sqrt{5}\,n_3 = 0$ (c)

$$\underline{n}(2) = \pm\frac{(-2,\ 1,\ \sqrt{5})}{\sqrt{10}} = \pm\left(-\sqrt{\frac{2}{5}}\,\underline{e}_1 + \sqrt{\frac{1}{10}}\,\underline{e}_2 + \sqrt{1/2}\,\underline{e}_3\right)$$

$\underline{\lambda = -c\sqrt{5}}$: $\underline{n}(3) = \pm\left(\sqrt{\frac{2}{5}}\,\underline{e}_1 - \sqrt{\frac{1}{10}}\,\underline{e}_2 + \sqrt{\frac{1}{2}}\,\underline{e}_3\right)$

(by observation of above)

See $\underline{n}(1)\cdot\underline{n}(2) = \underline{n}(2)\cdot\underline{n}(3) = \underline{n}(3)\cdot\underline{n}(1)$
$= 0$.

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Ex2 Given,

$$\underline{\underline{\sigma}} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ at } P.$$

Find: p-solution.

$$\lambda^3 - 6\lambda^2 + 9\lambda - \underbrace{(2*3 - 1*1 - 1*1)}_{4} = 0$$

$\Rightarrow \lambda = 1, 1, 4$

$\underline{\lambda = 4}$:

$$\left.\begin{aligned} -2n_1 + n_2 - n_3 &= 0 \\ n_1 - 2n_2 - n_3 &= 0 \\ -n_1 - n_2 - 2n_3 &= 0 \end{aligned}\right\} \rightarrow$$

after normalizing,

$$\underline{n}^{(1)} = \pm\frac{(-1, -1, 1)}{\sqrt{3}} = \pm(\underline{e}_1 + \underline{e}_2 - \underline{e}_3)/\sqrt{3}$$

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$\underline{\lambda = 1}$:
$$\left.\begin{aligned} n_1 + n_2 - n_3 &= 0 \\ n_1 + n_2 - n_3 &= 0 \\ -n_1 - n_2 + n_3 &= 0 \end{aligned}\right\}$$ all 3 eqns same, ie only one indep.

\# So we have freedom to choose 2 comp's of $\underline{n}$ independently, ie to choose 2 indep evectors or $\underline{n}$'s.

→ choose $n_2 = n_3 = 1$, ie $\underline{n}(2) = \pm\frac{(\underline{e}_2 + \underline{e}_3)}{\sqrt{2}}$

→ choose $n_1 = n_3 = 1 \Rightarrow \underline{n}(3) = \pm\frac{(\underline{e}_1 + \underline{e}_3)}{\sqrt{2}}$

Or, $\underline{n}(3) = \underline{n}(1) \times \underline{n}(2)$

$$= \pm \frac{(-2\underline{e}_1 + \underline{e}_2 - \underline{e}_3)}{\sqrt{6}}$$

observe $\underline{n}(1) . \underline{n}(2) = \underline{n}(1) . \underline{n}(3) = 0$;

$\underline{n}(1), \underline{n}(2), n(3)$, indep;

$\underbrace{\underline{n}(2) . \underline{n}(3) = 0}$ → second version (this pg)

LECTURE 5.

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STRESS ANALYSIS

- $\underline{\underline{\sigma}}$, $\underline{t}$, N, S referred to p-system.
- Invariants of Stress
- 2-D P-stress theory.
- Octahaedral Stresses.

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$\underline{\underline{\sigma}}$, $\underline{t}$, N, S, referred to principal coordinate system

Consider x', y', z' aligned along p-axes $\underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)}$, respectively. Thus we have a p-coord system. This pertains to pt. P.

$$\Rightarrow \underline{\underline{\sigma}}' = \begin{bmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{bmatrix} = \begin{bmatrix} \sigma'_{xx} & 0 & 0 \\ 0 & \sigma'_{yy} & 0 \\ 0 & 0 & \sigma'_{zz} \end{bmatrix}$$

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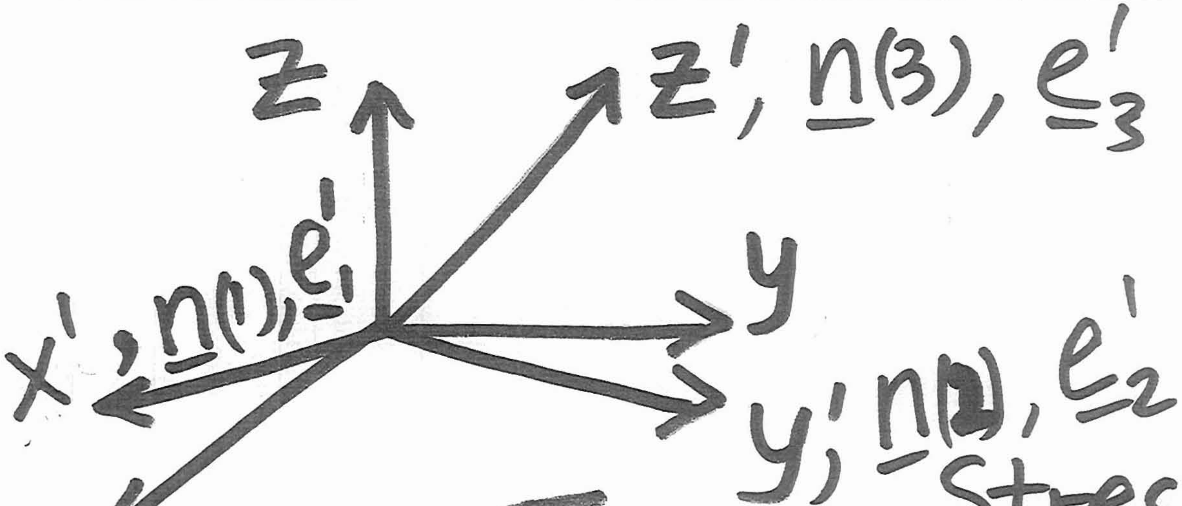
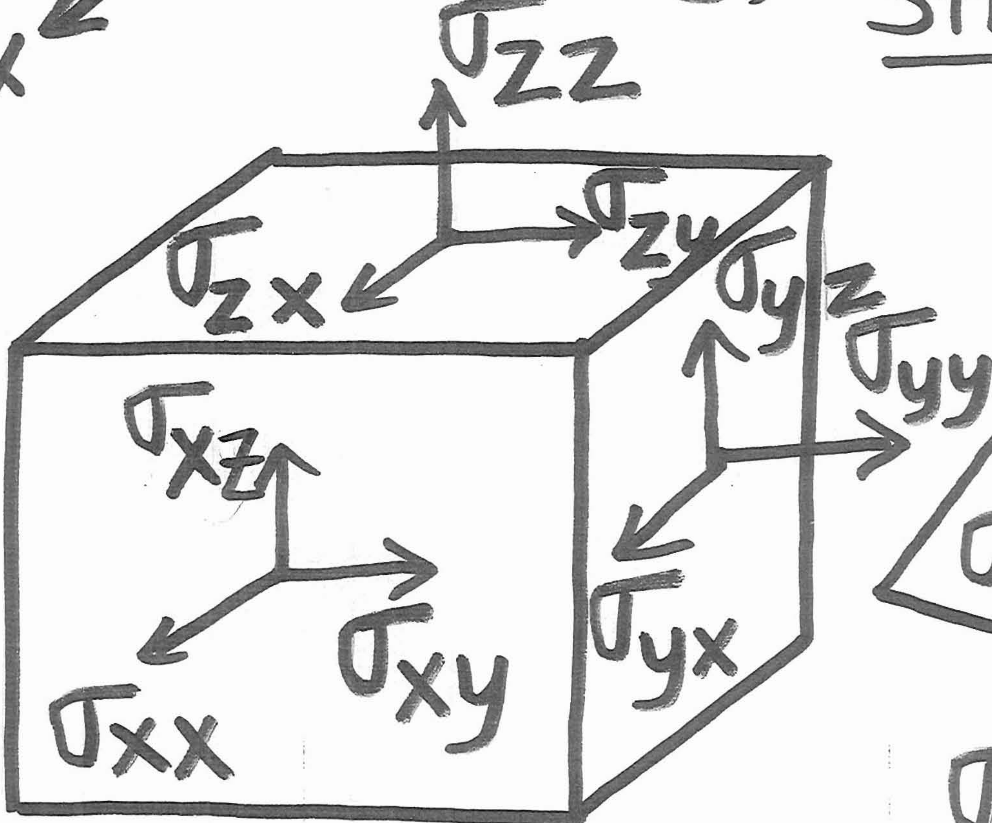
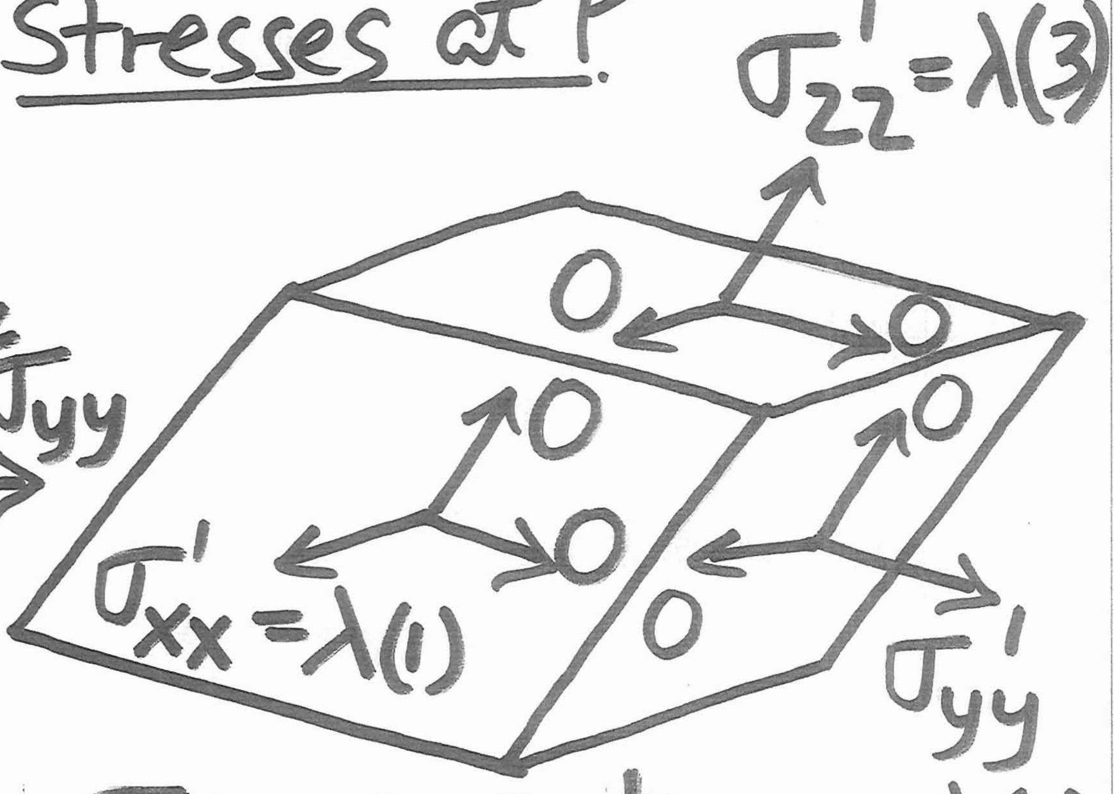


Stresses at P



$\underline{\underline{\sigma}}$ in x, y, z



$\underline{\underline{\sigma}}$ in P-System x', y', z'

L5

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Consider arbitrary plane defined by normal

$$\underline{n} = n_i \underline{e}_i = \underline{n}' = n'_i \underline{e}'_i$$

$$\Rightarrow N = \{\underline{n}'\}^T [\underline{\underline{\sigma}}'] \{\underline{n}'\} = \sigma'_{ij} n'_i n'_j$$

$$= \underbrace{\sigma'_{11}}_{\lambda(1)}(n'_1)^2 + \underbrace{\sigma'_{22}}_{\lambda(2)}(n'_2)^2 + \underbrace{\sigma'_{33}}_{\lambda(3)}(n'_3)^2$$

$$\boxed{N = \sum_{i=1}^{3} \lambda(i)(n'_i)^2}$$

($\underline{\Sigma \text{ convention}}$ $\underline{\text{supressed}}$).

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Order, $\lambda(1) > \lambda(2) > \lambda(3)$
(λ distinct)

$\Rightarrow N_{max} = \lambda(1)$, for $n_1' = 1$, $n_2' = n_3' = 0$, ie $\underline{n}(1)$

$N_{min} = \lambda(3)$, for $n_3' = 1$, $n_1' = n_2' = 0$, ie $\underline{n}(3)$

$\Rightarrow N$ extremum on p-planes.

$\underline{t} = [\underline{\underline{\sigma}}']\{\underline{n}'\} = \sigma_{ij}' n_j' \underline{e}_i'$

$$\boxed{\underline{t} = \sum_{i=1}^{3} \lambda(i)\, n_i' \underline{e}_i}$$ (Σ conv supressed).

$$\boxed{\begin{aligned} S^2 &= |\underline{t}|^2 - N^2 \\ S &= \Big\{ (\lambda(1) n_1')^2 + (\lambda(2) n_2')^2 + (\lambda(3) n_3')^2 \\ &\quad - \left(\lambda(1)(n_1')^2 + \lambda(2)(n_2')^2 + \lambda(3)(n_3')^2 \right)^2 \Big\}^{1/2} \end{aligned}}$$

Invariants of Stress.

I_1, I_2, I_3, are invariant with coord system. Hence p-stresses/axes invariant.

Proof:

$I_1 = \sigma_{ii}$

$$\sigma'_{i\cancel{j}_{\,i}} = \underbrace{a_{ir}\, a_{\cancel{j}_{\,i} s}}_{=\delta_{rs}} \sigma_{rs} = \sigma_{ss}$$

(ie do contraction, use $\underline{\underline{a}}\,\underline{\underline{a}}^T = \underline{\underline{I}}$)

(QED).

$$\underline{\underline{\sigma}}' = \underline{\underline{a}}\,\underline{\underline{\sigma}}\,\underline{\underline{a}}^T$$

$$I_3' = \det|\underline{\underline{\sigma}}'| = \det|\underline{\underline{a}}|\,\det|\underline{\underline{\sigma}}| \ast \det|\underline{\underline{a}}^T|$$

$$= \det|\underbrace{\underline{\underline{a}}\,\underline{\underline{a}}^T}_{\underline{\underline{I}}}|\, I_3$$

$$= I_3 \quad (\text{QED})$$

$$I_2 = \frac{1}{2}\big(\underbrace{\sigma_{ii}}_{I_1}\,\underbrace{\sigma_{jj}}_{I_1} - \sigma_{ij}\,\sigma_{ij}\big)$$ — expand to verify it is same as on p.12, L4

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$$I_2' = \frac{1}{2}\Big(\underbrace{\sigma_{ii}'\sigma_{jj}'}_{(I_1')^2 = (I_1)^2} - \sigma_{ij}'\sigma_{ij}'\Big)$$

$$\sigma_{ij}'\sigma'_{\underset{i}{\cancel{k}}\underset{j}{\cancel{k}}} = a_{ir}\,a_{js}\,a_{\underset{i}{\cancel{k}}t}\,a_{\underset{j}{\cancel{k}}u}\,\sigma_{rs}\,\sigma_{tu}$$

$$\delta_{rt} \qquad \delta_{su}$$

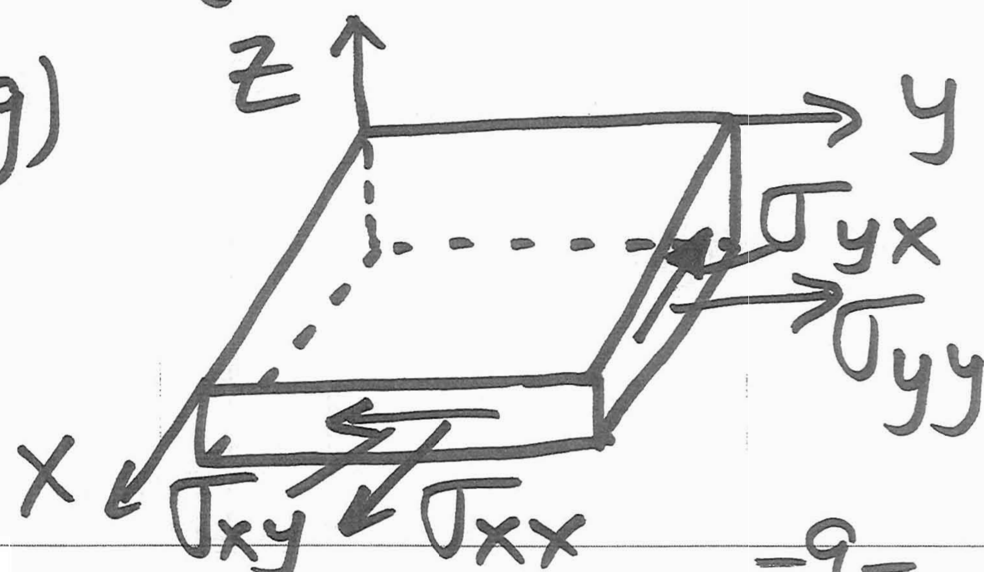
$$= \sigma_{rs}\,\sigma_{rs}$$

$$\Rightarrow I_2' = I_2 \quad (\text{QED})$$

Principal Stresses for 2D State of Stress.

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \sigma_{i3} = 0$$

(eg)



Thin plate with inplane edge loads which are functions of (x, y).

L5

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$I_1 = \sigma_{11} + \sigma_{22}$, $I_2 = \sigma_{11}\sigma_{22} - \sigma_{12}^2$

$I_3 = 0$

$$\lambda^3 - (\sigma_{11} + \sigma_{22})\lambda^2 + (\sigma_{11}\sigma_{22} - \sigma_{12}^2)\lambda = 0$$

$$\boxed{\lambda = 0, \quad \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}$$

$(-)$ for $\lambda(2)$

$(+)$ for $\lambda(3)$

$\underline{\lambda(1) = 0}$: $\sigma_{11} n_1 + \sigma_{12} n_2 = 0 \rightarrow (a)$

$\sigma_{12} n_1 + \sigma_{22} n_2 = 0 \rightarrow (b)$; $0 = 0 \rightarrow (c)$

If $\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} \neq 0$, $(n_1 = n_2 = 0,\ n_3 = 1)$

If $\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} = 0 = \sigma_{11}\sigma_{22} - \sigma_{12}^2$
$= I_2$

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$\Rightarrow \lambda = 0$ is double root. ie $\lambda(1) = \lambda(2) = 0$.

ie (a), (b) dependent.

(a) $\Rightarrow n_2 = -\dfrac{\sigma_{11}}{\sigma_{12}} n_1$

(c) $\Rightarrow n_3 =$ arbitrary.

(eg) $(0, 0, 1)$ & $\left(1, -\dfrac{\sigma_{11}}{\sigma_{12}}, 1\right)$

ie Choose arbitrary n_1, n_3, ie 2 evectors & then normalize.

<u>$\lambda(2), \lambda(3)$</u> : get evectors in usual manner ($I_2 \neq 0 \Rightarrow \lambda(2) \neq 0$).

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$$
\left.
\begin{aligned}
(\sigma_{11}-\lambda(3))\,n_1(3)+\sigma_{12}\,n_2(3)&=0\\
\sigma_{12}\,n_1(3)+(\sigma_{22}-\lambda(3))\,n_2(3)&=0
\end{aligned}
\right\}
$$

Only one of these two indep. . .

$$-\lambda(3)\,n_3(3)=0$$

$$\Rightarrow n_3(3)=0$$

use here

$$1=(n_1(3))^2+\left(\frac{\sigma_{11}-\lambda(3)}{\sigma_{12}}\right)^2(n_1(3))^2$$

$$n_1(3)=\pm\frac{\sigma_{12}}{\left[\sigma_{12}^2+(\sigma_{11}-\lambda(3))^2\right]^{1/2}};\quad n_2(3)=\frac{\mp(\sigma_{11}-\lambda(3))}{[\quad]^{1/2}}$$

Further on $\lambda(1) = \lambda(2) = 0$

For n_1, n_3 chosen as on p. 11,

$$\underline{n}(1) = (0, 0, 1)$$

$$\underline{n}(2) = \left(1, -\sigma_{11}/\sigma_{12}, 1\right) \Big/ \sqrt{2 + \left(\frac{\sigma_{11}}{\sigma_{12}}\right)^2}$$

use

$$\lambda(3) = \sigma_{11} + \sigma_{22}$$

$$\underline{n}(1) \cdot \underline{n}(3) = 0$$

$$\underline{n}(2) \cdot \underline{n}(3) = \left(\pm \sigma_{12} \mp \frac{\sigma_{11}\sigma_{22}}{\sigma_{12}}\right) \Big/ \left(\begin{matrix}\text{some}\\ \text{denom}\end{matrix}\right)$$

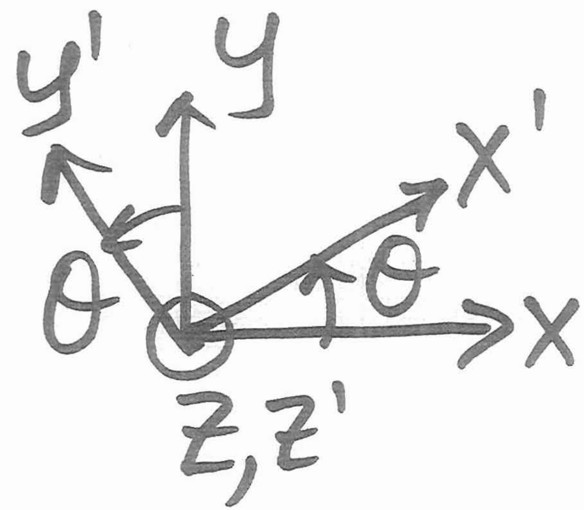
$$= 0$$

So $\underline{n}(1)$, $\underline{n}(2)$ always in plane $\perp^{ar}$ to $\underline{n}_3$

So whether $\lambda(2) \neq 0$ (unrepeated), or $\lambda(2) = 0$ (repeated roots), $\underline{n}(3) = (0, 0, 1)$ is valid evector.

$\Rightarrow$ Transf to p-system is rot. in $X_1 X_2$ plane, i.e.,

$$\underline{\underline{a}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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$$\underline{\underline{\sigma}}' = \underline{\underline{a}}\,\underline{\underline{\sigma}}\,\underline{\underline{a}}^T \text{ with } \sigma_{i3}=0, \text{ gives}$$

$$\sigma_{11}' = \left(\frac{1+c2\theta}{2}\right)\sigma_{11} + s2\theta\,\sigma_{12} + \left(\frac{1-c2\theta}{2}\right) * \sigma_{22}$$

$$\sigma_{22}' = \left(\frac{1-c2\theta}{2}\right)\sigma_{11} - s2\theta\,\sigma_{12} + \left(\frac{1+c2\theta}{2}\right)\sigma_{22}$$

$$\sigma_{12}' = \left(\frac{\sigma_{22}-\sigma_{11}}{2}\right)s2\theta + \sigma_{12}\,c2\theta$$

$$\sigma_{i3}' = 0$$

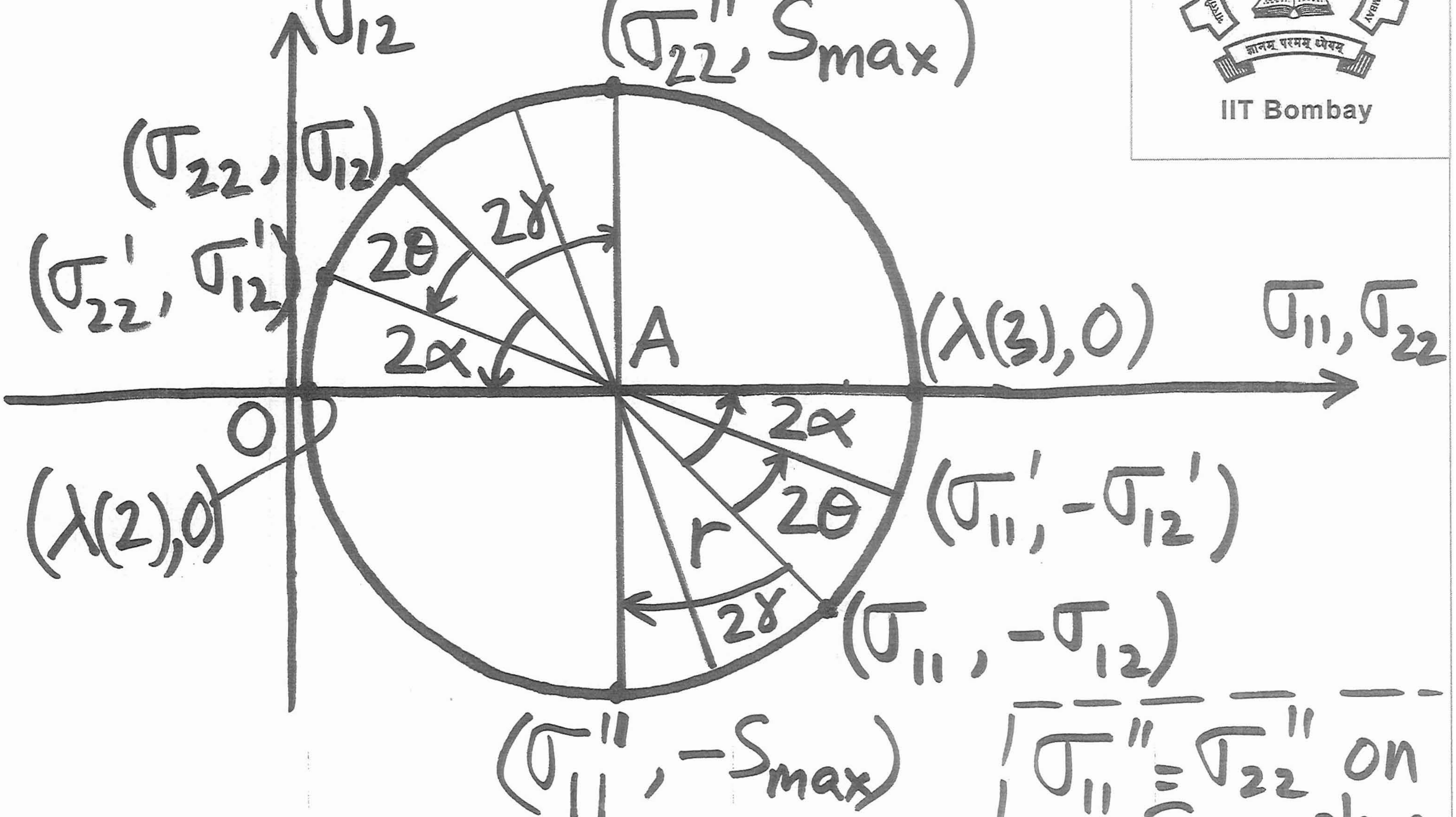
get $\sigma_{11}' = \lambda(2)$, $\sigma_{22}' = \lambda(3)$

p.10

subst

$$\Rightarrow \sigma_{12}' = 0 \text{ for } \tan 2\theta = \left(\frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}}\right)$$

Mohr's Circle – 2D



$\sigma_{11}'' = \sigma_{22}''$ on S_{max} plane

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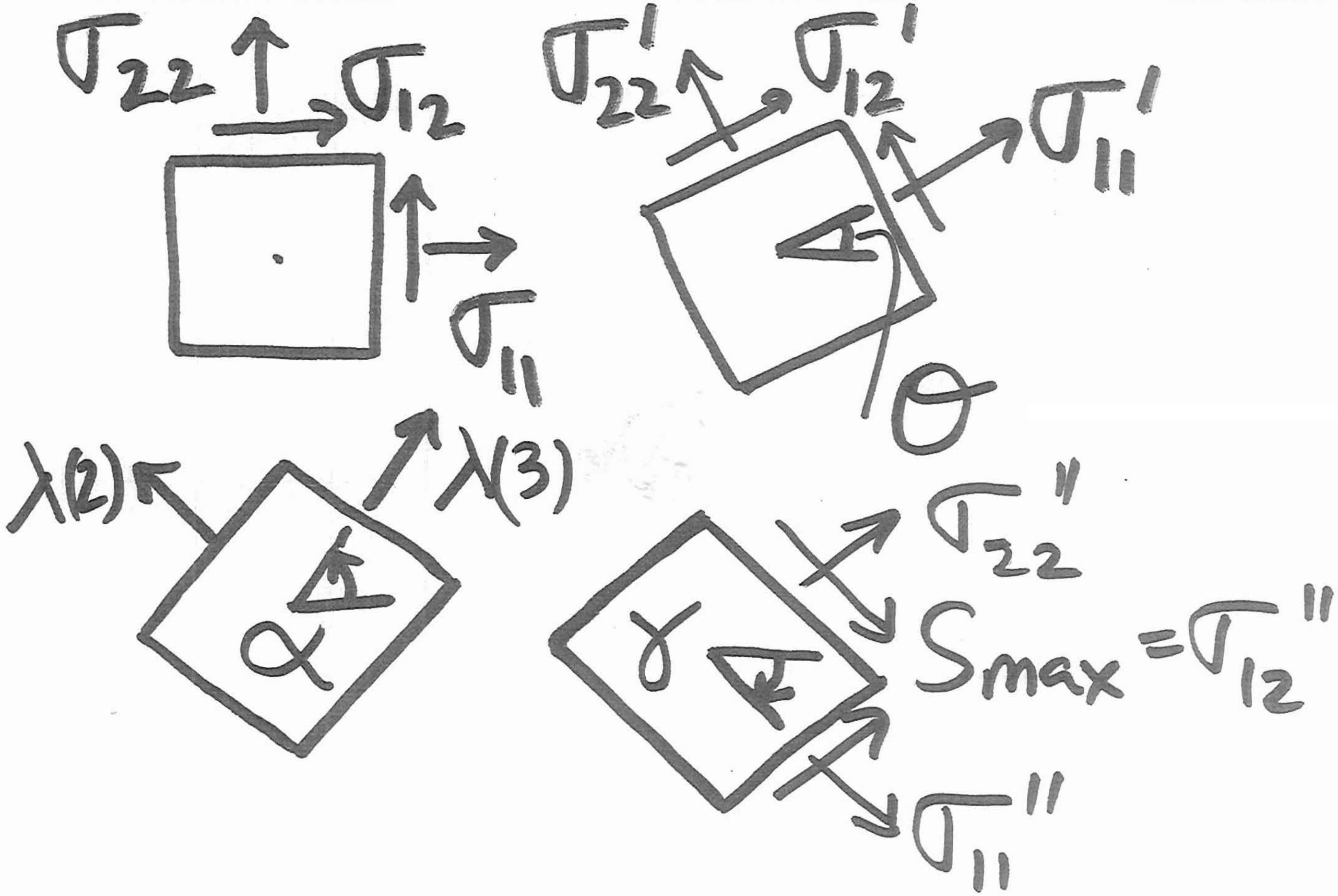

$$OA = \sigma_{22} + \frac{\sigma_{11} - \sigma_{22}}{2} = \frac{\sigma_{11} + \sigma_{22}}{2}$$

$$r^2 = (\sigma_{12})^2 + (\sigma_{11} - OA)^2 = \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2$$

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$$\sigma_{11}' = r\cos(2\alpha - 2\theta) + OA$$

$$\sigma_{22}' = OA - r\cos(2\alpha - 2\theta)$$

$$\sigma_{12}' = r\sin(2\alpha - 2\theta)$$

$$\cos 2\alpha = \frac{\sigma_{11} - \sigma_{22}}{2r}, \quad \sin 2\alpha = \frac{\sigma_{12}}{r}$$

Subst OA, r, $\cos 2\alpha$, $\sin 2\alpha$, get σ_{11}', σ_{22}', σ_{12}' as on p.14.

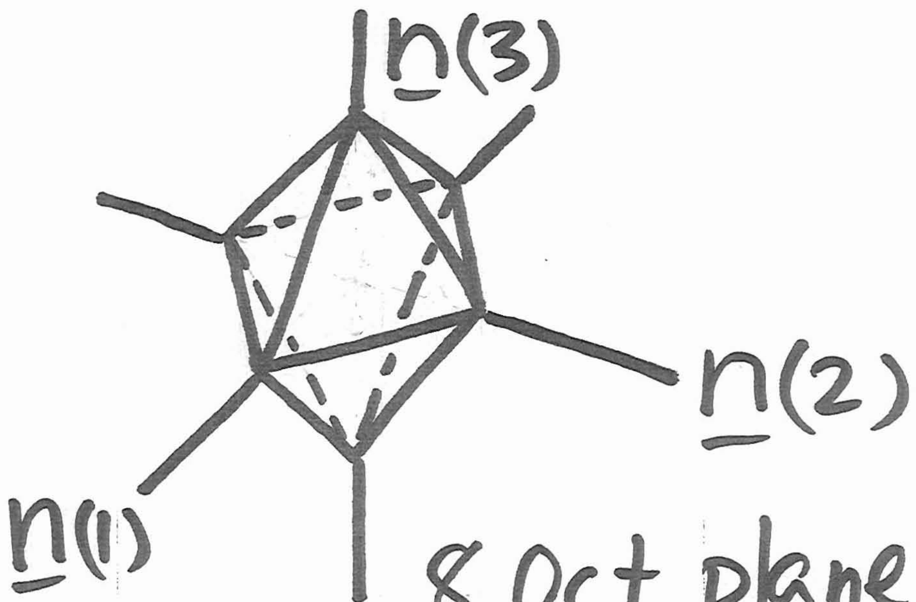
$$\left.\begin{aligned} \lambda(2) &= OA + r \\ \lambda(3) &= OA - r \end{aligned}\right\} \text{same as p.10.}$$

Mohr's circle verified.

Stresses on Octahaedral plane

Refer $\underline{n}$ to p-system. $\rightarrow \underline{e}_i = \underline{n}(i)$

$$\underline{n} = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right) \rightarrow \text{Oct. plane.}$$



8 Oct planes

$$N_{oct} = \frac{\lambda(1)+\lambda(2)+\lambda(3)}{3}$$

$$\boxed{N_{oct} = I_1/3}$$

$$\underline{t}_{oct} = \left(\frac{\lambda(1)}{\pm\sqrt{3}}, \frac{\lambda(2)}{\pm\sqrt{3}}, \frac{\lambda(3)}{\pm\sqrt{3}}\right)$$

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$$S_{oct}^2 = \left(\lambda_{(1)}^2 + \lambda_{(2)}^2 + \lambda_{(3)}^2\right)/3 - \left(I_1/3\right)^2$$

$$I_1^2 = \lambda_{(1)}^2 + \lambda_{(2)}^2 + \lambda_{(3)}^2 + 2\left(\lambda_{(1)}\lambda_{(2)} + \lambda_{(2)}\lambda_{(3)} + \lambda_{(3)}\lambda_{(1)}\right)$$

$$I_2 = \lambda_{(1)}\lambda_{(2)} + \lambda_{(2)}\lambda_{(3)} + \lambda_{(3)}\lambda_{(1)}$$

$$\boxed{S_{oct}^2 = \frac{\left(I_1^2 - 2I_2\right)}{3} - \left(I_1/3\right)^2} \boxed{= \frac{2}{9}I_1^2 - \frac{2}{3}I_2}$$

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LECTURE 6

STRESS ANALYSIS.

- Pure Shear State of stress
- Deviatoric stress
- Max shear stress.
- Equilibrium equations.

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Pure Shear State of Stress.

If $\underline{\underline{\sigma}} \longrightarrow \underline{\underline{\sigma}}'$:

$$\underline{\underline{\sigma}}' = \begin{bmatrix} 0 & \sigma_{12}' & \sigma_{13}' \\ & 0 & \sigma_{23}' \\ \text{symm} & & 0 \end{bmatrix},$$

then $\underline{\underline{\sigma}}$ is pure shear state of stress.

Necessary & sufficient condt for pure shear state to exist is

$$\sigma_{ii} = 0 = \text{trace}\left(\underline{\underline{\sigma}}\right) = I_1$$

Necessary: $\cancel{\sigma_{11}'}+\cancel{\sigma_{22}'}+\cancel{\sigma_{33}'}=I_1$

$\qquad\qquad 0 \qquad 0 \qquad 0$

$$= 0 = \sigma_{xx}+\sigma_{yy}+\sigma_{xx}$$

Sufficient:

$I_1 = 0 \Rightarrow$ at least one diagonal comp >0 & one <0.

Say $\sigma_{11}>0$, $\sigma_{22}<0$.

Do rot in 1-2 plane $\rightarrow \underline{\underline{a}} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so that $\sigma_{11}''=0$. Possible?

$$\sigma_{33}'' = a_{3r}\, a_{3s}\, \sigma_{rs} = \sigma_{33}$$

$$\sigma_{11}'' = a_{1r}\, a_{1s}\, \sigma_{rs} = c^2\theta\, \sigma_{11} + s^2\theta\, \sigma_{22} + 2c\theta s\theta\, \sigma_{12}$$

$$\| ?$$

$$0 = x\sigma_{11} + (1-x)\sigma_{22} \pm 2\sqrt{x}\sqrt{1-x}\ \sigma_{12}, \qquad \underline{x = \cos^2\theta}$$

$$\underbrace{\left(\sigma_{11}^2 + \sigma_{22}^2 + 4\sigma_{12}^2 - 2\sigma_{11}\sigma_{22}\right)}_{A} x^2 + \underbrace{\left(2\sigma_{11}\sigma_{22} - 2\sigma_{22}^2 - 4\sigma_{12}^2\right)}_{B} x + \underbrace{\sigma_{22}^2}_{C} = 0$$

$$x = \frac{1}{2A}\left[2\left(\sigma_{22}^2 + 2\sigma_{12}^2 - \sigma_{11}\sigma_{22}\right) \pm 4\sqrt{\sigma_{12}^4 - \sigma_{11}\sigma_{22}\sigma_{12}^2}\right]$$

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$\sigma_{22} < 0 \Rightarrow A > 0, \; -B > 0, \; C > 0$

$\Rightarrow -B > \sqrt{B^2 - 4AC}$

$\Rightarrow x > 0$, $\cos\theta$ real,

transformation: $\sigma_{11}'' = 0$ possible.

$\Rightarrow \sigma_{11}'' = 0$, σ_{22}'' & $\sigma_{33}'' (= \sigma_{33})$ have opp. sign $\because I_1 = 0$. Can Do 2-3 rotation

: $\sigma_{22}' = 0$ using same arguements.

$\therefore I_1 = 0$, $\sigma_{33}' = 0$ simultaneously.

[Note $\sigma_{11}' = \sigma_{11}'' = 0$ ($\because$ 2-3 rot)] QED

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L6

Deviatoric Stress.

$$\sigma_{ij} = \underbrace{\hat{\sigma}_{ij}}_{\text{Deviatoric}} + \underbrace{\frac{1}{3}\sigma_{kk}\delta_{ij}}_{\text{Cubical}}$$

Deviatoric - Cubical - State of stress

$$\underline{\underline{\sigma}} = \underline{\underline{\hat{\sigma}}} + \begin{bmatrix} I_1/3 & 0 & 0 \\ 0 & I_1/3 & 0 \\ 0 & 0 & I_1/3 \end{bmatrix}$$

— pure hydrostatic state

Contraction $\rightarrow$ $\sigma_{ii} = \hat{\sigma}_{ii} + \frac{1}{\cancel{3}}\sigma_{kk}\cancel{\delta_{ii}}^{\cancel{3}}$

$\hat{\sigma}_{ii} = 0$, $\underline{\underline{\hat{\sigma}}}$ pure shear state

Maximum Shear Stress

So far

- Principal Stresses/Max-min Normal Stress, $\underline{t} = \lambda \underline{n}$, $\sigma'_{ij} = 0$ $i \neq j$, (on $\underline{e}'_i$ planes) N extremum, $S = 0_\lambda$. Transf always exists
- Pure Shear State, $\sigma'_{ij} = 0$, $i = j$, on $\underline{e}'_i$ planes $N = 0$, $S \neq \text{extremum}_\lambda$ (in general), Transf. exists iff $I_1 = 0$.

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Now seek $\underline{e}_i'$ planes on which $S =$ extremum, $N \neq 0$ in general.

Refer to p-axes system $(\underline{e}_i)$

$$S^2 = (\lambda(1)\, n_1)^2 + (\lambda(2)\, n_2)^2 + (\lambda(3)\, n_3)^2 - \left[\lambda(1)\, n_1^2 + \lambda(2)\, n_2^2 + \lambda(3)\, n_3^2\right]^2$$

(L5, p.5)

$$\frac{\partial S^2}{\partial n_1} = \frac{\partial S^2}{\partial n_2} = \frac{\partial S^2}{\partial n_3} = 0$$ wont work $\because$ $\underbrace{|\underline{n}| = 1}_{\text{constraint.}}$

Can eliminate, say $n_3^2 = 1 - n_1^2 - n_2^2$, then

$\partial S^2/\partial n_1 = \partial S^2/\partial n_2 = 0$ will work — tedious.

So use Lagrange Multiplier approach.

$$G = S^2 + L(1 - n_1^2 - n_2^2 - n_3^2)$$

ie $G = S^2$ if constraint satisfied.

Now L (Lagr Mult), n_1, n_2, n_3 are indep ∵ constraint built in.

$$\frac{\partial G}{\partial L} = 1 - n_1^2 - n_2^2 - n_3^2 = 0 \rightarrow (a)$$

$$\frac{\partial G}{\partial n_1} = n_1\left\{\lambda^2(1) - 2\left[\lambda(1) n_1^2 + \lambda(2) n_2^2 + \lambda(3) n_3^2\right]\lambda(1) + L\right\} = 0 \rightarrow (b)$$

$$\frac{\partial G}{\partial n_2} = n_2\left\{\lambda^2(2) - 2\left[\lambda(1) n_1^2 + \lambda(2) n_2^2 + \lambda(3) n_3^2\right]\lambda(2) + L\right\} = 0 \rightarrow (c)$$

$$\frac{\partial G}{\partial n_3} = n_3\left\{\lambda^2(3) - 2\left[\lambda(1) n_1^2 + \lambda(2) n_2^2 + \lambda(3) n_3^2\right]\lambda(3) + L\right\} = 0 \rightarrow (d)$$

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Since (a) represents constraint, stationary values of G same as those of S^2, ie treating λ, n_1, n_2, n_3 as indepdent works !!

<u>Case(i)</u>: only one comp of $\underline{n}$ non-zero.

$n_1=1, n_2=n_3=0, \ L=\lambda^2(1), \ S^2=0$

$n_2=1, n_1=n_3=0, \ L=\lambda^2(2), \ S^2=0$

$n_3=1, n_1=n_2=0, \ L=\lambda^2(3), \ S^2=0$

same as P-stress soln, S is stationary but not max.

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Case (ii) : one comp of $\underline{n}$ zero, two non-zero.

$n_1 = 0,\ n_2 \neq 0,\ n_3 \neq 0$

(a), (c), (d), $\rightarrow$ $$\lambda^2(2) - 2\left[\lambda(2)n_2^2 + \lambda(3) - \lambda(3)n_2^2\right]\lambda(2) + L = 0$$

$$\lambda^2(3) - 2\left[\lambda(2)n_2^2 + \lambda(3) - \lambda(3)n_2^2\right]\lambda(3) + L = 0$$

$$\Rightarrow \left[\lambda(2) - \lambda(3)\right]\left[\lambda(2) + \lambda(3) - 2\left(\lambda(2)n_2^2 + \lambda(3) - \lambda(3)n_2^2\right)\right] = 0$$

Either $\lambda(2) = \lambda(3)$, $n_2 =$ arb, $L = \lambda^2(2)$, $S^2 = 0$

or $\lambda(2) \neq \lambda(3)$, $n_2 = \pm\frac{1}{\sqrt{2}}$, $n_3 = \pm\frac{1}{\sqrt{2}}$, $L = \lambda(2)\lambda(3)$

$$\left[\lambda(2)-\lambda(3)\right]\left[1-2n_2^2\right]=0$$

11a

L6

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$$S^2 = \left(\lambda^2(2) + \lambda^2(3) - 2\,\lambda(2)\,\lambda(3)\right)/4$$

$$S = \frac{|\lambda(2) - \lambda(3)|}{2}$$

Similarly for $n_2 = 0$, $n_1 \neq 0$, $n_3 \neq 0$

& for $n_3 = 0$, $n_1 \neq 0$, $n_2 \neq 0$

Summary case (ii):

$$\underbrace{n_1 = 0,\ n_2 = \pm\frac{1}{\sqrt{2}},\ n_3 = \pm\frac{1}{\sqrt{2}}}_{\lambda(2) \neq \lambda(3)};\quad \underbrace{n_2 = \text{arb},\ n_3 = \pm\sqrt{1-n_2^2}}_{\lambda(2) = \lambda(3)};$$

$$L = \lambda(2)\,\lambda(3)\ , \quad S = \frac{1}{2}\,|\lambda(2) - \lambda(3)|$$

$$n_2 = 0\,,\ \underbrace{n_1 = \pm\frac{1}{\sqrt{2}}\,,\ n_3 = \pm\frac{1}{\sqrt{2}}}_{\lambda(1)\neq\lambda(3)}\ ;$$

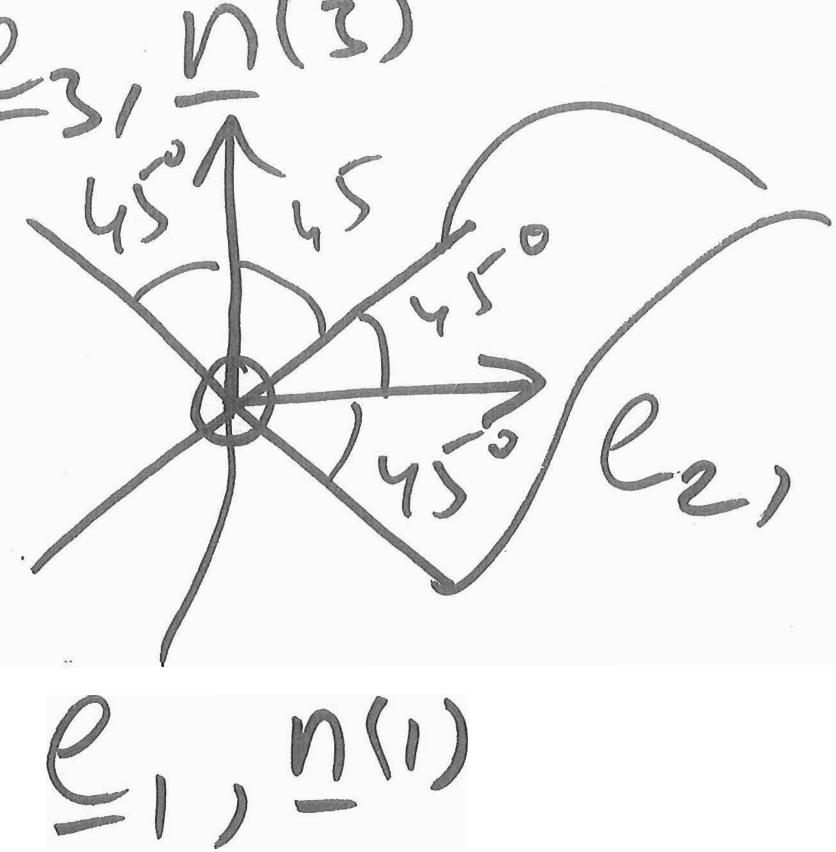
$$\underbrace{n_1 = arb\,,\ n_3 = \pm\sqrt{1-n_1^2}}_{\lambda(1)=\lambda(3)}\,,\quad L = \lambda(1)\lambda(3)\,,$$

$$S = \frac{1}{2}\left|\lambda(1)-\lambda(3)\right|$$

$$n_3 = 0\,,\ \underbrace{n_1 = \pm\frac{1}{\sqrt{2}}\,,\ n_2 = \pm\frac{1}{\sqrt{2}}}_{\lambda(1)\neq\lambda(2)}\ ;\ \underbrace{n_1 = arb\,,\ n_2 = \pm\sqrt{1-n_1^2}}_{\lambda(1)=\lambda(2)}$$

$$L = \lambda(1)\lambda(2)\,,\quad S = \frac{1}{2}\left|\lambda(1)-\lambda(2)\right|$$

If $S_{max} = \dfrac{|\lambda(2) - \lambda(3)|}{2}$



$\underline{n} = 0\,e_1 + \pm \frac{1}{\sqrt{2}}\underline{e}_2 \pm \frac{1}{\sqrt{2}}\underline{e}_3$

n_1

If $\lambda(2) = \lambda(3)$.

$$S_{max} = \left| \frac{\lambda(1) - \lambda(2)}{2} \right|$$

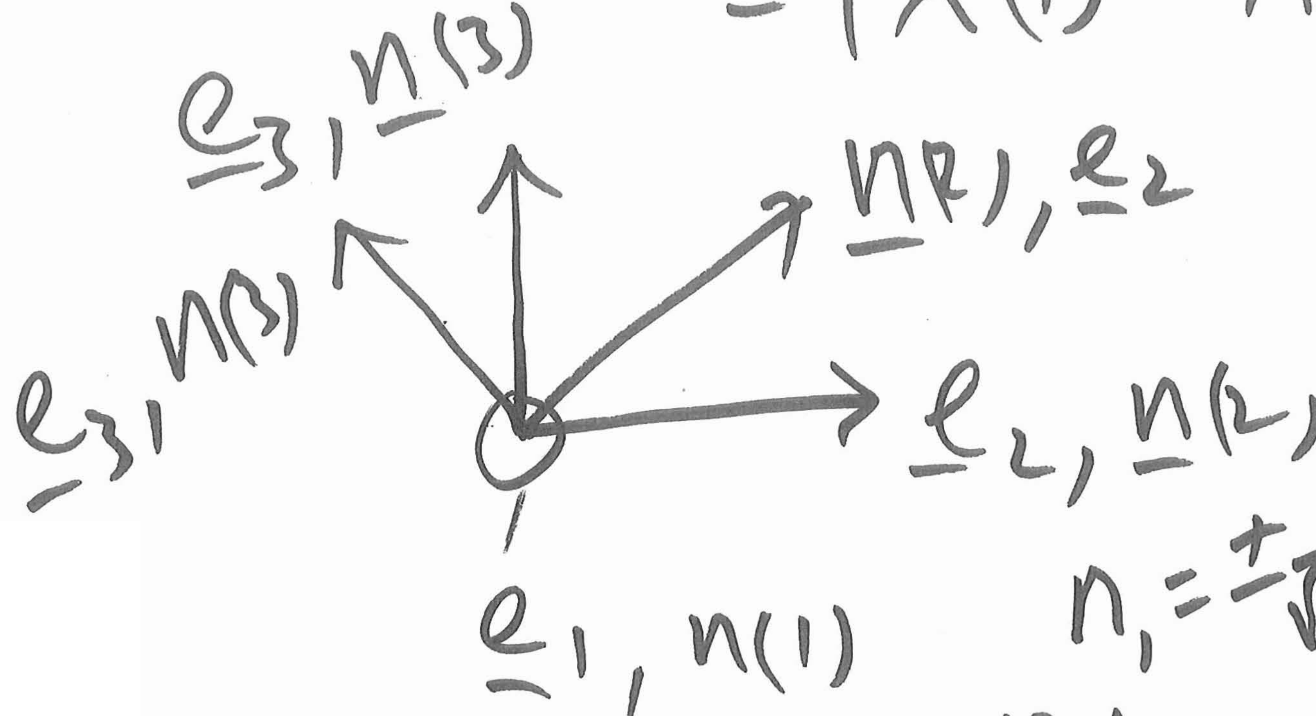
$$= |\lambda(1) - \lambda(3)|/2$$

$n_1 = \pm\frac{1}{\sqrt{2}}$

$n_2 = \pm\frac{1}{\sqrt{2}}$

$n_1 = \pm\frac{1}{\sqrt{2}}$

$n_3 = ?$



$n_2 = $ arb.

$n_1 = \pm\frac{1}{\sqrt{2}}, \; n_3^2 = 1 - n_2^2 - n_1^2$

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$S_{max} = \max\left(\frac{1}{2}|\lambda(1)-\lambda(2)|, \frac{1}{2}|\lambda(2)-\lambda(3)|, \frac{1}{2}|\lambda(1)-\lambda(3)|\right)$

S_{max} occurs on planes containing one p-axis & equally ($\pm 45^\circ$) inclined to other two p-axes, when (all) p-stresses are distinct.

If any two p-stresses same, say $\lambda(1)=\lambda(2)$, $S_{max} = \frac{1}{2}|\lambda(1) - \lambda(3)| = \frac{1}{2}|\lambda(2)-\lambda(3)|$, acts on plane with $n_3 = \pm\frac{1}{\sqrt{2}}$, $n_1 = \text{arb}$, $n_2 = \pm\sqrt{1-n_1^2-n_3^2}$

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$\because$ $\underline{n}(1)$, $\underline{n}(2)$ arbitrary but $\perp^{ar}$ to each other & to $\underline{n}(3)$.

Ref 2D Mohr's circle (L5, p.15) to see results recovered for 2D case.

<u>Case (iii):</u> $n_1 \neq 0$, $n_2 \neq 0$, $n_3 \neq 0$.

Here (b, c, d), p.9, give inconsistent system of eqns in n_1^2, n_2^2, n_3^2, unless $\lambda(1) = \lambda(2) = \lambda(3)$. <u>See det|coeff mat| = 0</u>

So only possible soln i.e. for equal p-stresses, yields $S = 0$ on all planes (as before)

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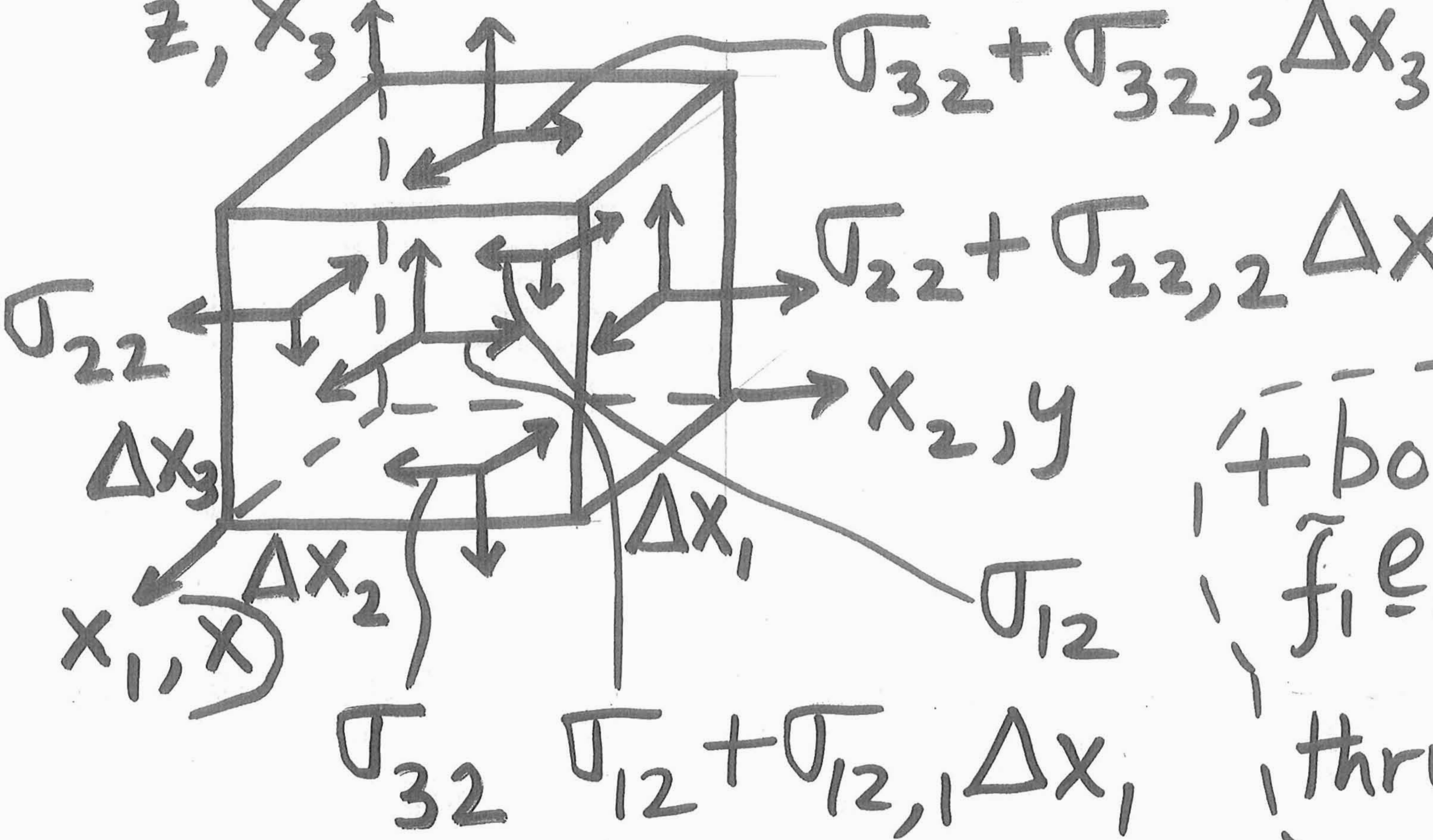
Equilibrium Equations.



+ body force $\tilde{f}_1 \underset{\sim}{e}_1 + \tilde{f}_2 \underset{\sim}{e}_2 + \tilde{f}_3 \underset{\sim}{e}_3$ thru centroid

Fig: Stresses on infinitesimal parallelopiped element at P.

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L6

$\sum F_y = 0$:

$$(\cancel{\sigma_{22}} + \sigma_{22,2}\Delta x_2 - \cancel{\sigma_{22}})\Delta x_1 \Delta x_3$$
$$+ (\cancel{\sigma_{12}} + \sigma_{12,1}\Delta x_1 - \cancel{\sigma_{12}})\Delta x_2 \Delta x_3$$
$$+ (\cancel{\sigma_{32}} + \sigma_{32,3}\Delta x_3 - \cancel{\sigma_{32}})\Delta x_1 \Delta x_2$$
$$+ f_2 \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} + \tilde{f}_2 = 0$$

Similarly for $\sum F_x = 0$, $\sum F_z = 0$.

$$\boxed{\sigma_{ji,j} + \tilde{f}_i = 0} \rightarrow \text{EQUIL EQ (CARTESIAN)} \quad (1)$$

Lecture 7

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STRESS ANALYSIS.

- Equilibrium in Curvilinear Coordinates
- Boundary conditions (stress)
- Problems.

—o—

L7

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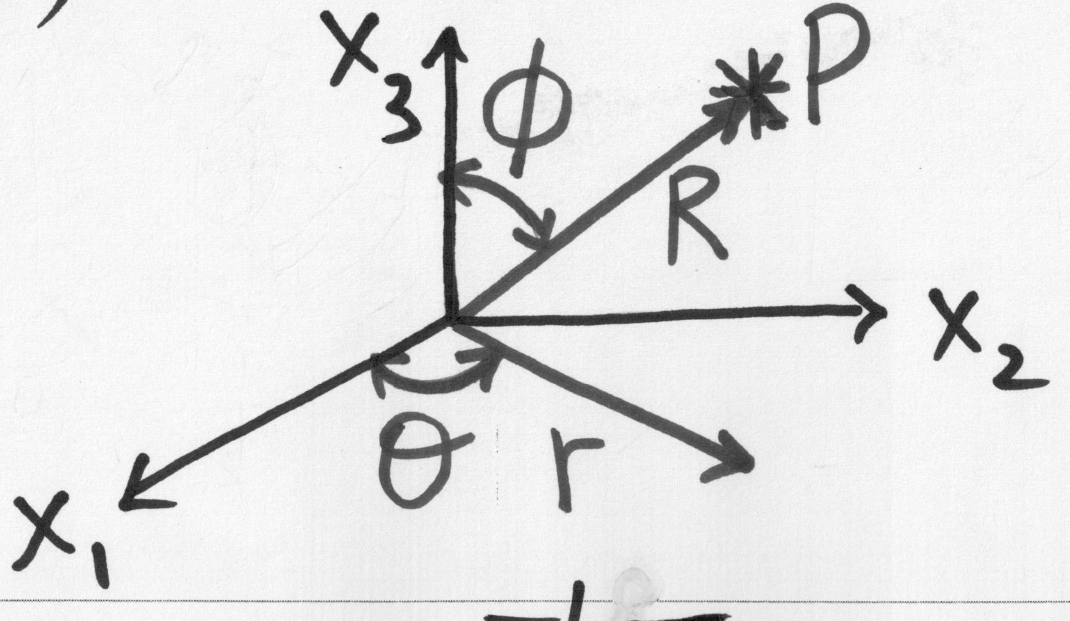
Equilibrium equations in curvilinear orthogonal coordinates

$(r, \theta, z) \rightarrow$ cylindrical coords.

$(R, \theta, \phi) \rightarrow$ spherical coords.



L7

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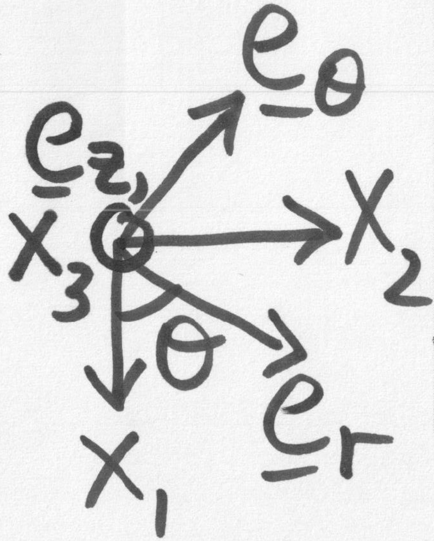
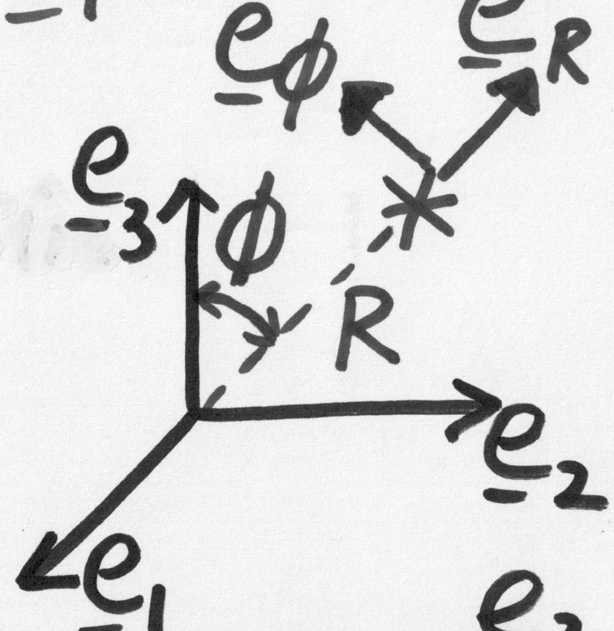
$$\underline{\underline{a}} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

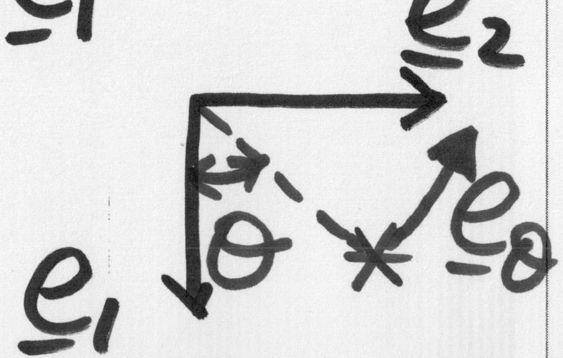
for cyl.



$$\underline{\underline{a}} = \begin{bmatrix} s\phi c\theta & s\phi s\theta & c\phi \\ -s\theta & c\theta & 0 \\ -c\phi c\theta & -c\phi s\theta & s\phi \end{bmatrix}$$

for spherical





so $\underline{e}_R \times \underline{e}_\theta = \underline{e}_\phi$

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$$\sigma_{ji} = a_{rj}\, a_{si}\, \sigma_{rs}'$$

$$\frac{\partial(\;)}{\partial x_j} = \frac{\partial(\;)}{\partial x_k'}\,\cancel{\frac{\partial x_k'}{\partial x_j}}$$

transf betwn a_{kj} for ⊥ RCC systems
else $\frac{\partial r}{\partial x}$ etc

Will give transf equil eqns

$$0 = \sigma_{ji,j} + \tilde{f}_i = \left(a_{rj}\, a_{si}\, \sigma_{rs}'\right)_{,k} \frac{\partial x_k'}{\partial x_j}$$

eg $\sigma_{rs}' \equiv \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$, $(\;)_{,k} = \begin{Bmatrix} (\;)_{,r} \\ (\;)_{,\theta} \\ (\;)_{,z} \end{Bmatrix}$

In general curvilinear coordinates (see eg. Boresi & Schmidt, Advanced Mechanics of Materials, 2003)

$$\frac{\partial(\beta\gamma\sigma_{xx})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{xy})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{xz})}{\partial z}$$

$$+ \gamma\sigma_{xy}\frac{\partial\alpha}{\partial y} + \beta\sigma_{xz}\frac{\partial\alpha}{\partial z} - \gamma\sigma_{yy}\frac{\partial\beta}{\partial x}$$

$$- \beta\sigma_{zz}\frac{\partial\gamma}{\partial x} + \alpha\beta\gamma\bar{f}_x = 0$$

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$$\frac{\partial(\beta\gamma\sigma_{xy})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yy})}{\partial y} + \frac{\partial(\alpha\beta\,\sigma_{yz})}{\partial z} + \gamma\frac{\partial\beta}{\partial x}\sigma_{xy} + \alpha\frac{\partial\beta}{\partial z}\sigma_{yz} - \gamma\,\frac{\partial\alpha}{\partial y}\sigma_{xx} - \alpha\frac{\partial\gamma}{\partial y}\sigma_{zz} + \alpha\beta\gamma\tilde{f}_y = 0$$

$$\frac{\partial(\beta\gamma\sigma_{xz})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yz})}{\partial y} + \frac{\partial(\alpha\beta\,\sigma_{zz})}{\partial z} + \beta\frac{\partial\gamma}{\partial x}\sigma_{xz} + \alpha\frac{\partial\gamma}{\partial y}\sigma_{yz} - \alpha\frac{\partial\beta}{\partial z}\sigma_{yy} - \beta\frac{\partial\alpha}{\partial z}\sigma_{xx} + \alpha\beta\gamma\hat{f}_z$$

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where

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2$$

$$= (\text{diagonal of element})^2$$

Cartesian: $\alpha = \beta = \gamma = 1$, $ds^2 = dx^2 + dy^2 + dz^2$

$x \to x$, $y \to y$, $z \to z$

Cylindrical: $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

$\alpha = 1$, $\beta = r$, $\gamma = 1$, $x \to r$, $y \to \theta$, $z \to z$

Spherical: $ds^2 = dR^2 + (R\sin\phi)^2 d\theta^2 + R^2 d\phi^2$

$\alpha = 1$, $\beta = R\sin\phi$, $\gamma = R$, $x \to R$, $y \to \theta$, $z \to \phi$

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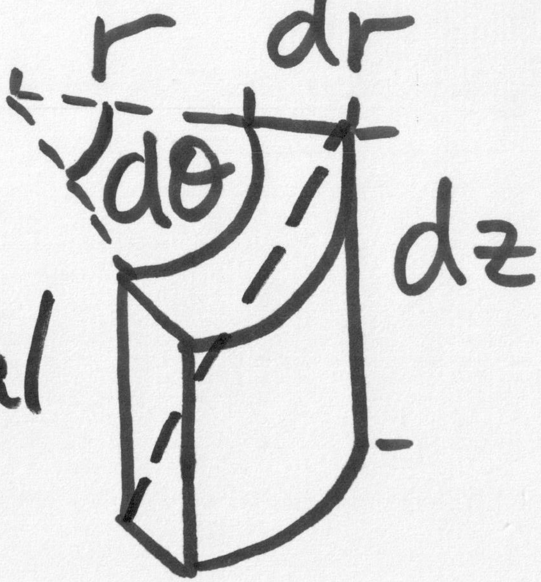
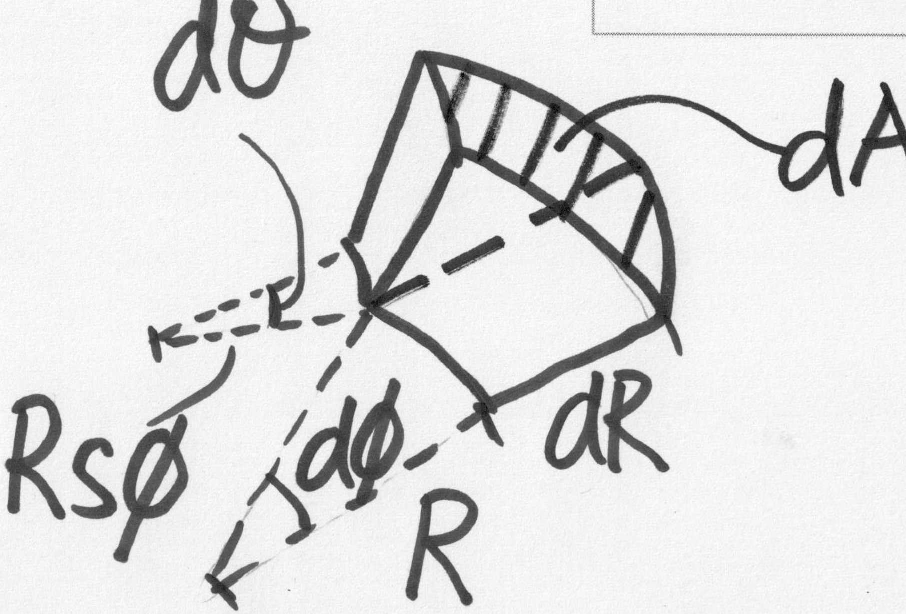
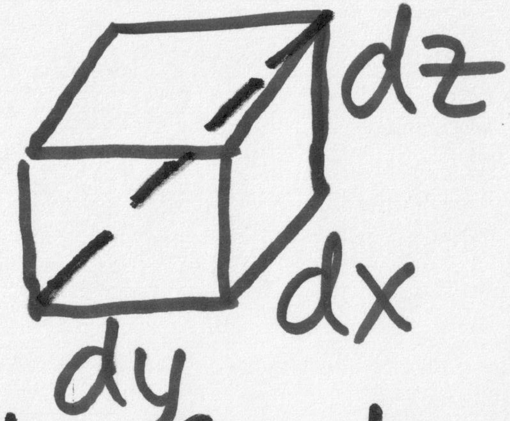


Cylindrical element



Spherical element



Rect. Cartesian element

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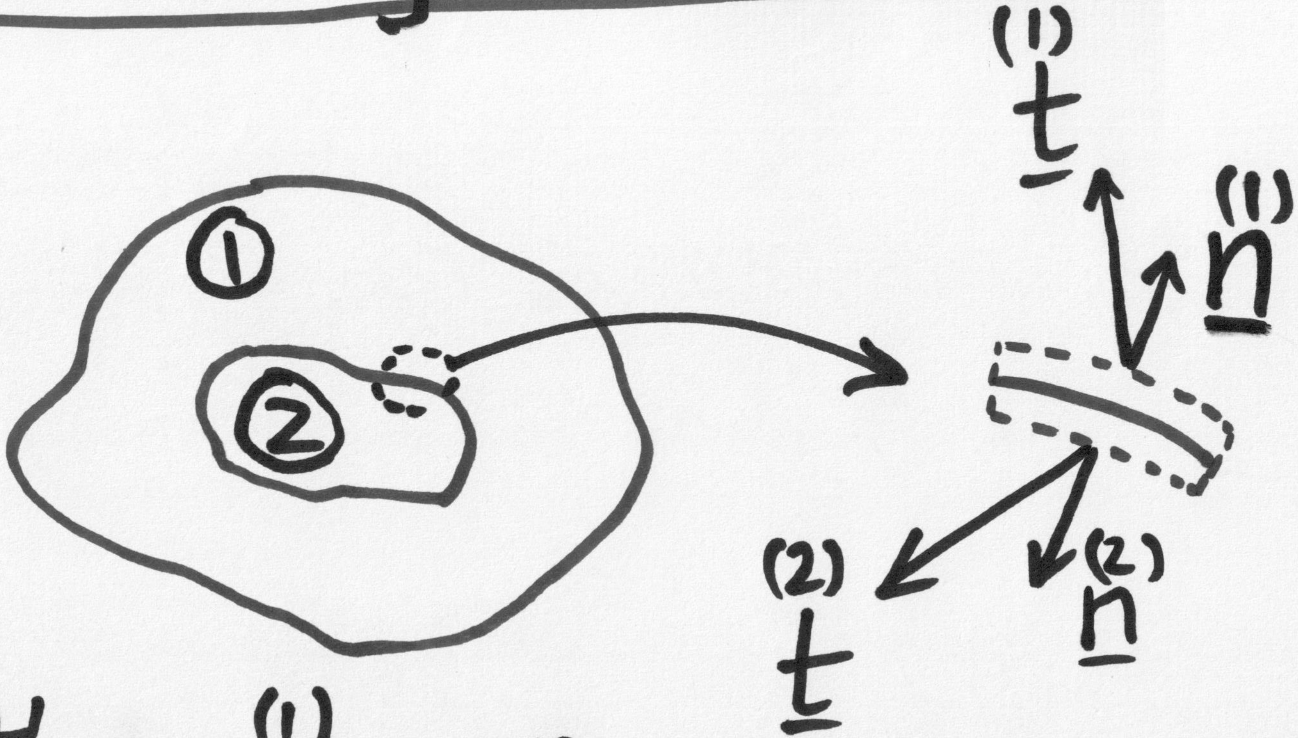
Boundary Conditions.



Thin slice around boundary.

$$\underline{n}^{(2)} = -\underline{n}^{(1)} = -\underline{n}$$

$$\sum \underline{F} = \underline{0} \Rightarrow \left(\underline{t}^{(1)} + \underline{t}^{(2)}\right) dA = 0 \quad \left(\begin{array}{l}\text{neglect} \\ \text{side faces}\end{array}\right)$$

$$\Rightarrow \underline{\underline{\sigma}}^{(1)} \underline{n}^{(1)} + \underline{\underline{\sigma}}^{(2)} \underline{n}^{(2)} = 0$$

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$$\left(\underline{\underline{\sigma}}^{(1)} - \underline{\underline{\sigma}}^{(2)}\right)\underline{n} = 0$$ (ie defines boundary)

$\therefore$ $\underline{n}$ not arbitrary,

$$\boxed{\underline{\underline{\sigma}}^{(1)}\,\underline{n} = \underline{\underline{\sigma}}^{(2)}\,\underline{n}} \;/\; \boxed{\sigma^{(1)}_{ij}\, n_j = \sigma^{(2)}_{ij}\, n_j}$$

ie.,

$$\sigma^{(1)}_{11} n_1 + \sigma^{(1)}_{12} n_2 + \sigma^{(1)}_{13} n_3 = \sigma^{(2)}_{11} n_1 + \sigma^{(2)}_{12} n_2 + \sigma^{(2)}_{13} n_3$$

$$\sigma^{(1)}_{21} n_1 + \sigma^{(1)}_{22} n_2 + \sigma^{(1)}_{23} n_3 = \sigma^{(2)}_{21} n_1 + \sigma^{(2)}_{22} n_2 + \sigma^{(2)}_{23} n_3$$

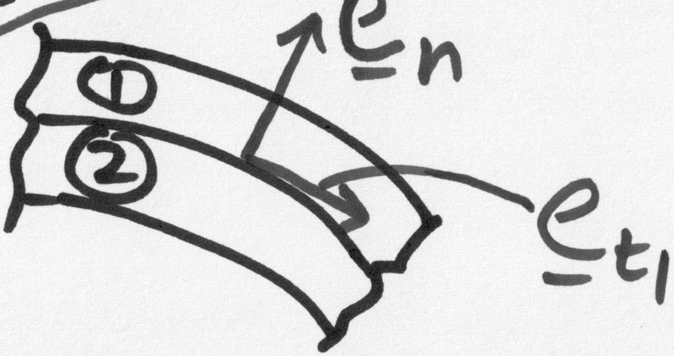
$$\sigma^{(1)}_{31} n_1 + \sigma^{(1)}_{32} n_2 + \sigma^{(1)}_{33} n_3 = \sigma^{(2)}_{31} n_1 + \sigma^{(2)}_{32} n_2 + \sigma^{(2)}_{33} n_3$$

(eg)

$$\underline{n} = \{0 \quad 0 \quad 1\}^T$$

$$\left\{ \overset{(1)}{\sigma}_{i3} = \overset{(2)}{\sigma}_{i3} \text{ on bndry} \right.$$

$i = 1, 2, 3$



$$\underline{n} = \underline{\nabla}\phi[x_1, x_2, x_3]$$

$$\overset{(1)}{\sigma}_{in} = \overset{(2)}{\sigma}_{in}$$

$i = 1 \to t_1$, $2 \to t_2$, $3 \to n$

ie $\overset{(1)}{\sigma}_{nn} = \overset{(2)}{\sigma}_{nn}$ $\to$ on bndry

$\overset{(1)}{\sigma}_{t_1 n} = \overset{(2)}{\sigma}_{t_1 n}$, $\sigma_{t_2 n} = \sigma_{t_2 n}$

for other stress comps no continuity across bndry.

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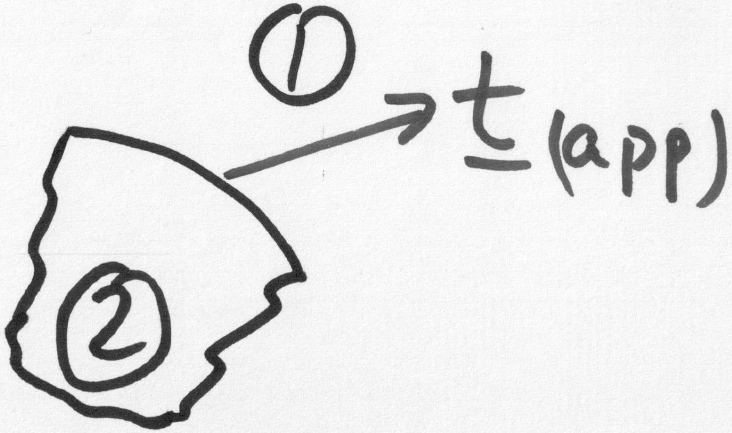

$$\overset{(2)}{\sigma}_{nn} = t_{n(app)}$$

$$\overset{(2)}{\sigma}_{t_1 n} = t_{t_1(app)}$$

$$\overset{(2)}{\sigma}_{t_2 n} = t_{t_2(app)}$$

If 1 & 2 same, then $\underline{n}$ arbitrary

$\Rightarrow \overset{(1)}{\sigma}_{ij} = \overset{(2)}{\sigma}_{ij}$ ie all comps continuous. (as expected).

L7

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P5 At a point in a solid principal stresses are

$\lambda(1) = 1$, $\lambda(2) = 4$, $\lambda(3) = -2$

p-axes are $\underline{n}(1) = \left(\frac{1}{2}, \frac{1}{2}, \sqrt{\frac{1}{2}}\right)$

$\underline{n}(2) = (0, \quad , \quad)$

$\boxed{\text{Find } \underline{\underline{\sigma}}}$.

$\underline{n}(1) \cdot \underline{n}(2) = 0 = n_2(2) \cdot \frac{1}{2} + n_3(2) \cdot \frac{1}{\sqrt{2}}$

$\underline{n}(2) \cdot \underline{n}(2) = 1 = n_2^2(2) + n_3^2(2)$

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$$\Rightarrow n_2(2) = \mp\sqrt{\frac{2}{3}}, \; n_3(2) = \pm\sqrt{\frac{1}{3}}$$

$$\underline{n}(3) = \underline{n}(1) \times \underline{n}(2)$$

$$= \left(\sqrt{\frac{1}{12}} + \sqrt{\frac{1}{3}}, \; -\sqrt{\frac{1}{12}}, -\sqrt{\frac{1}{6}}\right)$$

Let x_i' be p-axes system, ie

$\underline{e}_1' = \underline{n}(1)$, $\underline{e}_2' = n(2)$, $\underline{e}_3' = \underline{n}(3)$

$$\underline{\underline{a}} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 0 & -\sqrt{2/3} & \sqrt{1/3} \\ \sqrt{1/12}+\sqrt{1/3} & -\sqrt{1/12} & -\sqrt{1/6} \end{bmatrix}$$

L7

$\sigma_{ij} = a_{ri}\, a_{sj}\, \sigma'_{rs}$ / $\underline{\underline{\sigma}} = \underline{\underline{a}}^T \underline{\underline{\sigma}}' \underline{\underline{a}}$

$$\underline{\underline{\sigma}}' = \begin{bmatrix} \lambda(1) & 0 & 0 \\ 0 & \lambda(2) & 0 \\ 0 & 0 & \lambda(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} -1.25 & 0.75 & 1.061 \\ 0.75 & 2.75 & -1.768 \\ 1.061 & -1.768 & 1.5 \end{bmatrix}$$

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P.6. Given $\underline{\underline{\sigma}}$ at a point

$$\underline{\underline{\sigma}} = \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}$$

Find: (a) Principal Deviator stresses & corresponding planes

(b) N_{oct}, S_{oct}

(c) S_{max} & corresponding plane

Principal stress problem is

$$(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\,\underline{n} = \underline{0}$$

Also, $\hat{\underline{\underline{\sigma}}}$ defined as (L6, p.5)

$$\underline{\underline{\sigma}} = \hat{\underline{\underline{\sigma}}} + \frac{I_1}{3}\underline{\underline{I}}$$

$$\Rightarrow \left(\hat{\underline{\underline{\sigma}}} - \left(\lambda - \frac{I_1}{3}\right)\underline{\underline{I}}\right)\underline{n} = \underline{0}$$

$\hat{\lambda} = \lambda - \dfrac{I_1}{3}$ = p-deviator str, p-axes remain same.

Solving evp for $\lambda, \underline{n}$, (MATLAB)

L7

$\lambda^3 - 6\lambda^2 - 4\lambda + 24 = 0$

$\lambda(1) = -2$, $\lambda(2) = 2$, $\lambda(3) = 6$

$\underline{n}(1) = (1/\sqrt{2},\ 1/\sqrt{2},\ 0)^T$

$\underline{n}(2) = (-0.5\quad 0.5\quad 1/\sqrt{2})^T$

$\underline{n}(3) = (0.5\quad -0.5\quad 1/\sqrt{2})^T$

$\hat{\lambda}(1) = -2 - \dfrac{I_1}{3} = -2 - \dfrac{6}{3} = -4$, $\hat{\lambda}(2) = 0$

$\hat{\lambda}(3) = 6 - 2 = 4$

$N_{oct} = I_1/3 = 2$ (L5, p. 18)

L7

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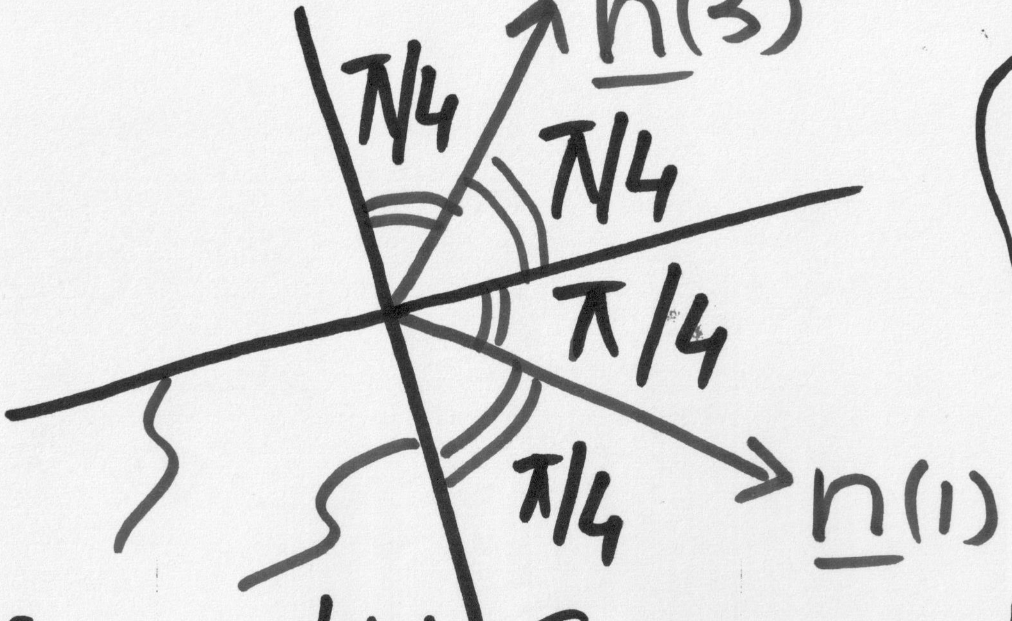
$$S_{oct}^2 = \frac{2}{9} I_1^2 - \frac{2}{3} I_2 = \frac{2}{9} \cdot 6^2 - \frac{2}{3}(-4)$$

$$S_{oct} = \sqrt{32/3}$$

$$S_{max} = (6-(-2))/2 = 4$$



Let S_{max} planes be defined by $\underline{n}$. Then

$$\begin{cases} \underline{n} \cdot \underline{n}(1) = 1/\sqrt{2} \\ \underline{n} \cdot \underline{n}(3) = \pm 1/\sqrt{2} \\ \underline{n} \cdot \underline{n} = 1 \end{cases} \quad \text{solve}$$

planes on which S_{max} acts:

P7 Rectangular plate, thickness $t = 1\,cm$, lying in region $0 \le x_1 \le 2b$, $-c \le x_2 \le c$, loaded in a manner that yields

$$\sigma_{11} = \frac{q}{2I}\left(x_1^2 x_2 - \frac{2}{3}x_2^3 + \frac{2}{5}c^2 x_2\right),$$

$$\sigma_{22} = \frac{q}{2I}\left(\frac{1}{3}x_2^3 - c^2 x_2 + \frac{2}{3}c^3\right), \quad \sigma_{i3} = 0$$

where $I = 2c^3/3$, q is constant.

Assume $\tilde{f}_b = \tilde{m}_b = 0$

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Find : (a) σ_{12} to ensure equil.
(b) BC's for which Soln of $\underline{\underline{\sigma}}$ valid.

Equilibrium eqns yield

$$\sigma_{11,1} + \sigma_{12,2} = 0 = \frac{q}{2I} 2x_1 x_2 + \sigma_{12,2}$$

$$\sigma_{12,1} + \sigma_{22,2} = 0 = \sigma_{12,1} + \frac{q}{2I}(x_2^2 - c^2)$$

$$\Rightarrow \sigma_{12} = -\frac{q}{2I} x_1 x_2^2 + f(x_1) + c_1$$

$$= -\frac{q}{2I}(x_2^2 - c^2) x_1 + g(x_2) + c_2$$

L7

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$$\Rightarrow f(x_1) = \frac{q}{2I} c^2 x_1, \quad g(x_2) = 0$$

$$c_1 = c_2 = R.$$

$$\sigma_{12} = -\frac{q}{2I}(x_2^2 - c^2)x_1 + R.$$

BC's: $x_1 = 0$, $(\sigma_{11})_{appl} = \frac{3q}{2c^3}\left(-\frac{x_2^3}{3} + \frac{c^2}{5}x_2\right)$

$$(\sigma_{12})_{appl} = R$$

$x_1 = 2b$, $(\sigma_{11})_{appl} = \frac{3q}{2c^3}\left(2b^2 x_2 - \frac{x_2^3}{3} + \frac{c^2}{5}x_2\right)$

$$(\sigma_{12})_{appl} = -\frac{3q}{2c^3}(x_2^2 - c^2)b + R$$

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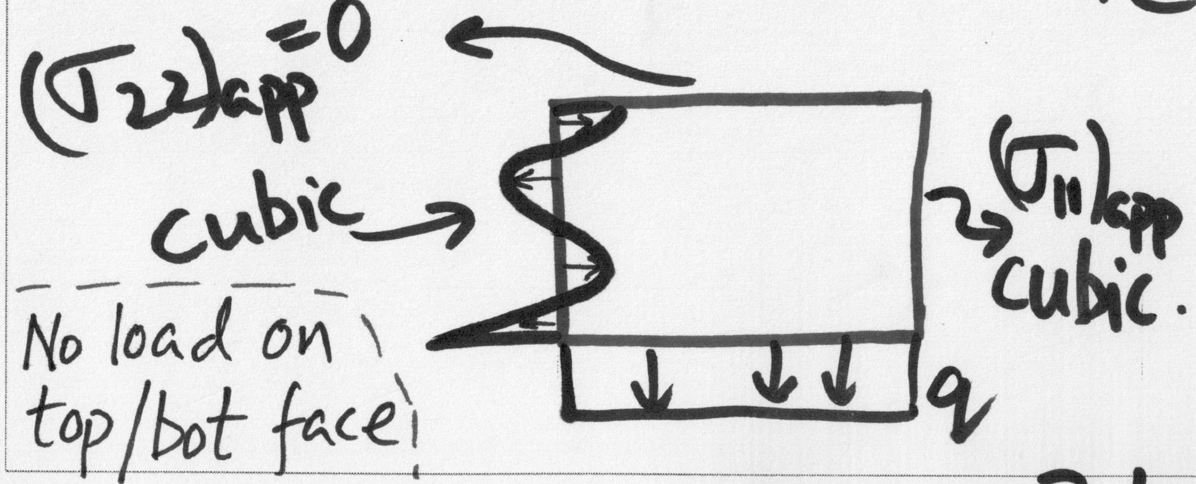
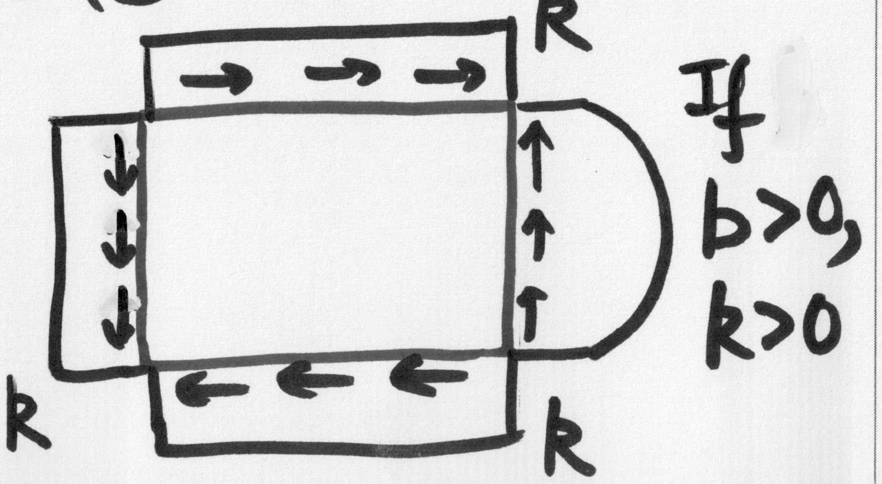
$$X_2 = c,\ (\sigma_{22})_{appl} = \frac{3q}{4c^3}\left(\frac{c^3}{3} - c^3 + \frac{2}{3}c^3\right)$$
$$= 0$$

$$X_2 = \pm c,\ (\sigma_{12})_{appl} = -\frac{3q}{4c^3}\left(c^2 - c^2\right)X_1 + R$$
$$= R.$$

$$X_2 = -c,\ (\sigma_{22})_{appl} = \frac{3q}{4c^3}\left(\frac{4}{3}c^3\right) = q$$





P8 Given: For $x_3 > 0$,

$$\sigma_{ij} = \frac{a\, x_i x_j}{r^5} x_3, \quad r \neq 0, a > 0$$

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$\boxed{\text{Find}}$: Total force on surface of hemisphere $r = a$.

Method-1: Work in terms of $\underline{t}$.

$\phi = x_1^2 + x_2^2 + x_3^2 - a^2 = 0$ defines hemisphere

$\underline{n} = \underline{\nabla}\phi / |\underline{\nabla}\phi|$, unit normal to hemisph $r = a$

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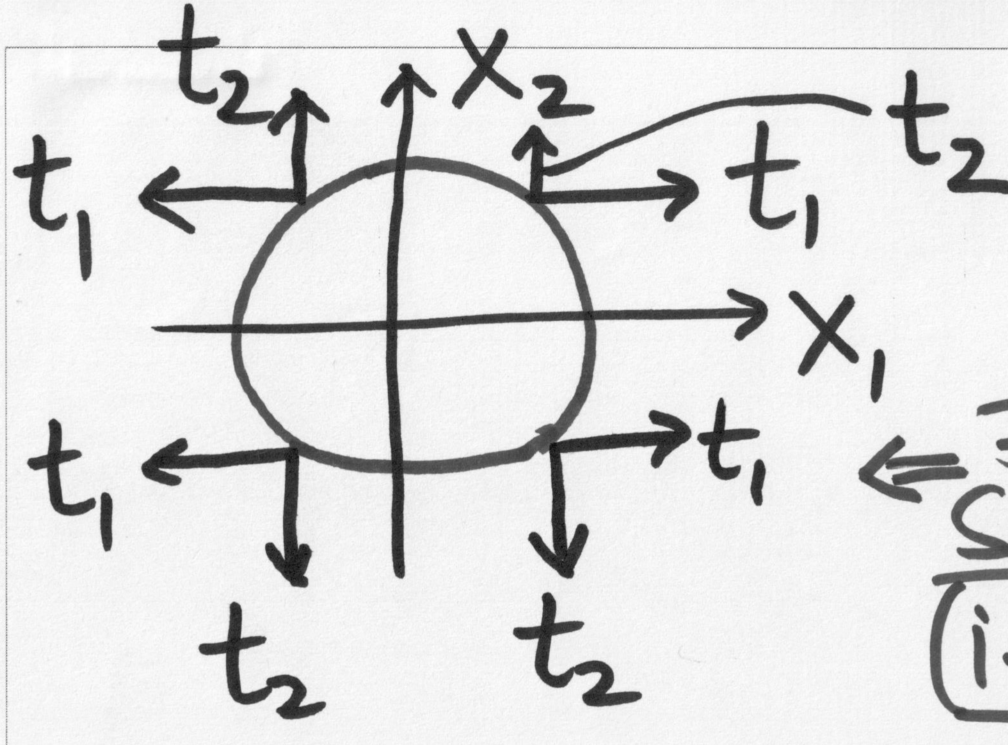
$$\underline{n} = \frac{2x_1\underline{e}_1 + 2x_2\underline{e}_2 + 2x_3\underline{e}_3}{2\underbrace{\sqrt{x_1^2 + x_2^2 + x_3^2}}_{a}}$$

$$= \frac{x_1}{a}\underline{e}_1 + \frac{x_2}{a}\underline{e}_2 + \frac{x_3}{a}\underline{e}_3$$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} = \frac{1}{a^5}\begin{bmatrix} x_1^2 x_3 & x_1x_2x_3 & x_1x_3^2 \\ x_1x_2x_3 & x_2^2x_3 & x_2x_3^2 \\ x_1x_3^2 & x_2x_3^2 & x_3^3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$= \frac{1}{a^3}\{\underbrace{x_1x_3}\quad \underbrace{x_2x_3}\quad x_3^2\}^T$$

t_1 odd in x_1, t_2 odd in x_2



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Plan of frustrum of Sphere (ie latitude ϕ, z, const)

For each annular strip of area $2\pi a \sin\phi \, a\,d\phi$, lying at latitude ϕ, contributions of t_1, t_2 cancel, i.e $F_x = \iint t_1 \, dA = 0$, $F_y = \iint t_2 \, dA = 0$

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$$F_z = \iint t_3\, dA = \frac{1}{a^3}\iint z^2\, dA$$

$$z = a\cos\phi, \quad dA = a\sin\phi\, d\theta.\, \underline{a\,d\phi}$$

$$F_z = \int_0^{\pi/2}\int_0^{2\pi}\left(\frac{1}{a^3} a^2\cos^2\phi \,.\, a^2\sin\phi\, d\theta\right) d\phi$$

$$= 2\pi a \int_0^{\pi/2} \cos^2\phi \sin\phi\, d\phi = \frac{2\pi a}{3}$$

$\underline{\text{Method-2}}$: $(\underline{\underline{\sigma}})_{\underset{x_1'}{R},\underset{x_2'}{\theta},\underset{x_3'}{\phi}} = \underline{\underline{a}}\,(\underline{\underline{\sigma}})_{x_1,x_2,x_3}\,\underline{\underline{a}}^T$

Use $\underline{\underline{a}}$, P.2.

L7

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To find forces on surface with $\underline{e}_R$ as normal, need only comp's $\sigma_{RR}, \sigma_{R\theta}, \sigma_{R\phi}$ → ie BC's

Transf gives (much algebra),

$$\sigma_{RR} = \frac{\cos\phi}{a}, \quad \sigma_{R\phi} = \sigma_{R\theta} = 0$$

$\because$ σ_{RR} depends on ϕ only, when integrating $\sigma_{RR}\underline{e}_R\, dA$, horizontal contributions cancel on opp sides of a latitudinal band $\Rightarrow F_x = F_y = 0$

$$F_z = \int_0^{\pi/2}\int_0^{2\pi}\Big(\sigma_{RR}\,\underbrace{a\,d\phi\, a\sin\phi\, d\theta}_{dA}\Big)\cdot\cos\phi$$

$$= \frac{2\pi a}{3}$$

In general,

$$F_z = \iint_A \left(\sigma_{RR}\, dA\cos\phi + \sigma_{R\phi}\, dA\sin\phi\right)$$

$$F_x = \iint_A \left(\sigma_{RR}\, dA\, s\phi\, c\theta - \sigma_{R\phi}\, dA\, c\phi\, c\theta - \sigma_{R\theta}\, dA\, s\theta\right)$$

$$F_y = \iint_A \left(\sigma_{RR}\, dA\, s\phi\, s\theta - \sigma_{R\phi}\, dA\, c\phi\, s\theta + \sigma_{R\theta}\, dA\, c\theta\right)$$