TYPES OF BOUNDARY VALUE PROBLEMS IN ELASTICITY

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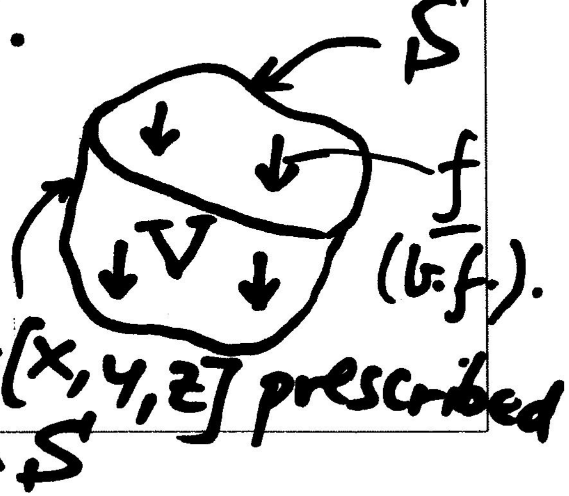
TYPE-I DISPLACEMENT FORMULATION

Given: displacements u_i (ie u_1, u_2, u_3) prescribed on boundary surface S', and body forces f_i at all points (x, y, z).

Find: u_i at all pts, ie $u_i[x,y,z]$ and $\sigma_{ij}[x,y,z]$ and $e_{ij}[x,y,z]$.

Equilibrium $\rightarrow$ $\sigma_{ij,j} + \underset{\text{b.f. per unit vol}}{f_i} = 0$



Subst Constitutive Law ($\underbrace{3a \text{ or } 4b}_{\text{of CL}}$)
in Equil eqn,

$$\frac{E}{1+\nu}\left[e_{ij,j} + \frac{\nu}{1-2\nu}\delta_{ij}e_{kk,j}\right] + f_i = 0$$

Subst Strain-Displ relations (1A of STR ANALYSIS)

$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ in above, get,

$\{e_{kk} = u_{k,k} = D\}$

$$\boxed{\begin{gathered}\frac{E}{2(1+\nu)}\Big[\underbrace{u_{i,jj}}_{\nabla^2 u_i} + \frac{1}{1-2\nu}\underbrace{u_{j,ji}}_{D_{,i}}\Big] + f_i = 0 \\ \text{or} \quad (\lambda+\mu)D_{,i} + \mu\nabla^2 u_i + f_i = 0\end{gathered}} \rightarrow (1)$$

NAVIER'S EQN'S

$\hookrightarrow$ 3 coupled PDE's (partial differential eqns) in unknowns u_1, u_2, u_3. Their solution must satisfy BC's on u_1, u_2, u_3!

So use the prescribed BC's on S on u_1, u_2, u_3 to solve for $u_i[x, y, z]$

NOTE:

- To derive NAVIER's eqn we used Equil, Constitutive, & Strain-Displ relations. So solution of NAVIER's eqn satisfies all these 3 sets of equations (ie 3-equil, 6-constitive, 6-str-displ eqns).
- What about satisfying the 6-Compatibility equations (ie ST. VENANT or B.M. compat eqs). If you put the str-displ eqns into the ST. VENANT COMPAT Eqs, you see that they are identically satisfied, as follows.

$$e_{ij,kl} + e_{kl,ij} = e_{ik,jl} + e_{jl,ik}$$

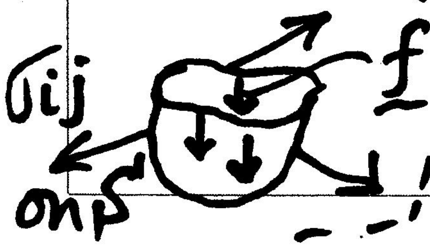
$$\cancel{u_{i,jkl}} + \cancel{u_{j,ikl}} + \cancel{u_{k,lij}} + \cancel{u_{l,kij}}$$

$$= \cancel{u_{i,kjl}} + \cancel{u_{k,ijl}} + \cancel{u_{j,lik}} + \cancel{u_{l,jik}}$$

So since we used str-displ to derive Navier's eqn, Soln of Navier's eqn automatically satisfies Compatibility. (both in terms of $\underline{\underline{e}}$ & $\underline{\underline{\sigma}}$).

TYPE-II — STRESS FORMULATION.

Given: σ_{ij} on boundary S (in terms of stress BC's involving applied loads), and b.f's $f_i[x, y, z]$ throughout V.



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Find: $\sigma_{ij}[X, Y, Z]$ in V and hence $e_{ij}[X, Y, Z]$ & $u_i[X, Y, Z]$ thruout V.

In general a set of stress components σ_{11}, σ_{12}, etc that satisfy only equilibrium eqns may not be correct since these stress comp's may violate BM compat or resulting strains may violate ST. VENANT compat.

So you must explicitly satisfy either BM or ST VENANT compat eqns (6 of them). Solution of these 6-compat eqns, along with stress BC's given, yield stresses $\sigma_{ij}(X, Y, Z)$. Then get $e_{ij}(x,y,z)$ from C.L., and get $u_i(x,y,z)$ by integrating Str-displ eqns (either analytically or numerically). Since Compat eqns were explicitly satisfied (as σ_{ij} solution is based on solving Compat PDE's), we are guaranteed that $u_i(x,y,z)$ thus obtained are single-valued.

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TYPE-III — MIXED FORMULATION.

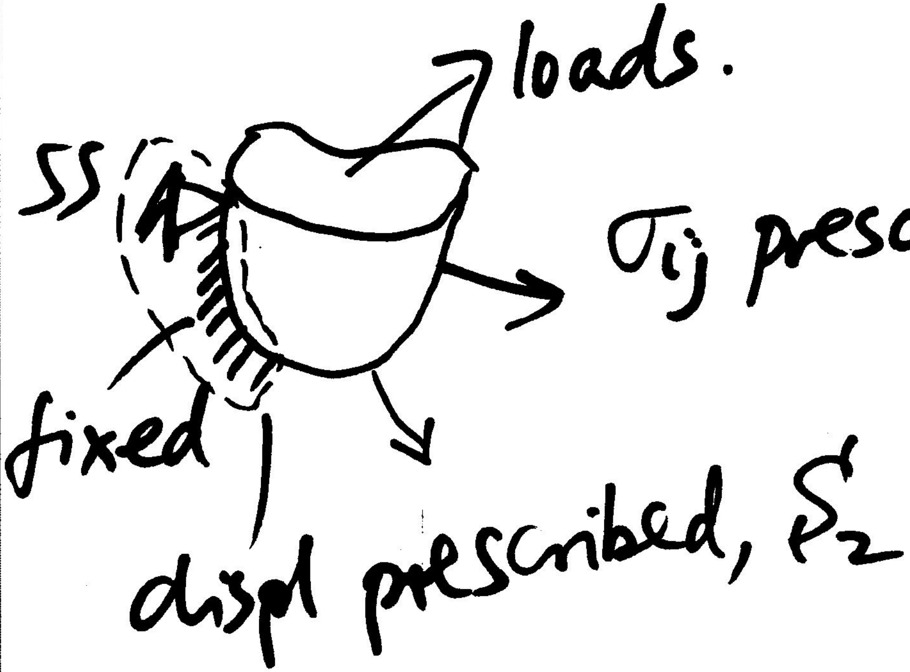
Given: $\underline{\underline{T}}$ prescribed on S_1, $\underline{u}$ prescribed on S_2, $S = S_1 \cup S_2$ is boundary, and $\underline{f}$ given throughout V.

Find: $\underline{\underline{\sigma}}$, $\underline{u}$, $\underline{\underline{e}}$ thruout V.

Here you need to use all 4-sets of equs ie,

Equil (3), CL (6), Str-displ (6), Compat (6), and do mixed formulation

(ie part $\underline{\underline{\sigma}}$, part $\underline{u}$)



We will do TYPE-I & TYPE-II formulations in this course.