TOPIC - II

BEAM - COLUMNS (BARS).

We will do different approaches using Energy Method & Equilibrium Method. This will include an introduction to Variational Methods in Mechanics, to be used later on in Plates, etc.

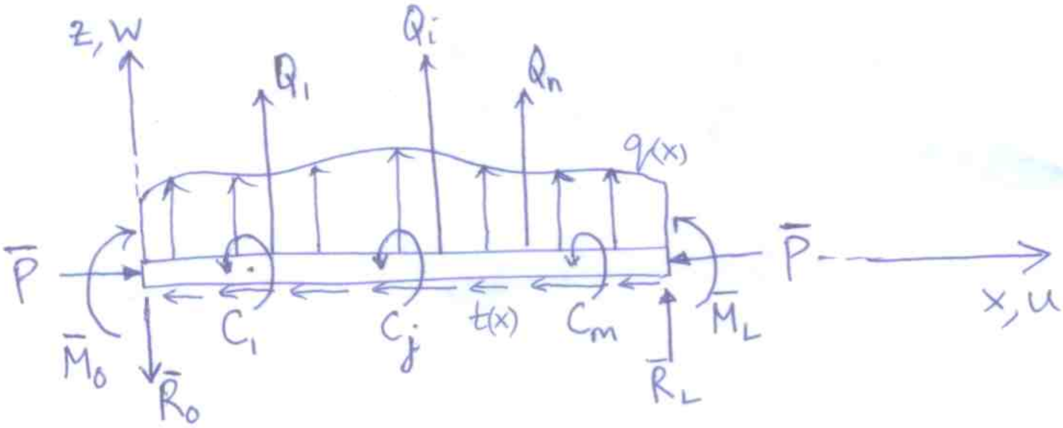
Assumptions:

(i) Homogenous, isotropic,

(ii) Euler - Bernoulli beam, ie Plane sections remain plane and normal to neutral plane. Hence transverse shear is neglected.

(iii) Loads act in a plane containing principal axis of cross-section, ie symmetric bending.

Consider beam with loading shown.



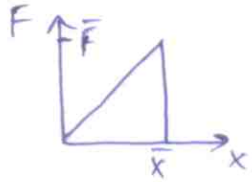
Aside: (I) Strain Energy, (II) Principles of Virtual Work & Stationary Potential Energy.

(I) For a deformable body, work done by external loads (W_e) in deforming body from strain state I to II equals strain energy (U_i or U_s, i = internal, s = strain).

$$W_e = U_i = \int_V \int_I^{II} \left[\sigma_{xx} d\epsilon_{xx} + \sigma_{yy} d\epsilon_{yy} + \sigma_{zz} d\epsilon_{zz} + \sigma_{xy} d\gamma_{xy} + \sigma_{yz} d\gamma_{yz} + \sigma_{xz} d\gamma_{xz} \right] dV$$

Note — $\gamma_{xy} = 2e_{xy}$, $\gamma_{yz} = 2e_{yz}$, $\gamma_{xz} = 2e_{xz}$, ie twice the tensorial strains are engineering strains. So $[\sigma_{xx}\, de_{xx} + \cdots] = \sum_{i,j=1}^{3}\sum \sigma_{ij}\, de_{ij}$ → i.e, analogous to differential of work done by elastic spring $(F dx)$ for every type of strain (normal, shear strains). For Hookean

$$\int_{I}^{II} F\,dx = \frac{1}{2}\bar{F}\bar{x}$$



Analogously for Hookean solid, strain energy density (ie strain energy per unit volume) is,

$$\bar{U}_i = \int_{I}^{II} [\sigma_{xx}\, de_{xx} + \cdots] = \frac{1}{2}[\sigma_{xx} e_{xx} + \cdots]$$

$$\Rightarrow \boxed{U_i = \int_V \bar{U}_i\, dV = \frac{1}{2}\int_V \left[\sigma_{xx} e_{xx} + \sigma_{yy} e_{yy} + \sigma_{zz} e_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz}\right] dV}$$

(II) Virtual displacement: it is a hypothetical displacement which is compatible with the constraints and approximations of the given problem. It is hypothetical since forces (external, internal) do not change during a virtual displacement, which is basically incompatible with the equilibrium and response of systems in general. The compatibility requirement means that (i) it obeys the composition of the solid (ie if no cracks or voids then actual displ is single valued, and so should be virtual displ); (ii) obeys kinematic constraints, kinematic equations (ie strain-displ eqns) and kinematic approximations or assumptions (e.g. if actual displ assumed to vary only in x-direction, then same holds for virtual displ).

Apart from these requirements, virtual displacements are arbitrary. When they are considered as infinitesimal, they are termed variations.

So actual displ occur as a result of loading & equilibrium and are one member of the infinite set of virtual displacements. Similarly, differential displacements (eg, dw) are actual infinitesimal displ's occurring due to differential changes in loads, and their counterparts are variational displ (or variations) (e.g., δw) which are hypothetical incremental displ at constant loads.

\# The principle of virtual work states that a body is in equilibrium if & only if the work done (W) by external forces (loads + reactions) during a virtual displ equals the strain energy due to that virtual displ.

$$\delta W = \delta U_i$$

If the system is conservative then W is expressed as $-U_p$, ie the potential of the ext. forces (eg. $U_p = mgz$, work done by gravity in rising thru z is $W = -mgz$).

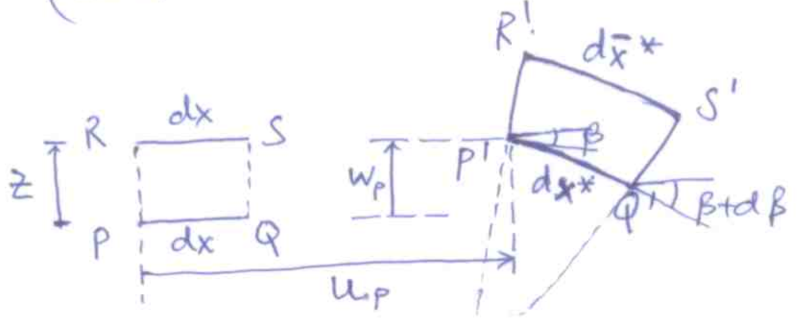
Thus $\delta W = -\delta U_p$ and principle of VW gives

$$-\delta U_p = \delta U_i \Rightarrow \delta(U_i + U_p) = \delta U = 0.$$

i.e, principle of stationary (minimum) potential energy (U), as we had for discrete systems.

x

(Back to Beam-Columns.)



PQ & RS are undeformed elements & P'Q' & R'S' are deformed elements after bending & extension.

P displaces by $u_P \underline{i} + w_P \underline{k}$.

Extensional strain of element PQ $= \varepsilon_{PQ} = \sqrt{1+2E_{xx}} - 1$

where $E_{xx} = u_{,x} + \frac{u_{,x}^2}{2} + \frac{w_{,x}^2}{2}$

Binomial/Taylors expansion gives

$$\Rightarrow \varepsilon_{pq} = \left[\cancel{1} + \tfrac{1}{2}(2E_{xx}) - \tfrac{1}{8}(2E_{xx})^2 + \text{H.O.T's}\right] - \cancel{1}$$

$$= u_{,x} + \frac{u_{,x}^2}{2} + \frac{w_{,x}^2}{2} - \frac{1}{2}\left[u_{,x}^2 + \text{H.O.T's}\right] + \text{H.O.T's}$$

$$= u_{,x} + \frac{w_{,x}^2}{2}$$

Here we have considered small strains, ie $u_{,x}$ retained, & moderate rotations, ie $w_{,x}^2$ considered same order as $u_{,x}$ (ie $w_{,x}^2 = O(u_{,x})$). Even without doing Taylor series above we can straight away consider that small strains $\Rightarrow u_{,x}^2$ to be dropped (in E_{xx}) compared to $u_{,x}$ & $w_{,x}^2$ since rotations are considered moderate. This is the intermediate class of deformations considered in the analysis. (If we do small strains & small rotations then $w_{,x}^2 << u_{,x}$ so $w_{,x}^2$ also would be dropped.)

$$\varepsilon_{RS} = \frac{d\bar{x}^* - dx}{dx} = \frac{dx^* + (d\beta)z - dx}{dx} = \frac{dx^* - dx}{dx} + z\frac{d\beta}{dx}$$

$$e_{xx} = \boxed{\varepsilon_{RS} = u_{,x} + \frac{w_{,x}^2}{2} + z\beta' = u_{,x} + \frac{w_{,x}^2}{2} - zw'' = \overset{\circ}{e}_{xx} - zw'' = \overset{\circ}{e}_{xx} - zK_{xx}}$$

(ie, $\beta = -w'$, u, w are displ of point on neutral axis)

Now, $\delta U = \delta U_i + \delta U_p = 0$.

$$\delta U_i = \frac{1}{2}\int_V (\delta\sigma_{xx} e_{xx} + \sigma_{xx}\delta e_{xx})\, dV$$

K_{xx} is radius of curvature of neutral axis

$\sigma_{xx} = E\, e_{xx}$ (Hooke's law, all other stress components σ_{yy}, σ_{xy} etc $= 0$).

$$\Rightarrow \delta U_i = \int_V \sigma_{xx}\, \delta e_{xx}\, dV = \int_0^L \int_A E\left(u' + \frac{w'^2}{2} - z w''\right) * \left(\delta u' + w'\delta w' - z\delta w''\right) dA\, dx$$

Now, $\int_A z\, dA = 0$, $\int_A z^2 dA = I_{yy} = I$

$$\Rightarrow \delta U_i = \int_0^L \int_A E\left(\mathring{e}_{xx} - z w''\right)\left(\delta \mathring{e}_{xx} - z\delta w''\right) dA\, dx$$

$$= A\int_0^L E\,\mathring{e}_{xx}\,\delta\mathring{e}_{xx}\, dx + EI\int_0^L w''\delta w''\, dx$$

$$= \int_0^L P\left(\delta u' + w'\delta w'\right) dx + EI\int_0^L w''\delta w''\, dx$$

where, $P = P(x)$ axial force at any cross section

$$= \int_A \sigma_{xx}\, dA = \int E\left(\mathring{e}_{xx} - z w''\right) dA = E\,\mathring{e}_{xx} A.$$

$$\Rightarrow \delta U_i = P\delta u\Big|_0^L + P w'\delta w\Big|_0^L + EI w''\delta w'\Big|_0^L - (EI w'')'\delta w\Big|_0^L$$

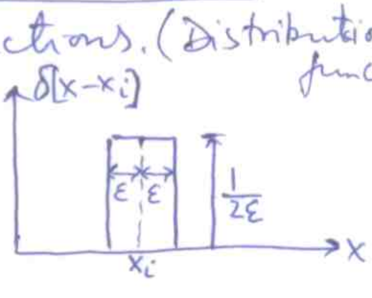
$$\int_0^L \left(-\left[P'\delta u + (P w')'\delta w\right] + (EI w'')''\delta w\right) dx$$

$$\delta U_p = -\Bigg[\int_0^L (q\,\delta w - t\,\delta u)\,dx + \sum_{i=1}^{n} Q_i\,\delta w_i + \sum_{j=1}^{m} C_j\,\delta w_j' - \bar{M}_0\,\delta w_0' + \bar{M}_L\,\delta w_L' - \bar{R}_0\,\delta w_0 + \bar{R}_L\,\delta w_L + \bar{P}(\delta u_0 - \delta u_L)\Bigg]$$

where $\delta w_i = \delta w[x_i]$, $\delta w'_i = \delta w'[x_i]$, $\delta w'_0 = \delta w'[0]$, $\delta w'_L = \delta w'[L]$, $\delta w_0 = \delta w[0]$, $\delta w_L = \delta w[L]$, $\delta u_0 = \delta u[0]$, $\delta u_L = \delta u[L]$

Aside: Dirac-δ and Doublet functions. (Distribution functions).

Dirac-δ $\rightarrow$ $$\delta[x - x_i] = \begin{cases} 0, & x < x_i - \varepsilon \\ 1/2\varepsilon, & x_i - \varepsilon \leqslant x \leqslant x_i + \varepsilon \\ 0, & x > x_i + \varepsilon \end{cases}$$



$$\int_{-\infty}^{\infty} \delta[x-x_i]\,dx = 1$$

$$\int_{-\infty}^{\infty} f[x]\,\delta[x-x_i]\,dx = f[x_i]$$ $\longrightarrow$ here $\lim_{\varepsilon\to 0}$ considered in $\delta[x-x_i]$ definition,

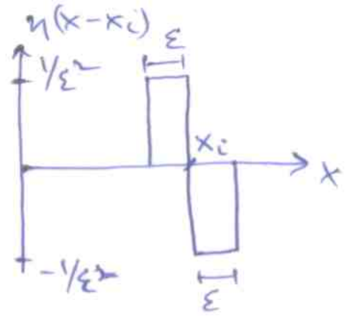
so $\delta[x-x_i] \to \infty$, $x = x_i$.

$\hookrightarrow$ So for point load Q_i at $x=x_i$ you can use the distribution function $Q_i\delta[x-x_i]$ to model it as distributed load $q[x]$. Thus Δ(shear force) as you pass the point load $= \Delta V = \int_{x_i^-}^{x_i^+} \frac{dV}{dx}\,dx = \int_{x_i^-}^{x_i^+} Q_i\,\delta[x-x_i]\,dx = Q_i$

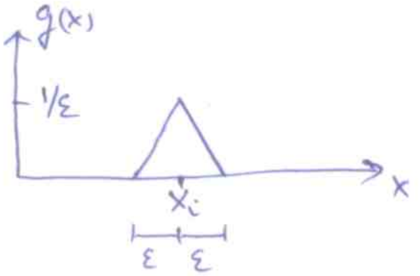
which fits in with the equilibrium of an element where point load Q_i acts.

Doublet $\rightarrow$

$$\eta[x-x_i] = \begin{cases} 0, & x < x_i - \varepsilon \\ 1/\varepsilon^2, & x_i - \varepsilon < x < x_i \\ 0, & x = x_i \\ -1/\varepsilon^2, & x_i < x < x_i + \varepsilon \\ 0, & x > x_i \end{cases}$$



let $g(x) = \int_{-\infty}^{x} \eta[\xi - x_i]\,d\xi =$



$= \delta(x-x_i)$ when $\lim \varepsilon \to 0$.

So in the limit as $\varepsilon \to 0$,

$$\frac{d}{dx}\delta[x-x_i] = \eta[x-x_i]$$

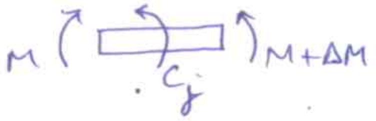
$$\Rightarrow \int_{-\infty}^{\infty} f[x]\,\eta(x-x_i) = \underbrace{f[x]\,\delta[x-x_i]\Big|_{-\infty}^{\infty}}_{=0 \text{ at } \infty \,\&\, -\infty} - \int_{-\infty}^{\infty} f'\,\delta[x-x_i]\,dx = -f'[x_i]$$

If we model a couple at x_j as $-C_j\,\eta[x-x_j]$ being an equivalent distributed load $q[x]$ then,

$$V[x] = \int_{-\infty}^{x} -C_j\,\eta[\xi - x_j]\,dx = -C_j\,\delta[x-x_j]$$

and $\Delta M = \int_{x_j^-}^{x_j^+} M\,dx = \int_{x_j^-}^{x_j^+} -C_j\,\delta[x-x_j]\,dx = -C_j$

Which is consistent with change in BM as you pass a station where couple is applied.



x

So, $$\delta U_p = -\left[\int_0^L \left(q\,\delta w - t\,\delta u + \sum_{i=1}^{n} Q_i\,\delta[x-x_i]\,\delta w + \sum_{j=1}^{m} C_j\,\delta[x-x_j]\,\delta w'\right)dx - \bar{M}_0\,\delta w_0' + \bar{M}_L\,\delta w_L' - \bar{R}_0\,\delta w_0 + \bar{R}_L\,\delta w_L + \bar{P}(\delta u_0 - \delta u_L)\right]$$

So $\delta U = \delta U_i + \delta U_p = 0$ gives one integral term $\left(\int_0^L (\;)\,dx\right)$ and the second term is a set of boundary terms (ie terms evaluated at $x=0, L$). Noting that $\delta w[x]$, $\delta u[x]$ in the integral term are arbitrary and independent, we set the coefficients of the of $\delta w(x)$ and $\delta u[x]$ in the integrand to zero, yielding the equilibrium equations. Similarly the quantities $\delta u_0, \delta u_L, \delta w_0, \delta w_L, \delta w_0', \delta w_L'$ are independent & arbitrary so the terms containing them, in the boundary term, can be independently made to vanish. This gives the boundary conditions. Before doing this step, lets look at the point moment term. There are two ways of handling this:

(i) $$\int_0^L \sum_{j=1}^{m} C_j\,\delta[x-x_j]\,\delta w'\,dx = \sum_{j=1}^{m} C_j\,\delta[x-x_j]\,\delta w\Big|_0^L - \sum_{j=1}^{m}\int_0^L C_j\,\eta[x-x_j]\,\delta w\,dx$$

(ii) Represent it as part of distributed load using doublet f^{n}

ie $\int_0^L \left(q - C_j\,\eta[x-x_j]\right)\delta w\,dx$ which is more direct.

They both yield same resulting term for the applied point couple in the equilibrium equation.

Thus we have, (see δU_i p. 5, δU_p p. 7), from $\delta U_i + \delta U_p = 0$:

Equilibrium Equations :

$$\delta w : \quad (EI\,w'')'' - (Pw')' = q + \sum_{i=1}^{n} Q_i\, \delta[x-x_i] - \sum_{j=1}^{m} C_j\, \eta(x-x_j]$$

$$\delta u : \quad P' = t$$

Boundary Conditions: At $x = 0, L$,

$P + \bar{P} = 0$ <u>or</u> $\delta u = 0$, i.e, $u = \bar{u}$

$Pw' - EI(w'')' - \bar{R} = 0$ <u>or</u> $\delta w = 0$, i.e, $w = \bar{w}$

$EIw'' - \bar{M} = 0$ <u>or</u> $\delta w' = 0$, ie $w' = \bar{w}'$

These BC's are termed Natural <u>or</u> Kinematic, respectively.

eg: Free edge, no load applied at edge,

$$P = 0, \quad \underbrace{\cancel{P}}_{0} w' - (EI\,w'')' \; \underbrace{\cancel{\bar{R}}}_{0} = 0, \quad EI\,w'' + \underbrace{\cancel{\bar{M}}}_{0} = 0$$

<u>Ideal Columns</u> (no geometric imperfection or loading eccentricity)

(1) Simply Supported.

$$\left.\begin{aligned} P' &= 0 \\ \text{at } x=L,\; P + \bar{P} &= 0 \end{aligned}\right\} \Rightarrow P = -\bar{P}, \; x = 0 - L.$$

$$EI\,w^{IV} + \bar{P}w'' = 0$$

$$w[0] = w[L] = w''[0] = w''[L] = 0$$

Define $\bar{P}/EI = k^2 \Rightarrow w^{IV} + k^2 w'' = 0 \longrightarrow (1)$

Solution is $w = e^{sx}$

$\Rightarrow s^4 + k^2 s^2 = 0 \Rightarrow s = 0, 0, \pm ik$

$$w[x] = C_1 + C_2 x + C_3 e^{ik} + C_4 e^{-ik} = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4$$

Apply BC's, seek non-trivial sol,

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin kL & \cos kL & L & 1 \\ 0 & -k^2 & 0 & 0 \\ -k^2 \sin kL & -k^2 \cos kL & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0 \Rightarrow \det(\text{coeff matrix}) = 0$$

$k^2(k^2 L \sin kL) = 0 \implies \sin kL = 0 \rightarrow$ Characteristic equation

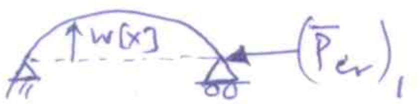
$kL = n\pi$, $n = 1, 2, \ldots$

$\bar{P}_{cr} = \frac{n^2\pi^2}{L^2} EI$, $\bar{P}_{buckling} = \bar{P}_{cr}\big|_{n=1} = \frac{\pi^2 EI}{L^2} \rightarrow$ buckling load. (Euler buckling load).

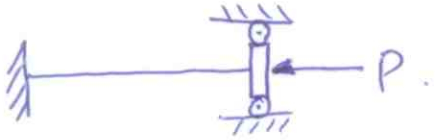
$\sigma_{cr} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 E}{(L/\rho)^2}$, ρ = rad. of gyration of section.

$W[x] = A_1 \sin kx$ (put $\sin kL = 0$, get $A_2 = A_4 = A_3 = 0$)

So buckling mode for $(\bar{P}_{cr})_1 = \pi^2 EI/L^2$ is first mode,

i.e $W[x] = A_1 \sin \frac{n\pi}{L} x \implies$ 

(2) Clamped - Clamped Column.



DE & its solution remain same.
Only BC's change.

$W(0) = W(L) = W'(0) = W'(L) = 0.$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin kL & \cos kL & L & 1 \\ k & 0 & 1 & 0 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0$$

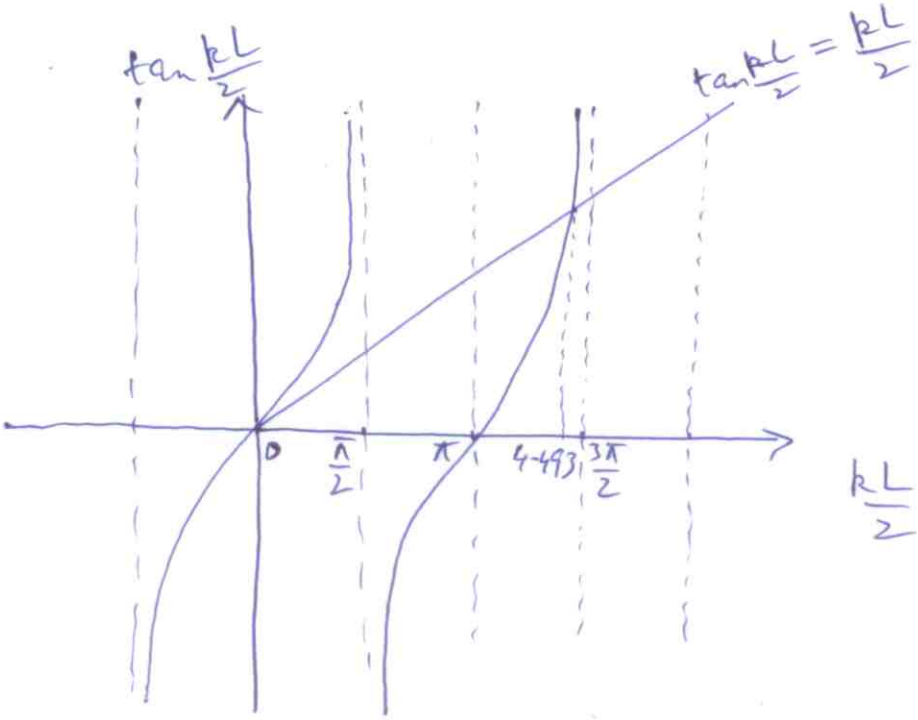
$$(-1)(1)(k - k\cos kL) - (1)\left[(-1)(-k\sin^2 kL - k\cos^2 kL) - k(\cos kL + kL\sin kL)\right] = 0 \rightarrow (CE)$$

$\Rightarrow k(\cos kL - 1) + k(-1 + \cos kL + kL\sin kL) = 0.$

$\Rightarrow 2k(\cos kL - 1) + k^2 L \sin kL = 0.$

$\Rightarrow \sin\frac{kL}{2}\left(-2\sin\frac{kL}{2} + kL\cos\frac{kL}{2}\right) = 0.$

ie, $\sin\frac{kL}{2} = 0$ or $\frac{kL}{2} = \tan\frac{kL}{2}$



So $\frac{kL}{2} = n\pi, \quad n = 1, 2 \ldots$

or $\frac{kL}{2} = 4.493, \ldots\ldots$ (higher roots, use numerical method, e.g. Newton Rhapson).

$$(\bar{P}_{cr})_1 = \bar{P}_{buckl} = \frac{4\pi^2}{L^2} EI \, .$$

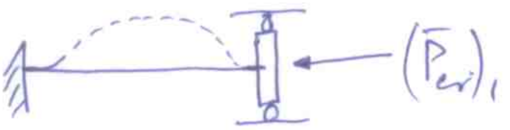
$$(\bar{P}_{cr})_2 = \frac{4 * (4.493)^2}{L^2} EI \rightarrow \text{next higher critical load.}$$

Solving for coefficients (with $\sin\frac{kL}{2} = 0$) we get,

$$\left.\begin{aligned} A_2 + A_4 &= 0 \\ A_2 + LA_3 + A_4 &= 0 \\ kA_1 + A_3 &= 0 \end{aligned}\right\} \Rightarrow A_3 = A_1 = 0$$

$$\Rightarrow w[x] = A_2(\cos kx - 1) \rightarrow \text{buckling mode}$$

Also $\sigma_{cr} = \frac{(\bar{P}_{cr})_1}{A} = \frac{4\pi^2 E}{(L/\rho)^2}$



(3) Clamped-Free column.

BC's are,

$$w(0) = w'(0) = (\bar{P}w' + EIw''')_{x=L} = EIw''[L] = 0.$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \cancel{(k^3\cos kL - k^3\cos kL)}^{\,0} & \cancel{(-k^3\sin kL + k^3\sin kL)}^{\,0} & k^2 & 0 \\ -k^2\sin kL & -k^2\cos kL & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0$$

$\Rightarrow A_3 = A_1 = 0.$
$A_4 = -A_2.$

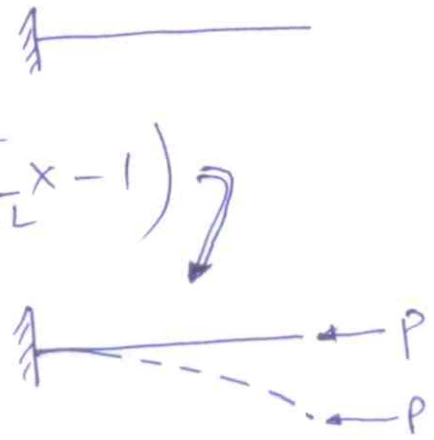
$\cos kL = 0 \longrightarrow$ (CE).

$\Rightarrow kL = (2n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \dots$

for $n=0$, $(\bar{P}_{cr})_1 = \bar{P}_{buckl} = \dfrac{\pi^2 EI}{4L^2}$

$W[X] = A_2(\cos kx - 1) = A_2\left(\cos \frac{\pi}{2L}x - 1\right)$

$\sigma_{cr} = \dfrac{(\bar{P}_{cr})_1}{A} = \dfrac{\pi^2 E}{4(L/\rho)^2}$



(4). Clamped – SS column.

BC's $\rightarrow$ $W(0) = W(L) = W'(0) = W''(L) = 0$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin kL & \cos kL & L & 1 \\ k & 0 & 1 & 0 \\ -k^2\sin kL & -k^2\cos kL & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0$$

$\Rightarrow A_2 + A_4 = 0$
$kA_1 + A_3 = 0$

$(-1)(1)(k^2 \sin kl) - 1\{(-1)(0) - k(k^2 l \cos kl)\} = 0 \longrightarrow$ (CE)

$\tan kl = kl \longrightarrow$ lowest root is $kl = 4.493$.

$(\bar{P}_{cr})_1 = P_{buck} = \dfrac{EI}{l^2}(4.493)^2 = \left(\dfrac{4.493}{\pi}\right)^2 \dfrac{\pi^2 EI}{L^2}$

From the CE we also get, $A_1 + \overbrace{\cancel{\tan kl}}^{=kL} A_2 = 0 = A_1 + kl A_2$

Energy Method for Stability (Energy Criterion)

$U = U_i + U_p$ if U_p is due to conservative forces (as is our case in CE619), else we only have $dU = dU_i + dU_p$, ie for non-conservative forces dU_p is not perfect differential & hence work done by P when traversing a closed loop is non-zero (ie non conservative, energy not preserved, as in damping or hysterisis).

$$U_i = \frac{1}{2}\int_0^L\int_A \sigma_{xx} e_{xx}\, dA\, dx = \frac{1}{2}\int_0^L\int_A E\left(u' + \frac{w'^2}{2} - zw''\right)^2 dA\, dx$$

$$= \frac{1}{2}\int_0^L\left(EA\left(u' + \frac{w'^2}{2}\right)^2 + EI(w'')^2\right)dx$$

$$U_p = -\int_0^L (q^* w - tu)\,dx + \bar{P}u\Big|_0^L - \bar{M}w'\Big|_0^L - \bar{R}w\Big|_0^L$$

For Beam-Column

$$U = U_i + U_p = \int_0^L F[x, u, u', w, w', w'']\,dx + f[u_0, w_0, w_0', u_L, w_L, w_L'] \longrightarrow \text{Semi-generalization}$$

(For Beam-Col, F = sum of integrands in U_i, U_p, and f = boundary terms in U_p.

Note that point loads/couples are absorbed in q^* via Dirac-delta & Doublet functions, ie (as on p.7, way(ii))

$$q^*_{(x)} = q[x] + \sum_{i=1}^{m} Q_i\, \delta[x - x_i] - \sum_{j=1}^{n} C_j\, \eta[x - x_j].$$

The x-dependency in F is due to prescribed $q[x]$, $t[x]$ loads that are x-dependent.

So U = integral of function F + boundary term = functional (function of a function F).

Let u_e, w_e denote equilibrium configuration (solution).

Then,

$$u[x] = u_e[x] + u_1[x] = u_e[x] + \varepsilon_1 \xi[x] = u_e[x] + \delta u[x]$$

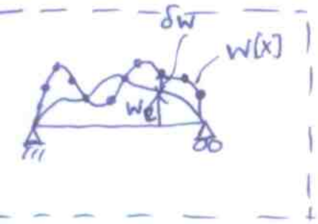
$$w[x] = w_e[x] + w_1[x] = w_e[x] + \varepsilon_2 \eta(x) = w_e[x] + \delta w[x].$$

(NOTE: We can also take $\varepsilon_1 = \varepsilon_2 = \varepsilon$ w/o loss of generality $\because \xi, \eta$ are arbitrary) ie, $u_1[x] = \varepsilon_1 \zeta[x] = \delta u[x] = \text{variation of } u[x] =$ arbitrary but kinematically consistent

$w_1[x] = \varepsilon_2 \eta[x] = \delta w[x] = \text{variation of } w[x] =$

where ε_1 and ε_2 are small, arbitrary $\Rightarrow$ variations are small.

Recall that loads remain constant when applying/performing a variation.



similarly

$$\left.\begin{aligned} u_0 = u[0] &= u_e[0] + \delta u[0] = u_{e0} + \delta u_0 \\ u_L = u[L] &= u_e[L] + \delta u[L] = u_{eL} + \delta u_L \\ w_0 = w[0] &= w_e[0] + \delta w[0] = w_{e0} + \delta w_0 \\ w_L = w[L] &= w_e[L] + \delta w[L] = w_{eL} + \delta w_L \end{aligned}\right\}$$

here also all variations, ie, $\delta u[0] (=\delta u_0)$, etc are small, arbitrary, kinematically consistent.

and hence $\delta u_0 = \varepsilon_1 \zeta[0]$, $\delta u_L = \varepsilon_1 \zeta[L]$, $\delta w_0 = \varepsilon_2 \eta[0]$, $\delta w_L = \varepsilon_2 \eta[L]$.

Thus, we can re-think U as $U = U[\varepsilon_1, \varepsilon_2]$, hence,

$$U\Big|_{(\varepsilon_1,\varepsilon_2)} = \underbrace{U\Big|_{(0,0)}}_{U_e} + \frac{\partial U}{\partial \varepsilon_1}\Big|_{(0,0)} \varepsilon_1 + \frac{\partial U}{\partial \varepsilon_2}\Big|_{(0,0)} \varepsilon_2 + \frac{1}{2}\left[\frac{\partial^2 U}{\partial \varepsilon_1^2}\Big|_{(0,0)} \varepsilon_1^2 + \frac{\partial^2 U}{\partial \varepsilon_2^2}\Big|_{(0,0)} \varepsilon_2^2 + 2\frac{\partial^2 U}{\partial \varepsilon_1 \partial \varepsilon_2}\Big|_{(0,0)} \varepsilon_1 \varepsilon_2 \right] + \text{HOT's}$$

$$= U_e + \delta U + \frac{1}{2}\delta^2 U + \text{H.O.T's}$$

where $U_e = \int_0^L F[x, u_e, u_e', w_e, w_e', w_e''] \, dx + f[u_{e0}, u_{eL}, w_{e0}, w_{eL}, w_{e0}', w_{eL}']$

and $U\Big|_{(\varepsilon_1,\varepsilon_2)} = U = \int_0^L F[x, u_e + u_1, w_e + w_1, \ldots] + f[u_{e0} + \delta u_0, \ldots]$

$$\delta U = \int_0^L \left(\frac{\partial F}{\partial \varepsilon_1}\Big|_{(0,0)} * \varepsilon_1 + \frac{\partial F}{\partial \varepsilon_2}\Big|_{(0,0)} * \varepsilon_2 \right) dx + \frac{\partial f}{\partial \varepsilon_1}\Big|_{(0,0)} \varepsilon_1 + \frac{\partial f}{\partial \varepsilon_2}\Big|_{(0,0)} \varepsilon_2$$

$$= \int_0^L \left[\left(\frac{\partial F}{\partial u}\Big|_{(0,0)} \underbrace{\frac{du}{d\varepsilon_1}}_{\zeta[x]} + \frac{\partial F}{\partial u'}\Big|_{(0,0)} \underbrace{\frac{du'}{d\varepsilon_1}}_{\zeta'[x]} \right) \varepsilon_1 + \left(\frac{\partial F}{\partial w}\Big|_{(0,0)} \underbrace{\frac{dw}{d\varepsilon_2}}_{\eta[x]} + \frac{\partial F}{\partial w'}\Big|_{(0,0)} \underbrace{\frac{dw'}{d\varepsilon_2}}_{\eta'[x]} + \frac{\partial F}{\partial w''}\Big|_{(0,0)} \underbrace{\frac{dw''}{d\varepsilon_2}}_{\eta''[x]} \right) \varepsilon_2 \right] dx$$

$$+ \left(\frac{\partial f}{\partial u_0}\Big|_{(0,0)} \underbrace{\frac{du_0}{d\varepsilon_1}}_{\zeta[0]} + \frac{\partial f}{\partial u_L}\Big|_{(0,0)} \underbrace{\frac{du_L}{d\varepsilon_1}}_{\zeta[L]} \right) \varepsilon_1 + \left(\frac{\partial f}{\partial w_0}\Big|_{(0,0)} \underbrace{\frac{dw_0}{d\varepsilon_2}}_{\eta[0]} + \frac{\partial f}{\partial w_L}\Big|_{(0,0)} \underbrace{\frac{dw_L}{d\varepsilon_2}}_{\eta[L]} + \frac{\partial f}{\partial w_0'}\Big|_{(0,0)} \underbrace{\frac{dw_0'}{d\varepsilon_2}}_{\eta'[0]} + \frac{\partial f}{\partial w_L'}\Big|_{(0,0)} \underbrace{\frac{dw_L'}{d\varepsilon_2}}_{\eta'[L]} \right) \varepsilon_2$$

$$= \int_0^L \left[\frac{\partial F}{\partial u_e}\delta u + \frac{\partial F}{\partial u'_e}\delta u' + \frac{\partial F}{\partial w_e}\delta w + \frac{\partial F}{\partial w'_e}\delta w' + \frac{\partial F}{\partial w''_e}\delta w''\right]dx$$

$$\frac{\partial f}{\partial u_{e0}}\delta u_0 + \frac{\partial f}{\partial u_{eL}}\delta u_L + \frac{\partial f}{\partial w_{e0}}\delta w_0 + \frac{\partial f}{\partial w_{eL}}\delta w_L + \frac{\partial f}{\partial w'_{e0}}\delta w'_0 + \frac{\partial f}{\partial w'_{eL}}\delta w'_L$$

(we used $\left.\frac{\partial F}{\partial u}\right|_{(0,0)} = \frac{\partial F}{\partial u_e}$, etc, and $\left.\frac{\partial f}{\partial u_0}\right|_{(0,0)} = \frac{\partial f}{\partial u_{e0}}$, etc, and $\varepsilon_1 S[x] = \delta u$, etc, and $\varepsilon_1 S[0] = \delta u[0] = \delta u_0$, etc).

$$= \int_0^L \left[\left(\frac{\partial F}{\partial u_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'_e}\right)\right)\delta u + \left(\frac{\partial F}{\partial w_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial w'_e}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w''_e}\right)\right)\delta w\right]dx$$

$$+ \delta u_0\left(\frac{\partial f}{\partial u_{e0}} - \left.\frac{\partial F}{\partial u'_e}\right|_{x=0}\right) + \delta u_L\left(\frac{\partial f}{\partial u_{eL}} + \left.\frac{\partial F}{\partial u'_e}\right|_{x=L}\right)$$

$$+ \delta w_0\left(\frac{\partial f}{\partial w_{e0}} - \left.\frac{\partial F}{\partial w'_e}\right|_{x=0} + \left.\frac{d}{dx}\left(\frac{\partial F}{\partial w''_e}\right)\right|_{x=0}\right) + \delta w_L\left(\frac{\partial f}{\partial w_{eL}} + \left.\frac{\partial F}{\partial w'_e}\right|_{x=L} - \left.\frac{d}{dx}\left(\frac{\partial F}{\partial w''_e}\right)\right|_{x=L}\right)$$

$$+ \delta w'_0\left(\frac{\partial f}{\partial w'_{e0}} - \left.\frac{\partial F}{\partial w''_e}\right|_{x=0}\right) + \delta w'_L\left(\frac{\partial f}{\partial w'_{eL}} + \left.\frac{\partial F}{\partial w''_e}\right|_{x=L}\right)$$

(we have integrated by parts to remove (') primes from variational quantities δu & δw)

$= 0$ for equilibrium.

So equate integral & boundary terms individually to zero and note that δu, δw, δu_0 etc are arbitrary. We get

$$\frac{\partial F}{\partial u_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'_e}\right) = 0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial w_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial w'_e}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w''_e}\right) = 0 \longrightarrow (2)$$

EULER-LAGRANGE EQUATIONS FOR EQUILIBRIUM

at $x=0$: $\frac{\partial f}{\partial u_{e0}} - \frac{\partial F}{\partial u'_e} = 0$ <u>or</u> $u = \bar{u}_0$

$\frac{\partial f}{\partial w_{e0}} - \frac{\partial F}{\partial w'_e} + \frac{d}{dx}\left(\frac{\partial F}{\partial w''_e}\right) = 0$ <u>or</u> $w = \bar{w}_0$

B.C's Natural or Kinematic, respectively.

$$\frac{\partial f}{\partial w'_{e0}} - \frac{\partial F}{\partial w''_e} = 0 \quad \underline{\underline{or}} \quad w' = \bar{w}'_0$$

at $x=L$:

$$\frac{\partial f}{\partial u_{eL}} + \frac{\partial F}{\partial u'_e} = 0 \quad \underline{\underline{or}} \quad u = \bar{u}_L$$

$$\frac{\partial f}{\partial w_{eL}} + \frac{\partial F}{\partial w'_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial w''_e}\right) = 0 \quad \underline{\underline{or}} \quad w = \bar{w}_L$$

$$\frac{\partial f}{\partial w'_{eL}} + \frac{\partial F}{\partial w''_e} = 0 \quad \underline{\underline{or}} \quad w' = \bar{w}'_L$$

BC's.

Example: Application to Beam - Column.

$$F = \frac{1}{2}\left[EA\left(u' + \frac{w'^2}{2}\right)^2 + EI(w'')^2\right] - q^* w + t u$$

$$f = \bar{P} u\Big|_0^L - \bar{M} w'\Big|_0^L - \bar{R} w\Big|_0^L$$

put subscript 'e' on u, w in F, f, thruout.

we get,

$$\frac{\partial F}{\partial u_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'_e}\right) = t - \frac{d}{dx}\underbrace{\left[EA\left(u'_e + \frac{w_e'^2}{2}\right)\right]}_{=P_e[x]} = \boxed{t - P'_e = 0} \rightarrow \text{(i)}$$

$$\frac{\partial F}{\partial w_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial w'_e}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w''_e}\right) = \boxed{-q^* - \left[P_e w'_e\right]' + \left[EI w''_e\right]'' = 0} \rightarrow \text{(ii)}$$

at $x=0$:

$$\frac{\partial f}{\partial u_{e0}} - \frac{\partial F}{\partial u'_e} = \boxed{-\bar{P} - P_e = 0 \quad \text{or} \quad u = \bar{u}_0}$$

$$\frac{\partial f}{\partial w_{e0}} - \frac{\partial F}{\partial w'_e} + \frac{d}{dx}\left(\frac{\partial F}{\partial w''_e}\right) = \boxed{\bar{R} - P_e w'_e + \left[EI w''_e\right]' = 0 \quad \text{or} \quad w = \bar{w}_0}$$

$$\frac{\partial f}{\partial w'_{e0}} - \frac{\partial F}{\partial w''_e} = \boxed{\bar{M} - EI w''_e = 0 \quad \text{or} \quad w' = \bar{w}'_0}$$

at $x=L$:

$$\frac{\partial f}{\partial u_{eL}} + \frac{\partial F}{\partial u'_e} = \boxed{\bar{P} + P_e = 0 \quad \text{or} \quad u = \bar{u}_L}$$

$$\frac{\partial f}{\partial w_{eL}} + \frac{\partial F}{\partial w'_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial w''_e}\right) = \boxed{-\bar{R} + P_e w'_e - \left[EI w''_e\right]' = 0 \quad \text{or} \quad w = \bar{w}_L}$$

$$\frac{\partial f}{\partial w'_{eL}} + \frac{\partial F}{\partial w''_e} = \boxed{-\bar{M} + EI w''_e = 0 \quad \text{or} \quad w' = \bar{w}'_L}$$

∴ So we get same equilibrium equations as we had on p.8 when we started from first principles.

NOTE: (i) All that we have achieved is a formalization of the variational procedure for a class of systems where F & f are functional forms as on p.13.

(ii) For F & f having different functional forms, eg $F[x, u, u', v, v', w, w', w'']$ we can derive corresponding (generalized) EULER LAGR equilibrium eqns and BC's, in the same manner. The definition of variational quantities is unchanged, only you will have $v[x] = v_e[x] + v_1[x] = v_e[x] + \varepsilon_3 \gamma[x] = v_e[x] + \delta v[x]$ as an additional variational definition.

(iii) Instead of working in terms of $\varepsilon_1, \varepsilon_2$, ie $U = U[\varepsilon_1, \varepsilon_2]$ you can work in terms of u_1, w_1 as follows.

$$\Delta U = \int_0^L \Big\{ F[x, u_e+u_1, u_e'+u_1', w_e+w_1, w_e'+w_1', w_e''+w_1''] - F[x, u_e, u_e', w_e, w_e', w_e''] \Big\} dx$$
$$+ f[u_{e0}+u_{10}, u_{eL}+u_{1L}, w_{e0}+w_{10}, w_{eL}+w_{1L}, w_{e0}'+w_{10}', w_{eL}'+w_{1L}']$$
$$- f[u_{e0}, u_{eL}, w_{e0}, w_{eL}, w_{e0}', w_{eL}']$$
$$= \delta U + \tfrac{1}{2}\delta^2 U + \text{H.O.T's}$$

Then use F, f for the specific problem (eg beam-column), and firstly collect terms linear in the variations ($u_1, w_1, u_1', w_1', w_1'', u_{10}, u_{1L}, w_{10}, w_{1L}, w_{10}', w_{1L}'$). These ~~correspond to~~ comprise δU (first variation). Do the integration by parts to remove derivatives from the variations, and the rest is the same as before. You get same equilibrium eqns and BC's <u>for the specific problem</u> at hand

<u>secondly</u>, collect terms quadratic/bilinear in variations (eg. u_1^2, $u_1' w_1$, $u_{10} w_{10}$, etc). These ~~correspond~~ comprise

to $\delta^2 U$, the second variation.

(iv) Our motive for this formalism is to now use the second variation $\delta^2 U$ to develop stability equations (criterion) via the energy method, like we had done for discrete systems.

(v) You can also obtain $\delta^2 U$ by working in terms of $\varepsilon_1, \varepsilon_2$, ie $U[\varepsilon_1, \varepsilon_2]$, ie, the formalization (semi-general approach). This is done below.

$$\delta^2 U = \left.\frac{\partial^2 U}{\partial \varepsilon_1^2}\right|_{(0,0)} \varepsilon_1^2 + \left.\frac{\partial^2 U}{\partial \varepsilon_2^2}\right|_{(0,0)} \varepsilon_2^2 + 2\left.\frac{\partial^2 U}{\partial \varepsilon_1 \partial \varepsilon_2}\right|_{(0,0)} \varepsilon_1 \varepsilon_2$$

$$\frac{\partial^2 U}{\partial \varepsilon_1^2} = \int_0^L \frac{\partial}{\partial \varepsilon_1}\left(\frac{\partial F}{\partial u}\varsigma + \frac{\partial F}{\partial u'}\varsigma'\right)dx + \frac{\partial}{\partial \varepsilon_1}\left[\frac{\partial f}{\partial u_0}\varsigma_0 + \frac{\partial f}{\partial u_L}\varsigma_L\right]$$

$$= \int_0^L \left(\frac{\partial^2 F}{\partial u^2}\varsigma^2 + 2\frac{\partial^2 F}{\partial u' \partial u}\varsigma\varsigma' + \frac{\partial^2 F}{\partial u'^2}\varsigma'^2\right)dx + \frac{\partial^2 f}{\partial u_0^2}\varsigma_0^2 + \cancel{2\frac{\partial^2 f}{\partial u_0 \partial u_L}\varsigma_0\varsigma_L} + \frac{\partial^2 f}{\partial u_L^2}\varsigma_L^2$$

$\rightarrow 0$ (see note below)

we used $\frac{\partial \varsigma}{\partial \varepsilon_1} = \frac{\partial \varsigma'}{\partial \varepsilon_1} = \frac{\partial \varsigma_0}{\partial \varepsilon_1} = \frac{\partial \varsigma_L}{\partial u_L} = 0$

$$\frac{\partial^2 U}{\partial \varepsilon_2^2} = \int_0^L \frac{\partial}{\partial \varepsilon_2}\left(\frac{\partial F}{\partial w}\eta + \frac{\partial F}{\partial w'}\eta' + \frac{\partial F}{\partial w''}\eta''\right)dx + \frac{\partial}{\partial \varepsilon_2}\left(\frac{\partial f}{\partial w_0}\eta_0 + \frac{\partial f}{\partial w_L}\eta_L + \frac{\partial f}{\partial w_0'}\eta'[0] + \frac{\partial f}{\partial w_L'}\eta'[L]\right)$$

$$= \int_0^L \left(\frac{\partial^2 F}{\partial w^2}\eta^2 + \frac{\partial^2 F}{\partial w'^2}\eta'^2 + \frac{\partial^2 F}{\partial w''^2}\eta''^2 + 2\frac{\partial^2 F}{\partial w \partial w'}\eta\eta' + 2\frac{\partial^2 F}{\partial w' w''}\eta'\eta'' + 2\frac{\partial^2 F}{\partial w \partial w''}\eta\eta''\right)dx$$

we used $\frac{d\eta}{d\varepsilon_2} = \frac{d\eta'}{d\varepsilon_2} = \frac{d\eta''}{d\varepsilon_2} = \frac{d\eta_0}{d\varepsilon_2} = \text{etc} = 0$

$$+ \frac{\partial^2 f}{\partial w_0^2}\eta_0^2 + \frac{\partial^2 f}{\partial w_L^2}\eta_L^2 + \frac{\partial^2 f}{\partial w_0'}\eta'[0]^2 + \frac{\partial^2 f}{\partial w_L'}\eta'[L]^2 + 2\frac{\partial^2 f}{\partial w_0 \partial w_0'}\eta_0\eta'[0]$$

$$+ 2\frac{\partial^2 f}{\partial w_L \partial w_L'}{}^{*}\eta_L\eta'[L] + 2\frac{\partial^2 f}{\partial w_0 \partial w_L}\eta_0\eta_L + 2\frac{\partial^2 f}{\partial w_0' \partial w_L'}\eta'[0]\eta'[L] + 2\frac{\partial^2 f}{\partial w_0 \partial w_L'}\eta_0\eta'[L] + 2\frac{\partial^2 f}{\partial w_0' \partial w_L}\eta'[0]\eta_L$$

0 (see note below)

Note: The mixed partials in boundary terms will be zero since work done by a boundary load will involve only u_0, or u_L, or w_0, or w_L, or $w'[0]$, or $w'[L]$.

$$\frac{\partial^2 U}{\partial\varepsilon_1\partial\varepsilon_2} = \frac{\partial}{\partial\varepsilon_2}\int_0^L\left(\frac{\partial F}{\partial u}\zeta + \frac{\partial F}{\partial u'}\zeta'\right)dx + \underbrace{\frac{\partial}{\partial\varepsilon_2}\left(\frac{\partial f}{\partial u_0}\Big|_{(0,0)}\zeta(0) + \frac{\partial f}{\partial u_L}\Big|_{(0,0)}\zeta(L)\right)}_{=0 \ (\because \text{mixed partials} = 0 \text{ in bndry terms}).}$$

$$= \int_0^L\left(\frac{\partial^2 F}{\partial u\partial w}\zeta\eta + \frac{\partial^2 F}{\partial u\partial w'}\zeta\eta' + \frac{\partial^2 F}{\partial u\partial w''}\zeta\eta'' + \frac{\partial^2 F}{\partial u'\partial w}\zeta'\eta + \frac{\partial^2 F}{\partial u'\partial w'}\zeta'\eta' + \frac{\partial^2 F}{\partial u'\partial w''}\zeta'\eta''\right)dx$$

$$\therefore \delta^2 U = \int_0^L\left(\frac{\partial^2 F}{\partial u_e^2}u_1^2 + 2\frac{\partial^2 F}{\partial u_e'\partial u_e}u_1u_1' + \frac{\partial^2 F}{\partial u_e'^2}u_1'^2\right)dx$$

$$+ \int_0^L\left(\frac{\partial^2 F}{\partial w_e^2}w_1^2 + \frac{\partial^2 F}{\partial w_e'^2}w_1'^2 + \frac{\partial^2 F}{\partial w_e''^2}w_1''^2 + 2\frac{\partial^2 F}{\partial w_e\partial w_e'}w_1w_1' + 2\frac{\partial^2 F}{\partial w_e w_e''}w_1w_1'' + 2\frac{\partial^2 F}{\partial w_e'\partial w_e''}w_1'w_1''\right)dx \longrightarrow (3)$$

$$+2\int_0^L\left(\frac{\partial^2 F}{\partial u_e\partial w_e}u_1w_1 + \frac{\partial^2 F}{\partial u_e\partial w_e'}u_1w_1' + \frac{\partial^2 F}{\partial u_e\partial w_e''}u_1w_1'' + \frac{\partial^2 F}{\partial u_e'\partial w_e}u_1'w_1 + \frac{\partial^2 F}{\partial u_e'\partial w_e'}u_1'w_1' + \frac{\partial^2 F}{\partial u_e'\partial w_e''}u_1'w_1''\right)dx$$

$$+ \frac{\partial^2 f}{\partial u_{e0}^2}u_{10}^2 + \frac{\partial^2 f}{\partial u_{eL}^2}u_{1L}^2 + \frac{\partial^2 f}{\partial w_{e0}^2}w_{10}^2 + \frac{\partial^2 f}{\partial w_{eL}^2}w_{1L}^2 + \frac{\partial^2 f}{\partial w_{e0}'^2}w_{10}'^2 + \frac{\partial^2 f}{\partial w_{eL}'^2}w_{1L}'^2$$

(recall: $u_{10} = u_1[0]$, $w_{10}' = w_1'[0]$, $\frac{\partial^2 f}{\partial w_{e0}'^2} = \frac{\partial^2 f}{\partial w_0'^2}\Big|_{(0,0)}$, etc).

$\delta^2 U > 0 \Rightarrow$ Stable.

Instead of finding sign definiteness of $\delta^2 U$ for all possible arbitrary variations $(u_1[x], w_1[x])$ along equilibrium path $(u_e[x], w_e[x])$, we seek conditions, ie stability equations, that make $\delta^2 U \geq 0$ (P.S.D.). Note that in general $\delta^2 U > 0$ for small loads and it becomes PSD at certain value of load. That is, $\delta^2 U > 0$ for all possible $(u_1[x], w_1[x])$ when load is small, and $\delta^2 U < 0$ for some set of $(u_1[x], w_1[x])$ when load is large. Hence at a certain critical load $\delta^2 U$ becomes zero for the first time corresponding to a certain $(u_1^*[x], w_1^*[x])$ and $\delta^2 U$ 0 for all other $(u_1[x], w_1[x])$. Hence at this critical load $\delta^2 U$ is a local minima (with zero value) as we span the infinite set of variations $(u_1[x], w_1[x])$. Thus,

TREFFTZ'S CRITERION $\rightarrow$ $\boxed{\delta(\delta^2 U) = 0 \quad \text{for } u_1 = u_1^*[x],\ w_1 = w_1^*[x]}$ $\rightarrow$ gives STABILITY EQNS which give buckling load and modes (u_1^*, w_1^*)

A more rigorous proof of Trefftz's criterion is as follows.

Consider $\alpha = \dfrac{\delta^2 U}{T}$, let $\delta^2 U = R$

$T[u_1, w_1]$ = positive definite functional, e.g, $T = \int_0^L u_1^2\, dx$

Let, $\left.\begin{aligned} u_1[x] &= u_1^*[x] + \varepsilon_u \alpha[x] \\ w_1[x] &= w_1^*[x] + \varepsilon_w \beta(x] \end{aligned}\right\}$ α, β arbitrary, ie, we are doing variation of variation.

$$\alpha = \alpha^* + \Delta\alpha = \frac{R[u_1^* + \varepsilon_u \alpha, w_1^* + \varepsilon_w \beta]}{T[u_1^* + \varepsilon_u \alpha, w_1^* + \varepsilon_w \beta]}$$

From p. 18, 19, note that R is quadratic in $\zeta, \zeta', \eta, \eta', \eta''$, ie quadratic in $u_1, u_1', w_1, w_1', w_1''$ (this means it contains terms like u_1^2, $u_1 u_1'$, etc — 2^{nd} order terms in variations & their derivatives). Also choose T as quadratic.

$$\alpha^* + \Delta\alpha = \frac{R[u_1^*, w_1^*] + \varepsilon_u \delta_u R + \varepsilon_w \delta_w R + O(\varepsilon^2)}{T[u_1^*, w_1^*] + \varepsilon_u \delta_u T + \varepsilon_w \delta_w T + O(\varepsilon^2)}$$

where $\delta_u R \overset{\& \delta_u T}{=}$ terms that a linear in $\alpha[x]$, $\delta_w R \overset{\& \delta_u T}{=}$ terms linear in $\beta[x]$, and $O(\varepsilon^2)$ are $\varepsilon_u^2, \varepsilon_w^2, \varepsilon_u \varepsilon_w$ terms.

$$\alpha^* + \Delta\alpha = \left[\frac{R}{T}\Big|_{(u_1^*, w_1^*)} + \varepsilon_u \frac{\delta_u R}{T[u_1^*, w_1^*]} + \varepsilon_w \frac{\delta_w R}{T[u_1^*, w_1^*]} + O(\varepsilon^2) \right] *$$
$$\left[1 + \varepsilon_u \frac{\delta_u T}{T[u_1^*, w_1^*]} + \varepsilon_w \frac{\delta_w T}{T[u_1^*, w_1^*]} + O(\varepsilon^2) \right]^{-1}$$

Note that $\delta_u R, \delta_w R, \delta_u T, \delta_w T$ also contain terms linear in u_1^*, w_1^*

$$\text{eg: } u_1 w_1' = u_1^* w_1^{*\prime} + \underbrace{\varepsilon_u \alpha w_1^{*\prime}}_{\substack{\text{part of} \\ \delta_u R}} + \underbrace{\varepsilon_w \beta u_1^*}_{\substack{\text{part of} \\ \delta_w R}} + \underbrace{\varepsilon_u \varepsilon_w \alpha \beta}_{O(\varepsilon^2)}$$

Let $T[u_1^*, w_1^*] = T^*$, $R[u_1^*, w_1^*] = R^*$

$$\alpha^* + \Delta\alpha = \frac{R}{T}\bigg|_{(u_1^*, w_1^*)} + \frac{\varepsilon_u \delta_u R + \varepsilon_w \delta_w R}{T^*} - \left(\frac{\varepsilon_u \delta_u T + \varepsilon_w \delta_w T}{T^*}\right)\frac{R^*}{T^*}$$

For α^* to be a min/max value, $\Delta\alpha = 0$. Now let $\varepsilon_u = \varepsilon_w = \varepsilon$ (we could have done it in beginning w/o loss of generality since α, β are arbitrary).

$\Rightarrow \delta R - \alpha^* \delta T = 0$ for α^* min/max.

($\delta R = \delta_u R + \delta_w R$ = all linear terms in α, β, similarly for δT).

First time $\delta^2 U = 0$ for some $u_1, w_1 = u_1^*, w_1^*$ (as we increase load) gives critical buckling load at which $\alpha = \left(\frac{\delta^2 U}{T}\right)_{[u^*, w^*]} = \alpha^* = 0$

So when this happens,

$$\boxed{\delta R = \delta(\delta^2 U) = 0, \quad u_1 = u_1^*, \; w_1 = w_1^*}$$

(buckling modes).

QED

<u>Example</u>: Apply Trefftz's criterion to Beam-Column to get its Stability Equations

(1) Find $\delta^2 U$ using equation (3) developed on p. 19.

$$\delta^2 U = \int_0^L \left(EA\, u_1'^2 + EA\left(u_e' + \tfrac{3}{2} w_e'^2\right) w_1'^2 + EI\, w_1''^2 + 2EA\, w_e' u_1' w_1'\right) dx$$

$$\frac{\partial^2 F}{\partial u_e'^2} = EA, \quad \frac{\partial^2 F}{\partial w_e''^2} = EI$$

$$\frac{\partial^2 F}{\partial w_e'^2} = EA\left(u_e' + \tfrac{3}{2} w_e'^2\right)$$

$$\frac{\partial^2 F}{\partial w_e' \partial u_e'} = EA\, w_e'$$

$$\delta(\delta^2 U) = 2\int_0^L \Big[EA\, u_1' \delta u_1' + EA\left(u_e' + \tfrac{3}{2} w_e'^2\right) w_1' \delta w_1' + EI\, w_1'' \delta w_1'' + EA\, w_e'\left(u_1' \delta w_1' + w_1' \delta u_1'\right)\Big] dx$$

$$\Rightarrow 0 = \int_0^L \left\{ \left(EA\left[u_1' + w_e' w_1'\right]\right)' \delta u_1 + \left(\left[EI\, w_1''\right]'' - \left[EA\left(\frac{P_e}{EA} w_1' + w_e' u_1' + w_e'^2 w_1'\right)\right]'\right) \delta w_1 \right\} dx$$

$$+ EA\left(u_1' + w_e' w_1'\right)\delta u_1\Big|_0^L - \left(EI\, w_1''\right)' \delta w_1\Big|_0^L + EI\, w_1'' \delta w_1'\Big|_0^L + EA\left(\frac{P_e}{EA} w_1' + w_e' u_1' + w_e'^2 w_1'\right)\delta w_1\Big|_0^L$$

Let $P_1 \triangleq EA(u_1' + w_e' w_1')$ → this definition comes naturally from
$P = P_e + P_1$, put $u = u_e + u_1$, $w = w_e + w_1$,
P_e contains term of $(u_e)^0, (w_e)^0$,
P_1 " " " $(u_e)^1, (w_e)^1$

$$\Rightarrow \left.\begin{array}{l} P_1' = 0 \\ (EI w_1'')'' - (P_e w_1' + P_1 w_e')' = 0 \end{array}\right\}$$

Stability eqns in u_1, w_1 for equilibrium solution w_e, u_e. Soln w_1, u_1 are mode shapes of buckling.

BC's at $X = 0, L$,

$P_1 = 0$ or $u_1 = \bar{u}_1 = 0$

$P_e w_1' + P_1 w_e' - (EI w_1'')' = 0$ or $w_1 = \bar{w}_1 = 0$

$\underbrace{EI w_1''}_{= -M_1} = 0$ or $w_1' = \bar{w}_1' = 0$

(only P loading)

Observation: For columns, for trivial equilibrium solution $w_e[x] = 0$ you get (after inserting $P_1[x] = 0$, which is always true irrespective of $w_e[x]$, since $P_1' = 0$ and $P_1 = 0$ on at least one boundary where load applied):

$$\left\{\begin{array}{ll} (EI w_1'')'' - (P_e w_1')' = 0 & \\ P_e w_1' - (EI w_1'')' = 0 \text{ or } w_1 = 0 & \text{at } x = 0, L \\ EI w_1'' = 0 \text{ or } w_1' = 0 & \text{at } x = 0, L \end{array}\right.$$

→ Identical to equilibrium eqns and bc's for column (not beam-col) with only compressive load P. So solution of w_1 same as w_e

(2) Find $\delta^2 U$ by first principles (ie expand Taylor series and collect quadratic terms in u_1, w_1): as on p.17 (iii) and collect quadratic terms in u_1, w_1 to second order terms.

Only U_i contributes to second order terms.

$$\frac{EA}{2}\left(u' + \frac{w'^2}{2}\right)^2 \longrightarrow \left(u_1'^2 + \frac{6}{4}(w_1'^2 w_e'^2) + 2u_1' w_1' w_e' + u_e' w_1'^2\right) * \frac{EA}{2} = (A)$$

$$\frac{EI}{2}(w'')^2 \longrightarrow w_1''^2 * \frac{EI}{2} = (B)$$

(arrows indicate 2^{nd} order terms produced).

$$\delta^2 U = \int_0^L 2(\text{A} + \text{B})\,dx$$ → same as what we obtained from 'formula' on p.19 (see p.22).

$\Rightarrow \delta(\delta^2 U) = 0$ → will get same stability eqns and b.c's as on p.24.

NOTE:

(i) Can also use EULER-LAGRANGE EQNS (p15-16) with replacements $F \to$ integrand of $\delta^2 U = F[u_1, w_1]$, $(u_e, w_e) \to (u_1, w_1)$ and get Stability Eqns and BC's directly. Result will be same as on p.24.

(ii) Solution of stability eqns give buckling load (P_{cr}) & $(u_1, w_1) \to$ buckling mode shapes.

(iii) For column, stability equations for trivial equilibrium soln $w_e[x] = 0$ are same as the equilibrium equations, and so are the BC's. This is unique to column only. We also get $u_1[x] = 0$ from $P_1(x) = 0$, $u_1[0] = 0$ and $P_1[x] = EA(u_1' + \underbrace{w_e'}_{0} w_1')$, ie,

$$u_1 = \int \frac{\overset{\nearrow 0}{P_1}}{EA}\,dx + \underbrace{c_1}_{=0} \left(\because u_1[0] = 0\right).$$

We had $P_e' = 0$ and $P_e[L] = -\bar{P} \Rightarrow P_e[x] = -\bar{P}$

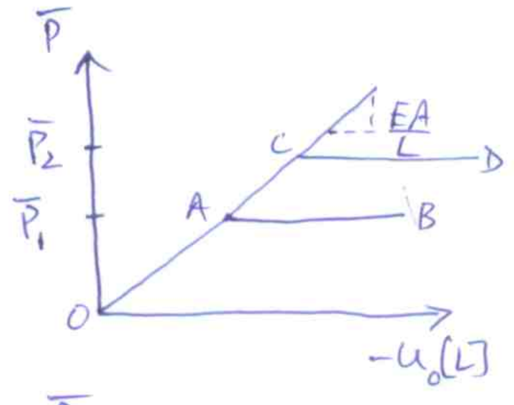
$$P_e[x] = EA\left[u_e' + \frac{\overset{\nearrow 0}{w_e'^2}}{2}\right] \Rightarrow u_e[x] = -\frac{\bar{P}}{EA}x \to \text{pre-buckling } (w_e[x] = 0).$$

For simply-supported case we had first buckling mode,

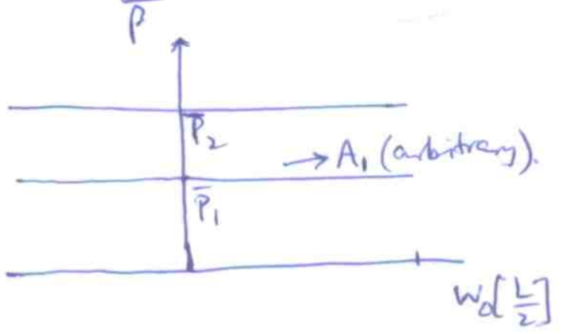
$$w_e[x] = A_1 \sin\frac{\pi x}{L} = w_1[x] \quad (\because \text{equil \& stability eqns and BC's are same}).$$

$$\Rightarrow u_e[x] = -\frac{\bar{P}_1 x}{EA} - \int_0^x \frac{A_1^2}{2}\cos^2\left(\frac{\pi x}{L}\right) * \frac{\pi^2}{L^2}\,dx = -\frac{\bar{P}_1 x}{EA} - \frac{A_1^2}{2}\frac{\pi^2}{L^2}\frac{\left(\frac{L}{2\pi}\sin\frac{2\pi x}{L} + x\right)}{2}$$

$$u_e[L] = -\frac{\bar{P}_1 L}{EA} - \frac{1}{4}\frac{\pi^2}{L}A_1^2$$



OC, OA → pre-buckling
CD, AB → post-buckling.



(iv) Since $\delta^2 U$ is quadratic in u_1, w_1, stability equations $(\delta(\delta^2 U) = 0)$ will always be linear in (u_1, w_1) unlike equilibrium equations which are non-linear in general (only linear in (u_e, w_e) for column).

(v) For stability at the bifurcation point we need to examine sign of $\delta^3 U$, $\delta^4 U$, etc. Not of importance for buckling considerations, but it is important if we want to know the initial-postbuckling behavior.

Reduced (Second) Order Differential Equation.

$$(EIw'')'' - (Pw')' = 0 \qquad (P' = 0,\ P = \text{const} = -\bar{P})$$

$\Rightarrow \ EIw'' + \bar{P}w = C_1 x + C_2$ (after $\iint$ double integration)

Measure displacement w from left end, i.e.,

$w(0) = 0$.

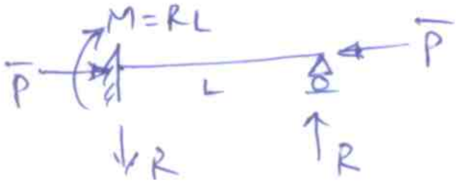
$\Rightarrow \ C_2 = EIw''(0) = M_0$

$C_1 = \left[(EIw'')' + \bar{P}w' \right]_{x=0} = -R_0$

$\Rightarrow \ EIw'' + \bar{P}w = -R_0 x + M_0$

(eg) SS-SS: R_0, M_0 are zero (ie, no load applied at left end).

CL-SS : $R_0 = R$, $M_0 = RL$.

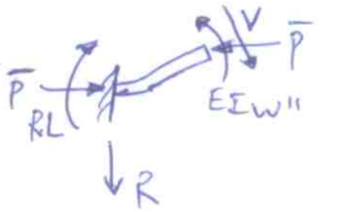


You also get these by first principles by equating moments of FBD in deformed configuration.

(eg) SS-SS

$EIw'' + \bar{P}w = 0$.

SS-CL



$EIw'' + \bar{P}w = -Rx + RL$.

Convention

$M = EIw''$

$V = EIw'''$

When using second order DE, you must satisfy (invoke) only the kinematic BC's (ie on w, w'). Invoking the shear & moment BC's (ie on w'', w''') will lead to redundant equations which will not help in obtaining the solution (ie constants of integration).

Loading ecentricity Imperfection.

$$w^{IV} + k^2 w'' = 0, \quad k^2 = P/EI$$

BC's:

$$w(0) = w(L) = 0$$

Inhomogenous BC's so we can solve for $A_1 \cdots A_4$ constants. (ie not an eigenvalue problem in this case).

$$\begin{cases} EI\,w''(0) = Pe \\ EI\,w''(L) = Pe \end{cases}$$

$$w(0) = A_2 + A_4 = 0$$

$$w(L) = A_1 \sin kL + A_2 \cos kL + A_3 L + A_4 = 0$$

$$w''(0) = -A_2 k^2 = k^2 e$$

$$w''(L) = -A_1 k^2 \sin kL - A_2 k^2 \cos kL = k^2 e$$

$$\Rightarrow A_2 = -e, \quad A_4 = e, \quad A_1 = -e\left(\frac{1-\cos kL}{\sin kL}\right) = -e\tan\left(\frac{kL}{2}\right)$$

$$A_3 = -\frac{1}{L}\left[e\left(1-\cos kL + \cos kL - 1\right)\right] = 0.$$

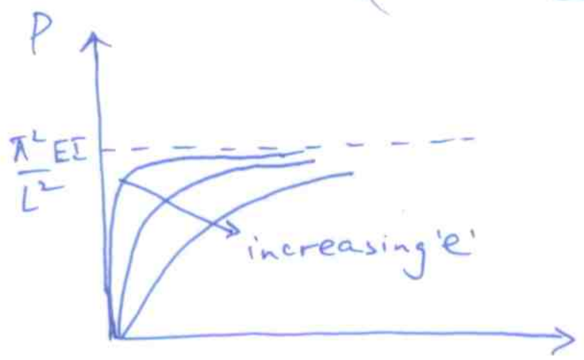
$$w(x) = e\left(-\tan\frac{kL}{2}\sin kx - \cos kx + 1\right)$$

$$w \to \infty \text{ as } \tan\frac{kL}{2} \to \infty, \text{ ie } kL = (2n+1)\pi$$

$$\text{So } (P_{cr})_1 = \frac{\pi^2 EI}{L^2}$$

$$w(L/2) = -\delta = e\left(-\tan\frac{kL}{2}\sin\frac{kL}{2} - \cos\frac{kL}{2} + 1\right) = e\left(\frac{-1+\cos\frac{kL}{2}}{\cos\frac{kL}{2}}\right)$$

$$= e\left(1 - \sec\frac{kL}{2}\right)$$

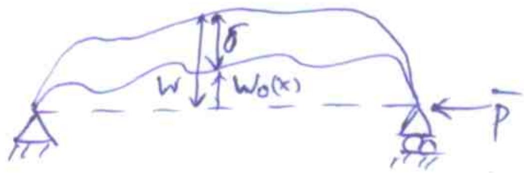


As $e \to 0$, $w(x) \to 0$ for $\frac{kL}{2} < \frac{\pi}{2}$

When $\frac{kL}{2} = \frac{\pi}{2}$, $w(x) \to \infty$ for infinitesimal 'e'.

※ So we get behavior of perfect column as limiting case $e \to 0$.

Geometric Imperfections



$w_0(x) =$ imperfection

$= \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$ (can always represent as a Fourier series)

δ = displacement from unstressed (undeformed) state

The bending moment depends only on change in curvature, i.e, $(w-w_0)''$, ie, it is $EI(w-w_0)''$. The moment due to $\bar{P}$ depends on w, ie Pw. So we have,

$$EI(w-w_0)'' + \bar{P}w = 0.$$

$$\Rightarrow EIw'' + \bar{P}w = -EI\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 a_n \sin\frac{n\pi x}{L}$$

Let $w = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L}$

$$\Rightarrow \sum_{m=1}^{\infty}\left[-EI\left(\frac{m\pi}{L}\right)^2 + \bar{P}\right] b_m \sin\frac{m\pi x}{L} = -\sum_{n=1}^{\infty} EI\left(\frac{n\pi}{L}\right)^2 a_n \sin\frac{n\pi x}{L}$$

Multiply both sides by $\sin\frac{j\pi x}{L}$, integrate from $0-L$, note orthogonality of sine function — or just equate terms on either side,

$$b_j = -\frac{P_j}{\bar{P}-P_j}\,a_j \quad , \quad P_j = \left(\frac{j\pi}{L}\right)^2 EI$$

$$w = \sum_{m=1}^{\infty}\frac{P_m}{P_m-\bar{P}}\,a_m \sin\frac{m\pi x}{L} \quad , \quad w - w_0 = \sum_{m=1}^{\infty}\frac{\bar{P}}{P_m-\bar{P}}\,a_m \sin\frac{m\pi x}{L} = \delta$$

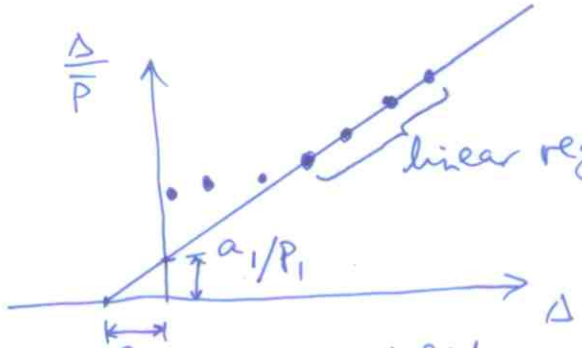
When $\bar{P} \to P_1$, ie $\bar{P} = \frac{\pi^2}{L^2}EI$, first term dominates, ie, it buckles in first mode provided the imperfection also contains a component of the first mode, ie,

$a_1 \neq 0$. In a test, if you measure $\delta(L/2) = \Delta$. and $\bar{P}$, then if $a_1 \neq 0$ and other $a_m = 0$ (ie, first term dominates $\equiv$ we are near first critical load $(P_{cr})_1 = \frac{\pi^2}{L^2} EI$), we have

$$\Delta = \delta(L/2) = \frac{\bar{P}}{P_1 - \bar{P}} \, a_1 \, \overbrace{\cancel{\sin\left(\frac{\pi L/2}{L}\right)}}^{=1.}$$

$$\Rightarrow \quad P_1 \left(\frac{\Delta}{\bar{P}}\right) - a_1 = \Delta$$

$\rightarrow$ ie $\frac{\Delta}{\bar{P}}$ v/s Δ plot is linear in vicinity of $(P_{cr})_1 = P_1$



Southwell Plot

linear region when P_1 value is approached from below. (ie $\bar{P} < P_1$, $\bar{P} \rightarrow P_1$)

So ratio of intercepts gives P_1, the buckling load.

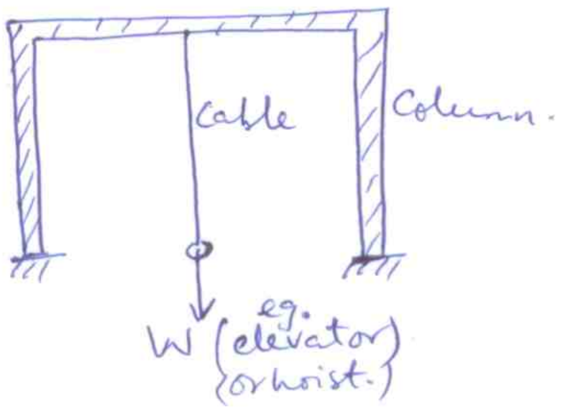
This experimental determination of $P_1 = (P_{cr})_1$ is due to Southwell.

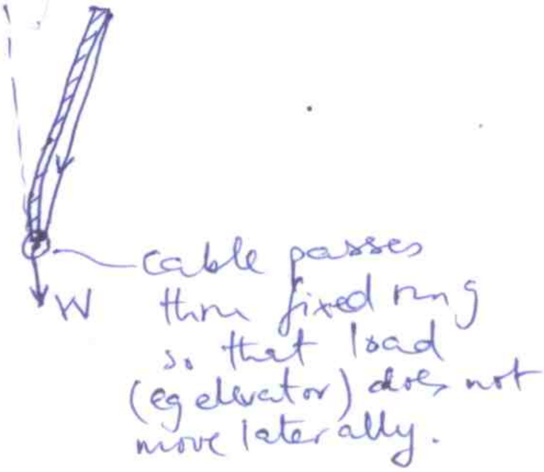
Other Examples

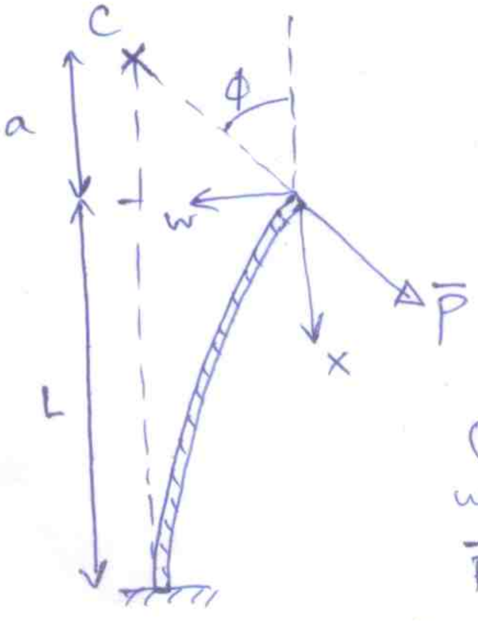
(1) Tilt Buckling.

Applied load passes thru a fixed point

Example:







Do by reduced (2^{nd}) order ODE.

Fix origin at free end, w measured wrt origin.

$$EI w'' + \bar{P}\cos\phi \, w + \bar{P}\sin\phi \, x = 0.$$

(obtained by just Σ moments, or using reduced order ODE with $\bar{P} \to \bar{P}\cos\phi$, $\bar{R}_0 = \bar{P}\sin\phi$).

Assume ϕ = small, almost constant (since $a \gg w(0)$)

valid if rope/cable is long from tip to ring in above Fig.

$$w'' + k^2 w = -k^2 \phi x$$

Soln, $w = A_1 \sin kx + A_2 \cos kx - \phi x$

* Note: ϕ = small = almost const means that it is treated as constant in the ODE although it depends on $w(0)$. Thus P is not a follower force (non-conservative) so we can solve by equilibrium method.

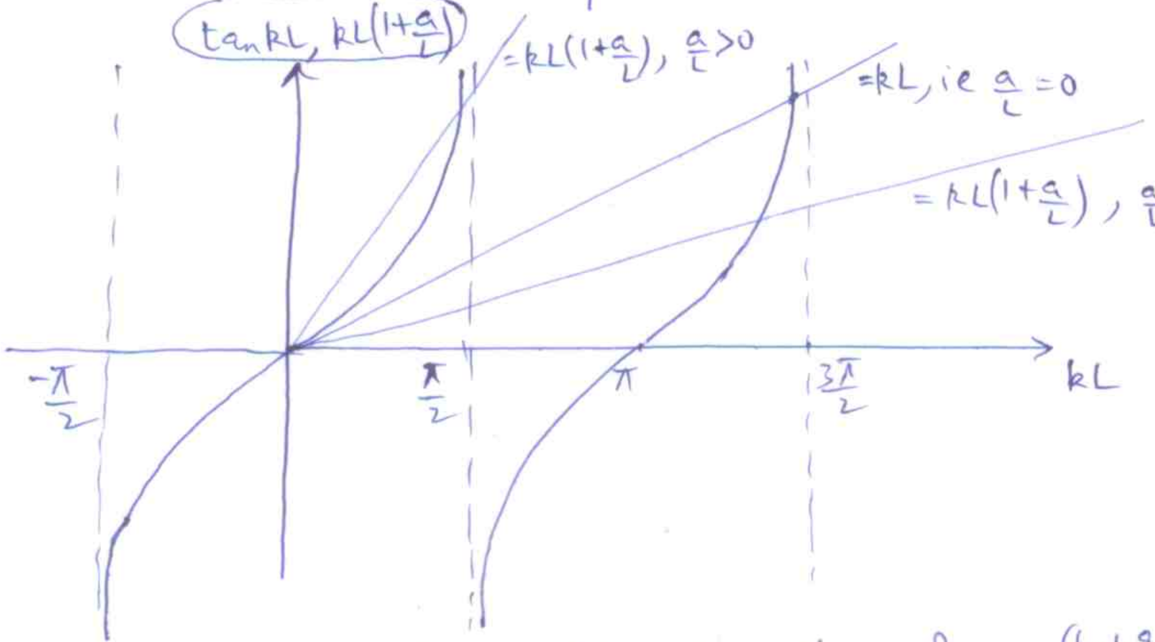
BC's: $w(0) = w'(L) = 0$, $w(L) = a\tan\phi \cong a\phi$

$A_2 = 0$, $A_1 k \cos kL - \phi = 0$

$$W[x] = \phi\left(\frac{\sin kx}{k\cos kL} - x\right)$$

$$\phi\left(\frac{\sin kL}{k\cos kL} - L - a\right) = 0 \longrightarrow \underline{CE}$$

Nontrivial $w \Rightarrow \phi \neq 0 \Rightarrow \tan kL = k(L+a) = kL\left(1+\frac{a}{L}\right)$



For $\frac{a}{L}=0$, $kL = 4.493$ is root, $P_{cr} = (4.493)^2\frac{EI}{L^2}$ (no meaning since $a >> w(0)$ assumed).

$\frac{a}{L} = -1$, $kL = \pi$ " , $P_{cr} = \pi^2\frac{EI}{L^2}$

$\frac{a}{L} \to \pm\infty$, $kL \to \pm\frac{\pi}{2}$ " , $P_{cr} = \frac{\pi^2}{4}\frac{EI}{L^2}$ ($\equiv$ clamped-free case).

only these two are valid cases.

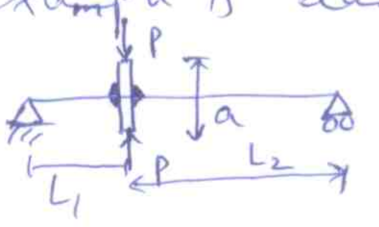
\# Using 4th order ODE and its solution, $W(x) = A_1\sin kx + A_2\cos kx + A_3x + A_4$, with BC's (here origin at fixed end)

$W(0) = W'(0) = 0$, $EIW''[L] = \cancel{\overline{M}_L} \to 0$,

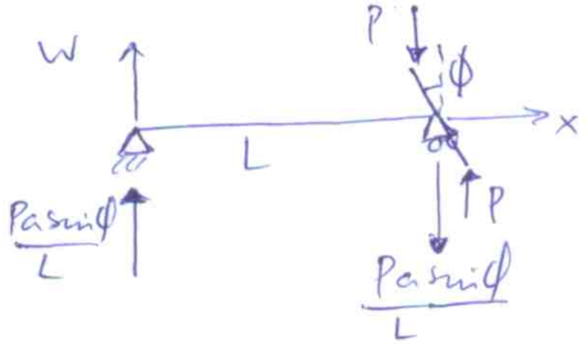
$-EIW'''[L] - \overline{P}\underbrace{\cos\phi}_{\simeq 1} W'[L] = \overline{R}_L = -\overline{P}\underbrace{\sin\phi}_{\simeq \tan\phi = \left(-\frac{W(L)}{a}\right)}$.

we get the same CE as above, ie, same solution.

Another example is beam with rigid transverse strut welded on beam.



For case when $l_1 = L$, $l_2 = 0$, equilibrium gives



$$EI\,W'' = \frac{Pa\sin\phi}{L}\,x = \frac{Pa\phi}{L}\,x$$

$$W = \frac{Pa\phi}{EIL}\left[\frac{x^3}{6} + C_1 x + C_2\right]$$

BC's: $W(0) = 0 \rightarrow C_2 = 0$, $W(L) = 0$, $C_1 = -\frac{L^2}{6}$.

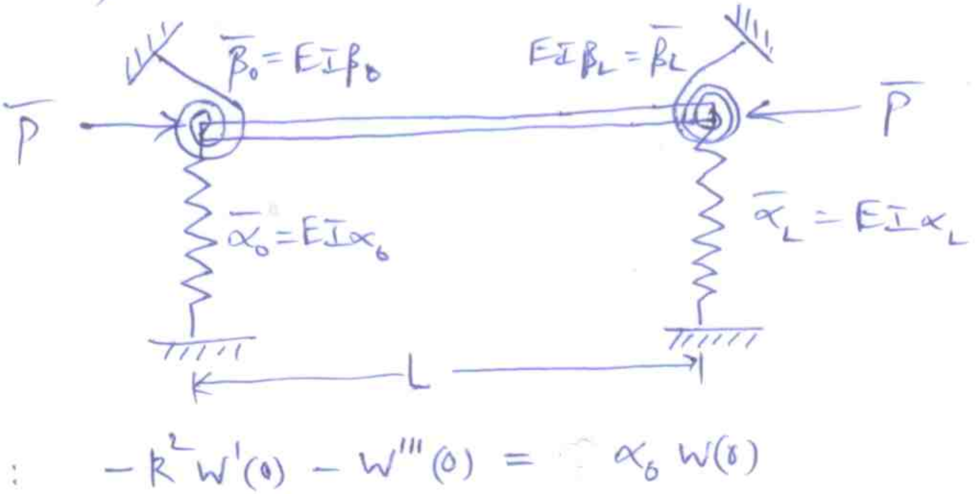
$$W = \frac{Pa\phi}{6EIL}\left(x^3 - L^2 x\right)$$

BC: $W'(L) = \tan\phi \simeq \phi \Rightarrow \phi = \frac{Pa\phi}{6EIL}\left(3L^2 - L^2\right)$

non-trivial $\phi \Rightarrow P_{cr} = \frac{3EI}{aL}$

(2) <u>Elastically supported Columns.</u>

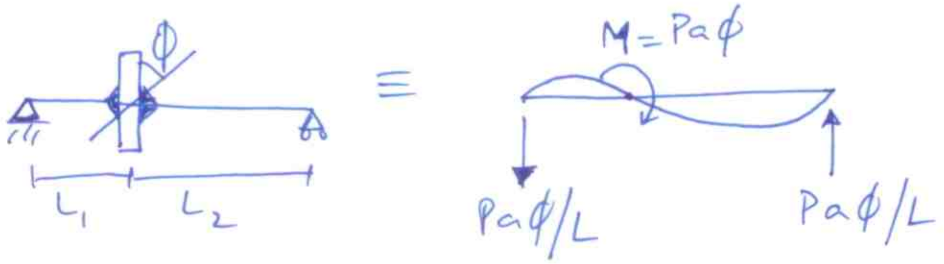
Columns connected to beams (etc) at ends which provide rotational restraint. Rotational spring constant (equivalent to restraint provided) easily computed from structural configuration (moment distribution or other methods).



BC's:
$$-k^2 W'(0) - W'''(0) = \alpha_0 W(0)$$
$$W''(0) = \beta_0 W'(0)$$
$$-k^2 W'(L) - W'''(L) = -\alpha_L W(L)$$
$$W''(L) = -\beta_L W'(L)$$

Sol^n: $W = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4$

Generalized case of tilt buckling example when disk welded inside the span.



$$EIw'' + \frac{Pa\phi}{L}x - Pa\phi\, h(x-L_1) = 0$$

unit step function.

$$w' + \frac{k^2 a\phi}{L}\frac{x^2}{2} - k^2 a\phi (x-L_1)\, h(x-L_1) = C_1$$

$$w + \frac{k^2 a\phi}{L}\frac{x^3}{6} - k^2 a\phi \frac{(x-L_1)^2}{2} h(x-L_1) = C_1 x + C_2$$

BC's : $w(0) = w(L) = 0$, auxillary condition $w'(L_1) = -\phi$

$$w(0) = C_2 = 0$$

$$w(L) = C_1 L + k^2 a\phi\left(\frac{L_2^2}{2} - \frac{L^2}{6}\right) = 0$$

$$w'(L_1) = k^2 a\phi\left(\frac{L}{6} - \frac{L_2^2}{2L}\right) - k^2 a\phi\left(\frac{L_1^2}{2L}\right) = -\phi$$

$(\because \phi \neq 0)$

$$\Rightarrow P_{cr} = -\frac{EI}{a}\,\frac{6L}{L^2 - 3L_2^2 - 3L_1^2}$$

$$= 3\frac{EI}{a}\,\frac{L_1 + L_2}{L_1^2 + L_2^2 - L_1 L_2}$$

$$\begin{bmatrix} \cancel{k^3}-\cancel{k^3} & \alpha_0 & k^2 & \alpha_0 \\ -\beta_0 k & -k^2 & -\beta_0 & 0 \\ (\cancel{k^3\cos u}-\cancel{k^3\cos u} - \alpha_L \sin u) & (-\cancel{k^3\sin u}+\cancel{k^3\sin u} - \alpha_L\cos u) & k^2-\alpha_L L & -\alpha_L \\ (-k^2\sin u+\beta_L k\cos u) & (-k^2\cos u-\beta_L k\sin u) & \beta_L & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0.$$

where $u = kL$.

This gives a 6th order transcedental eqn in u $(=kL)$, which you have to solve to get k_{cr}, hence P_{cr}.

Special cases:

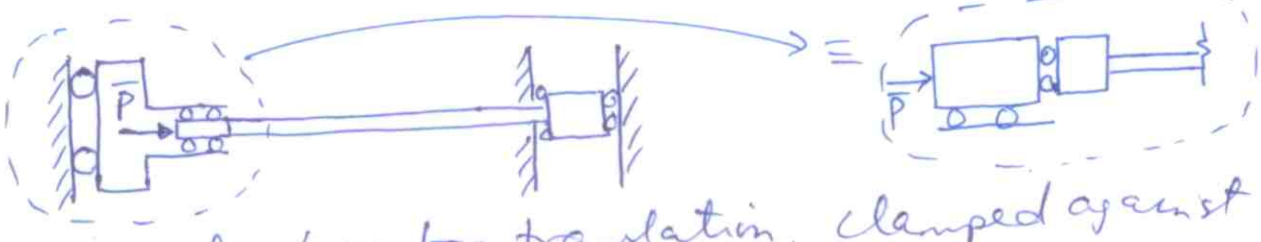
(i) $\alpha_0 = \alpha_L = \infty$, $\beta_0 = \beta_L = 0$ → pinned pinned col.

(ii) $\alpha_0 = \alpha_L = \beta_0 = \beta_L = \infty$ → clamped- clamped.

(iii) $\alpha_0 = \beta_0 = \infty$, $\alpha_L = \beta_L = 0$ → clamped- free.

(iv) $\alpha_0 = \beta_0 = \infty = \alpha_L$, $\beta_L = 0$ → clamped- pinned.

In cases (i)-(iv) we recover the familiar results

(v) 

Both ends free for translation, clamped against rotation → $\alpha_0 = \alpha_L = \varepsilon$, $\beta_0 = \beta_L = \infty$.

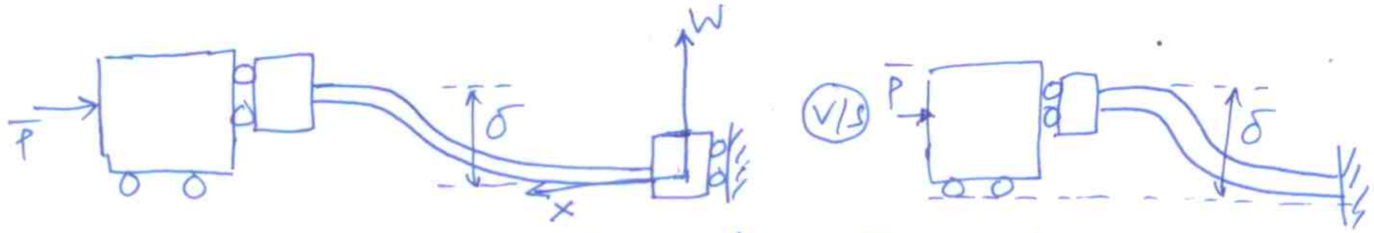
$$\det \begin{bmatrix} 0 & \varepsilon & k^2 & \varepsilon \\ -k & 0 & -1 & 0 \\ -\varepsilon\sin u & -\varepsilon\cos u & k^2-\varepsilon L & -\varepsilon \\ k\cos u & -k\sin u & 1 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \varepsilon\left[-k\left(-\varepsilon\cos u + k^3\sin u - k\sin u * \varepsilon L\right) - 1\left(\varepsilon k\sin^2 u + \varepsilon k\cos^2 u\right)\right] - \varepsilon\left[-\varepsilon(-k+k\cos u) + k^2(k^2\sin u)\right] = 0$$

(We used $\frac{1}{(\beta_0, \beta_L)} = 0$)

Let $\varepsilon \to 0$, $\Rightarrow k^4 \sin u = 0$, ie $kL = n\pi$. $\Rightarrow (\bar{P}_{cr})_1 = \frac{\pi^2}{L^2} EI$

If you take right end fixed (ie $\alpha_L = \infty$) we get same same result ($\because$ same characteristic equation). This is expected if we consider the bc's in the two cases:

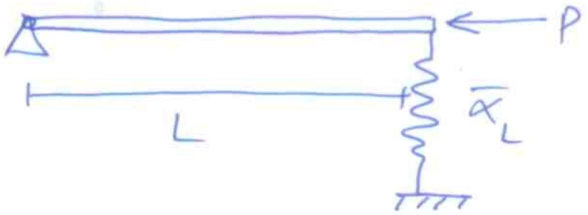


$w' = 0$, SF $= 0$ at both ends, δ = relative displ of ends, is same. So 2nd order method (which required only $w'\big|_{x=0,\, x=L} = 0$ and $w(L) = \delta$, W measured from right support as shown.

The general case has C.E. (see det on top of p. 33),

$$
\begin{aligned}
&\Big\{-(\alpha_0+\alpha_L)\frac{u^6}{L^6} + \Big[\beta_0\beta_L(\alpha_0+\alpha_L) + \alpha_0\alpha_L L\Big]\frac{u^4}{L^4} \\
&\quad + \alpha_0\alpha_L(\beta_0+\beta_L-\beta_0\beta_L L)\frac{u^2}{L^2}\Big\}\sin u \\
&+ \Big\{(\alpha_0+\alpha_L)(\beta_0+\beta_L)\frac{u^5}{L^5} - \alpha_0\alpha_L L(\beta_0+\beta_L)\frac{u^3}{L^3} - 2\alpha_0\alpha_L\beta_0\beta_L\frac{u}{L}\Big\}\cos u \\
&+ 2\alpha_0\alpha_L\beta_0\beta_L\frac{u}{L} = 0
\end{aligned}
$$

<u>Example</u>



$\alpha_L = \frac{\bar{\alpha}_L}{EI}$

Put $\alpha_0 = \infty$, $\beta_0 = \beta_L = 0$ in above 'general CE', divide by α_0.

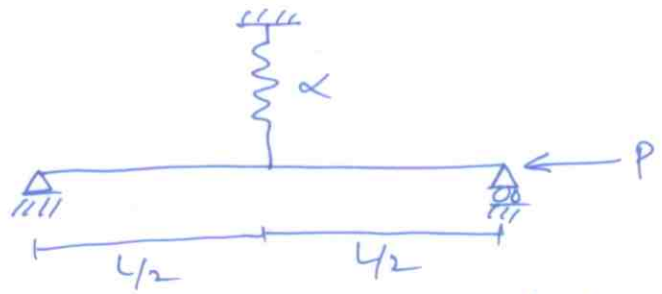
$$\left(-\frac{u^6}{L^6} + \alpha_L L \frac{u^4}{L^4}\right)\sin u = 0$$

$\Rightarrow \sin u = 0$ or $\alpha_L L - \frac{u^2}{L^2} = 0$.

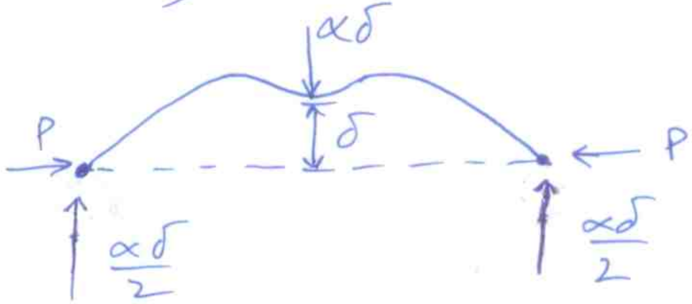
$\Longrightarrow k = \frac{n\pi}{L}$ or $k = \sqrt{\alpha_L L}$

For small α_L, $k = \sqrt{\alpha_L L}$ is the critical condition.
As α_L is increased, at $\alpha_L L = \frac{\pi^2}{L^2}$, both conditions are equivalent, beyond which $k = \frac{\pi}{L}$ is critical.
So no use in increasing α_L beyond $\frac{\pi^2}{L^3}$, ie
$\boxed{(\bar{\alpha}_L)_{critical} = \frac{\pi^2}{L^3} EI}$. Beyond this value of spring stiffness the column will buckle in first Euler mode $\left(\sin \frac{\pi x}{L}\right)$ so no use in making the spring any stiffer since P_{cr} is limited to $\frac{\pi^2 EI}{L^2}$, beyond this <u>critical spring stiffness</u>.

<u>Example</u>



As P increased, column will buckle in 1^{st} mode, ie,



$$EIw'' + Pw = \frac{\alpha\delta}{2} x, \quad 0 < x < \frac{L}{2}$$

$$w = A_1 \sin kx + A_2 \cos kx + \frac{\alpha\delta}{2P} x$$

BC's: $w(0) = 0$, $w\left(\frac{L}{2}\right) = \delta$, $w'\left(\frac{L}{2}\right) = 0$.

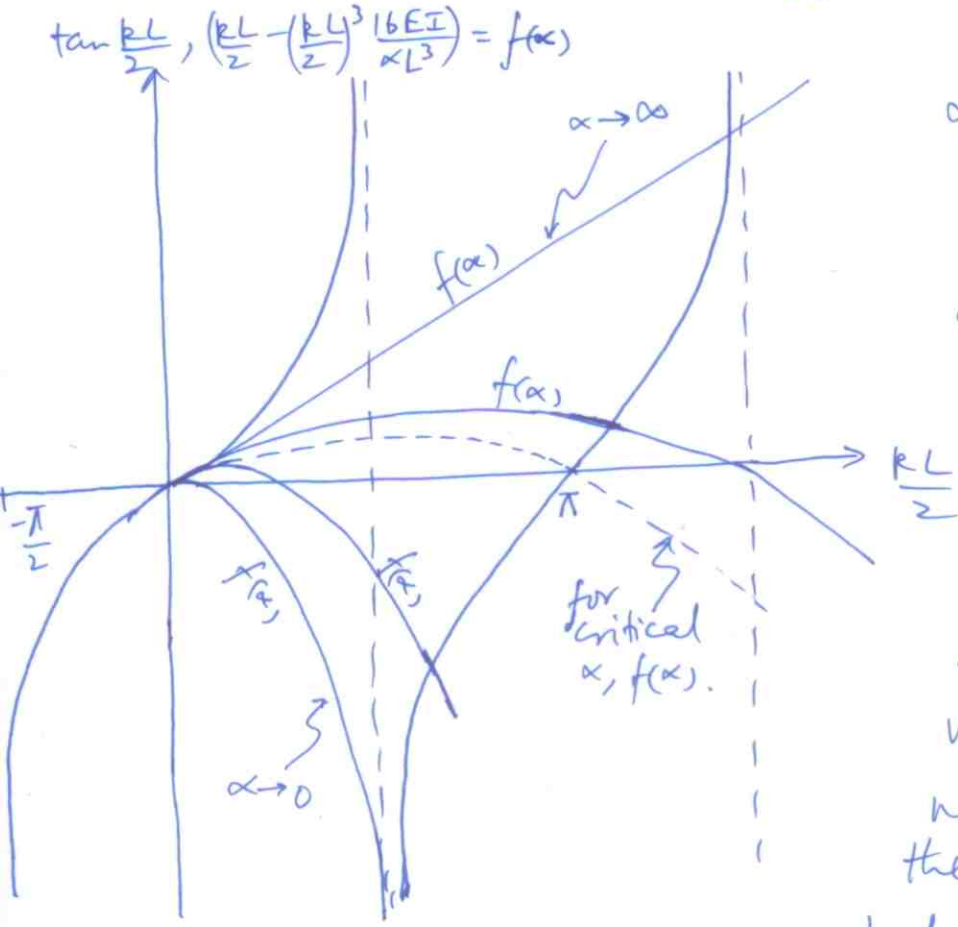
$$A_2 = 0$$

$$A_1 \sin \frac{kL}{2} + \frac{\alpha\delta L}{4P} = \delta$$

$$A_1 k \cos\frac{kL}{2} + \frac{\alpha\delta}{2P} = 0$$

$$\Rightarrow \frac{1}{k}\tan\frac{kL}{2} = \frac{(\alpha L/4P - 1)}{\alpha/2P}$$

$$\tan\frac{kL}{2} = \frac{kL}{2} - \frac{2k^3EI}{\alpha} = \frac{kL}{2} - \left(\frac{kL}{2}\right)^3\frac{16EI}{\alpha L^3}$$



$\alpha \to 0$, $\tan\frac{kL}{2} \to -\infty$,
$\frac{kL}{2} \to \pi/2$

$\alpha \to \infty$, $\frac{kL}{2} \to 4.493$

For 2nd buckling mode without spring, we have $kL = 2\pi$

So if you increase α, the value $\frac{kL}{2} = 4.493$ will never be achieved for the symmetric mode, since before that it will buckle in 2nd (antisymmetric) mode at $kL = 2\pi$.

Thus critical α is for $f(\alpha)$ intersecting $\tan\frac{kL}{2}$ at $kL = 2\pi$ (ie $\frac{kL}{2} = \pi$),

ie, $\left.\frac{kL}{2} - \left(\frac{kL}{2}\right)^3\frac{16EI}{\alpha L^3}\right|_{\frac{kL}{2}=\pi} = 0 = f(\alpha)\Big|_{\frac{kL}{2}=\pi}$ $\longrightarrow$ gives α_{cr}.

$$\Rightarrow \quad \alpha = \boxed{\frac{\pi^2\, 16EI}{L^3} = \alpha_{cr}}$$