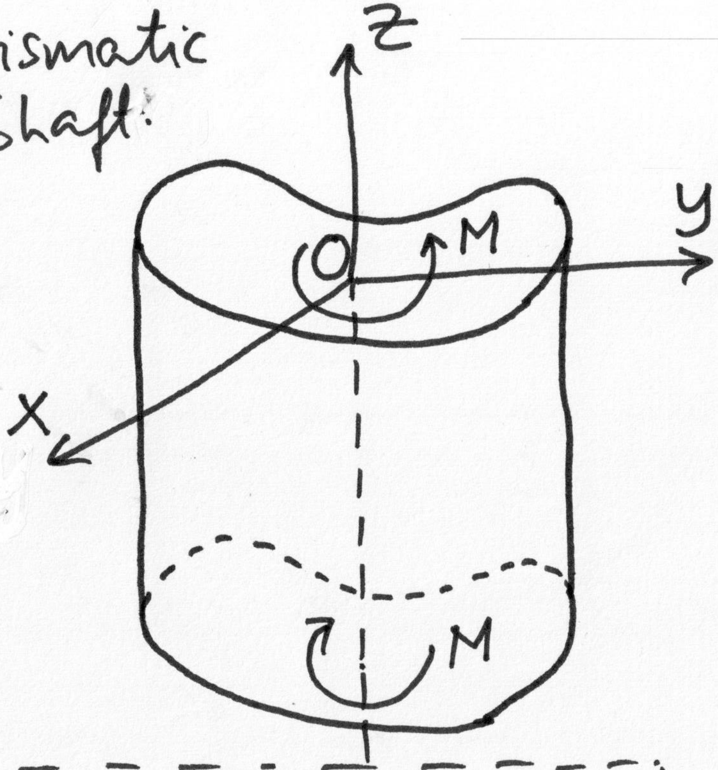
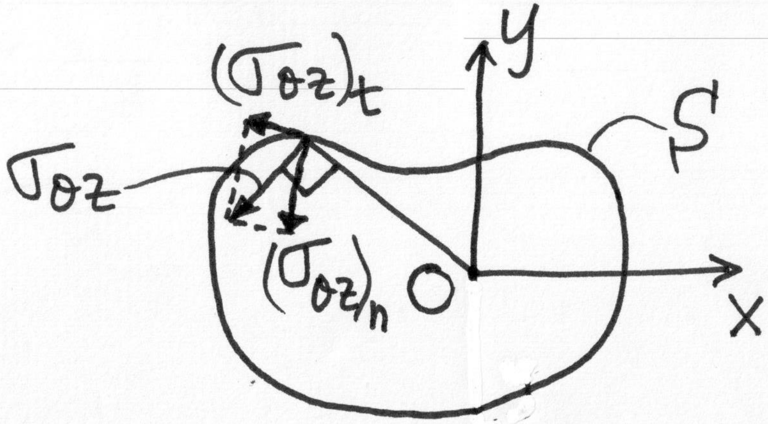
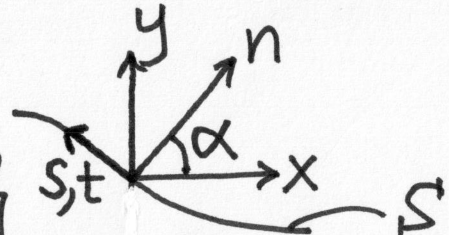
TORSION (Non-circular sections).

Prismatic Shaft.





For circular sections $\tau_{\theta z}$ is the only stress. If we assume the same for non-circular sections, then $(\tau_{\theta z})_t$ & $(\tau_{\theta z})_n$ are components tangential & normal to boundary S, respectively, of $\tau_{\theta z}$. Due to complimentarity of shear stresses, $(\tau_{\theta z})_n$ is present on longitudinal face also. This violates traction free BC on longitudinal face.



$\underset{\sim}{n} = \begin{Bmatrix} l \\ m \end{Bmatrix}$

$\cos\alpha = l = dy/ds = \partial x/\partial n$

$\sin\alpha = m = -dx/ds = \partial y/\partial n$

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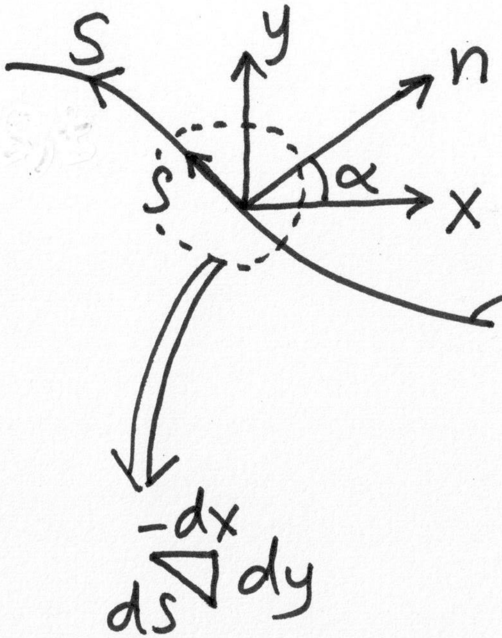


Boundary S

on S, $x = x(s)$

$y = y(s)$

$\cos\alpha = l = dy/ds$

$\sin\alpha = m = -dx/ds$

off S, $x = x(s,n)$

$y = y(s,n)$

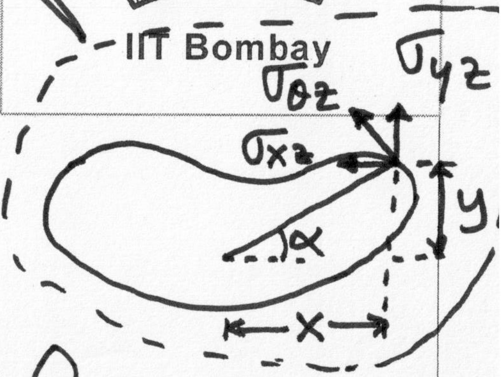
$\cos\alpha = l = \partial x/\partial n$

$\sin\alpha = m = \partial y/\partial n$

Alternate way to see this is as follows. If $\sigma_{\theta z}$ is only stress,

$$\sigma_{xz} = -\frac{y}{r}\,\sigma_{\theta z} \; ; \quad \sigma_{yz} = \frac{x}{r}\,\sigma_{\theta z}$$

BC on long. face is $(\underline{\underline{\sigma}}\,\underline{n})_s = \underline{t} = \underline{0}$ (traction free),



$$\Rightarrow \begin{pmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & 0 \end{pmatrix} \begin{Bmatrix} l \\ m \\ 0 \end{Bmatrix} = \underline{0} \implies \sigma_{xz} \underbrace{l}_{\frac{dy}{ds}} + \sigma_{yz} \underbrace{m}_{-\frac{dx}{ds}} = 0$$

$$\Rightarrow \underbrace{\frac{\sigma_{\theta z}}{r}}_{\neq 0 \text{ for non-circular sections}} \underbrace{(-y\,dy - x\,dx)}_{=0 \text{ for circle only}} = 0$$

\# Thus, $\sigma_{\theta z}$ & σ_{rz} both exist such that their components in n-direction cancel out, thus satisfying traction free BC's on long. face.

\# Easier to work with σ_{xz}, σ_{yz} instead of $\sigma_{rz}, \sigma_{\theta z}$ for non-circular sections.

Method I — PRANDTL STRESS FUNCTION (ϕ) FORMULATION.

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Assume only σ_{xz}, σ_{yz} non-zero stresses (guided by basic Solid Mech.). Assume zero b.f. We now attempt to satisfy all governing field equations & BC's.

Equilibrium:
$$\left.\begin{aligned}\sigma_{zx,z} &= 0\\ \sigma_{zy,z} &= 0\end{aligned}\right\} \rightarrow \sigma_{zx}, \sigma_{zy} \text{ functions of } (x,y) \text{ only.}$$

$$\sigma_{zx,x} + \sigma_{zy,y} = 0 \rightarrow \text{satisfied by } \phi \text{ defined below.}$$

Define ϕ such that $\boxed{\sigma_{xz} = \phi_{,y}\ ,\ \sigma_{yz} = -\phi_{,x}}$, $\phi = \phi(x,y)$ (1)

Compatibility: B-M compatibility eqs reduce to,

$$\nabla^2 \sigma_{xz} = 0\ ;\ \nabla^2 \sigma_{yz} = 0 \Rightarrow \frac{\partial}{\partial x}(\nabla^2 \phi) = 0\ ;\ \frac{\partial}{\partial y}(\nabla^2 \phi) = 0$$

$$\Rightarrow \boxed{\nabla^2 \phi = K \text{ (const)}} \rightarrow \text{Poisson's eqn.} \quad (2)$$

BC's on lateral (ie longitudinal) face:

$$\left(\underline{\underline{\sigma}}\,\underline{n}\right)_s = \underline{0} \Rightarrow l\,\sigma_{xz} + m\,\sigma_{yz} = 0 \quad \text{(other two are } \underline{\underline{\text{i.s}}}\text{)}$$

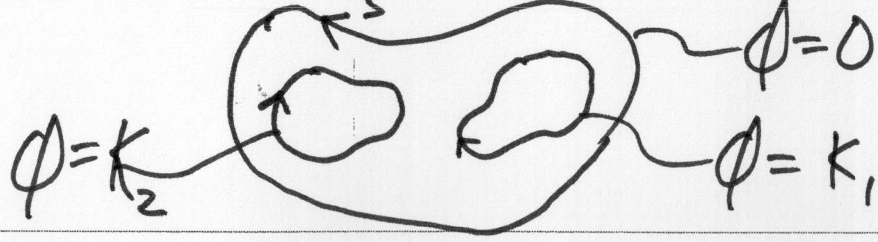
$$\frac{dy}{ds}\left(\frac{\partial\phi}{\partial y}\right)_s + \frac{dx}{ds}\left(\frac{\partial\phi}{\partial x}\right)_s = \left(\frac{d\phi}{ds}\right)_s = 0$$

$$\Rightarrow \boxed{(\phi)_s = \text{Const.}}$$

$\because$ addition of const to ϕ doesent affect stresses,

$$3 \leftarrow \boxed{(\phi)_s = 0} \rightarrow \text{valid for simply connected domain (shaft with no holes)}$$

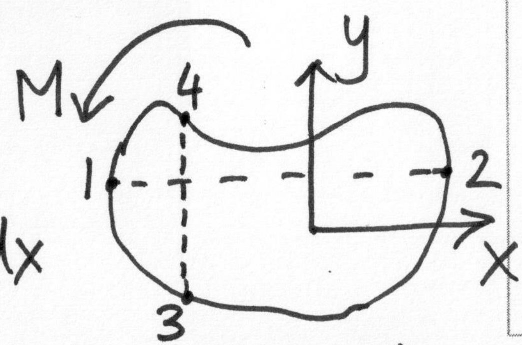
For multiply connected domains, take $\phi = 0$ on one boundary. Then find non-zero constant values of ϕ on other boundaries by imposing condition that displacements are single-valued. (done later).



Cross-section with holes

B.C's on end faces (cross-sections)

$$\Sigma F_x = \iint_A \sigma_{xz}\,dx\,dy = 0$$



$$= \iint_A \frac{\partial \phi}{\partial y}\,dx\,dy = \int \underbrace{\cancel{(-\phi(3)+\phi(4))}}_{=0\ (\because\ \phi = \text{const on } S)} dx$$

So this BC is i.s.

$$\Sigma F_y = \iint_A \sigma_{yz}\,dx\,dy = \iint_A \frac{\partial \phi}{\partial x}\,dx\,dy = \int \underbrace{\cancel{(\phi(2)-\phi(1))}}_{0} dy = 0, \quad \text{i.s.}$$

$$\iint (-y\sigma_{xz} + x\sigma_{yz})\,dx\,dy = M$$ (ie moment due to internal stresses equals applied M).

$$\Rightarrow M = \iint_A (-y\phi_{,y} - x\phi_{,x})\,dx\,dy = -\iint [(y\phi)_{,y} + (x\phi)_{,x} - 2\phi]\,dx\,dy$$

$$\boxed{M = 2\iint_A \phi\,dx\,dy - \oint_S \underbrace{\cancel{(xl+ym)\phi}}_{=0} ds} \quad (4)$$

for simply connected

used Divergence theorem

$$\oint_S \underline{V}\cdot\underline{n}\,ds = \iint_A \underline{\nabla}\cdot\underline{V}\,dA$$

$$\underline{V} = (x\phi)\underline{i} + (y\phi)\underline{j}$$

$$\underline{n} = l\underline{i} + m\underline{j}$$

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Displacements: Using C.L. & S-D equns, (note only τ_{xz}, τ_{yz} non-zero),

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}=0 \;;\quad \frac{\partial v}{\partial y}=0 \;;\quad \frac{\partial w}{\partial z}=0 \\
&\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=-\frac{1}{G}\frac{\partial \phi}{\partial x} \;;\quad \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\frac{1}{G}\frac{\partial \phi}{\partial y} \\
&\frac{\partial v}{\partial x}+\frac{\partial w}{\partial y}=0
\end{aligned}\right\} \rightarrow A$$

$$A_{1,2,3} \rightarrow u=f_1(y,z) \;;\; v=f_2(x,z) \;;\; w=f_3(x,y) \rightarrow B$$

$$A_{4,5,6}\ \&\ B \rightarrow \left.\begin{aligned}
&\frac{\partial f_3}{\partial y}+\frac{\partial f_2}{\partial z}=-\frac{1}{G}\frac{\partial \phi}{\partial x} \\
&\frac{\partial f_1}{\partial z}+\frac{\partial f_3}{\partial x}=\frac{1}{G}\frac{\partial \phi}{\partial y} \\
&\frac{\partial f_2}{\partial x}+\frac{\partial f_1}{\partial y}=0
\end{aligned}\right\} \rightarrow C$$

$$\frac{\partial f_3}{\partial y}+\frac{\partial f_2}{\partial z}=-\frac{1}{G}\frac{\partial \phi}{\partial x} \rightarrow \frac{\partial^2 f_2}{\partial z^2}=0$$

$$\frac{\partial f_1}{\partial z}+\frac{\partial f_3}{\partial x}=\frac{1}{G}\frac{\partial \phi}{\partial y} \rightarrow \frac{\partial^2 f_1}{\partial z^2}=0$$

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_1}{\partial y}=0 \rightarrow \frac{\partial^2 f_2}{\partial x^2}=0 \;;\; \frac{\partial^2 f_1}{\partial y^2}=0$$

$$\Rightarrow \left.\begin{aligned} f_1 &= ayz + by + cz + d \\ f_2 &= exz + fx + gz + h \end{aligned}\right\} \rightarrow \text{D}$$

$$\text{C}_3 \ \& \ \text{D} \rightarrow (e+a)z + f + b = 0 \rightarrow e = -a, \ f = -b$$

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$$u = f_1 = ayz + by + cz + d \ ; \ v = f_2 = -axz - bx + gz + h$$

BC's: $u = v = u_{,z} = v_{,z} = u_{,y} - v_{,x} = 0$ at $z = x = y = 0$,

i.e no translation & rotation at pt. O (origin).

$$\Rightarrow d = h = c = g = b = 0$$

Could have got this directly by recognizing that constant & linear terms represent rigid body translation & rotation, respectively, which we remove by dropping these terms.

$$5 \leftarrow \ \Rightarrow \boxed{u = ayz \ ; \ v = -axz} \rightarrow 5$$

$$u_r = v\sin\theta + u\cos\theta = az(-\underbrace{r\cos\theta}_{x}\sin\theta + \underbrace{r\sin\theta}_{y}\cos\theta) = 0$$

$$u_\theta = v\cos\theta - u\sin\theta = -azr = \alpha z r = \beta r$$

β = twist of section $(\alpha \equiv -a,\ \beta \equiv \alpha z)$

$$\alpha = \frac{d\beta}{dz} = \text{rate of twist, ie twist per unit length}$$

$$(A)_{4,5} \ \& \ (5) \longrightarrow \boxed{\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{G}\frac{\partial \phi}{\partial y} + \alpha y \\ \frac{\partial w}{\partial y} &= -\frac{1}{G}\frac{\partial \phi}{\partial x} - \alpha x \end{aligned}} \longrightarrow (5a) \longrightarrow \text{use to determine } w \rightarrow \text{warping displ.}$$

$$(5a) \longrightarrow \boxed{\nabla^2 \phi = -2G\alpha} = K \longrightarrow (2) \text{ (repeated).}$$

Thus kinematics is inplane u, v, resulting from rotation of points (in a section) about O thru angle β $(\because u_r = 0,\ u_\theta = \beta r)$, and out-of-plane w superposed. Thus plane sections warp (ie dont remain plane) unlike the case of circular shaft. However no distortion occurs in the plane of section.

This means that if you project deformed points onto original plane of section, the projected shape is exactly same as original section.

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Solve (2), subject to (3) & (4), for ϕ & α for given M applied

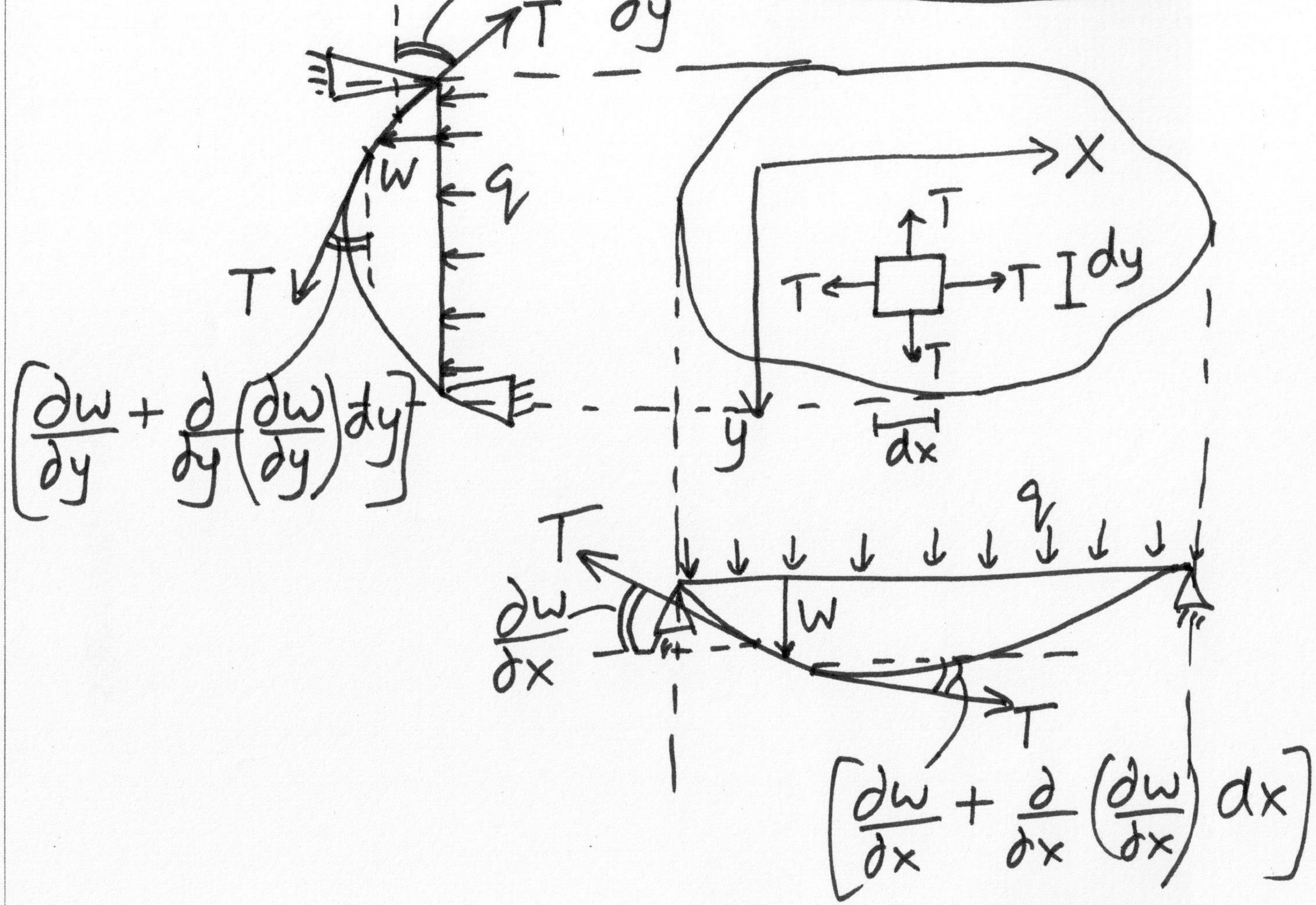
Then use (1) to obtain τ_{xz}, τ_{yz}

use (5), (5a) to obtain displacements.

<u>Membrane Analogy</u> (due to Prandtl).

Analogy exists between torsion problem & problem of a uniformly tensioned membrane subject to uniform (but small) transverse pressure.

Membrane Analogy



Uniformly tensioned membrane, $T(x,y) = \text{Const}$ before deformation. Load (pressure) q is uniform but small, so $T(x,y) \simeq \text{const}$ after deformation.

$$\left(\Sigma F_z\right)_{element} = 0 = -T dy \frac{\partial w}{\partial x} + T dy\left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)dx\right)$$

$$-T dx \frac{\partial w}{\partial y} + T dx\left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)dy\right) + q\, dx dy = 0$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{q}{T} \Rightarrow \nabla^2\left(\frac{T}{q} w\right) = -1 \;;\; \left(\frac{T}{q} w\right)_S = 0$$

Torsion $\rightarrow \nabla^2\left(\frac{\phi}{2G\alpha}\right) = -1 \;;\; \left(\frac{\phi}{2G\alpha}\right)_S = 0$

$$\Rightarrow \phi \equiv \left(\frac{2G\alpha T}{q}\right) w$$

$$M = 2\iint_A \phi\, dx\, dy \equiv \frac{2G\alpha T}{q} \cdot 2 \overbrace{\iint_A w\, dx\, dy}^{\text{vol. displaced by membrance} = V} = \frac{2G\alpha T}{q} \cdot 2V$$

$$\tau_{zx} = \phi_{,y} \equiv \left(\frac{2G\alpha T}{q}\right) w_{,y} \;;\; \tau_{zy} = \phi_{,x} \equiv \left(\frac{2G\alpha T}{q}\right) w_{,x}$$

If q, T adjusted so that $\frac{2G\alpha T}{q} = 1$, then,

$$\boxed{\phi \equiv w \;;\; M \equiv 2V \;;\; \tau_{zx} \equiv \frac{\partial w}{\partial y} \;;\; \tau_{zy} \equiv \frac{\partial w}{\partial x}} \rightarrow (6).$$

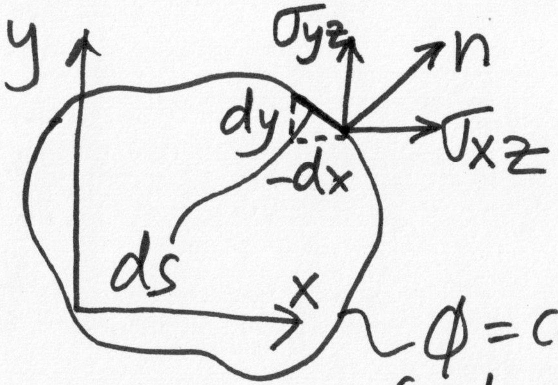
Above analogy very useful in solving torsion problems, as we will see.

Lines of shearing stress.

$$\underline{\nabla}\phi\cdot\left(\sigma_{xz}\,\underline{i}+\sigma_{yz}\,\underline{j}\right) = \left(\phi_{,x}\,\phi_{,y} - \phi_{,y}\,\phi_{,x}\right) = 0$$

$\Rightarrow$ on $\phi = \text{const}$ curves, the total shear stress $(\sigma_{xz}, \sigma_{yz})$ is tangential to curve $\because \underline{\nabla}\phi = \text{normal to curve}$.

$\Rightarrow \boxed{\sigma_{nz}=0}$



$\phi = \text{const}$ curve (not necessarily boundary curve).

$$l = \frac{dy}{ds} = \frac{\partial x}{\partial n} = \frac{\sigma_{yz}}{\sqrt{\sigma_{yz}^2+\sigma_{xz}^2}}\;;\quad m = -\frac{dx}{ds} = \frac{\partial y}{\partial n} = \frac{-\sigma_{xz}}{\sqrt{\sigma_{yz}^2+\sigma_{xz}^2}}$$

NOT IMP.

direction cosines of n which is normal to $\phi = \text{const}$ curve.

$$\sigma_{sz} = \tau = \text{total shear stress} = \sigma_{yz}\frac{dy}{ds} - \sigma_{xz}\left(-\frac{dx}{ds}\right) = -\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial n} - \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial n} = -\frac{\partial\phi}{\partial n}$$

$$\Rightarrow \boxed{\sigma_{sz} = \tau = -\frac{\partial\phi}{\partial n} = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2}}$$

Now consider

$0 \rightarrow (\because \nabla^2\phi = \text{const})$

$$\nabla^2\tau^2 = 2\Big(\phi_{,xx}^2 + \underline{\phi_{,x}\,\phi_{,xxx}} + \phi_{,xy}^2 + \underline{\underline{\phi_{,y}\,\phi_{,xxy}}} + \phi_{,yy}^2 + \underline{\phi_{,x}\,\phi_{,xyy}} + \underline{\underline{\phi_{,y}\,\phi_{,yyy}}}\Big)$$

$0 \rightarrow (\because \nabla^2\phi = \text{const})$

$\Rightarrow \nabla^2\tau^2 \geqslant 0$

From calculus we have result

If $\tau^2 \neq$ const in domain and: $\nabla^2\tau^2 = 0 \Rightarrow$ max & min of τ^2 on boundary & not in domain

$\nabla^2\tau^2 \geqslant 0 \Rightarrow$ max(τ^2) occurs on boundary & not in domain

$\nabla^2\tau^2 \leqslant 0 \Rightarrow$ min(τ^2) occurs on boundary & not in domain

Thus max shear stress occurs on boundary $\Rightarrow$ shear failure possible on lateral face.

METHOD – II St. VENANT WARPING FUNCTION (ψ)

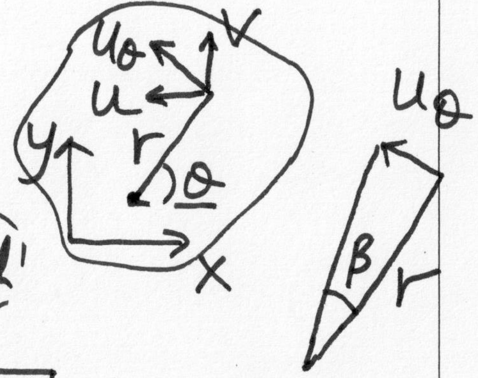
FORMULATION

Its a displacement formulation.

St. Venant assumed inplane kinematics same as circular shaft, ie,

$$u = -u_\theta \sin\theta = -r\alpha z \sin\theta = -\alpha z y$$
$$v = u_\theta \cos\theta = \alpha z x$$

same as circular shaft, & result of ϕ approach



$$w = \alpha\, \psi(x,y) \longrightarrow \text{WARPING FUNCTION.}$$

$\beta = \alpha z$

Stresses: (use CL & S-D eqn)

$$\boxed{\begin{aligned}\tau_{xz} &= G(u_{,z} + w_{,x}) = G\alpha(\psi_{,x} - y) = \phi_{,y}\\ \tau_{yz} &= G(v_{,z} + w_{,y}) = G\alpha(\psi_{,y} + x) = -\phi_{,x}\end{aligned}} \longrightarrow (1)$$

other stresses zero.

$\because e_x = e_y = e_{xy} = 0 \Rightarrow$ <u>no in-plane distortion of section.</u>

(ie deformed section when projected to plane of undeformed section looks exactly same as undeformed section)

Equilibrium eqns:

1^{st}, 2^{nd} eqns are <u>i.s.</u>

$3^{rd} \rightarrow \boxed{\psi_{,xx} + \psi_{,yy} = 0} \longrightarrow$ (2)

BC's on lateral face:

$$\left(l\,\sigma_{xz} + m\,\sigma_{yz}\right)_{S'} = 0 \Rightarrow \boxed{\frac{\partial x}{\partial n}\left(\psi_{,x} - y\right) + \frac{\partial y}{\partial n}\left(\psi_{,y} + x\right) = 0}$$

equivalent.

$$\Rightarrow \boxed{\underbrace{\frac{\partial \psi}{\partial n}}_{\underline{n}\cdot\underline{\nabla}\psi} = ly - mx \quad \text{on } S'} \longrightarrow (3)$$

BC's on end faces:

$$\iint_A \sigma_{xz}\, dA = \iint_A \left[(x\sigma_{xz})_{,x} + (x\sigma_{yz})_{,y}\right] dA = \oint_{S'} x \overbrace{\left(l\,\sigma_{xz} + m\sigma_{yz}\right)}^{0\ \text{(lateral BC)}} ds = 0$$

add 3^{rd} equil. eqn.

Gauss Div Theorem.

ie <u>i.s.</u>

Similarly $\iint_A \sigma_{yz}\, dA = 0$ is <u>i.s</u>

$$M = \iint_A (-y\,\sigma_{xz} + x\,\sigma_{yz})\,dA = G\alpha \underbrace{\iint_A (x^2+y^2+x\psi_{,y} - y\psi_{,x})\,dA}_{\text{"}C/G\text{"}}$$

$$\Rightarrow \boxed{\begin{array}{l} M = C\alpha. \\ C = \text{torsional rigidity} = G\iint (x^2+y^2+x\psi_{,y}-y\psi_{,x})\,dA \end{array}} \rightarrow 4$$

Consider, $I = \iint_A \left(x\dfrac{\partial\psi}{\partial y} - y\dfrac{\partial\psi}{\partial x}\right) dA$

$$= \iint_A \left(\frac{\partial(x\psi)}{\partial y} - \frac{\partial(y\psi)}{\partial x}\right) dA \;\overset{\text{(Div. Thrm)}}{=}\; \oint_S (-ly + mx)\,\psi\, ds$$

$$\underset{\text{(lateral BC)}}{=} -\oint_S \left(l\frac{\partial\psi}{\partial x} + m\frac{\partial\psi}{\partial y}\right)\psi\, ds \;\underset{\text{(Div Thrm \& 2)}}{=}\; -\iint_A \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 \right] dA \le 0.$$

$$\Rightarrow \frac{C}{G} \le \iint_A (x^2+y^2)\,dA \longrightarrow \boxed{C \le GJ}$$

$$\boxed{C = GJ - G\iint_A \left[(\psi_{,x})^2 + (\psi_{,y})^2\right] dA} \rightarrow (5)$$

$$(4),(5) \rightarrow \; C = 2C - C = G\iint \left(x^2+y^2+(\psi_{,x})^2+(\psi_{,y})^2+2x\psi_{,y}-2y\psi_{,x}\right) dA$$

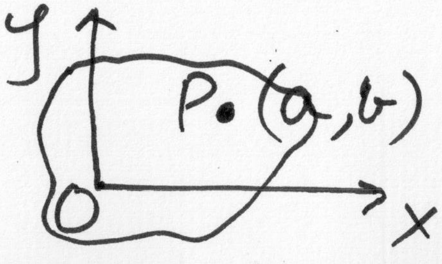
$$= \iint \left[(\psi_{,y}+x)^2 + (\psi_{,x}-y)^2\right] dA$$

$$\boxed{C = \frac{1}{\alpha^2 G}\iint_A \left(\sigma_{yz}^2 + \sigma_{xz}^2\right) dA} \rightarrow (6)$$

—— X —— X —— X ——

Change in torsion axis (ie reference origin).

New axis thru $P(a,b)$, old axis thru O.



$u = -\alpha(y-b)z$, $v = \alpha(x-a)z$, $w = \alpha\psi_1(x,y)$

$$\sigma_{xz} = G\alpha\left(\frac{\partial\psi_1}{\partial x} - y + b\right); \quad \sigma_{yz} = G\alpha\left(\frac{\partial\psi_1}{\partial y} + x - a\right) \xrightarrow[\text{3rd equil}]{} \nabla^2\psi_1 = 0 \rightarrow (a)$$

Lateral BC $\longrightarrow \dfrac{\partial \psi_1}{\partial n} = l(y-b) - m(x-a)$

$\{ l = x_{,n} \; ; \; m = y_{,n} \}$

$$\Rightarrow \frac{\partial}{\partial n}(\psi_1 + bx - ay) = ly - mx \text{ on } S \longrightarrow (b)$$

$\Rightarrow \psi_1 = \psi - bx + ay + \text{const}$ solves (a), (b)

where ψ is soln to original problem with axis thru O.

$$\left.\begin{aligned} \Rightarrow \sigma_{xz} &= G\alpha(\psi_{,x} - b - y + b) = G\alpha(\psi_{,x} - b) \\ \sigma_{yz} &= G\alpha(\psi_{,y} + a + x - a) = G\alpha(\psi_{,y} + x) \end{aligned}\right\} \Longrightarrow \text{stresses invariant to change of torsion axis.}$$

So only displ's altered by rigid body comp. due to change of torsion axis.

Thus C is invariant to change in axis of torsion (see (6) or $M = C\alpha$)

$\Rightarrow C \leq GJ$

$\quad\leq GJ_{min}$

J centroidal axis

Summary:

(I) Prandtl stress function approach. (ϕ)

Solve $\nabla^2\phi = -2G\alpha$ subject to $\phi = 0$ on Γ and $M = 2\iint \phi \, dA$ (simply connected domain), for ϕ & α in terms of M applied.

Then, $\sigma_{xz} = \phi_{,y}$, $\sigma_{yz} = -\phi_{,x}$

(II) St. Venant's Warping function approach (ψ)

Solve $\nabla^2\psi = 0$ subject to $\dfrac{\partial\psi}{\partial n} = ly - mx$ on Γ, for ψ.

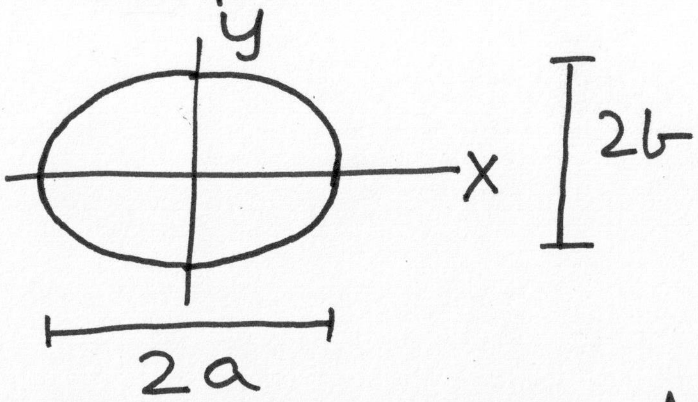
Then $C = GJ - \iint_A \left[(\psi_{,x})^2 + (\psi_{,y})^2\right] dA$ = torsional rigidity

and $\alpha = \dfrac{M}{C}$. Stresses are $\sigma_{xz} = G\alpha(\psi_{,x} - y)$; $\sigma_{yz} = G\alpha(\psi_{,y} + x)$

Displacements are $u = -\alpha yz$, $v = \alpha xz$, $w = \alpha\psi(x,y)$.

Some simple closed form solutions:

(i) Elliptic section.



$a > b$

Prandtl's stress fn approach — choose $\phi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

Such that $\phi_s = 0$ and $\nabla^2\phi = \text{const.}$

$\nabla^2\phi = -2G\alpha \Rightarrow m = \frac{a^2 b^2}{2(a^2+b^2)}(-2G\alpha)$

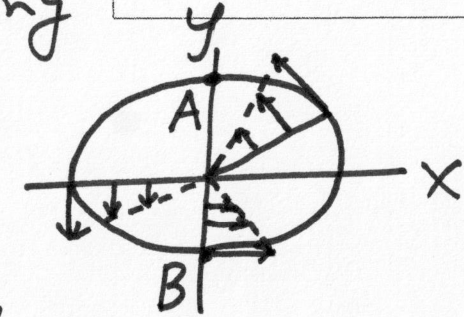
$$M = 2\iint \phi \, dA = \underbrace{2\,\frac{a^2b^2}{2(a^2+b^2)}(-2G\alpha)}_{=K} \iint_A \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dA = K\left(\frac{I_y}{a^2} + \frac{I_x}{b^2} - A\right)$$

For ellipse $I_y = \frac{\pi}{4}a^3 b$, $I_x = \frac{\pi}{4}ab^3$, $A = \pi ab$

$$\Rightarrow M = \frac{\pi a^3 b^3}{(a^2+b^2)} G\alpha \quad ; \text{ ie } C = \frac{\pi a^3 b^3}{a^2+b^2} G$$

$$\phi = -\frac{M}{ab}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

$$\sigma_{xz} = -\frac{2My}{\pi ab^3} \; ; \; \sigma_{yz} = \frac{2Mx}{\pi a^3 b}$$



$\frac{\sigma_{xz}}{\sigma_{yz}} \propto \frac{y}{x} \Rightarrow$ direction of τ is constant along radial line, and it should coincide with boundary (i.e $\phi = 0 =$ const curve).

$$\left(\tau^2\right)_S = \frac{4M^2}{\pi^2 a^2 b^2}\left(\frac{y^2}{b^4}+\frac{x^2}{a^4}\right) = \frac{4M^2}{(\pi ab)^2}\left[\frac{1}{b^2} - x^2\underbrace{\left(\frac{1}{a^2b^2}-\frac{1}{a^4}\right)}_{>0}\right]$$

$\sigma_{xz} = 0$ on x-axis

$\sigma_{yz} = 0$ on y-axis.

$\Rightarrow (\tau^2)_{max}$ occurs on S for $x=0$, ie at A, B. $\tau_{max} = \frac{2M}{\pi ab^2}$

You can get this result from membrane analogy.

$\tau = -\frac{\partial \phi}{\partial n}$ on S ($\because$ its a $\phi =$ const curve). Since $\left(\frac{\partial w}{\partial n}\right)_S$ for membrane is max at A & B (by observation), then τ max at A & B.

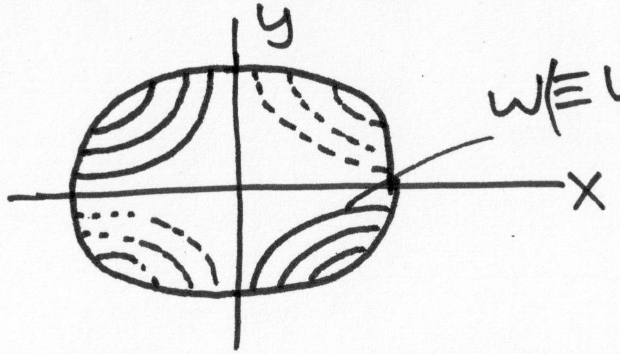
$$u = -\alpha z y = -\frac{(a^2+b^2)}{\pi a^3 b^3 G} M y z$$

$$v = \alpha z x = \frac{(a^2+b^2)}{\pi a^3 b^3 G} M x z$$

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{G}\frac{\partial \phi}{\partial y} + \alpha y = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G}\, y \\ \frac{\partial w}{\partial y} &= -\frac{1}{G}\frac{\partial \phi}{\partial x} - \alpha x = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G}\, x \end{aligned}\right\} \Rightarrow w = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G}\, xy + w_0$$

w_0 → const, ie RB motion so neglect.

w = const lines are hyperbolas



$w(\equiv \psi)$ const curves

...... concave

—— convex

Warping of section
convex in 2nd, 4th quad
concave in 1st, 3rd quad.

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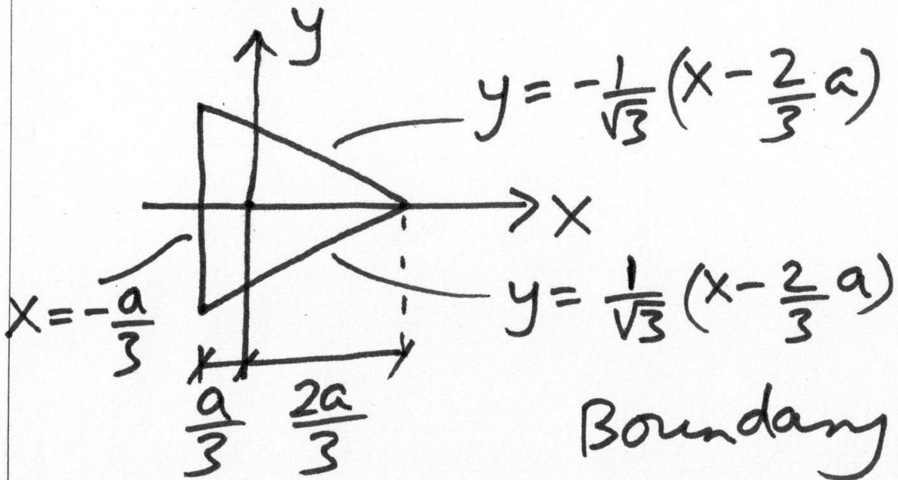
(ii) Equilateral Triangle.



Boundary curve $\rightarrow F(x,y) = \left(x + \frac{a}{3}\right)\left(x - \frac{2}{3}a + \sqrt{3}y\right) * \left(x - \frac{2}{3}a - \sqrt{3}y\right)$

Try $\phi = mF(x,y)$ so that $\phi_s = 0$. Check if $\nabla^2\phi = \text{const.}$

$$\phi = m\left(x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right) \longrightarrow \nabla^2\phi = m(-4a) = -2G\alpha \Rightarrow m = \frac{G\alpha}{2a}$$

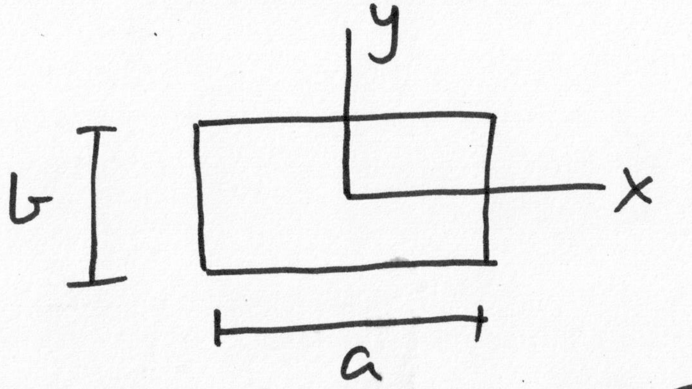
$$\tau_{xz} = \phi_{,y} = -\frac{G\alpha y}{a}(3x+a) \;;\; \tau_{yz} = -\phi_{,x} = -\frac{G\alpha}{2a}\left(3x^2 - 2ax - 3y^2\right)$$

- $\tau_{xz} = 0$ on $x = -\frac{a}{3}$, as it should be since τ is tangential to $x = -\frac{a}{3}$ curve, ie $\tau = \tau_{yz}$ on $x = -\frac{a}{3}$.
- $\tau_{xz} = \tau_{yz} = 0$ at corners — as it should be since tangent not unique at corners so $\tau = 0$ at corners.

$$M = 2\iint \frac{G\alpha}{2a}\left(x^3 - 3xy^2\right)dx\,dy - G\alpha J_0 + \frac{4}{27}G\alpha a^2 A$$

$$\underbrace{2\iint \frac{G\alpha}{2a}\left(x^3 - 3xy^2\right)dx\,dy}_{\text{do integral to get } M = C\alpha.}$$

(iii) <u>Rectangle</u>



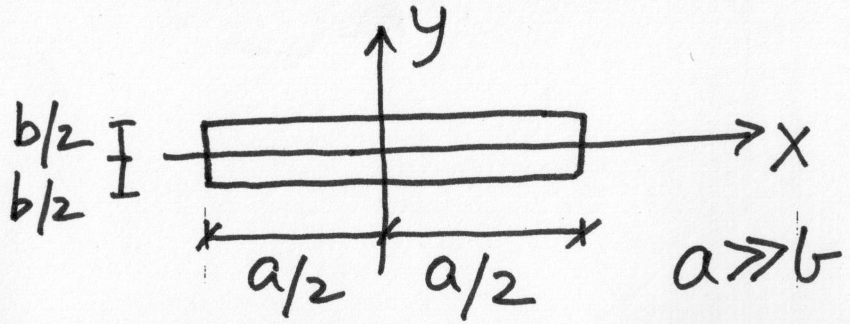
Try $\phi = \left(y^2 - \frac{b^2}{4}\right)\left(x^2 - \frac{a^2}{4}\right)$ so that $\phi_s = 0$.

But $\nabla^2\phi \neq$ const.

So this wont work.

see Timoshenko & Goodier for Fourier series solution.

(iv) <u>Narrow Rectangle</u>



For ellipse we had,

$$\phi = m\left[\frac{x^2}{(a/2)^2} + \frac{y^2}{(b/2)^2} - 1\right] \underset{a \gg b}{\simeq} \cancel{m}\left[y^2 - \frac{b^2}{4}\right]\frac{4}{b^2}$$

$-\frac{b^2}{4}G\alpha$

$$\simeq -G\alpha\left(y^2 - \frac{b^2}{4}\right)$$

Aside: You can get same result using membrane analogy. Using this for cylindrical bending ie, $w = w(y)$ $(\because a \gg b) \Rightarrow \tau_{zy} = 0$, $\phi = \phi(y)$

$$\Rightarrow \nabla^2 \phi = \frac{d^2\phi}{dy^2} = -2G\alpha \longrightarrow \phi = K_1 y + \frac{(-2G\alpha)}{2} y^2 + K_2$$

BC's: $\phi\big|_{y=\pm b/2} = 0 \longrightarrow K_1 = 0,\ K_2 = G\alpha \frac{b^2}{4} \longrightarrow \phi = -G\alpha\left(y^2 - \frac{b^2}{4}\right)$

—x——— x ——— x ——— x—

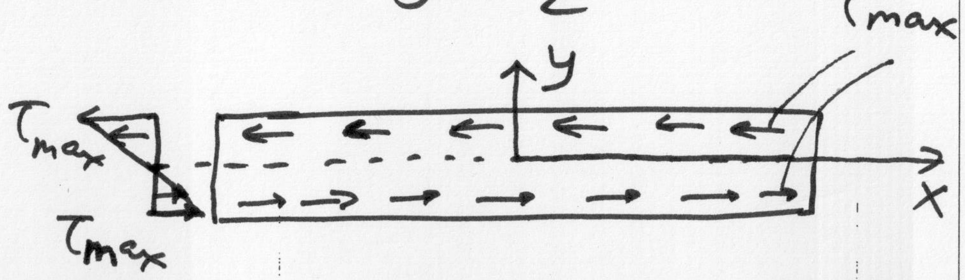
$$M = 2\iint_A \phi \, dx\,dy = 2\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} -G\alpha\left(y^2 - \frac{b^2}{4}\right)dx\,dy = G\alpha\frac{ab^3}{3} \Rightarrow \boxed{\alpha = \frac{3M}{ab^3 G}}$$

$$\boxed{C = \frac{Gab^3}{3}}$$

$$\Rightarrow \phi = \frac{3M}{ab^3}\left(\frac{b^2}{4} - y^2\right) \longrightarrow \tau_{zx} = \phi_{,y} = -\frac{6M}{ab^3}y = 2G\alpha y$$

Membrane analogy $\rightarrow w_{,y} \equiv \tau_{zx} = \tau$ is max at $y = \pm\frac{b}{2}$

$$\Rightarrow \boxed{\tau_{max} = \frac{3M}{ab^2} = bG\alpha}$$



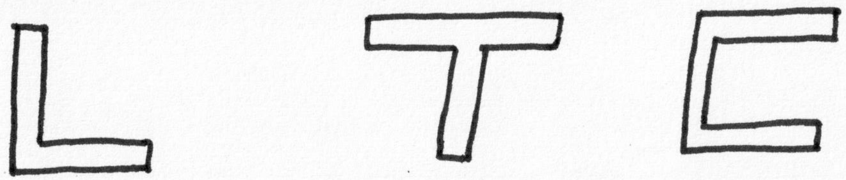
Note: If you calculate M using approx shear stresses ie $\tau_{zy} \simeq 0$,

$$M = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} \left(x\underbrace{\cancel{\tau_{yz}}}_{\simeq 0} - y\,\tau_{xz}\right) dx\,dy \cong \frac{1}{6} G\alpha a b^3$$

ie; half of actual M.

Thus we conclude that although $\tau_{zy} \simeq 0$ it contributes to 50% of M $\because$ lever arm x ($\equiv a/2$) is large.

Rolled Sections — Open Thin Walled Members.



Membrane analogy: If narrow rectangular membrane is loaded and then bent, the volume displaced (bounded) & the slopes will essentially remain unchanged (except at corners). Example is thin cylindrical balloon bent into various shapes. Thus narrow rectangular bar when bent into a curved cross-section bar will have the same torsional moment & shear stresses. So treat all curved narrow rectangles as straight rectangles when solving such problems.

Let,

$a_i, b_i \rightarrow$ length & width of i^{th} narrow rectangle in the cross-section

$M_i \rightarrow$ twisting moment on (or carried by) the i^{th} narrow rectangle

$\tau_i \rightarrow$ <u>max</u> shear stress in i^{th} narrow rectangle

$\alpha \rightarrow$ twist per unit length of bar (Note $\alpha_i = \alpha$, since no inplane distortion).

$$\Rightarrow \tau_i = \frac{3M_i}{a_i b_i^2} \; ; \quad \alpha = \frac{3M_i}{a_i b_i^3 G} = \alpha_i \; ; \quad M = \sum M_i = \frac{G\alpha}{3}\sum a_i b_i^3 \; ;$$

$$M_i = \frac{a_i b_i^3}{\sum a_i b_i^3} M \; ; \quad \boxed{\tau_i = \frac{3M}{\sum a_i b_i^3} b_i \; ; \quad \alpha = \frac{3M}{G\sum a_i b_i^3}} \; ; \quad C = \frac{G\sum a_i b_i^3}{3}$$

Above formula for τ_i (i.e. $(\tau_{max})_i$) does not hold at corners where we have stress concentrations.

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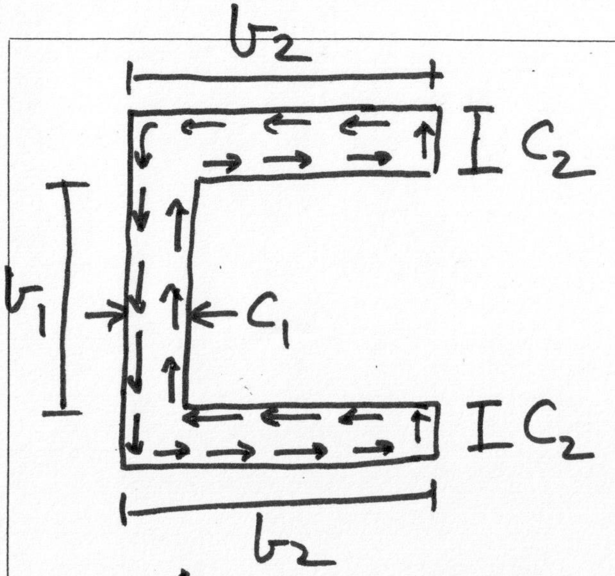
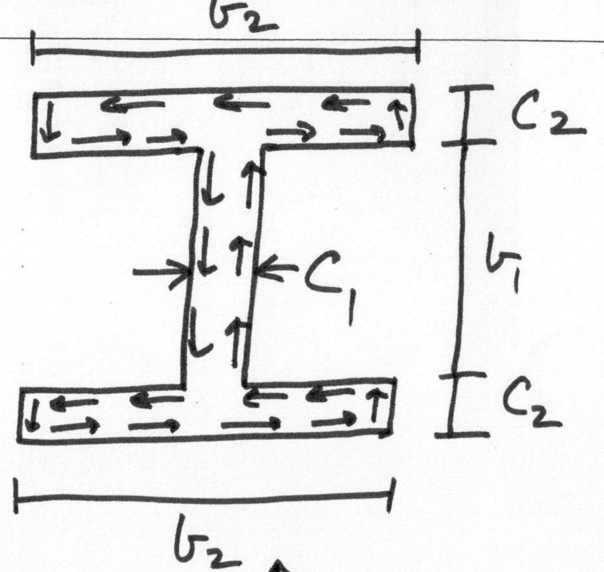
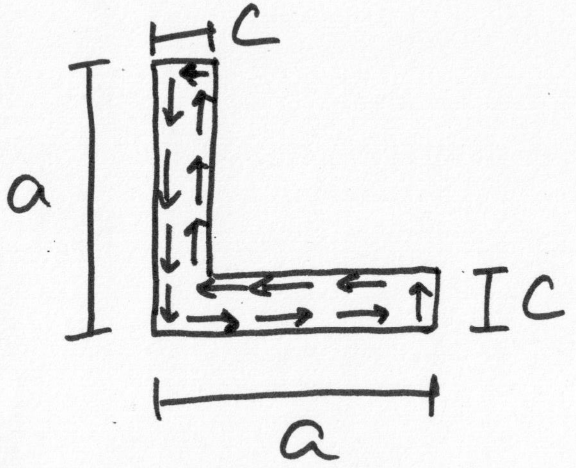
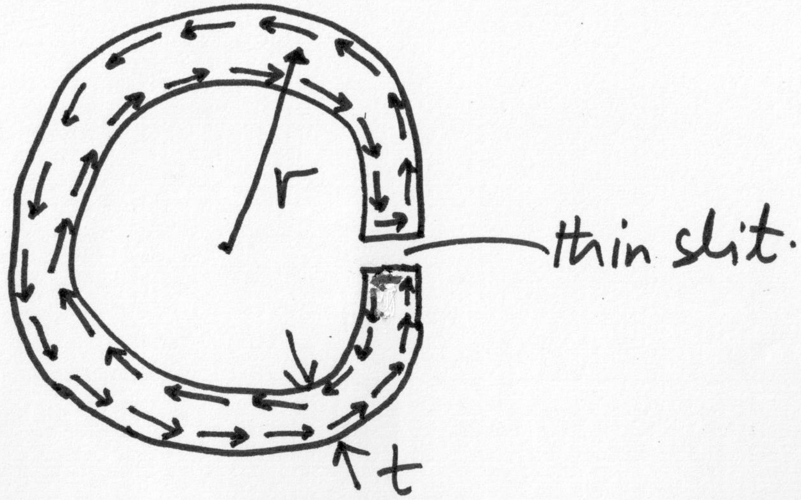




$$\alpha = \frac{3M}{(b_1 c_1^3 + 2 b_2 c_2^3)\, G}$$



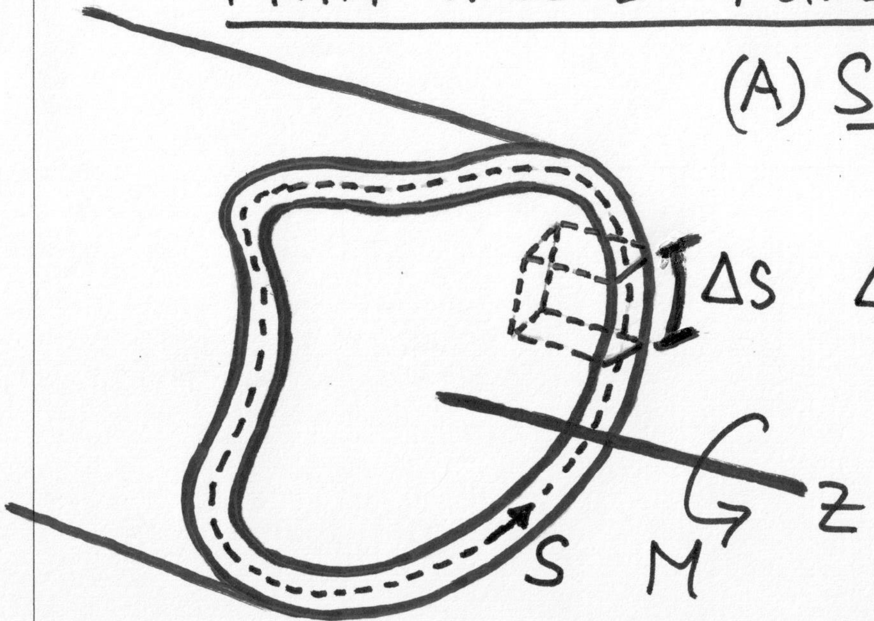
$$\alpha = \frac{3M}{(2a-c)c^3 G}$$

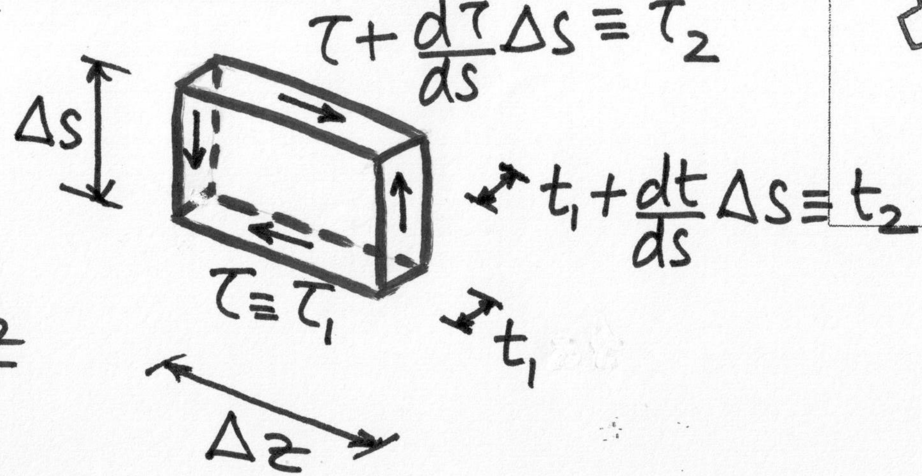


$$\alpha = \frac{3M}{2\pi r t^3 G}$$

THIN WALLED TUBES

(A) Single celled t.w. tubes

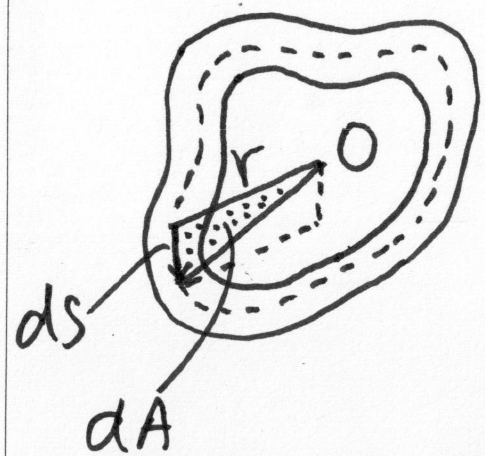




Assumption: Since wall is thin, $\tau = \sigma_{zs}$ = constant thru thickness and τ directed along ℄ even when $t = t(s)$.

$$\Sigma F_z = 0 \implies (\tau_2 t_2 - \tau_1 t_1)\Delta z = 0 \text{ , ie } d(\tau t) = 0$$

i.e, $\boxed{\tau t = q = \text{const (wrt } s) \quad \hookrightarrow \text{shear flow}} = \tau_1 t_1 = \tau_2 t_2 \rightarrow$ (1)



$$M\underline{k} = \oint_S \underline{r}\times \tau t\, d\underline{s} = q\oint_s \underline{r}\times d\underline{s} = q\int_A 2\,dA\,\underline{k}$$

$$\Rightarrow \boxed{M = 2qA} \rightarrow (2)$$

→ Bredt's formula

$$\boxed{\tau = \sigma_{zs} = \frac{M}{2tA}} \rightarrow (3)$$

→ A = area enclosed by ℄
≃ area encl by inner perimeter

Castigliano's theorem

Apply load F_i which generates strain energy U. Then,

$$d_i = \frac{\partial U}{\partial F_i} = \text{displ. in direction of } F_i$$

$$U_T = \text{strain energy in torsion} = \frac{1}{2}\iiint_V \sigma_{ij}e_{ij}\,dV = \frac{1}{2}\iiint_V 2\sigma_{sz}e_{sz}\,dV$$

$$= \frac{1}{2}\iiint_V \sigma_{sz}^2\,\frac{2(1+\nu)}{E}\,dV = \frac{1}{2}\iiint_V \frac{\tau^2}{G}\,dV = \frac{1}{2}\int_0^z\oint_S \frac{M^2}{4t^2A^2G}\,t\,ds\,dz$$

$$= \frac{1}{2}\,\frac{z}{4GA^2}\oint_S \frac{M^2}{t}\,ds$$

$i \equiv s$
$j \equiv z$
other stresses zero

for single-celled q = const wrt 's'
$\Rightarrow M$ = const wrt 's' $\Rightarrow U_T = \frac{1}{2}\,\frac{M^2 z}{4GA^2}\oint \frac{1}{t}\,ds$

Using Castigliano's theorem

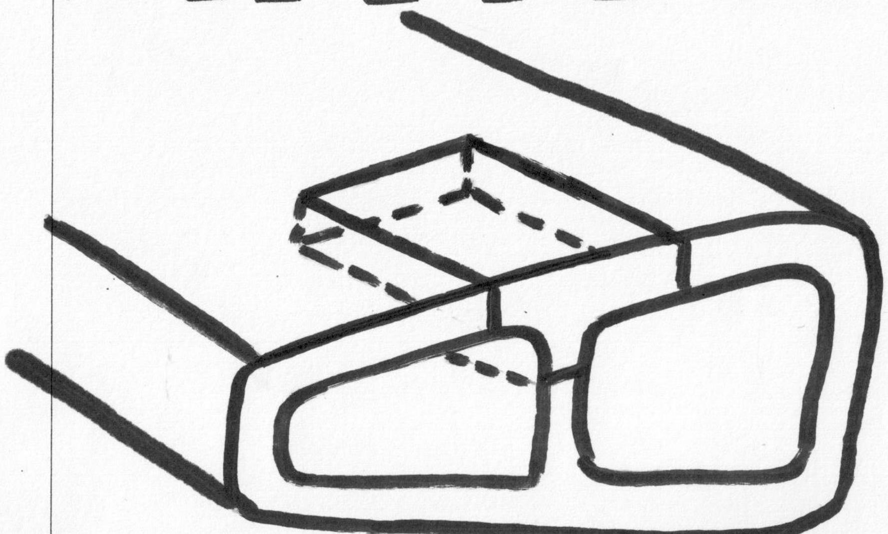
$$\frac{1}{2}\frac{\partial U_T}{\partial M} = \alpha = \frac{1}{4GA^2}\oint_s \frac{M}{t}\,ds = \boxed{\frac{1}{2GA}\oint_s \frac{q}{t}\,ds = \alpha} \quad (4)$$

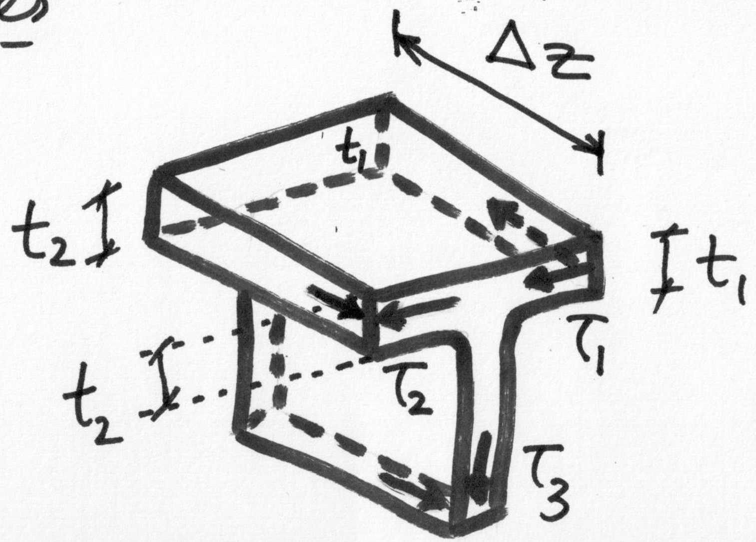
When q changes along ℄ of cell. (for use in multi-celled case)

For q = const over s, $\boxed{\alpha = \frac{q}{2GA}\oint \frac{1}{t}\,ds}$

(for use in single-celled case only)

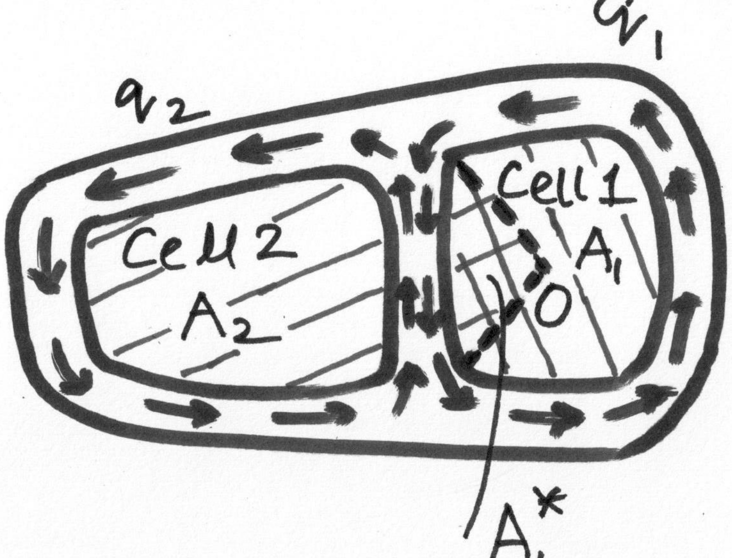
(B) Multi-celled t.w. tubes





analogous to current/fluid flow.

$$\sum F_z = 0 \implies (\tau_2 t_2 + \tau_3 t_3 - \tau_1 t_1)\Delta z = 0 \implies \boxed{q_2 + q_3 = q_1} \quad (5)$$



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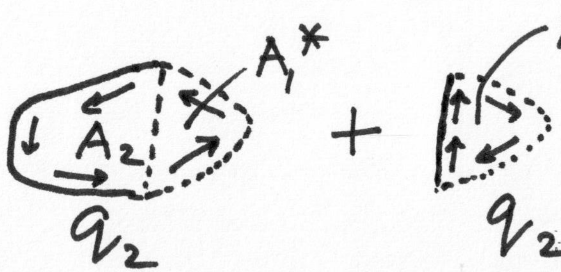
Cell 1 $\rightarrow$ $M_1 = 2q_1 A_1$

M_1 = moment about O due to q_1

Cell 2



$$M_2 = 2q_2(A_2 + A_1^*) - 2q_2 A_1^*$$

moment abt O due to q_2 $\leftarrow$ $\boxed{M_2 = 2q_2 A_2}$

$$\boxed{M = M_1 + M_2 = 2q_1A_1 + 2q_2A_2} \rightarrow (2a)$$

Now rate of twist is same for all cells (compatibility).

Using (4) $\rightarrow$ for cell 1 $\rightarrow$ $2G\alpha = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$ $\rightarrow$ (4a)

for cell 2 $\rightarrow$ $2G\alpha = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)$

Where $a_1 = \oint \frac{ds}{t}$ for cell 1 (including web); $a_2 = \oint \frac{ds}{t}$ for cell 2 (incl. web)

$a_{12} = \int \frac{ds}{t}$ for web only.

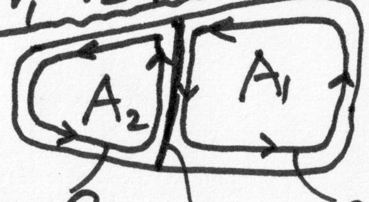
Rigorous way of deriving (4a):

Here we directly applied (4) to each cell. A better way of deriving (4a) from first principles is as follows.

$$U_T = \frac{1}{2}\int_V \frac{\tau^2}{G}\, dV = \frac{1}{2} z \oint \frac{q^2}{Gt^2}\, t\, ds = \frac{1}{2} z \oint \frac{q^2}{Gt}\, ds$$

Used $q = q_1$ for C_1
$= q_2$ for C_2
$= q_3 = q_1 - q_2$ for web

$$= \frac{1}{2} z \left[\oint_{C_1} \frac{q_1^2}{Gt}\, ds + \oint_{C_2} \frac{q_2^2}{Gt}\, ds - 2\int_{web} \frac{q_1 q_2}{Gt}\, ds \right.$$



q_2, C_2 web, C_1, q_1

$q_3 = q_1 - q_2$

$$= \frac{1}{2} z \left[\oint_{C_1} \frac{M_1^2}{Gt 4A_1^2}\, ds + \oint_{C_2} \frac{M_2^2}{Gt 4A_2^2}\, ds - 2\int_{web} \frac{M_1 M_2}{Gt 4A_1 A_2}\, ds \right]$$

Used $M_1 = 2q_1 A_1$
$M_2 = 2q_2 A_2$

Note: $M = M_1 + M_2$ applied. M_1 causes α_1 in cell 1. M_2 causes α_2 in cell 2. Using Castigliano's theorem,

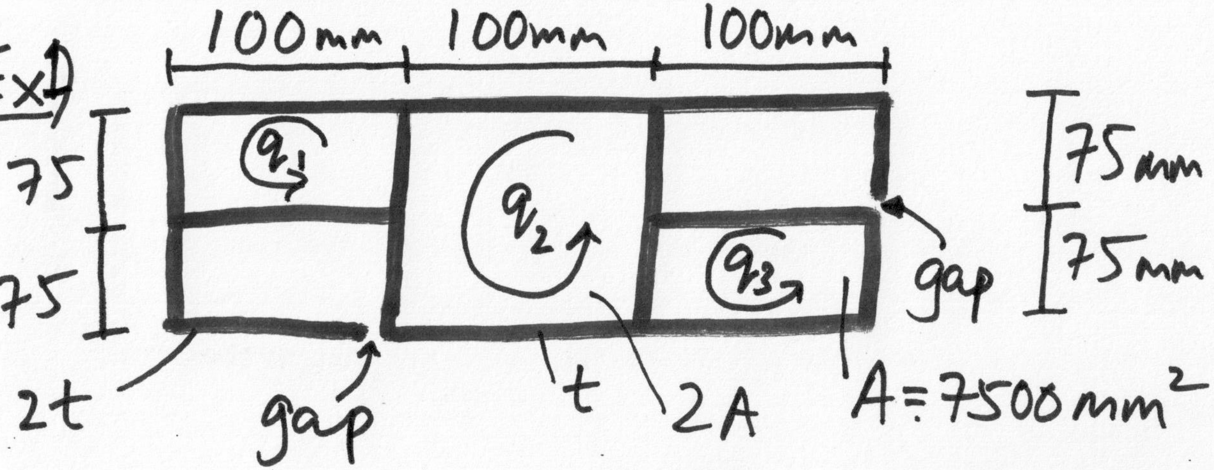
$$\frac{\partial U_T}{\partial M_1} = \alpha_1 z = z\left[\oint_{C_1} \frac{M_1}{Gt 4A_1^2}\, ds - \int_{web} \frac{M_2}{Gt 4A_1 A_2}\, ds \right]$$

$$\frac{\partial U_T}{\partial M_2} = \alpha_2 z = z\left[\oint_{C_2} \frac{M_2}{Gt 4A_2^2}\, ds - \int_{web} \frac{M_1}{Gt 4A_1 A_2}\, ds \right]$$

Compatibility $\Rightarrow \alpha_1 = \alpha_2 = \alpha$

$$\Rightarrow \boxed{\begin{aligned} \alpha &= \frac{1}{2GA_1}\left[q_1 \oint_{C_1} \frac{ds}{t} - q_2 \int_{web} \frac{ds}{t}\right] \\ \alpha &= \frac{1}{2GA_2}\left[q_2 \oint_{C_2} \frac{ds}{t} - q_1 \int_{web} \frac{ds}{t}\right] \end{aligned}} \rightarrow (4a) \text{ (repeated)}$$

(Ex1)



Thickness of open legs = 6mm

closed loop legs = 3mm

Find: (a) Torsional rigidity

(b) Max τ & leg(s) where it occurs.

Loops:

$$\left.\begin{aligned} 2G\alpha A &= \frac{1}{t}(q_1 \cdot 350 - q_2 \cdot 75) \\ 2G\alpha 2A &= \frac{1}{t}(q_2 \cdot 500 - q_1 \cdot 75 - q_3 \cdot 75) \\ 2G\alpha A &= \frac{1}{t}(q_3 \cdot 350 - q_2 \cdot 75) \end{aligned}\right\} \rightarrow \text{Solution is}$$

$$q_1 = q_3 = \frac{325}{81875}(2G\alpha At)$$

$$q_2 = \frac{425}{81875}(2G\alpha At)$$

$$M_1 = M_{loops} = 2 * 2q_1 A + 2q_2(2A) = 4A(q_1 + q_2)$$
$$= 12.3664 * 10^6 G\alpha$$

Open legs :

$$M_2 = M_{open\ legs}$$

$$\alpha = \frac{3M_2}{G \sum a_i b_i^3} = \frac{3M_2}{G(2 * 175 * 6^3)}$$

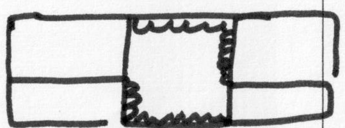
$$\text{Torsional rigidity} = C = \frac{M}{\alpha} = \frac{M_1 + M_2}{\alpha} = \underline{\underline{12.392 * 10^6 G}} \ N.mm^2$$
(if G in N/mm^2)

Max τ_{sz} in loops corresponds to q_2 shear flow

$$(\tau_{sz})_{max\ in\ loops} = \frac{q_2}{t} = 77.86\, G\alpha \ N/mm^2$$

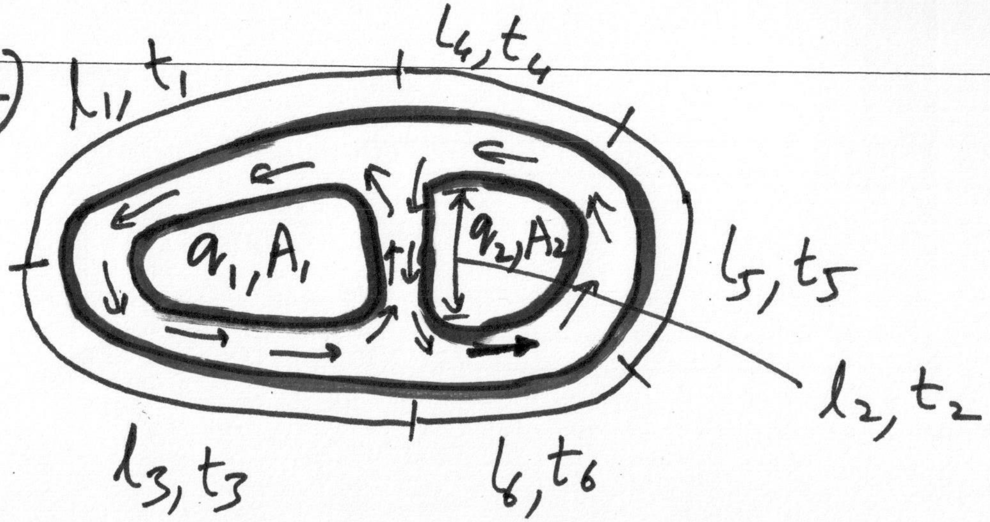
$$(\tau_{sz})_{max,\ in\ open\ legs} = G\alpha (b_i)_{max} = 6G\alpha \ N/mm^2$$

So $(\tau_{sz})_{max}$ occurs in loops. as indicated by ~~~ →



$$(\tau_{sz})_{max} = 77.86\, G\alpha = \frac{77.86}{12.392} M * 10^{-6} = 6.283 * 10^{-6} M, \ N/mm^2$$

(Ex 2)



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$$\left\{\begin{aligned} M &= 2q_1A_1 + 2q_2A_2 \\ 2G\alpha &= \frac{1}{A_1}\left(a_1q_1 - a_{12}q_2\right) = \frac{1}{A_1}\left(\frac{l_1}{t_1} + \frac{l_3}{t_3} + \frac{l_2}{t_2}\right)q_1 - \frac{l_2}{t_2}q_2 \\ 2G\alpha &= \frac{1}{A_2}\left(a_2q_2 - a_{12}q_1\right) = \frac{1}{A_2}\left(\frac{l_6}{t_6} + \frac{l_5}{t_5} + \frac{l_4}{t_4} + \frac{l_2}{t_2}\right)q_2 - \frac{l_2}{t_2}q_1 \end{aligned}\right.$$

$\rightarrow$ Solve q_1, q_2, α.

Later We will compare result derived from Hollow Thick Walled Torsion, specialized for thin-wall case, with the above result obtained from thin-walled torsion theory — see (Ex 4) pp. 44–46.

FINITE DIFFERENCE METHOD FOR TORSION.

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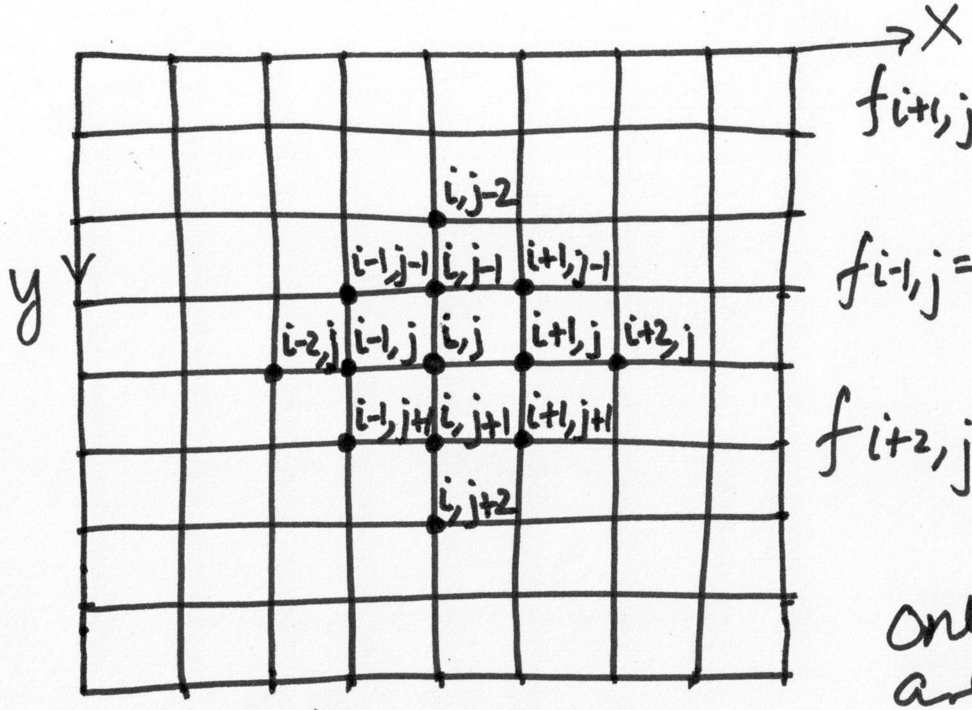


$$f_{i+1,j} = f_{i,j} + h\left(\frac{\partial f}{\partial x}\right)_{i,j} + \frac{h^2}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} \rightarrow \text{(i)}$$

$$f_{i-1,j} = f_{i,j} - h\left(\frac{\partial f}{\partial x}\right)_{i,j} + \frac{h^2}{2}\left(\frac{\partial f}{\partial x^2}\right)_{i,j} \longrightarrow \text{(ii)}$$

$$f_{i+2,j} = f_{i,j} + 2h\left(\frac{\partial f}{\partial x}\right)_{i,j} + 2h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} \longrightarrow \text{(iii)}$$

Only formulae relevant to Torsion are presented here.

Central Difference

$$\text{(i, ii)} \Rightarrow \frac{\partial f}{\partial x}_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} \rightarrow (1) \quad ; \quad \frac{\partial f}{\partial y}_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2h} \longrightarrow (1a)$$

$$\frac{\partial^2 f}{\partial x^2}_{i,j} = \frac{f_{i+1,j} + f_{i-1,j} - 2f_{i,j}}{h^2} \quad ; \quad \frac{\partial^2 f}{\partial y^2}_{i,j} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{h^2}$$

Forward / Backward Difference

(i),(iii) $\rightarrow$ $$\frac{\partial f}{\partial x}_{i,j} = \frac{4f_{i+1,j} - 3f_{i,j} - f_{i+2,j}}{2h}$$

$$\frac{\partial f}{\partial x}_{i,j} = \frac{-4f_{i-1,j} + 3f_{i,j} + f_{i-2,j}}{2h}$$

$$\frac{\partial f}{\partial y}_{i,j} = \frac{4f_{i,j+1} - 3f_{i,j} - f_{i,j+2}}{2h}$$

$$\frac{\partial f}{\partial y}_{i,j} = \frac{-4f_{i,j-1} + 3f_{i,j} + f_{i,j-2}}{2h}$$

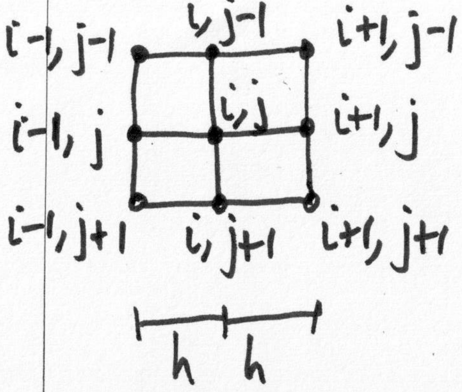
use only for stresses at boundary nodes. $\rightarrow$ 1(b, c, d, e)

$$\boxed{\begin{gathered}\nabla^2\phi_{i,j} = \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -2G\alpha h^2 \rightarrow ② \\ (\phi)_{\substack{\text{boundary}\\ \text{node}}} = (\phi)_s = 0 \rightarrow ③\end{gathered}}$$

Write ② for all interior nodes. Using ②, ③, solve ϕ at all interior nodes. Then use ① for stresses. $(f \equiv \phi)$.

Solution of ②, ③ gives $(\phi)_{\substack{\text{interior}\\ \text{node}}} = \text{number} * G\alpha h^2$

To get α in terms of applied M, use $M = 2\iint_A \phi\,dx\,dy$. ④

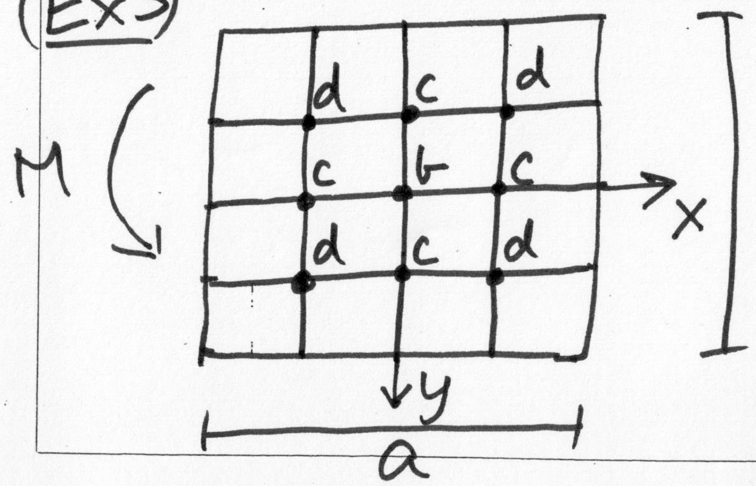


Using Simpsons rule,

$$\boxed{\int_{x_i-h}^{x_i+h}\int_{y_i-h}^{y_i+h} \phi\,dx\,dy = \frac{h^2}{9}\Big[16\phi_{i,j} + 4\left(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}\right) + \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\Big]}$$

Adding integrals over squares of size 2h×2h you get
$M = \text{number} * G\alpha h^4 \longrightarrow$ solve for α from this in terms of M.

(Ex3)



Square (a×a) section. Use $h = a/4$.
Using symmetry you can identify nodes 'd' and 'c' where ϕ is identical, ie ϕ_d & ϕ_c, respectively.
Thus ϕ_b, ϕ_c, ϕ_d are the distinct ϕ's.

From ②,

$$\left.\begin{aligned} 4\phi_b - 4\phi_c &= 2G\alpha h^2 \\ 4\phi_c - \phi_b - 2\phi_d &= 2G\alpha h^2 \\ 4\phi_d - 2\phi_c &= 2G\alpha h^2 \end{aligned}\right\} \Rightarrow \begin{aligned} \phi_b &= \frac{9}{4} G\alpha h^2 \\ \phi_c &= \frac{7}{4} G\alpha h^2 \\ \phi_d &= \frac{11}{8} G\alpha h^2 \end{aligned}$$

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$$\frac{M}{2} = 4 * \int_{x_d-h}^{x_d+h} \int_{y_d-h}^{y_d+h} \phi\, dx\, dy = \left(16\phi_d + 4(2\phi_c) + \phi_b\right)\frac{h^2}{9} * 4 = 17 G\alpha h^4$$

$$\alpha = \frac{M}{34 G h^4} = \frac{M}{0.133 G a^4}$$

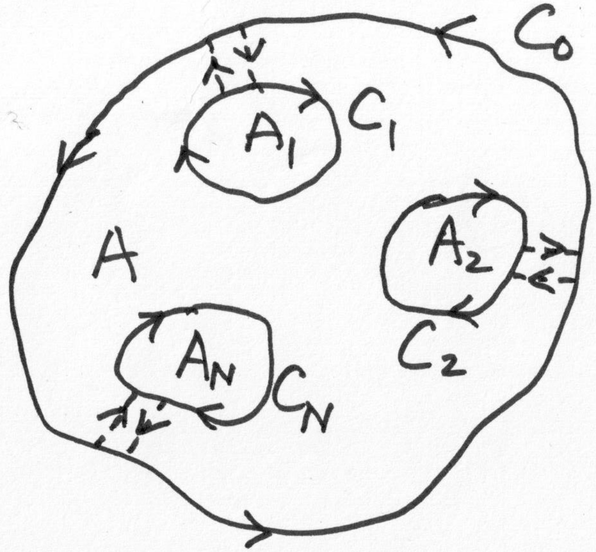
(exact value of coeff in denom is 0.141)

$$\tau_{max} = \sigma_{zy}\Big|_{\substack{x=a/2 \\ y=0}} = -\frac{\partial\phi}{\partial x}\Big|_{\substack{x=a/2 \\ y=0}} = -\frac{(-4\phi_c + \phi_b)}{2h} = \frac{19}{32} G\alpha a$$

$$= \frac{19}{32} \frac{M}{0.133 a^3} = \frac{M}{0.224 a^3}$$

(exact value is 0.208).

TORSION OF HOLLOW MULTICELLED THICK WALLED SHAFTS



$C_0 \rightarrow$ external boundary

$C_1, \cdots C_N \rightarrow$ internal bndry's

$$\nabla^2 \phi = -2G\alpha \quad \text{in } A$$

$$\phi = 0 \quad \text{on } C_0$$

$$\phi = K_m \quad \text{on } C_m, \quad m = 1, \cdots N.$$

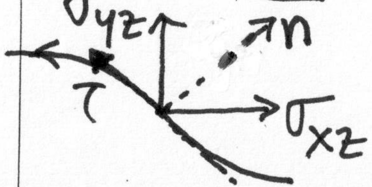
K_m determined from single-valuedness of $\psi \equiv w$. Note u, v, are single valued by their definition (ie $-y\alpha z$ & $x\alpha z$)

$$\Rightarrow G\alpha \oint_{C_m} d\psi = 0 \qquad \text{(ie } \psi \text{ single valued over } C_m \text{ closed loop.)}$$

$$G\alpha \oint_{C_m} (\psi_{,x} dx + \psi_{,y} dy) = \oint \left[(\phi_{,y} + G\alpha y) dx - (\phi_{,x} + G\alpha x) dy \right] = 0$$

$$\oint_{C_m} \left(\phi_{,y} \frac{dx}{ds} - \phi_{,x} \frac{dy}{ds} \right) ds = G\alpha \oint_{C_m} \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right) ds$$

$$\oint_{C_m} \underbrace{(-\sigma_{xz} m + \sigma_{yz} l)}_{\equiv\, \tau} ds = G\alpha \oint (lx + my)\, ds$$



(∵ $\phi = \text{const}$ on C_m, hence τ tangential to C_m. Note +ve τ along +ve s since CCW +ve for M)

(Alternatively $\phi_{,y} \dfrac{dx}{ds} - \phi_{,x} \dfrac{dy}{ds} = -\phi_{,y} \dfrac{\partial y}{\partial n} - \phi_{,x} \dfrac{\partial x}{\partial n} = -\dfrac{\partial \phi}{\partial n} = \tau$ on C_m on which $\phi = \text{const}$, see p.13)

$$\Rightarrow \frac{1}{G\alpha} \oint_{C_m} \tau\, ds = \oint_{C_m} (lx + my)\, ds = \iint_{A_m} (1+1)\, dA = 2A_m$$

$$\boxed{\oint_{C_m} \tau\, ds = 2 A_m G\alpha}$$

→ gives N equations (∵ N holes).

→ (1) Here A_m is area of the m^{th} hole bounded by C_m.

Also,

$$M = 2 \iint_A \phi\, dx\, dy - \oint_{C \text{ or } S} (xl + ym) \phi\, ds$$

, $C \equiv S = C_0 \cup C_1 \cdots \cup C_N$ traversed as shown, ie C_0 traversed CCW, $C_1, \cdots C_N$ trav. CW

A = area of **solid portion** of the hollow shaft.

Choose $\phi = 0$ on C_0, $\phi = K_m$ on C_m

$$M = 2\iint_A \phi\, dA + \sum_{i=1}^{N} K_i \underbrace{\oint_{C_i} (lx + my)\, ds}_{=2A_i}$$

$$\boxed{M = 2\iint_A \phi\, dA + \sum_{i=1}^{N} 2K_i A_i} \rightarrow (2) \leftarrow$$ used $\phi = 0$ on C_0 and $\oint_{C_m} = -\oint$ $\rightarrow$ see details on p 43a.

Solve (1), (2) for α, K_i ($N+1$ unknowns).

By membrane analogy, M = vol. displaced by membrane with flat rigid plates over holes A_m, ie.,

$$2\iint_A \phi\, dA = 2 * (\text{vol displaced by membrane portion}).$$

$$2\sum_{i=1}^{N} K_i A_i = 2 * (\text{vol. displaced by flat rigid plate over holes } A_m).$$

ie., K_i is analogous to vertical displacement of i^{th} flat rigid plate.

Details of sign change in $\oint_{C_i}$ term in eqn (2) p.43.

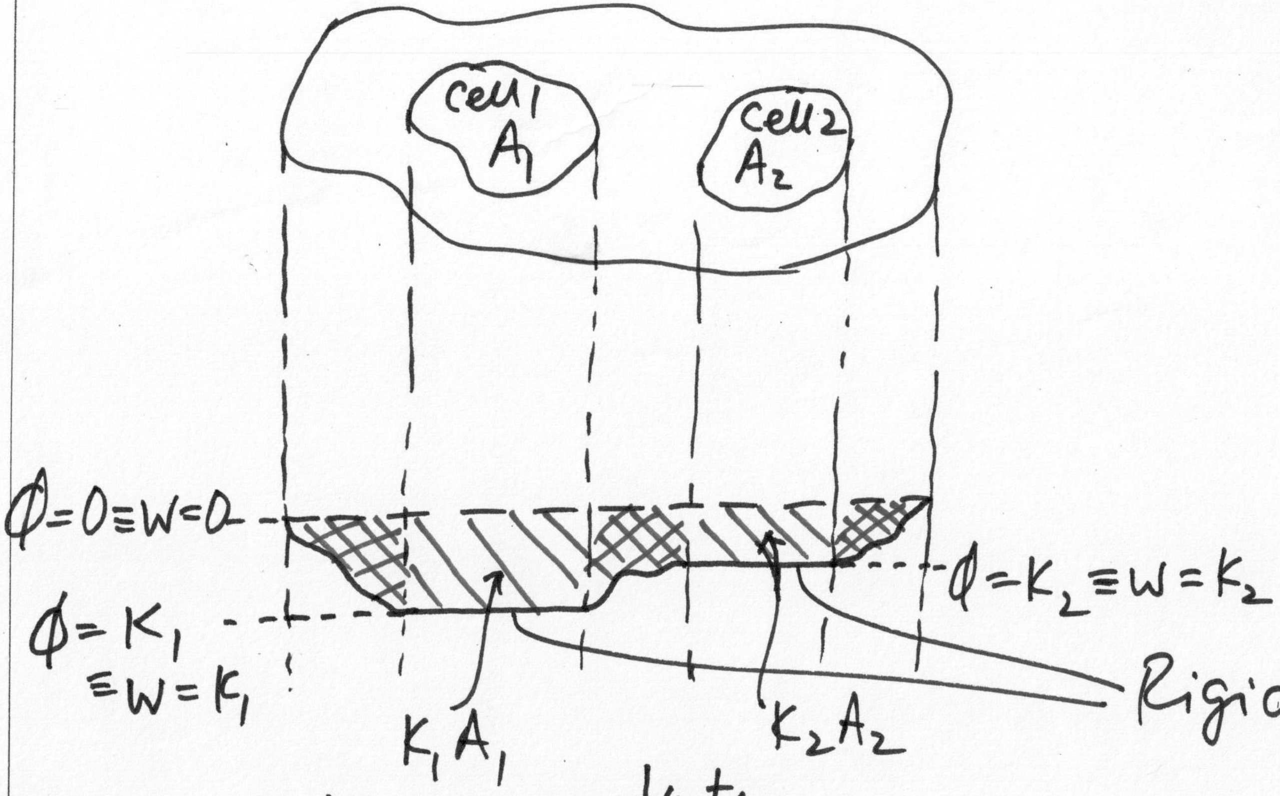
We have 2^{nd} term on p42 bottom,

$$-\oint_C (xl+ym)\phi\, ds = -\left[\oint_{C_0}(xl+ym)\cancelto{0}{\phi}\, ds + \oint_{C_1}(xl+ym)\cancelto{K_1}{\phi}\, ds + \cdots \oint_{C_N}(xl+ym)\cancelto{K_N}{\phi}\, ds\right]$$

Note that C traversed so that solid part lies to the left, ie C_0 traversed CCW & $C_1, \ldots, C_N$ traversed CW. (p.41).

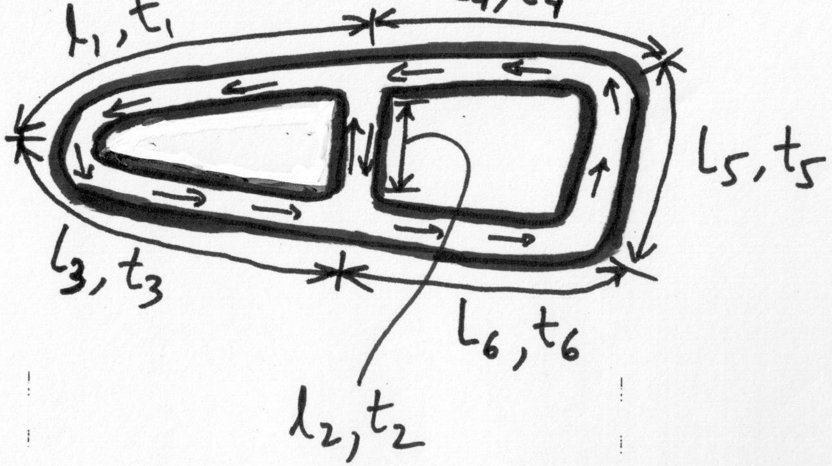
$$\Rightarrow -\oint_C (xl+ym)\phi\, ds = -\left[-K_1\oint_{C_1}(xl+ym)\,ds - \cdots - K_N\oint_{C_N}(xl+ym)\,ds\right]$$

$$= +\sum_{i=1}^{N} K_i \oint_{C_i} (lx+my)\, ds.$$



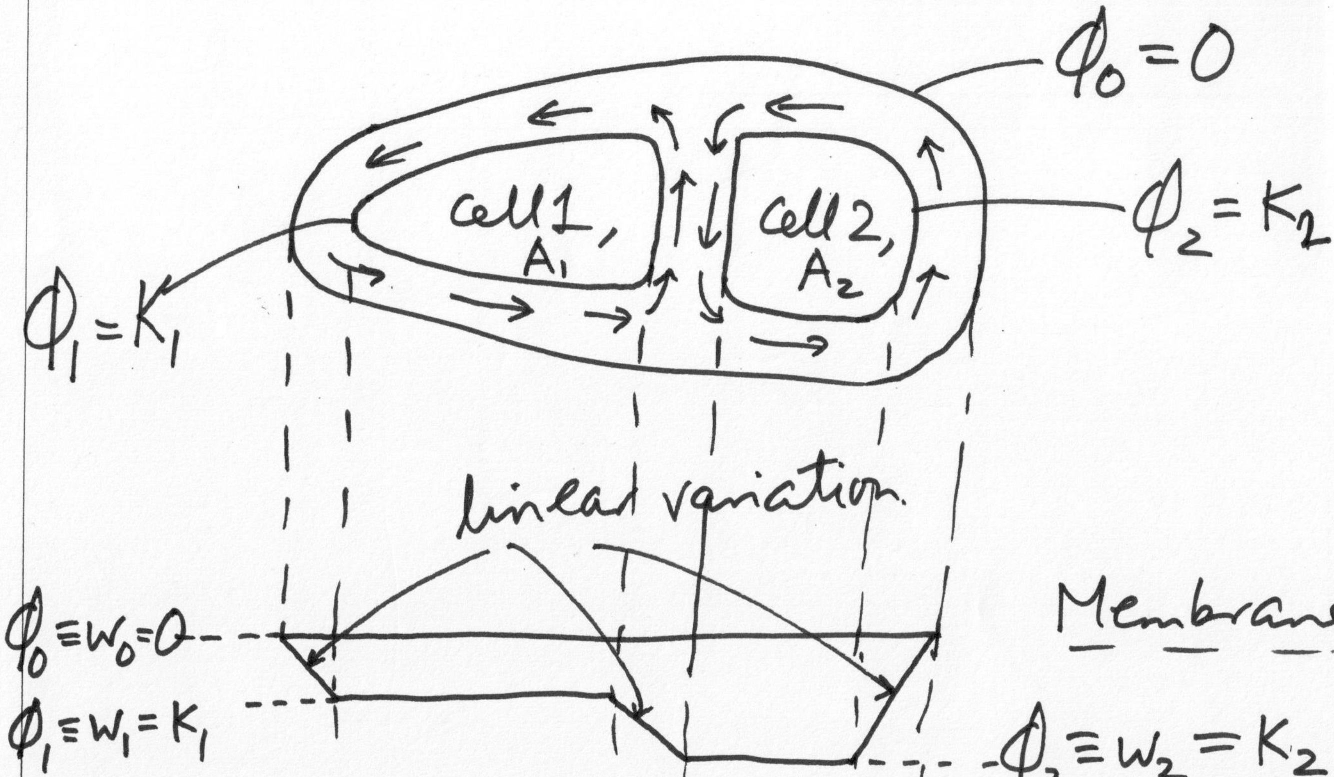
Cross-hatched vol $= \iint_A \psi \, dA$, $\psi \equiv \phi$.

(Ex 4)



Multi-celled thin walled tube with dimensions shown.

By specializing using eqns (1), (2) for thick walled multi-celled torsion, obtain same results as obtained using thin-walled closed tubes formulae on p. 32, 33, ie Ex2



Membrane analogy

Since thin walled, assume ~~from~~ using membrane analogy that $\phi \, (\equiv w)$ varies linearly thru wall thickness.

Use $\tau = -\dfrac{\partial \phi}{\partial n}$ ($\because$ $\tau =$ const thru thk so $\tau = \tau_s = -\phi_{,n}$).

leg l_1: $\tau_1 = -\left(\dfrac{\phi_0 - \phi_1}{t_1}\right) = \dfrac{K_1}{t_1}$; leg l_2: $\tau_2 = -\dfrac{(\phi_2 - \phi_1)}{t_2} = \dfrac{K_1 - K_2}{t_2}$

(ie., τ_2 assumed +ve upward $\uparrow$)

leg l_3: $\tau_3 = -\dfrac{(\phi_0-\phi_1)}{t_3} = \dfrac{K_1}{t_3}$; leg l_4: $\tau_4 = -\dfrac{(\phi_0-\phi_2)}{t_4} = \dfrac{K_2}{t_4}$

leg l_5: $\tau_5 = -\dfrac{(\phi_0-\phi_2)}{t_5} = \dfrac{K_2}{t_5}$; leg l_6: $\tau_6 = -\left(\dfrac{\phi_0-\phi_2}{t_6}\right) = \dfrac{K_2}{t_6}$

$$C_1: \oint_{C_1} \tau\, ds = 2G\alpha A_1 = \frac{K_1}{t_1} l_1 + \frac{(K_1-K_2)}{t_2} l_2 + \frac{K_1}{t_3} l_3 \longrightarrow (A)$$

$$C_2: \oint_{C_2} \tau\, ds = 2G\alpha A_2 = \frac{K_2}{t_4} l_4 + \frac{K_2}{t_5} l_5 + \frac{K_2}{t_6} l_6 + \frac{K_2-K_1}{t_2} l_2 \longrightarrow (B)$$

$$M = 2\underbrace{\iint_A \phi\, dA}_{\simeq 0} + 2K_1A_1 + 2K_2A_2 \longrightarrow (C)$$

$\simeq 0$ $\because$ A small due to thin wall.

Solve K_1, K_2, α, from (A), (B), (C).

Comparing with sol$^{\underline{n}}$ using thin-walled theory (Ex2), it matches with $K_1 \equiv q_1$, $K_2 \equiv q_2$ ie shear flows.