CE619 STRUCTURAL STABILITY

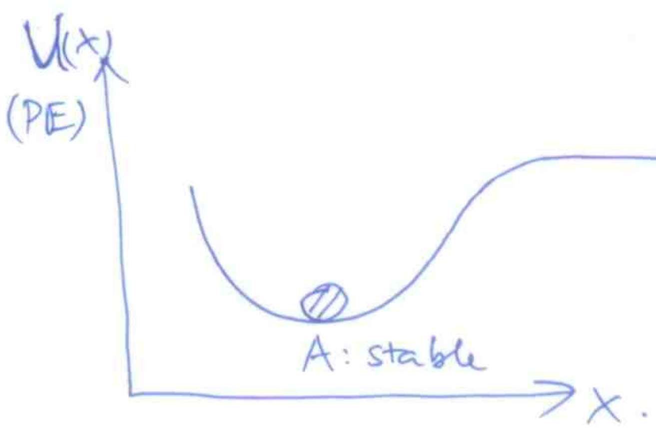


FIG: 1

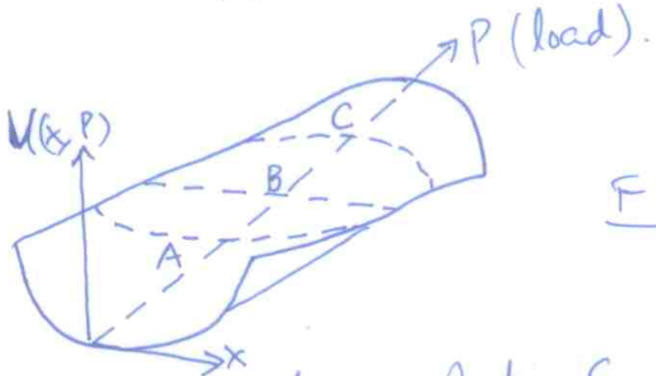


FIG: 2

In Fig 2, transition from A to C (stable to unstable) occurs as load changes along line AC. The cross-sections denote potential v/s displ for given load, ie U[x, P]. Neutral equilibrium (B) represents critical load, P_{cr}, when transition occurs.

Stability – Definition 1:

Study of "character" of response (ie static or dynamic equilibrium). By character we mean:

(i) whether response remains bounded or whether it goes unbounded

(ii) whether response increases incrementally for incremental increase in load or whether it increases in a finite (large) manner for incremental change in load.

Stability – <u>Definition 2.</u>

Study of whether response grows or stays bounded when <u>small</u> perturbations in displacement and/or velocity are applied. (see Fig. 1). If damping present response either grows or decays, which decides stability or instability.

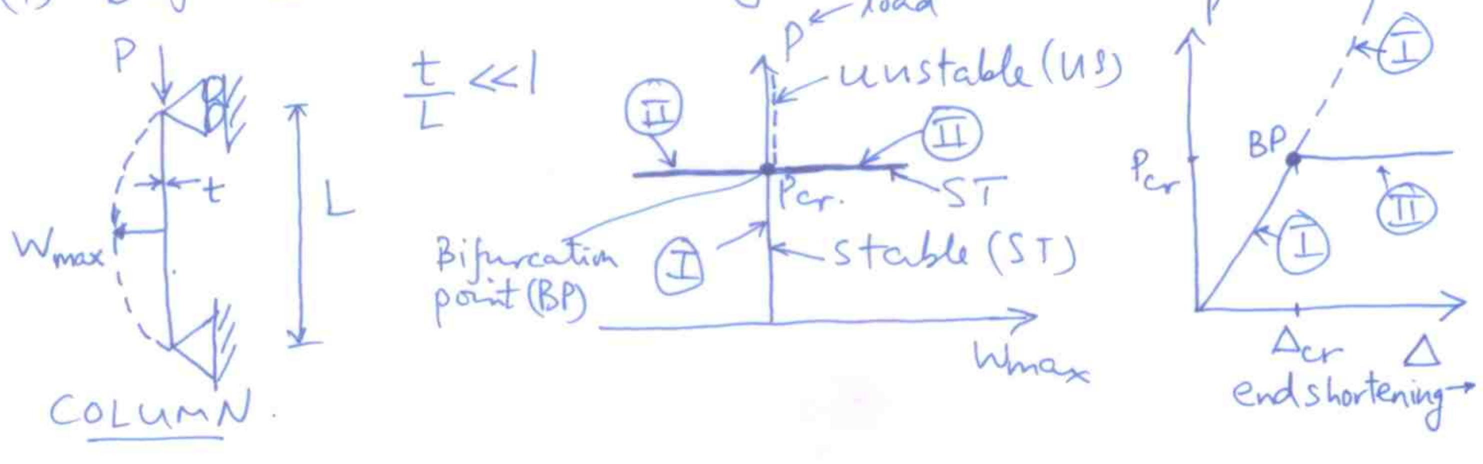
<u>Stability concept applied to structures</u>: (<u>see Defn. 1</u>).

Here we are concerned with stability of equilibrium configuration(s) as the applied load (characterized by P) increases. For small P, structures are stable in most practical situations. As the load is increased, the BUCKLING LOAD is the first (lowest) value of P for which equilibrium becomes unstable. This corresponds to the lowest CRITICAL LOAD calculated from the mathematical analysis.

In most structural elements, loss of stability is associated with a qualitative change in the deformation characteristics as seen later.

<u>Types of Buckling</u>. (for elastic structures with conservative loading).

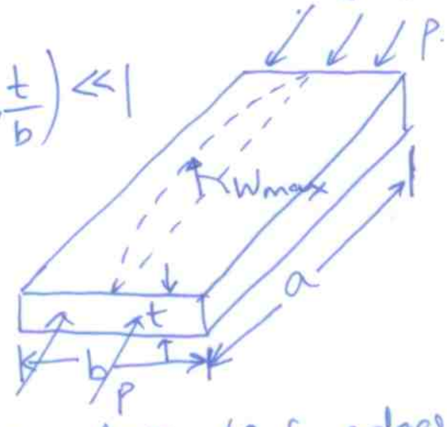
(i) Bifurcation Buckling.

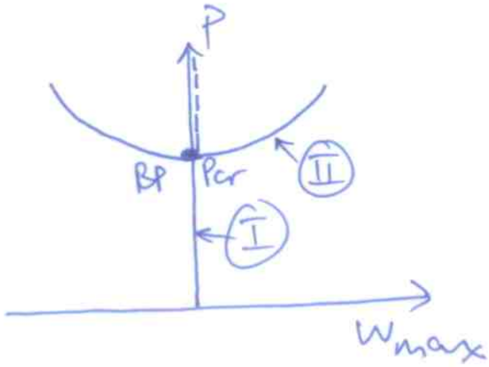


③ I – Pure compression, II → Compr + Buckling.

Columns are neutral in post-buckling (see flat portion of (P, W_{max}) or (P, Δ) curves).

$\left(\frac{t}{a}, \frac{t}{b}\right) \ll 1$





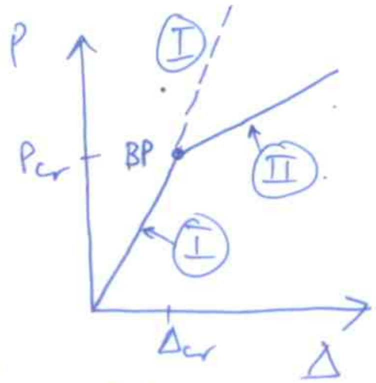
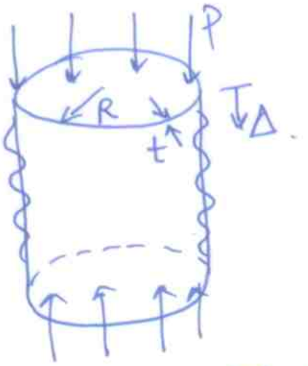
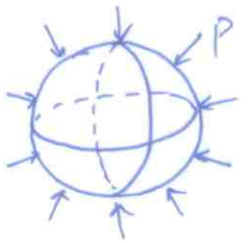


PLATE (S.S. edges). → Stable in post-buckling (ie can carry post-buckling load).

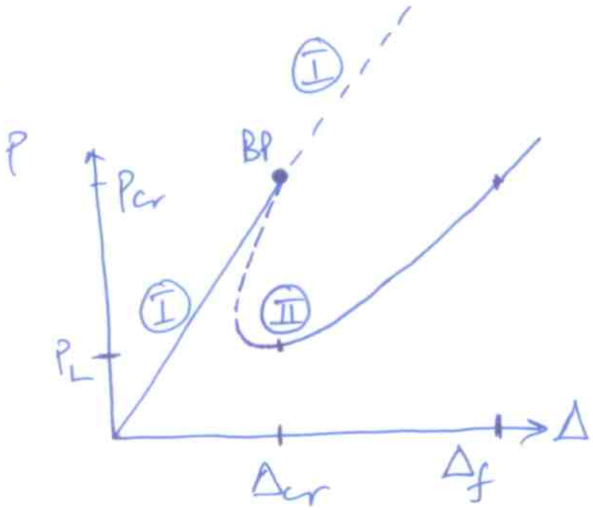
(ii) Finite-Disturbance Buckling.



THIN CYLINDRICAL SHELL

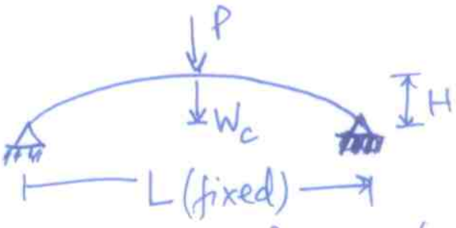


THIN SPHERICAL SHELL

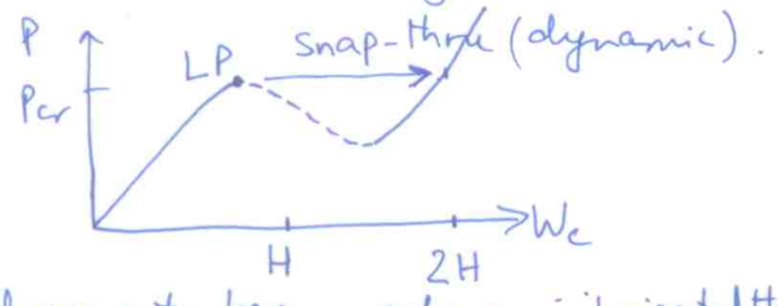


At B.P., if Δ_{cr} to be maintained then we must return to lower value of P (ie P_L), or else there will be finite jump ("disturbance") to Δ_f.

(iii) Limit Point / Snap-thru Buckling



SHALLOW ARCH



(analogous to beam-column ∵ horizontal thrust induced).

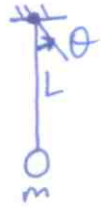
Some Basic Approaches to Stability Analysis.

(i) Equilibrium / Classical / Bifurcation approach:
Here we find level of load at which a transition from only-one equilibrium configuration to multiple (usually two) infinitesimally close equilibrium configurations exist. You end up with an eigenvalue problem whose eigenvalues are the critical loads.

(ii) Kinetic / Dynamic approach:
Perturb system about equilibrium keeping load constant and analyse whether response is bounded (oscillatory) or grows (unbounded). At some load value the transition takes place. The small perturbations should include all kinematically admissible modes / d.o.f.'s. (degrees-of-freedom).

(iii). Energy approach:
Write potential energy in terms of generalized coordinates. Find equilibrium by stationary values of potential. If potential is minimum for these stationary values then equilibrium stable. — eg. $U(\theta) = -mgl\cos\theta$, $\frac{dU}{d\theta} = mgl\sin\theta = 0$,

$\theta_{equil} = 0, \pi$, $\left.\frac{\partial^2 U}{\partial\theta}\right|_{\substack{\theta=0^\circ - ST \\ \theta=\pi - US}} = \pm mgl$ for pendulum.



Note: For a conservative system, at an equilibrium configuration,

$$\text{Min PE} \iff \text{STABLE}$$

Let $PE = U(q_1, q_2, \ldots q_N; P)$, $q_1 \ldots q_N$ are the generalized coordinates that completely define the structure's (systems) configuration — eg two angles θ, ϕ for double pendulum. 

Taylor series about equilibrium $(q_1, \cdots q_N)_e = (0, \cdots 0)$ gives,

$$U(q_1, \cdots q_N) = U(q_{1e} \cdots q_{ne}) + \sum_{i=1}^{n} \frac{\partial U}{\partial q_i}\Big|_{\underline{q}_e} \Delta q_i$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 U}{\partial q_i \partial q_j}\Big|_{\underline{q}_e} \Delta q_i \Delta q_j + H.O.T.'s$$

(higher order terms)

1st variation of U

2nd variation of U

$$= U(q_{1e}, \cdots q_{ne}) + \delta U + \frac{1}{2}\delta^2 U + H.O.T.'s.$$

NOTE: Without loss of generality, we can write $\underline{q}_e = (0, \cdots 0)$, ie even if equilibrium position is non-trivial, you can translate the coordinate system, ie $\underline{q}_{new} = \underline{q}_{old} - (\underline{q}_e)_{old}$ so that $(\underline{q}_e)_{new} = (0, \cdots 0)$. Then define $(q_i)_{new} \leftarrow (\Delta q_i)_{old}$ and drop the notations 'new', 'old'.

$$\Rightarrow U(q_1, \cdots q_N) = U(0, \cdots 0) + \underbrace{\sum_{i=1}^{N} \frac{\partial U}{\partial q_i}\Big|_{\underline{q}_e = (0, \cdots 0)} q_i}_{=0}$$

$=0$ by definition of equilibrium; ie $\delta U = 0$ is necessary condition for equilibrium

$$+ \frac{1}{2} \sum_{i,j=1}^{N} \sum \frac{\partial^2 U}{\partial q_i \partial q_j}\Big|_{\underline{q}_e} q_i q_j$$

so from here on, q denotes perturbation about equilibrium

$$\Rightarrow U(q_1, \cdots q_n) - U(0, \cdots 0) = \frac{1}{2} \sum_{i,j=1}^{N} \sum c_{ij} q_i q_j$$

where $c_{ij} = \dfrac{\partial^2 U}{\partial q_i \partial q_j}\Big|_{\underline{q}_e}$

For stable equilibrium at $\underline{q}_e$ we need $U(q_1, \cdots q_n) - U(0, \cdots 0) > 0$, ie, $\delta^2 U > 0$

ie $U(0, \cdots 0) = U(q_{1e}, \cdots q_{Ne})$ is a relative minimum.

Hence, $\sum\sum_{i,j=1}^{N} c_{ij} q_i q_j = \underbrace{\underset{(1\times N)}{\tilde{q}^T} \, \underset{(N\times N)}{\underline{\underline{C}}} \, \underset{(N\times 1)}{\underline{q}}}_{\text{symmetric quadratic form}} > 0$ for $\underline{q} \neq 0$ (ie non-trivial perturbation)

NOTE: $\underline{\underline{C}} = \underline{\underline{C}}^T \rightarrow$ symmetric

From linear algebra (e.g., see Simitses + Hodges for proof) we have the result that the symmetric quadratic form $q^T \underline{\underline{C}} \underline{q}$ is +ve iff $\underline{\underline{C}}$ is positive definite, i.e., if every principal-minor of $\underline{\underline{C}}$ has determinant >0.

$$\underline{\underline{C}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1N} \\ c_{12} & c_{22} & c_{23} & \cdots & c_{2N} \\ c_{13} & c_{23} & c_{33} & \cdots & c_{3N} \\ \vdots & & & & \\ c_{1N} & c_{2N} & c_{3N} & \cdots & c_{NN} \end{bmatrix}$$

, ie $c_{11} > 0$

$c_{11}c_{22} - c_{12}^2 > 0$

$\vdots$

$\det|\underline{\underline{C}}| > 0$.

where $\underline{\underline{C}} = \left[\left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{q_e} \right]$

Thus a bifurcation point satisfies

$$\underbrace{\frac{\partial U}{\partial q_i} = 0, \quad i = 1, \cdots N}_{\text{gives } \underline{q_e} \text{ as solution}}, \quad \text{and } \det\left[\frac{\partial^2 U}{\partial q_i \partial q_j}\right]_{\underline{q_e}} = 0$$

ie, $\underline{\underline{C}}$ is no longer positive definite.

Note that $U = U[q_1, \cdots q_N, P]$ so as you change load P, $\underline{q_e}$ changes, and at some P, bifurcation point (BP) occurs, ie, two or more branches of equilibrium ($\underline{q_e}$) meet. For the same equilibrium branch, we have stability on one side of the BP and instability on the other side.

Examples of sign definiteness:

$$\delta^2 U = q_1^2 + q_2^2 \longrightarrow \text{Positive Definite } (\delta^2 U > 0 \text{ for } \underset{\sim}{q} \neq \underline{0})$$

$$= (q_1+q_2)^2 \longrightarrow \text{Pos-semi-def } (\delta^2 U \geq 0, \ \underset{\sim}{q} \neq 0).$$

$$= -q_1^2 - q_2^2 \longrightarrow \text{Negative definite } (\delta^2 U < 0, \ \underset{\sim}{q} \neq 0)$$

$$= -(q_1+q_2)^2 \longrightarrow \text{Neg-semi def } (\delta^2 U \leq 0, \ \underset{\sim}{q} \neq 0).$$

$$= q_1 q_2 \longrightarrow \text{sign Indefinite}$$ (cant say anything about sign definite).

Another test for sign definiteness is based on eigenvalues of $\underline{\underline{C}}$, as follows.

$$\delta^2 U = \underline{q}^T \underline{\underline{C}}\, \underline{q}$$

seek transformation $\underline{q} = \underline{\underline{T}}\, \underline{y}$, $\det(\underline{\underline{T}}) \neq 0$

$$\delta^2 U = \underline{y}^T \underline{\underline{T}}^T \underline{\underline{C}}\, \underline{\underline{T}}\, \underline{y} = \underline{y}^T \underline{\underline{B}}\, \underline{y}, \quad B = B^T \longrightarrow (*)$$

Now EVP of $\underline{\underline{C}}$ is $\underline{\underline{C}}\, \underline{z} = \mu \underline{z}$, $\mu = \mu_1, \mu_2, \ldots \mu_N$

corresponding evectors are $\underset{\sim}{z}^{(1)}, \ldots z^{(N)}$

Property I: If $T = \begin{bmatrix} \underset{\sim}{z}^{(1)} & \underset{\sim}{z}^{(2)} & \cdots & z^{(N)} \\ \vdots & \vdots & & \vdots \end{bmatrix}$ = $N \times N$ matrix with columns as the N evectors of $\underline{\underline{C}}$

then $\underline{\underline{B}}$ = diagonal matrix = $\begin{bmatrix} \mu_1 & & 0 \\ & \mu_2 \ddots & \\ 0 & & \mu_N \end{bmatrix}$

Property II: $\mu_1, \ldots \mu_N$ are real.

Property III: evectors are orthogonal, ie,

$$\underline{z}^{(i)T} z^{(j)} = \delta_{ij} = 1, \ i = j \longrightarrow \text{make it orthonormal for uniqueness.}$$
$$= 0, \ i \neq j$$

Aside: Proofs of Properties I, II, III

Property II: Assume evalues are complex. Since $\underline{\underline{C}}$ is real, so the characteristic polynomical equation arising from $\det(\underline{\underline{C}} - \mu \underline{\underline{I}}) = 0$ has real coefficients. So if $\overset{(i)}{\mu}$ is a complex root (ie evalue) with corresponding evector $\overset{(i)}{\underline{z}}$ then $\overline{\overset{(i)}{\mu}}$ and $\overline{\overset{(i)}{z}}$ are also evalue & evector solutions, i.e, complex conjugates are also solution. Thus,

$$\underline{\underline{C}}\,\overset{(i)}{\underset{\sim}{z}} = \overset{(i)}{\mu}\,\overset{(i)}{\underset{\sim}{z}} \longrightarrow ①$$

$$\underline{\underline{C}}\,\overline{\overset{(i)}{\underset{\sim}{z}}} = \overline{\overset{(i)}{\mu}}\,\overline{\overset{(i)}{\underset{\sim}{z}}} \rightarrow ②$$

Do $\overline{\overset{(i)}{\underline{z}}}^T * ① - \overset{(i)}{\underset{\sim}{z}}^T * ②$,

$$\Rightarrow \left(\underbrace{\overline{\overset{(i)}{\underline{z}}}^T \underline{\underline{C}}\,\overset{(i)}{\underline{z}}} - \underbrace{\overset{(i)}{\underset{\sim}{z}}^T \underline{\underline{C}}\,\overline{\overset{(i)}{\underset{\sim}{z}}}}\right) = \overset{(i)}{\mu}\underbrace{\overline{\overset{(i)}{\underline{z}}}^T \overset{(i)}{\underline{z}}} - \overline{\overset{(i)}{\mu}}\underbrace{\overset{(i)}{\underline{z}}^T \overline{\overset{(i)}{\underline{z}}}}$$

Scalars so respective transpose equals term itself.

$$\Rightarrow \left(\overset{(i)}{\underline{z}}^T \underline{\underline{C}}^T\, \overline{\overset{(i)}{\underline{z}}} - \overset{(i)}{z}^T \underline{\underline{C}}\,\overline{\overset{(i)}{z}}\right) = \left(\overset{(i)}{\mu} - \overline{\overset{(i)}{\mu}}\right)\overset{(i)}{z}^T\,\overline{\overset{(i)}{z}}$$

$= \underline{\underline{C}} \because \underline{\underline{C}}$ symmetric

$$\Rightarrow 0 = \left(\overset{(i)}{\mu} - \overline{\overset{(i)}{\mu}}\right)\overset{(i)}{z}^T\,\overline{\overset{(i)}{z}} \quad \Leftrightarrow \quad \underline{\overset{(i)}{\mu} = \text{real.}}$$

Property III :

$$\left.\begin{aligned} \underline{\underline{C}}\,\overset{(i)}{\underline{z}} &= \overset{(i)}{\mu}\,\overset{(i)}{\underline{z}} \longrightarrow ③ \\ \underline{\underline{C}}\,\overset{(k)}{\underline{z}} &= \overset{(k)}{\mu}\,\overset{(k)}{\underline{z}} \rightarrow ④ \end{aligned}\right\}$$

$\rightarrow \overset{(k)}{\underline{z}}^T * ③ - \overset{(i)}{\underline{z}}^T * ④$ gives, after noting the scalar (triple products) on LHS & the scalar (dot products) on RHS & also $\underline{\underline{C}} = \underline{\underline{C}}^T$, as above,

$$\overset{(k)}{\underline{z}}^T \underline{\underline{C}}\,\overset{(i)}{\underline{z}} - \overset{(i)}{\underline{z}}^T \underline{\underline{C}}\,\overset{(k)}{\underline{z}} = \overset{(i)}{\mu}\,\overset{(k)}{\underline{z}}^T \overset{(i)}{\underline{z}} - \overset{(k)}{\mu}\,\overset{(i)}{\underline{z}}^T \overset{(k)}{\underline{z}}$$

$$0 = \left(\overset{(i)}{\mu} - \overset{(k)}{\mu}\right)\overset{(k)}{\underline{z}}^T \overset{(i)}{\underline{z}}$$

$\because \overset{(i)}{\mu} \neq \overset{(k)}{\mu}$ (ie distinct evalues assumed)

$$\Rightarrow \overset{(k)}{\underline{z}}^T \overset{(i)}{\underline{z}} = 0 \text{ for } i \neq k.$$

Since evectors are non-unique upto multiplicative constant, we make every evector a unit vector (i.e. orthonormalize), so

$$\underline{z}^{(i)T}\underline{z}^{(k)} = \delta_{ik} \begin{cases} = 0, & i \neq k \\ = 1, & i = k \end{cases} \text{ orthonormality property.}$$

Property I : $\underline{\underline{T}}^T \underline{\underline{C}}\, \underline{\underline{T}} = \begin{bmatrix} \underline{z}^{(1)T} \rightarrow \cdots \\ \underline{z}^{(2)T} \rightarrow \cdots \\ \vdots \\ \underline{z}^{(N)T} \rightarrow \cdots \end{bmatrix} [C] \begin{bmatrix} \underline{z}^{(1)} & \underline{z}^{(2)} & \cdots & \underline{z}^{(N)} \\ \downarrow & \downarrow & & \downarrow \\ \vdots & \vdots & & \vdots \end{bmatrix}$

$$= \begin{bmatrix} \underline{z}^{(1)T} \rightarrow \\ \underline{z}^{(2)T} \rightarrow \\ \vdots \\ \underline{z}^{(N)T} \rightarrow \end{bmatrix} \begin{bmatrix} \mu^{(1)}\underline{z}^{(1)} & \mu^{(2)}\underline{z}^{(2)} & \cdots & \mu^{(N)}\underline{z}^{(N)} \\ \downarrow & \downarrow & & \downarrow \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \mu^{(1)}\underline{z}^{(1)T}\underline{z}^{(1)} & \mu^{(2)}\underline{z}^{(1)T}\underline{z}^{(2)} & \cdots & \mu^{(N)}\underline{z}^{(1)T}\underline{z}^{(N)} \\ \mu^{(1)}\underline{z}^{(2)T}\underline{z}^{(1)} & \mu^{(2)}\underline{z}^{(2)T}\underline{z}^{(2)} & \cdots & \mu^{(N)}\underline{z}^{(2)T}\underline{z}^{(N)} \\ \vdots & & & \\ \mu^{(1)}\underline{z}^{NT}\underline{z}^{(1)} & \cdots & & \mu^{(N)}\underline{z}^{(N)T}\underline{z}^{(N)} \end{bmatrix}$$

Here all off-diagonal terms vanish due to orthogonality (Property-II), & diagonals are $\mu^{(1)}, \mu^{(2)}, \ldots \mu^{(N)}$ due to orthonormality.

$$\Rightarrow \underline{\underline{T}}^T \underline{\underline{C}}\, \underline{\underline{T}} = \begin{bmatrix} \mu^{(1)} & 0 & 0 \cdots & 0 \\ 0 & \mu^{(2)} & 0 & 0 \\ 0 & 0 & \ddots & \mu^{(N)} \\ 0 & 0 & 0 & \mu \end{bmatrix} \rightarrow \text{diagonal matrix with evalues along diagonals.}$$

So, from ⊛ p. 6a, $\boxed{\delta^2 U = \mu^{(1)} y_1^2 + \mu^{(2)} y_2^2 + \cdots \mu^{(N)} y_N^2 \rightarrow \text{PD if all } \mu^{(i)} > 0.}$

So, from ⊛ p. 6a,

$$\delta^2 U = \overset{(1)}{\mu} y_1^2 + \overset{(2)}{\mu} y_2^2 + \cdots + \overset{(N)}{\mu} y_N^2$$

$$= \text{P.D if} \quad \overset{(k)}{\mu} > 0, \quad k = 1, \ldots N.$$

Also note that

$$\underline{\underline{T}}^T \underline{\underline{T}} = \begin{bmatrix} \underline{\underline{z}}^{(1)T} \longrightarrow \cdots \\ \underline{\underline{z}}^{(2)T} \longrightarrow \cdots \\ \vdots \\ \underline{\underline{z}}^{(N)T} \longrightarrow \cdots \end{bmatrix} \begin{bmatrix} \underline{\underline{z}}^{(1)} & \underline{\underline{z}}^{(2)} & \cdots & \underline{\underline{z}}^{(N)} \\ \downarrow & \downarrow & & \downarrow \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$= \underline{\underline{I}} \quad \text{from orthonormality property.}$$

ie $\underline{\underline{T}}^T = \underline{\underline{T}}^{-1}$

<u>Continuous structures</u>: Analysis involves

(i) Equilibrium equations (PDE's arising from $\sum \underline{F} = 0$)

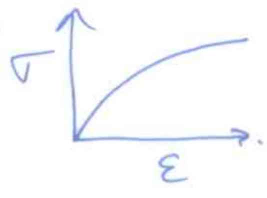
(ii) Constitutive Law $\xrightarrow{(CL)}$ $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}[\underline{\underline{\varepsilon}}]$.

(iii) Strain displacement $^{(SD)}$ relations $\rightarrow$ $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$, etc.

<u>Linear analysis</u>: Here small displacements (u_x, u_y, u_z) and rotations (ie displacement gradients, eg $\frac{\partial u_x}{\partial y}$, etc) assumed. So constitutive and SD relations are linear. Equilibrium condition applied on undeformed structure

<u>Geometrically nonlinear analysis</u>: Used for slender or thin-walled structures. Here small strains and moderate rotations assumed. This yields linear CL but nonlinear SD relations.

<u>Physically non-linear</u>: Here large strains, small/large rotations, so CL must be used as nonlinear. Used for massive structures or nonlinear elastic structures.



<u>Categorization of structures/structural elements based on dimensions</u>:

(i) All three dimensions comparable (spheres, moderate length cylinders). $\rightarrow$ No issue of stability for these.

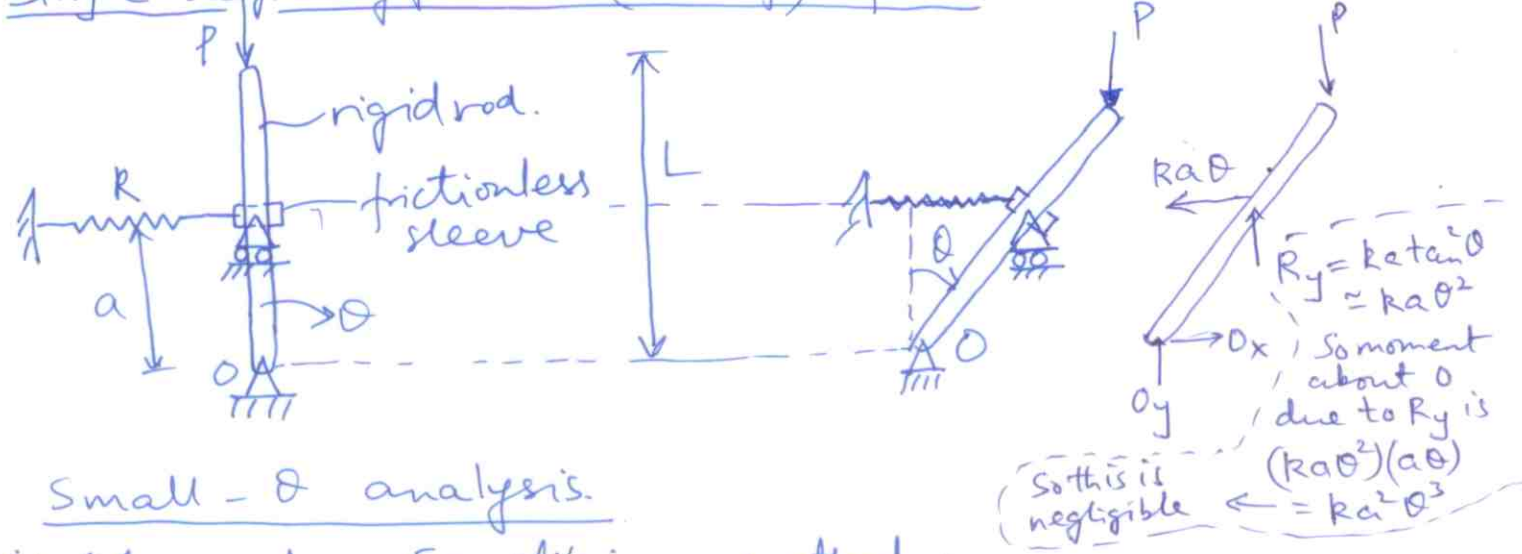
(ii) One dimension much larger than other two (eg beams, shafts).

(iii) One dim << other two (eg plates).

(iv) All three dimensions of different order (eg thin walled beams).

STABILITY OF DISCRETE (LUMPED) MECHANICAL MODELS.

Single degree of freedom (1-d.o.f) system



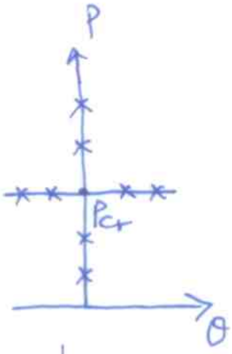
Small - θ analysis.

(i) Classical or Equilibrium method:

$$\Sigma M = 0 \Rightarrow -Pl\theta + (ka\theta)a = 0$$

$$\Rightarrow \theta = 0 \quad \text{or} \quad P = \frac{ka^2}{l}$$

$$\Rightarrow P_{cr} = \frac{ka^2}{l} \rightarrow \text{bifurcation point.}$$



× only equilibrium pts shown, stability information not known by this approach.

(ii) Dynamic method:

$$\frac{ml^2}{3}\ddot{\theta} = Pl\theta - ka^2\theta \Rightarrow \frac{ml^2}{3}\ddot{\theta} + (ka^2 - Pl)\theta = 0.$$

Natural freq, $\omega_n^2 = ka^2 - Pl$

ω_n real $\Rightarrow$ stable (bounded oscillations) $\Rightarrow P < \frac{ka^2}{l}$

ω_n imag $\Rightarrow$ unstable (exponential growth in θ with time) $\Rightarrow P > \frac{ka^2}{l}$.

$\omega_n = 0 \Rightarrow \theta$ increases linearly with time

$\Rightarrow P_{cr} = \frac{ka^2}{l}$, stable for $P < P_{cr}$. else unstable.

(iii) Energy method:

$$PE = U = \frac{1}{2}k(a\theta)^2 - Pl(1-\cos\theta)$$

Equilibrium $\Rightarrow$ $\dfrac{dU}{d\theta} = 0 = ka^2\theta - Pl\underset{\theta}{\cancel{\sin\theta}} = 0$

$$\Rightarrow \theta = 0 \text{ or } P = \frac{ka^2}{l}.$$

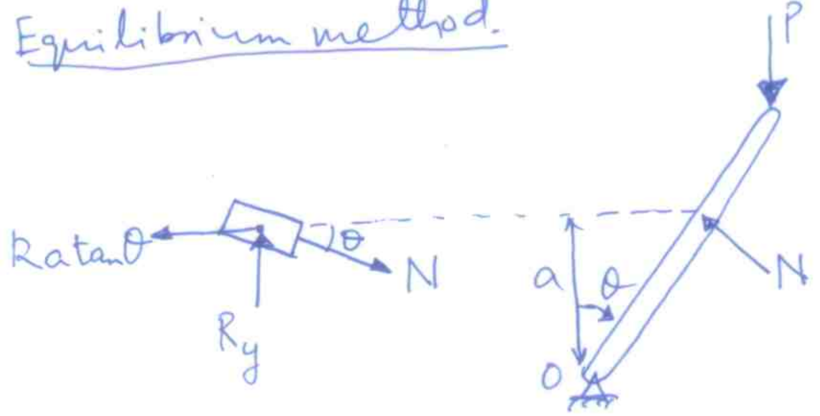
Stability $\Rightarrow$ $\dfrac{d^2U}{d\theta^2} > 0 \Rightarrow ka^2 - Pl > 0 \rightarrow$ stable

ie $ka^2 - Pl < 0 \rightarrow$ U.S.

Note: linear analysis cant give us stability of bifurcation point $ka^2 - Pl = 0$ since all higher derivatives (more than 2^{nd}) are identically zero.

Large θ analysis.

(i) Equilibrium method.



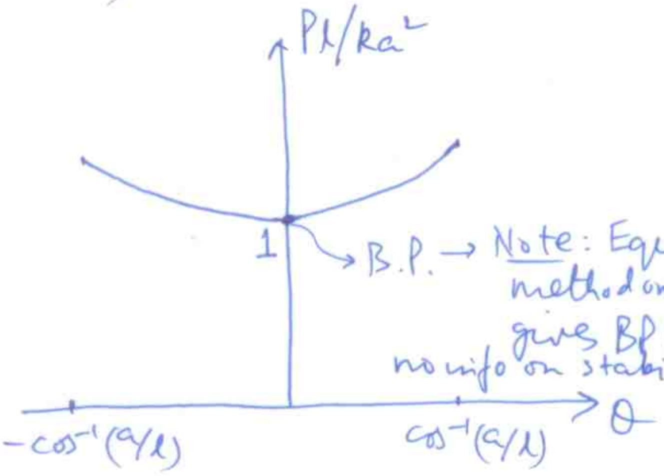
For sleeve:
$\Sigma F_y = 0 \Rightarrow R_y = N\sin\theta$
$= ka\tan^2\theta$
not reqd here, but see small θ analysis p.8.

$$\Sigma F_x: N\cos\theta = ka\tan\theta \quad ; \quad \Sigma M_0: N\frac{a}{\cos\theta} - Pl\sin\theta = 0$$

$$\Rightarrow \left(\frac{ka^2}{\cos^3\theta} - Pl\right)\sin\theta = 0 \longrightarrow (1)$$

$$\Rightarrow \theta = 0, \quad \frac{Pl}{ka^2} = \frac{1}{\cos^3\theta}$$

Note: $\cos\theta > \dfrac{a}{l}$ else ring/sleeve flies of rigid rod.



$\rightarrow$ B.P. $\rightarrow$ Note: Equil method only gives BP but no info on stability.

(ii) <u>Dynamic method</u>

$$\frac{ml^2}{3}\ddot{\theta} + \left(\frac{ka^2}{\cos^3\theta} - Pl\right)\sin\theta = 0 \longrightarrow (1)$$

To find θ_e, set all dynamic terms $(\dot{\theta}, \ddot{\theta})$ to zero. So as in method (i), $\theta_e = 0$, $\sec^3\theta_e = \frac{Pl}{ka^2}$.

Stability analysis

Put $\theta = \theta_e + \phi$ (small perturbation about θ_e) in (1) & expand Taylor series upto <u>first order</u> terms.

$$\frac{ml^2}{3}\ddot{\phi} + \left(\frac{ka^2}{\cos^3\theta_e} - Pl\right)\sin\theta_e + \left(\frac{ka^2}{\cos^2\theta_e} + \frac{3ka^2}{\cos^4\theta_e}\sin^2\theta_e - Pl\cos\theta_e\right)\phi = 0 \longrightarrow (2)$$

$$+\ \text{H.O.T's}$$

For $\theta_e = 0$, from (2),

$$\Rightarrow \frac{ml^2}{3}\ddot{\phi} + (ka^2 - Pl) = 0$$

$\Rightarrow$ Stable for $ka^2 > Pl$, unstable for $ka^2 < Pl$.

For $\theta_e = \sec^{-1}(Pl/ka^2)^{1/3}$, from (2),

$$\Rightarrow \frac{ml^2}{3}\ddot{\phi} + \frac{3ka^2}{\cos^4\theta_e}\sin^2\theta_e\,\phi = 0$$

$\because 3ka^2\sin^2\theta_e/\cos^4\theta_e > 0$ for $\theta_e \neq 0$, it is stable for the non-trivial θ_e branch.

What remains is stability of $\theta_e = 0$ for $Pl = ka^2$. Go to H.O.T's in Taylor series.

2nd order term: $$\frac{1}{2}\left[2ka^2\frac{\sin\theta_e}{\cos^3\theta_e} + 12ka^2\frac{\sin^3\theta_e}{\cos^5\theta_e} + 6ka^2\frac{\sin\theta_e}{\cos^3\theta_e} + Pl\sin\theta_e\right]_{\substack{\theta_e=0\\ Pl=ka^2}} * \phi^2 = 0.$$

3rd order term: Can see directly from 2nd order term that only parts of derivative of 1st, 3rd, 4th term survive for $\sin\theta_e = 0$. Thus we have, for 3rd order term, (11)

$$\frac{1}{6}\left[2ka^2 + 6ka^2 + Pl\right]_{Pl=ka^2}\phi^3 = \frac{3}{2}ka^2\phi^3$$

$$\Rightarrow \frac{ml^2}{3}\ddot{\phi} + \frac{3}{2}ka^2\phi^3 = 0 \rightarrow \text{nonlinear stiffness term for } \theta_e = 0,\ Pl = ka^2$$

<u>Aside</u>: Consider $\ddot{x} + f(x) = 0 \rightarrow$ indicates conservative system.

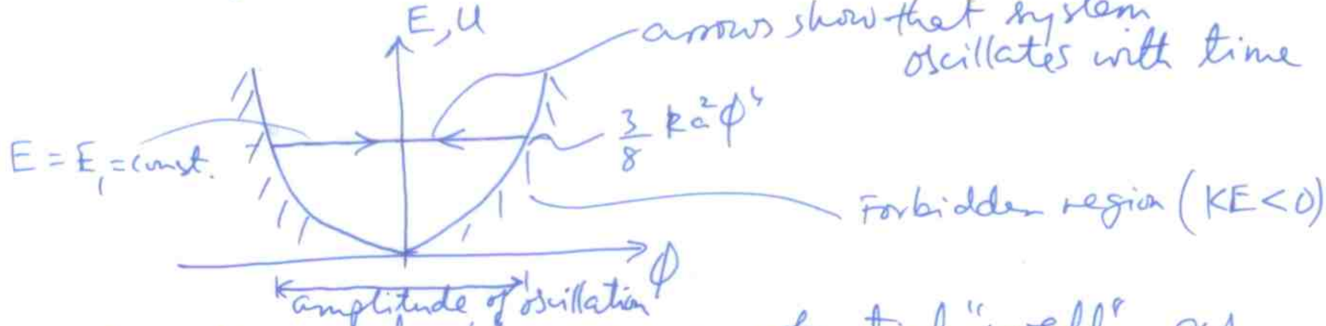
First integral of eqn. of motion $\rightarrow$ $\int \left[\ddot{x} + f(x)\right] dx = \text{const} \rightarrow$ 1st integral of motion

$$\int \frac{d}{dt}\left(\frac{dx}{dt}\right)\frac{dx}{dt}\,dt + \int f(x)\,dx = \text{const}$$

$$\int \dot{x}\,d(\dot{x}) + \int f(x)\,dx = \underbrace{\frac{\dot{x}^2}{2}}_{\text{KE for } m=1} + \underbrace{\int f(x)\,dx}_{=\text{PE}} = \text{const} = \underbrace{E}_{\text{Total energy}}$$

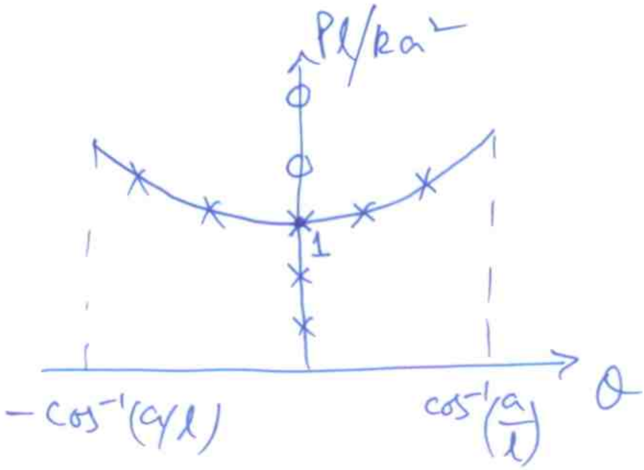
Apply to above nonlinear equation. So,

$$PE = U = \int \frac{3}{2}ka^2\phi^3\,d\phi = \frac{3}{8}ka^2\phi^4$$



whenever you get $U(x)$ as a potential "well", as above, system oscillates about $\phi = 0$. The amplitude of oscillation depends on the initial condition, ie, initial energy level (E_1 in this case) as shown.

Conclusion: $\theta_e = 0$, $ka^2 = Pl$ is STABLE.



o → US (unstable)
x → ST (stable.

(iii) Energy method.

$$U = \frac{1}{2}k(a\tan\theta)^2 - Pl(1-\cos\theta)$$

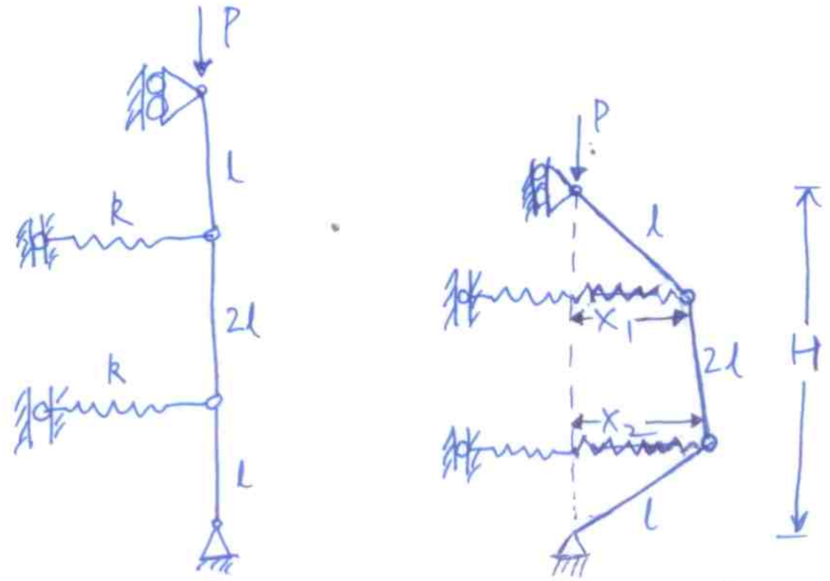
Equilibrium $\Rightarrow$ $\dfrac{dU}{d\theta} = 0$

$$ka\tan\theta\sec^2\theta - Pl\sin\theta = 0 \longrightarrow (1)$$

Eq(1) here is same as eq(1) on p.9 & eq(1) on p.10. This will always be the case since on p.11 we saw that first $\int$ of EOM is energy for conservative system.

The stability analysis amounts to a repeat of the Dynamic method (see last term of eq.(2) p.11, ie $\dfrac{d^2U}{d\theta^2}$, and onwards, ie $\dfrac{d^3U}{d\theta^3}$ and $\dfrac{d^4U}{d\theta^4}$). So we note that $\Sigma M_0 \equiv \dfrac{dU}{d\theta}$ and so on for higher derivatives.

2-Degree of Freedom (2-d.o.f.) System



$N = 2$

$\underset{\sim}{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \equiv \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$

(I) ENERGY METHOD.

$$U[x_1, x_2; P] = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 - P\left(4l - \sqrt{l^2 - x_1^2} - \sqrt{l^2 - x_2^2} - \sqrt{4l^2 - (x_2 - x_1)^2}\right)$$

Divide by kl^2, non-dimensionalize, i.e., $\bar{U} = \frac{U}{kl^2}$, $\bar{x}_1 = \frac{x_1}{l}$, $\bar{x}_2 = \frac{x_2}{l}$, $\lambda = \frac{P}{kl}$, then drop overbars for convenience,

$$U[x_1, x_2; \lambda] = \frac{x_1^2}{2} + \frac{x_2^2}{2} - \lambda\left(4 - \sqrt{1 - x_1^2} - \sqrt{1 - x_2^2} - \sqrt{4 - (x_2 - x_1)^2}\right)$$

Equilibrium:

$$\begin{cases} \dfrac{\partial U}{\partial x_1} = 0 = x_1 + \lambda\left(\dfrac{-x_1}{\sqrt{1 - x_1^2}} + \dfrac{x_2 - x_1}{\sqrt{4 - (x_2 - x_1)^2}}\right) \\[2ex] \dfrac{\partial U}{\partial x_2} = 0 = x_2 + \lambda\left(\dfrac{-x_2}{\sqrt{1 - x_2^2}} + \dfrac{x_1 - x_2}{\sqrt{4 - (x_2 - x_1)^2}}\right) \end{cases}$$

→ Nonlinear equations – possess multiple solutions. One of these is trivial solution. We examine its stability. (Note: other solutions may also be important, in general, and some of them may not be real).

→ NOTE: These equations don't have direct physical meaning, since they are (represent) a combination of the physical equilibrium equations (i.e. $\Sigma F_x = \Sigma F_y = \Sigma M = 0$). This is usually the case when using energy method.

Stability of $\underset{\sim}{x}_e = [0 \quad 0]^T = [x_{1e} \quad x_{2e}]^T$

$$\underline{\underline{C}} = \begin{bmatrix} \dfrac{\partial^2 U}{\partial x_1^2} & \dfrac{\partial^2 U}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 U}{\partial x_1 \partial x_2} & \dfrac{\partial^2 U}{\partial x_2^2} \end{bmatrix}_{\underset{\sim}{x}_e = [0,\, 0]^T}$$

$$\left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\underset{\sim}{x}_e = \underset{\sim}{0}} = 1 + \lambda\left[\frac{-1}{\sqrt{1-x_1^2}} - \frac{x_1^2}{(1-x_1^2)^{3/2}} - \frac{1}{\sqrt{4-(x_2-x_1)^2}} - \frac{(x_2-x_1)^2}{\left(\sqrt{4-(x_2-x_1)^2}\right)^3}\right] = 1 - \frac{3}{2}\lambda$$

$$\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\underset{\sim}{x}_e = 0} = 1 + \lambda\left[\frac{-1}{\sqrt{1-x_2^2}} - \frac{x_2^2}{(1-x_2^2)^{3/2}} - \frac{1}{\sqrt{4-(x_2-x_1)^2}} - \frac{(x_1-x_2)^2}{\left(\sqrt{4-(x_2-x_1)^2}\right)^3}\right] = 1 - \frac{3}{2}\lambda$$

$$\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\underset{\sim}{x}_e = 0} = \lambda\left(\frac{1}{\sqrt{4-(x_2-x_1)^2}} + \frac{(x_2-x_1)^2}{\left(\sqrt{4-(x_2-x_1)^2}\right)^3}\right) = \frac{\lambda}{2}$$

$$C = \begin{bmatrix} 1-\frac{3}{2}\lambda & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1-\frac{3}{2}\lambda \end{bmatrix} \Rightarrow \det\left[\underline{\underline{C}} - \mu \underline{\underline{I}}\right] = \det\begin{bmatrix} 1-\frac{3}{2}\lambda-\mu & \lambda/2 \\ \lambda/2 & 1-\frac{3}{2}\lambda-\mu \end{bmatrix} = 0 \to \text{EVP.}$$

$$\Rightarrow \left(1-\frac{3}{2}\lambda-\mu\right)^2 - \frac{\lambda^2}{4} = 0$$

$$1 - \frac{3}{2}\lambda - \mu = \pm\frac{\lambda}{2} \Rightarrow \left.\begin{aligned} \mu^{(1)} &= 1-2\lambda \\ \mu^{(2)} &= 1-\lambda \end{aligned}\right\} \text{evalues.}$$

$\mu^{(1)} > 0$, $\mu^{(2)} > 0$ for stability.

So $\lambda < \frac{1}{2}$ for stability.

$\delta^2 U$ = P.D for $\lambda < \frac{1}{2}$ $(\delta^2 U > 0) \longrightarrow$ stable equil.

correspond to unstable equilibrium $\left\{ \begin{array}{ll} \text{PSD for } \lambda = \frac{1}{2} & (\delta^2 U \geq 0) \\ \text{IND for } \frac{1}{2} < \lambda < 1 & (\delta^2 U \text{ can be any sign}). \\ \text{NSD for } \lambda = 1 & (\delta^2 U \leq 0) \\ \text{ND for } \lambda > 1 & (\delta^2 U < 0) \end{array} \right.$

Eigenvectors of $\underline{\underline{C}}$:

$$\begin{bmatrix} 1 - \frac{3}{2}\lambda - \overset{(k)}{\mu} & \lambda/2 \\ \lambda/2 & 1 - 3/2\,\lambda - \overset{(k)}{\mu} \end{bmatrix} \begin{Bmatrix} \overset{(k)}{z}_1 \\ \overset{(k)}{z}_2 \end{Bmatrix} = 0 \longrightarrow \text{EVP.}$$

Note: determinant of this = 0 for $\mu = \overset{(1)}{\mu}$ or $\overset{(2)}{\mu}$. In fact that is how we solved for evalues $\overset{(1)}{\mu}, \overset{(2)}{\mu}$.

$$\Rightarrow \left.\begin{array}{l} \left(1 - \frac{3}{2}\lambda - \overset{(k)}{\mu}\right) \overset{(k)}{z}_1 + \frac{\lambda}{2} \overset{(k)}{z}_2 = 0 \\ \frac{\lambda}{2} \overset{(k)}{z}_1 + \left(1 - \frac{3}{2}\lambda - \overset{(k)}{\mu}\right) \overset{(k)}{z}_2 = 0 \end{array}\right\}$$

only one of these can be used since they are not independent equations, since $\det \left| \underline{\underline{C}} - \overset{(k)}{\mu} \underline{\underline{I}} \right| = 0$.

so use first equation,

For $\overset{(1)}{\mu} = 1 - 2\lambda$, $\overset{(1)}{z}_1 = -z_2$

For $\overset{(1)}{\mu} = 1 - \lambda$, $\overset{(1)}{z}_1 = z_2$

So $\overset{(1)}{\underset{\sim}{z}} = [c \;\; -c]^T$, $\overset{(2)}{\underset{\sim}{z}} = [c \;\; c]^T$, c = arbitrary const.

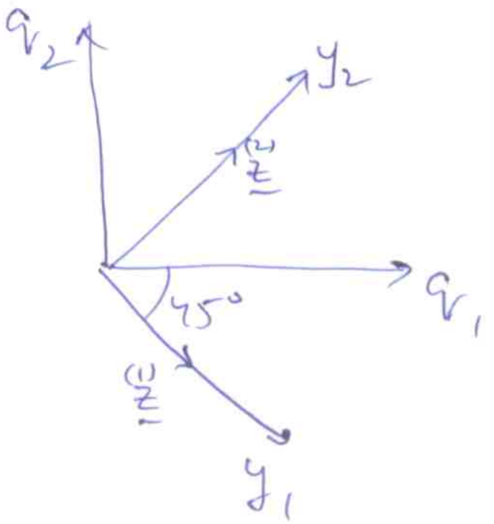
check: $\overset{(1)}{\underset{\sim}{z}}{}^T \overset{(2)}{\underset{\sim}{z}} = 0 \rightarrow$ ie orthogonality.

Make $\overset{(1)}{\underset{\sim}{z}}, \overset{(2)}{\underset{\sim}{z}}$ unit vectors (orthonormalize), so $c = \frac{1}{\sqrt{2}}$,

$$\overset{(1)}{\underset{\sim}{z}} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{Bmatrix}, \quad \overset{(2)}{\underset{\sim}{z}} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix}$$

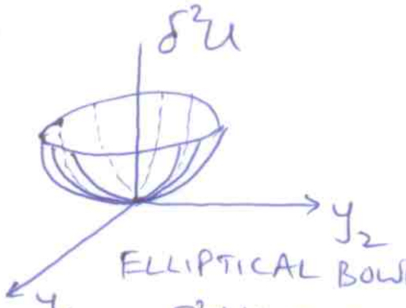
Now see p. 6a – 6d.

$$\underset{\sim}{y} = \underline{\underline{T}}^T \underset{\sim}{q} \Rightarrow \begin{array}{l} y_1 = \frac{1}{\sqrt{2}} q_1 - \frac{1}{\sqrt{2}} q_2 \\ y_2 = \frac{1}{\sqrt{2}} q_1 + \frac{1}{\sqrt{2}} q_2 \end{array}$$



Examine $\delta^2 U = \overset{(1)}{\mu} y_1^2 + \overset{(2)}{\mu} y_2^2$

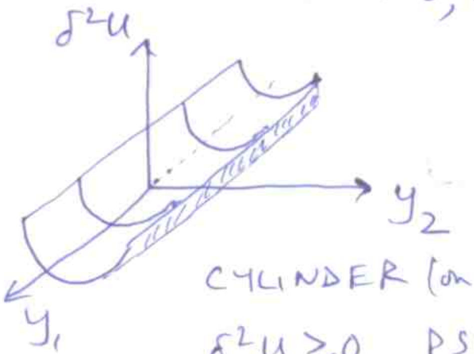
(i) $0 < \lambda < \frac{1}{2} \Rightarrow \overset{(1)}{\mu} > 0, \overset{(2)}{\mu} > 0 \longrightarrow$



ELLIPTICAL BOWL
$\delta^2 U > 0$, P.D.

(ii) $\lambda = \frac{1}{2}, \Rightarrow \overset{(1)}{\mu} = 0, \overset{(2)}{\mu} > 0$

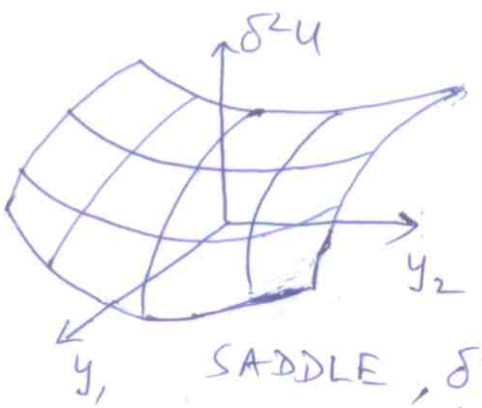
$\delta^2 U = 0 . y_1^2 + \overset{(2)}{\mu} y_2^2$



CYLINDER (on y_1 axis)
$\delta^2 U \geq 0$, PSD.

(iii) $\frac{1}{2} < \lambda < 1 \Rightarrow \overset{(1)}{\mu} < 0, \overset{(2)}{\mu} > 0$

Cross-sections parallel to y_1 axis are inverted parabolas, & sections parallel to y_2 axis are parabolas. So disturbance in y_1 direction leads to instability.

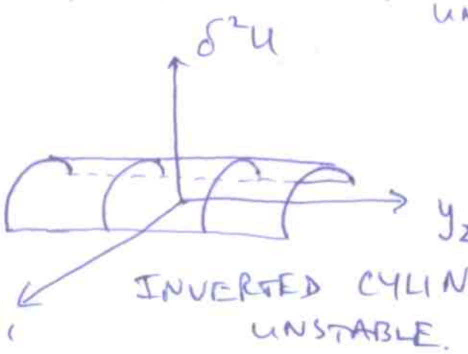


SADDLE, $\delta^2 U$ = IND, UNSTABLE

(iv) $\lambda = 1, \Rightarrow \overset{(1)}{\mu} < 0, \overset{(2)}{\mu} = 0$

$\delta^2 U = \overset{(1)}{\mu} y_1^2 + 0 \cdot y_2^2$

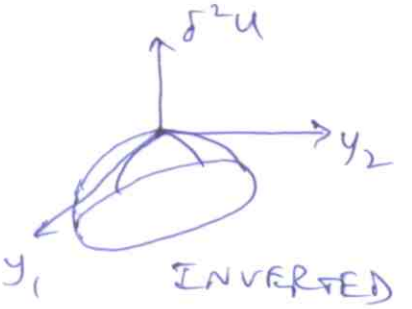
$\delta^2 U \leq 0$



INVERTED CYLINDER, UNSTABLE.

(v) $\lambda > 1, \Rightarrow \mu^{(1)} < 0, \ \mu^{(2)} < 0$

$\delta^2 U < 0$

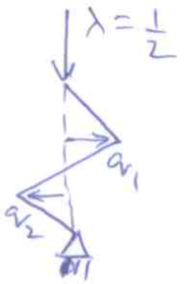


INVERTED ELLIPTICAL BOWL, UNSTABLE

Critical points are at $\lambda = \frac{1}{2}$, $\lambda = 1$, ie when $\delta^2 U$ can equal zero.

$\underline{\lambda = \frac{1}{2}}$: Critical direction is y_1, since perturbation along that direction remains as such (ie persists). Also $\delta^2 U = 0$ if $y_2 = 0$. Use $\underset{\sim}{q} = \underline{\underline{T}}\,\underline{y}$

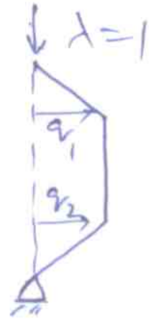
$$\left.\begin{aligned} q_1 &= \frac{y_1}{\sqrt{2}} + \cancel{\frac{y_2}{\sqrt{2}}}^{\,0} = y_1/\sqrt{2} \\ q_2 &= -\frac{y_1}{\sqrt{2}} + \cancel{\frac{y_2}{\sqrt{2}}}^{\,0} = -y_1/\sqrt{2} \end{aligned}\right\} \Rightarrow q_1 = -q_2$$



ANTISYMMETRIC MODE OF BUCKLING

$\underline{\lambda = 1}$: Critical direction is y_2, and $\delta^2 U = 0$ for $y_1 = 0$.

$q_1 = q_2 = y_2/\sqrt{2}$



SYMMETRIC MODE OF BUCKLING

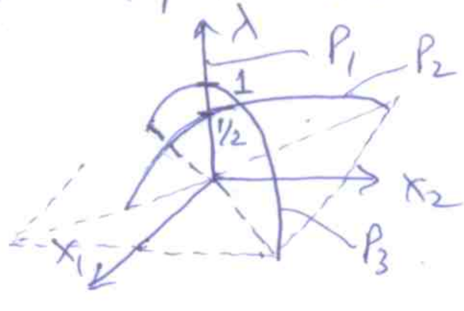
Critical loads $\lambda = \frac{1}{2}$, $\lambda = 1$ correspond to bifurcation pts in (λ, x_1, x_2) space.

NOTE: Other solutions of the equil eqns on p. 13 are:

(already analyzed for stability) P_1 : $x_1 = x_2 = 0$ for all λ

(can do stability analysis of these also) P_2 : $x_2 = -x_1$, $\lambda = \sqrt{1 - x_1^2}/2$

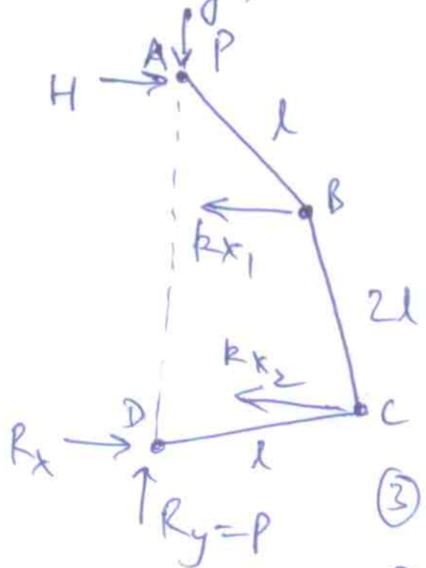
P_3 : $x_2 = x_1$, $\lambda = \sqrt{1 - x_1^2}$



(II) (CLASSICAL) EQUILIBRIUM METHOD (BIFURCATION METHOD):

Seek load when two or more solutions of equilibrium, that are infinitesimally close (ie bifurcation point) exist. We seek bifurcation pt on trivial solution branch $(x_1, x_2 = 0, 0)$. Thus we can linearize the resulting equilibrium equations for small (x_1, x_2) which represent the bifurcated, infinitesimally close equilibrium branch.

NOTE: Using this method to seek bifurcation points on non-trivial equilibrium branches will get very messy, but possible.



For infinitesimally small (x_1, x_2),

$$\Sigma M_D = 0 = H4l - kx_1 3l - kx_2 l = 0 \rightarrow ①$$

$$\Sigma M_A = 0 = R_x 4l - kx_2 3l - kx_1 l = 0 \rightarrow ②$$

Since B, C are smooth pins,

③ $\Sigma M_B = 0 = Px_1 - Hl$ (FBD on top side of pin_B)

④ $\Sigma M_C = 0 = \underbrace{R_y}_{=P \text{ (from } \Sigma F_y = 0)} x_2 - R_x l$ (FBD on bot. side of pin C).

Subst ③, ④ in ①, ②,

$$\left.\begin{aligned} 4P\frac{x_1}{l} - 3kx_1 - kx_2 &= 0 \\ 4P\frac{x_2}{l} - 3kx_2 - kx_1 &= 0 \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} (4\lambda - 3) & -1 \\ -1 & (4\lambda - 3) \end{bmatrix}}_{[A]} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

where $\lambda \triangleq P/kl$.

For non-trivial (x_1, x_2), $\det[A] = 0$

$$\Rightarrow (4\lambda - 3)^2 - 1 = 0 \Rightarrow \lambda = 1, \frac{1}{2} \rightarrow \text{same as from energy method.}$$

The solutions of (x_1, x_2) are,

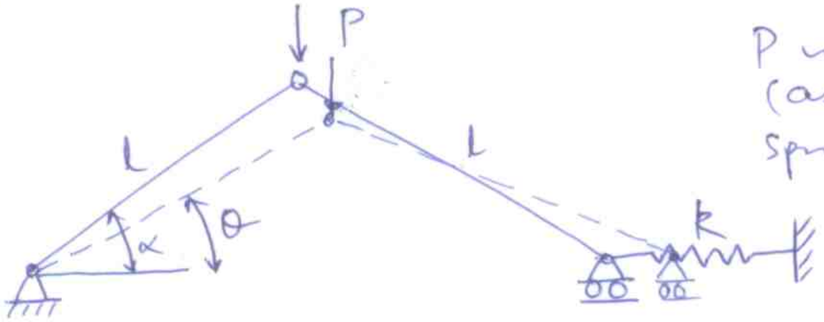
Buckling modes ⇓

$\lambda = \frac{1}{2} \Rightarrow -x_1 - x_2 = 0$, $x_1 = -x_2 \rightarrow$ antisymmetric

$\lambda = 1 \Rightarrow x_1 - x_2 = 0$, $x_1 = x_2 \rightarrow$ symmetric

——— x ———

Snap-through buckling model.



P increased quasistatically (as in previous cases). Spring unstretched when $\theta = \alpha$.

By nature of this problem, only nonlinear analysis is possible since θ changes continuously with P.

(i) Energy approach.

$$U = \frac{1}{2}k\,(2l\cos\theta - 2l\cos\alpha)^2 - Pl(\sin\alpha - \sin\theta)$$

$$\frac{dU}{d\theta} = 0 = -2kl^2 \cdot 2(\cos\theta - \cos\alpha)\sin\theta + Pl\cos\theta$$

$$\Rightarrow \frac{P}{4kl} = \sin\theta - \cos\alpha\tan\theta \longrightarrow (1)$$

(1) is a nonlinear equation, to be solved numerically.

$$\frac{d^2U}{d\theta^2} = -4kl^2\left[-\sin^2\theta + (\cos\theta - \cos\alpha)\cos\theta\right] - Pl\sin\theta \longrightarrow (2)$$

* Subst (1) in (2), → common practice when evaluating $\left.\frac{d^2U}{dq_i\,dq_j}\right|_{q_e}$

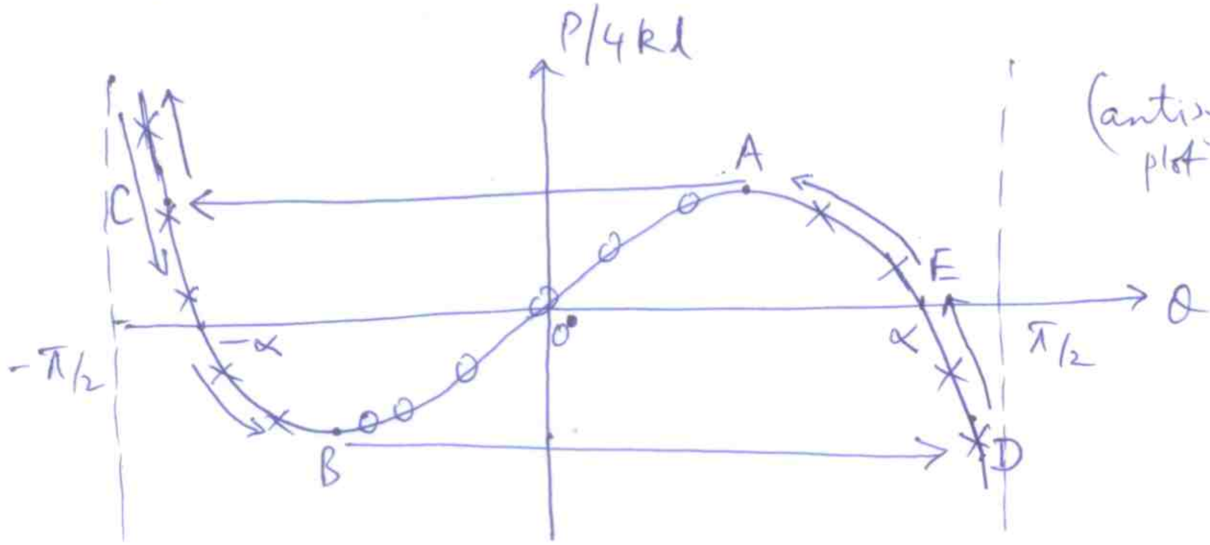
$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta_e} = -4kl^2\left[-\cancel{\sin^2\theta} + \cos^2\theta - \cos\theta\cos\alpha + \cancel{\sin^2\theta} - \cos\alpha\frac{\sin^2\theta}{\cos\theta}\right]$$

$$= -4kl^2\left[-\frac{\cos\alpha}{\cos\theta} + \cos^2\theta\right]$$

$\dfrac{d^2U}{d\theta^2} > 0$ when $\cos^3\theta < \cos\alpha$, ie, $\theta > \cos^{-1}\left[(\cos\alpha)^{1/3}\right]$
or $\theta < -\cos^{-1}\left[(\cos\alpha)^{1/3}\right]$

$\dfrac{d^2U}{d\theta^2} < 0$ when $-\cos^{-1}\left[(\cos\alpha)^{1/3}\right] < \theta < \cos^{-1}\left[(\cos\alpha)^{1/3}\right]$.

Plot of solution of ① is



At A, B, $\dfrac{dP}{d\theta} = 0 = 4kl\left(\cos\theta - \cos\alpha \sec^2\theta\right) = 0$
(using ①).

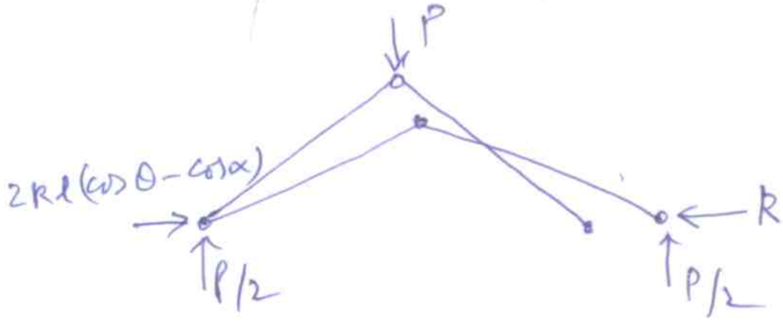
$\Rightarrow \cos^3\theta = \cos\alpha$, ie $\theta_A = \cos^{-1}\left[(\cos\alpha)^{1/3}\right]$

$\theta_B = -\theta_A$

So from above stability analysis, equilibrium path between A & B is UNSTABLE, rest is STABLE

Starting at E ($P=0$, $\theta=\alpha$) and increasing P, you reach A where stable & unstable segments meet. Increasing P further, the system snaps-thru to C, and increasing P further, $\theta \to -\frac{\pi}{2}$. From C onwards, decreasing P you reach point where $P=0$, $\theta=-\alpha$. Decreasing P further (ie P now upward) you reach B and then snap-back to upright configuration with $\theta > \alpha$, ie spring is in extension.

(ii) Equilibrium approach.



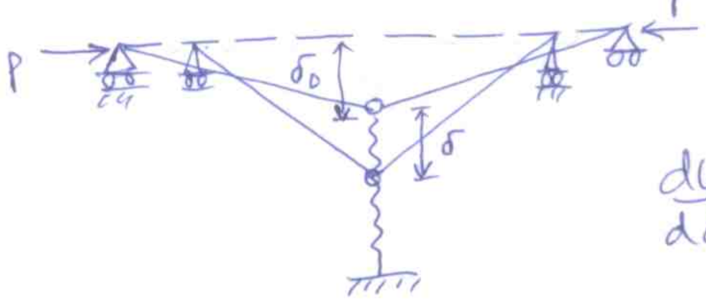
SDOF so we seek one equation relating P & θ.

so $\sum M_{Top} = 0$ for FBD of right or left link since top hinge is smooth.

$$\frac{P}{2} l\cos\theta - 2kl(\cos\theta - \cos\alpha)\, l\sin\theta = 0$$

↳ same as eqn ① p.19, so we stop here.

Model with Imperfection in geometry



Energy method:

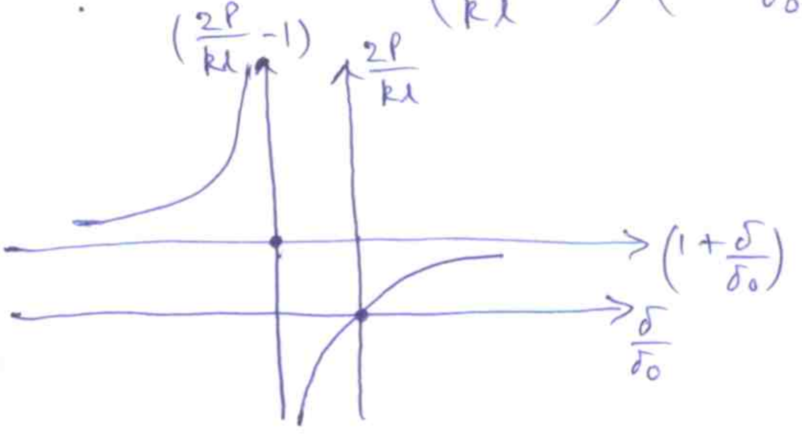
$$U = -2P\left(\sqrt{l^2-\delta_0^2} - \sqrt{l^2-(\delta+\delta_0)^2}\right) + \frac{k\delta^2}{2}$$

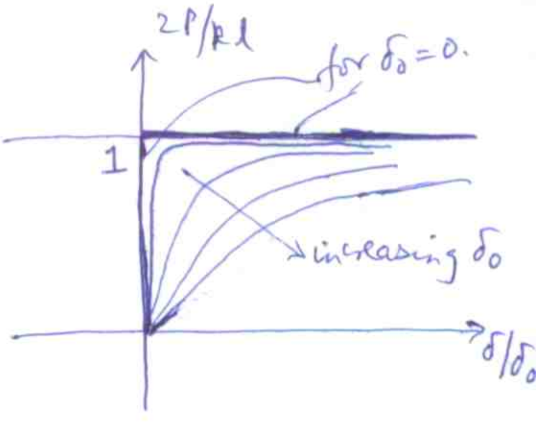
$$\frac{dU}{d\delta} = 2P\frac{\left(-[\delta+\delta_0]\right)}{\sqrt{l^2-(\delta_0+\delta)^2}} + k\delta = 0$$

⇒ Our aim here is to predict behavior of perfect system by studying the imperfect system's response. Thus we will eventually let $\delta_0 \to 0$, and hence we consider small δ and δ_0 $\left(\frac{\delta}{l}, \frac{\delta_0}{l} << 1\right)$, ie linearized analysis.

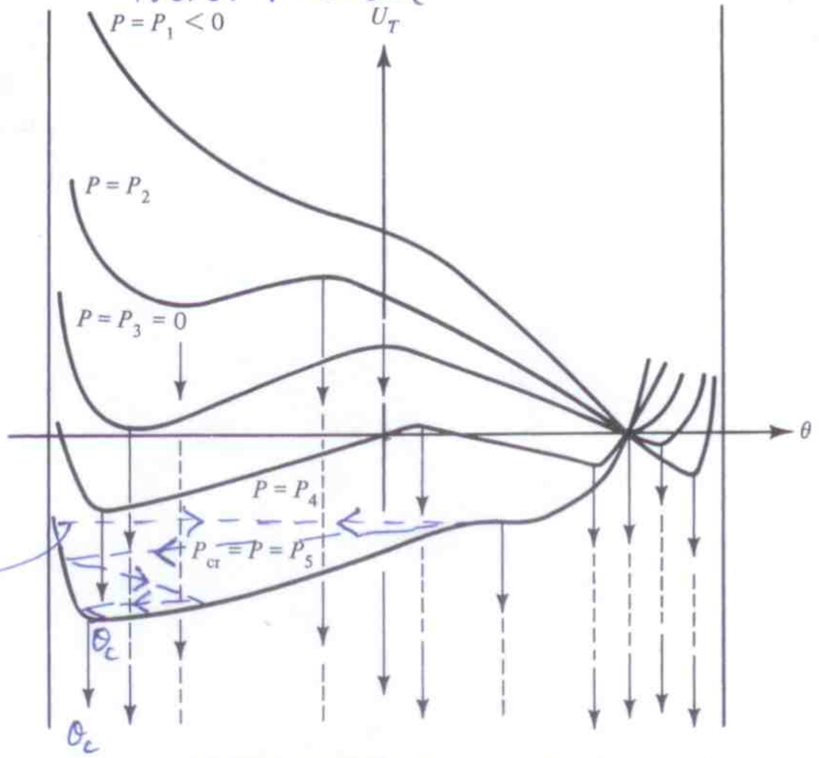
So, $\frac{dU}{d\delta} = 0 \Rightarrow P(\delta+\delta_0) - \frac{k\delta l}{2} = 0$

$$\Rightarrow \left(\frac{2P}{kl} - 1\right)\left(1 + \frac{\delta}{\delta_0}\right) = -1. \text{ (see plots below)}$$

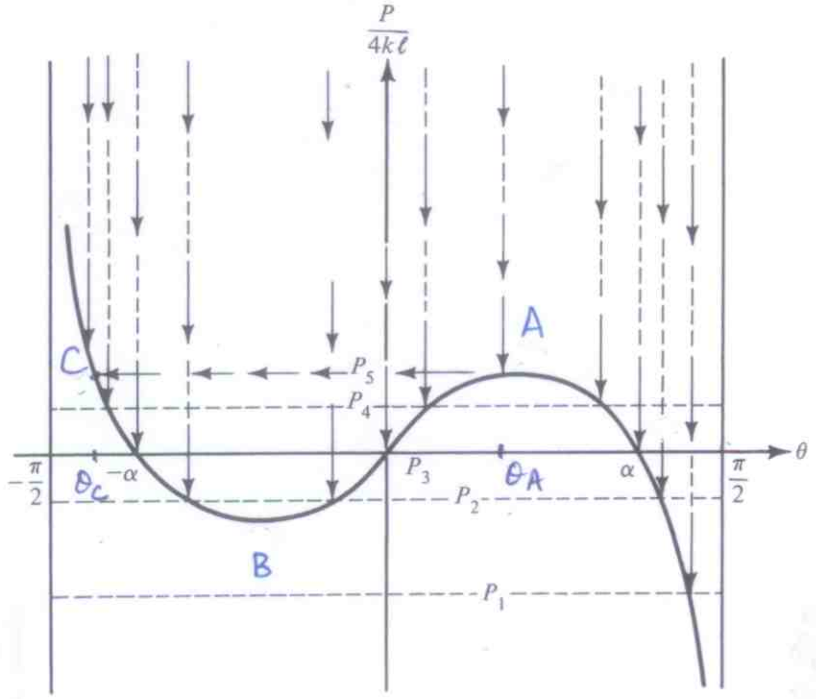




Further explanation of Snap-Thru Model



(a) Total potential curves at constant P



(b) Load-deflection curve

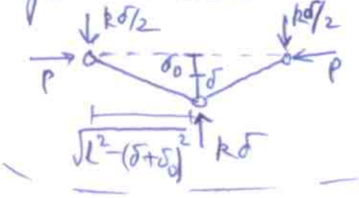
Energy level at which nonlinear oscillations between θ_A & angle beyond θ_c occur if no damping. If damping present then energy reduces & we move along downward spiral to settle at θ_c

Reproduced from Simitses & Hodges 'Fundamentals of Structural Stability', 2006.

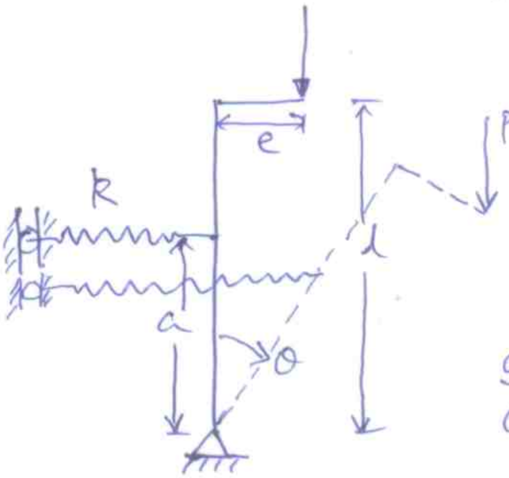
NOTE: (1) The valley's and peaks and corresponding equilibrium points from fig (a) and (b), respectively.

(2) Assuming no damping, if $P_{cr} = P_5$ maintained at point A (ie θ_A) then system oscillates nonlinearly between θ_A and some angle beyond θ_c, assuming spring stays elastic. If damping present then it will fall in potential well and converge to θ_c (equilibrium position after snap thru).

Plot in $\left(\frac{2P}{kl}-1\right)$ v/s $\left(1+\frac{\delta}{\delta_0}\right)$ plane is hyperbola. In the $\frac{2P}{kl}$ v/s $\frac{\delta}{\delta_0}$ plane the curve asymptote $\frac{\delta}{\delta_0}=-1$ and $\frac{2P}{kl}=1$. As $\frac{2P}{kl}\to 1^-$, $\frac{\delta}{\delta_0}\to +\infty$ and as $\frac{2P}{kl}\to 1^+$, $\frac{\delta}{\delta_0}\to -\infty$. However, these are out of the range of the linearized theory so they have no meaning. As $\delta_0\to 0$ we get $\left(\frac{2P}{kl}\right)_{cr}=1$ where trivial branch bifurcates to non-trivial one. Equilibrium method gives same result



Model with Loading Imperfection.



Energy method

$$U=\frac{1}{2}ka^2\sin^2\theta - Pl\left(1-\cos\theta+\frac{e}{l}\sin\theta\right)$$

$$\frac{dU}{d\theta}=ka^2\sin\theta\cos\theta - Pl\left(\sin\theta+\frac{e}{l}\cos\theta\right)$$

$$\Rightarrow Pl/ka^2=\sin\theta/\left(\tan\theta+e/l\right) \quad (1)$$

$\Rightarrow$ Here the aim is to find the effect of eccentricity on P_{cr}.

$$\frac{d^2U}{d\theta^2}=ka^2\cos 2\theta - Pl\left(\cos\theta-\frac{e}{l}\sin\theta\right)\longrightarrow (2)$$

Subst (1) in (2),

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta_e}=ka^2\left[\cos 2\theta-\left(\frac{\sin\theta}{\tan\theta+\frac{e}{l}}\right)\left(\cos\theta-\frac{e}{l}\sin\theta\right)\right]$$

$$=\frac{ka^2}{\left(\tan\theta+\frac{e}{l}\right)}\left[\cancel{\sin\theta\cos\theta}-\frac{\sin^3\theta}{\cos\theta}+\frac{e}{l}\cos 2\theta-\cancel{\sin\theta\cos\theta}+\frac{e}{l}\sin^2\theta\right]$$

$$=\frac{ka^2}{\left(\tan\theta+\frac{e}{l}\right)}\cos^2\theta\left[\frac{e}{l}-\tan^3\theta\right]$$

For $0<\theta<\frac{\pi}{2}$, $\left.\frac{d^2U}{d\theta^2}\right|_{\theta_e}>0$ for $\tan^3\theta<\frac{e}{l}\to$ stable.

(assume this range of operation)

<0 " $\frac{e}{l}<\tan^3\theta\to$ unstable

$$P_{cr} = \frac{ka^2}{l}\left(\frac{\sin\theta}{\tan\theta + \frac{e}{l}}\right)_{\theta_{cr}} = \frac{ka^2}{l}\,\frac{\left(\sqrt{1+\cos^2\theta}\right)^{-1}}{\left(\frac{e}{l}\right)^{1/3} + \frac{e}{l}}$$

$$\frac{P_{cr}}{(ka^2/l)} = \frac{\left[1+(l/e)^{2/3}\right]^{-1/2}}{\left(\frac{e}{l}\right)^{1/3} + \left(\frac{e}{l}\right)} = \frac{1}{\sqrt{\left[\left(\frac{e}{l}\right)^{1/3} + \frac{e}{l}\right]^2\left[1+\left(\frac{l}{e}\right)^{2/3}\right]}}$$

$$= \left\{\left[(e/l)^{2/3} + (e/l)^2 + 2(e/l)^{4/3}\right]\left[1+\left(\frac{l}{e}\right)^{2/3}\right]\right\}^{-1/2}$$

$$= \left\{(e/l)^{2/3} + 1 + (e/l)^2 + \left(\frac{e}{l}\right)^{4/3} + 2(e/l)^{4/3} + 2(e/l)^{2/3}\right\}^{-1/2}$$

$$P_{cr} = \left[1+\left(\frac{e}{l}\right)^{2/3}\right]^{-3/2}(ka^2/l)$$

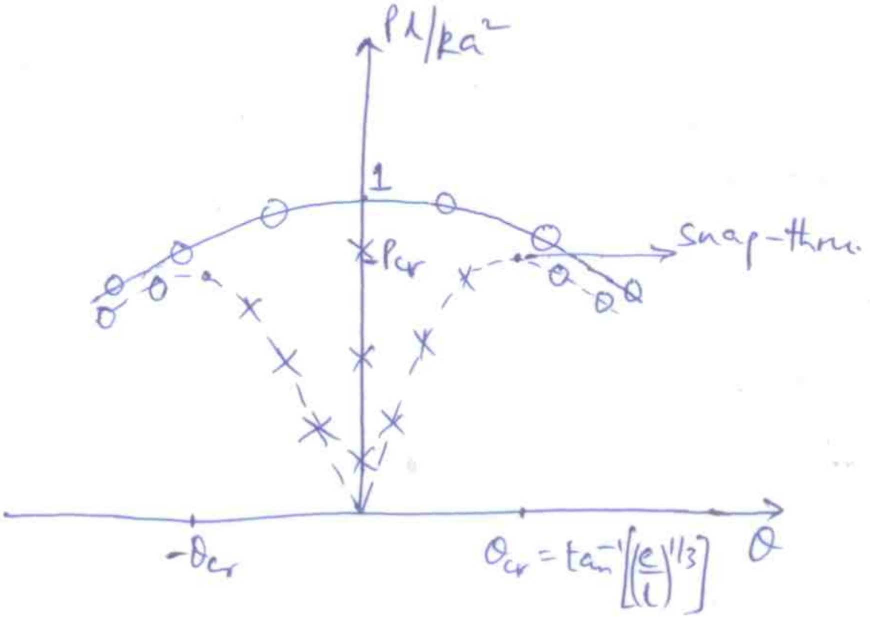
For $e \to 0$, $\frac{Pl}{ka^2} = \frac{\sin\theta}{\tan\theta}$, ie $\theta = 0$ or $\underbrace{\frac{Pl}{ka^2} = \cos\theta}$

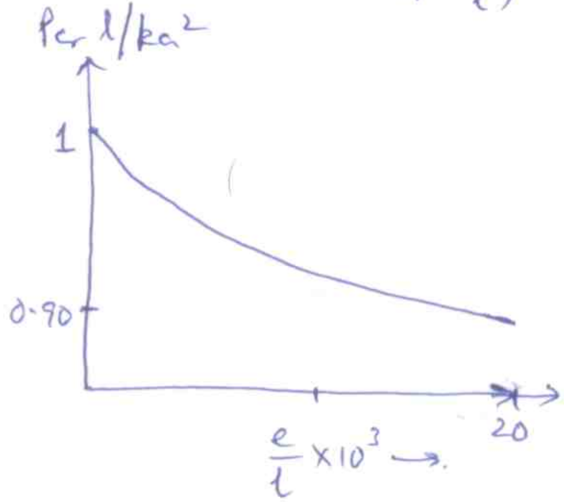
and $P_{cr} = 1$.

$\downarrow$ unstable

$\because \theta_{cr} = 0°$.

(~~from~~ $\tan^3\theta_{cr} < \frac{e}{l}$)





Dotted → eccentric loading (imperfect)
Solid → central loading (perfect)

Result: (1) Snap-thru buckling for column with eccentric loading.

(2) P_{cr} for imperfect system is reduced (ie $< \frac{ka^2}{l}$).

STABILITY EQUATIONS — DISCRETE SYSTEMS.

Recall, $\delta^2 U = \underline{q}^T \underline{\underline{C}}\, \underline{q} = \sum\sum_{i,j=1}^{N} C_{ij} q_i q_j$,

$\underline{q} \rightarrow$ perturbation of equilibrium $\underline{q}_e$.

If load λ is small enough, $\delta^2 U$ is P.D.
Increase λ, and the first value (ie, λ_{cr}) when $\delta^2 U$ ceases to be PD is called critical load.
We seek to develop equations that directly give the critical load & buckling mode shapes.

Consider $\alpha = \dfrac{\delta^2 U}{\Delta}$, $\Delta = \sum_{i=1}^{N} q_i^2 \longrightarrow$ (1)

Seek stationary values of α, ie

$$\frac{\partial \alpha}{\partial q_k} = 0 = \frac{1}{\Delta}\left[\frac{\partial(\delta^2 U)}{\partial q_k} - \alpha \frac{\partial \Delta}{\partial q_k}\right], \quad \Delta \neq 0$$

$$\Rightarrow \frac{\partial}{\partial q_k}\left(\sum\sum_{i,j=1}^{N} C_{ij} q_i q_j\right) - \alpha(2 q_k) = 0$$

$$2\sum_{i=1}^{N} C_{ik} q_i - 2\alpha q_k = 0 \qquad (C_{ik} = C_{ki})$$

$k = 1, \dots N.$

$$\Rightarrow \underline{\underline{C}}\, \underline{q} = \alpha\, \underline{q} \longrightarrow (2)$$

So stationary (min/max) value of α are same as eigenvalues of $\underline{\underline{C}}$.
Now we know that $\delta^2 U$ is PD if all its eigenvalues are positive ($\mu_i > 0$, $\mu_1 < \mu_2 < \cdots < \mu_N$ ordering) and PSD when $\mu_1 = 0$, ie $\alpha = 0$.
Also by definition, α and $\delta^2 U$ have same sign-definiteness property (see (1)).

Hence, when $\delta^2 U$ is PSD, ie $\alpha = \mu_1 = 0$, ② gives

$$\boxed{\underline{\underline{C}}\,\underline{q} = \frac{\partial(\delta^2 U)}{\partial q_k} = 0 \quad " = \delta(\delta^2 U)"} \longrightarrow \text{STABILITY EQNS (SE's) (TREFFTZ's CRITERIA).}$$

<u>Conclusion</u>: stability eqns are conditions for $\delta^2 U$ to be PSD, and $\delta^2 U$ is PSD when it is stationary.

Equilibrium $\Rightarrow \delta(U) = 0$ (first variation), Stability $\Rightarrow \delta(\delta^2 U) = 0$ (first var of second var.)

NOTE: $\delta(\delta^2 U) \neq \delta^3 U$ ← third var, ie 3rd order terms in Taylor series.

\# Application of SE's to 2-DOF problem of p.13:

Recall, $\underline{\underline{C}} = \begin{bmatrix} 1-3/2\,\lambda & \lambda/2 \\ \lambda/2 & 1-3/2\,\lambda \end{bmatrix}$ for $\underline{q}_e = \underline{0}$

S.E. $\rightarrow \underline{\underline{C}}\,\underline{q} = 0 \Rightarrow \det|\underline{\underline{C}}| = 0 \Rightarrow (1-\frac{3}{2}\lambda)^2 - \frac{\lambda^2}{4} = 0$

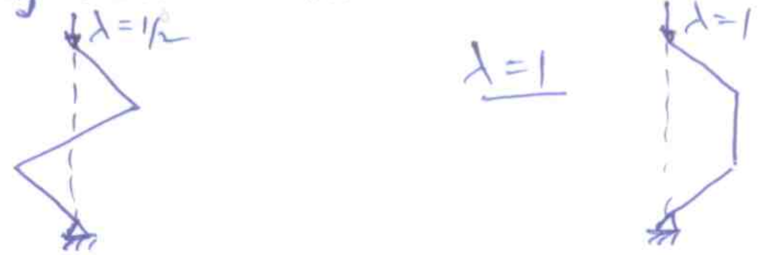
$\lambda = 1/2, 1, \lambda_{cr} = 1/2$

For $\lambda = 1/2$, $q_1 = -q_2$ (from SE.)

$\lambda = 1$, $q_1 = q_2$ (")

So buckling modes are

<u>$\lambda = 1/2$</u> <u>$\lambda = 1$</u>



NOTE: (1) STABILITY EQS are always linear.

(2) Above example shows advantage for direct evaluation of buckling loads & modes by Energy method as opposed to the working of p.15–17, also by Energy method.

(3) Equilibrium approach (p.18) gives $\underline{A}$ which is same as $\underline{\underline{C}}$ here.

Summary:

(1) When bifurcation buckling occurs the Classical Equilibrium method can predict critical load, ie where trivial & non-trivial solutions co-exist nearby. However it cannot predict stability of the solution branches. For that we need to use energy method.

(2) When snap-thru buckling occurs, Classical Equilibrium method can only predict the equilibrium path and strictly speaking it does not predict P_{cr} (although by visual inspection of equilibrium path we can guess snap-thru & hence P_{cr}). For example in Fig on p.20, using only Classical method, we cannot say whether it wont follow path between A & B if we reduce load at A. From energy method we know that this is not possible since path between A & B is unstable. So you need to use the energy method.

(3) Linear (small-deflection) analysis is useful only as far as finding bifurcation buckling loads & mode shapes of perfect systems. Introduction of small imperfections can greatly reduce P_{cr} of imperfect systems in certain cases (see p.23 bottom), while in other cases it does not have much effect (e.g. add small imperfection θ_0 to SDOF system on p.9 and do non-linear analysis — see fig). So for imperfect systems, nonlinear analysis is reccommended. For systems with snap-thru, it is essential. Also to find stability of bifurcation point, it is essential.