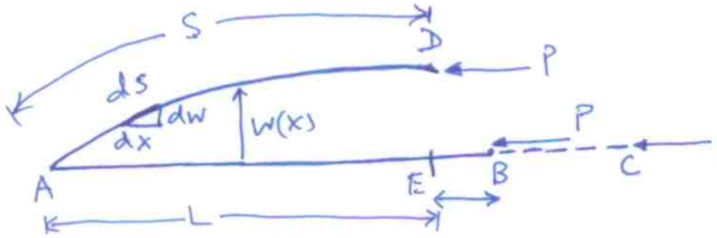
APPROXIMATE METHODS. — TOPIC III

(I) RAYLEIGH QUOTIENT, TIMOSHENKO QUOTIENT, RITZ METHOD.



C-B : Axial deformation

B-D : Bending deformation.

Assume axial deformation occurs and then bending deformation occurs with W(x) small, such that the stretching energy due to axial def. remains constant during bending.

$$U_{BC} = U_{axial} = \frac{1}{2} P.\overline{BC}$$

$$U_{Total} = U_{axial} + \Delta U \qquad (\Delta U = U_{BD})$$

$$\Delta U = \frac{1}{2}\int_0 EI(w'')^2 dx - P.\overline{EB}$$

$$ds^2 = dw^2 + dx^2 \Rightarrow \frac{ds}{dx} = 1 + \frac{(w')^2}{2} \quad (\text{for } w' \ll 1)$$

$$\int_0^S ds - \int_0^L dx = \frac{1}{2}\int_0^L (w')^2 dx = \overline{EB}.$$

Another way to see this is from U_i, T-II p. 13,

$$U_i = \frac{1}{2}\int_0^L \left[EA\left(u' + \frac{w'^2}{2}\right)^2 + EI(w'')^2\right] dx$$

$$EA\left(u' + \frac{w'^2}{2}\right) = P(x) \quad (\text{here } P(x) \text{ is +ve in tension}).$$

$$U_i = \frac{1}{2}\int_0^L \left(P\left(u' + \frac{w'^2}{2}\right) + EI(w'')^2\right) dx$$

$$= \frac{1}{2}\int_0^L \Big(P\frac{du}{dx}dx + \underbrace{Pw'^2 + EI(w'')^2\Big)dx}_{\Delta U}$$

$$= \frac{1}{2}P.\overline{BC} + \Delta U.$$

$\because$ stable Equilibrium $\Rightarrow U_{Total}$ is a minimum, so if $\Delta U < 0$ then it is not stable equilibrium since minimum potential energy is not yet achieved by system. So condition for buckling (onset of unstable equilibrium) is

$\Delta U = 0$

$$\Rightarrow \boxed{P_{cr} = \frac{\int_0^L EI (w'')^2\, dx}{\int_0^L (w')^2\, dx}} \equiv R[w] \longrightarrow \underline{\text{Rayleigh Quotient}}$$

(original condition proposed by Timoshenko).

For approximate solution, use

$$w_{approx} = w_a = \sum_{i=1}^{N} a_i w_i(x), \quad w_i = \text{trial functions - admissible.}$$

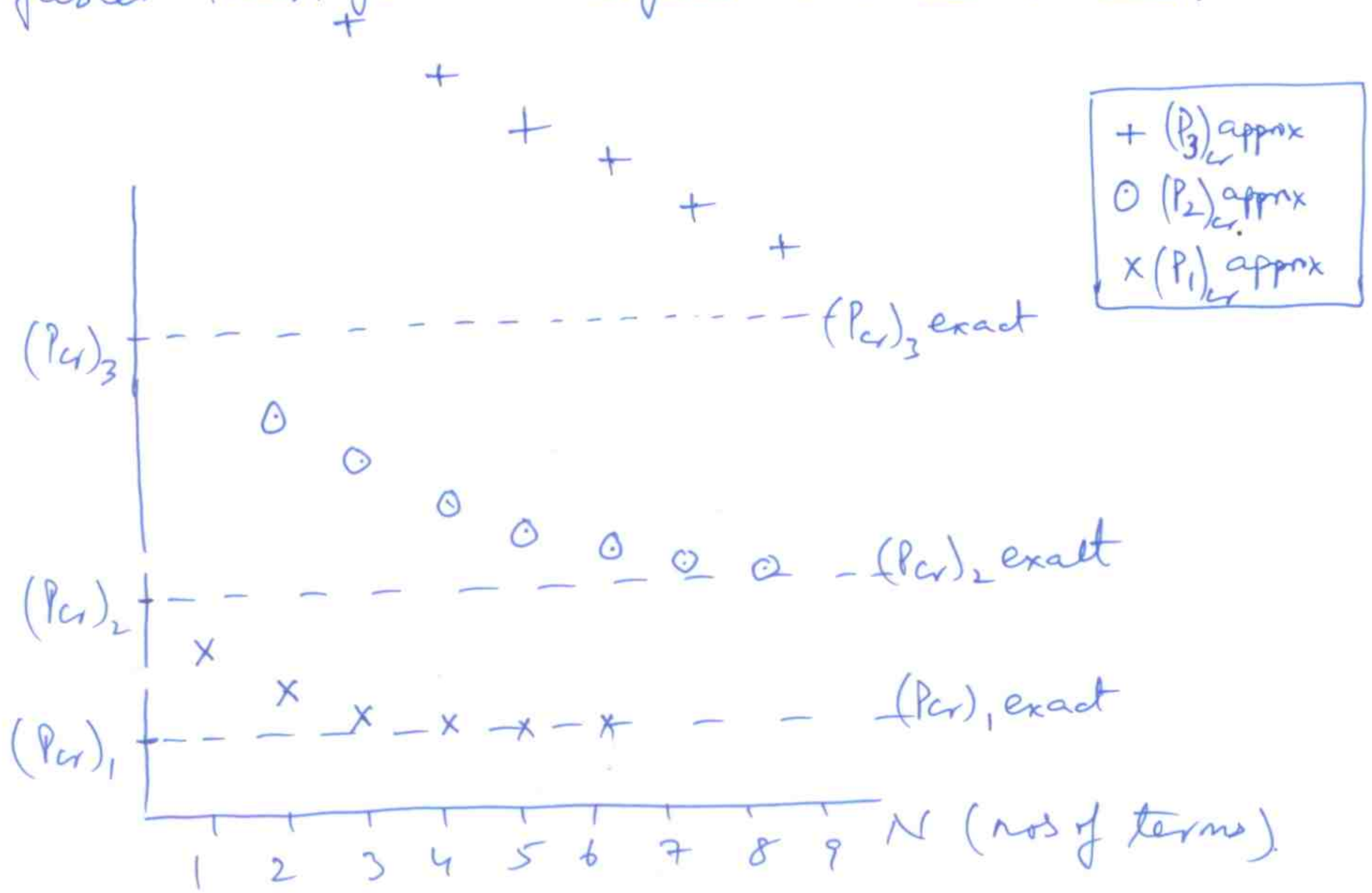
$$\Rightarrow P = \frac{\int_0^L EI \sum_{i,j=1}^{N}\sum a_i a_j w_i'' w_j''\, dx}{\int_0^L \sum_{i,j=1}^{N}\sum a_i a_j w_i' w_j'\, dx} = \frac{N(a_1, \dots a_N)}{D(a_1, \dots a_N)}$$

$$\frac{\partial P}{\partial a_i} = 0 \Rightarrow \left(\frac{\partial N}{\partial a_i}\frac{1}{D} - \frac{N}{D^2}\frac{\partial D}{\partial a_i}\right) = 0$$

$$\because D \neq 0, \quad \boxed{\frac{\partial N}{\partial a_i} - P_{cr}\frac{\partial D}{\partial a_i} = 0} \longrightarrow \text{RITZ EQUATIONS (RITZ METHOD)}$$

$i = 1, \dots N.$

Thus the *stationary value of P (*wrt coeffs a_i) is P_{cr}. You solve Ritz equations for a_i; ie by adjusting a_i suitably, the Rayleigh quotient evaluated for w_a equals the critical load, ie $R[w_a] = P_{cr}$. For an N-term approximation w_a, we get N values of P_{cr}. All these converge to the exact respective critical loads from above. Convergence for lower critical loads, ie $(P_1)_{cr}$, $(P_2)_{cr}$, etc is faster than for the higher critical loads.



<u>NOTE</u>: Following Guidelines for Choosing trial functions, $W_i(x)$

(1) More terms in series implies that (lower) critical loads are approximated well.

(2) All trial functions must be admissible, ie they must satisfy all kinematic (w, w') b.c.'s.

(3) Choose <u>complete</u> set of trial functions so that any function satisfying kinematic bc's (including the exact $W(x)$) can be approximated, as close as we desire, by choosing N large. So dont miss out any term in the series of trial functions. Thus, for example, $x(L-x)$, $x^2(L-x)$, ... $x^{N+1}(L-x)$ is a complete set satisfying zero-displacement bc's at $x=0, L$ (ie SS-SS)

(4) Trial functions could be chosen as the exact buckling modes of a simpler problem — for eg.,

for beam with EI varying and clamped-clamped, can choose $w_1 = \cos\left(\frac{2\pi}{L}x - 1\right)$.

(5) Trial functions can also be chosen to satisfy natural b.c's. This would yield faster convergence to exact solution. For eg., you can choose trial function for simply-supported beam as $w_i = x^{i+1}(L^3 - 2Lx^2 + x^3)$. For $i=0$, this is the solution for a S.S. beam with u.d.l. Note that trial function $x^{i+1}(L-x)^2$ satisfies all b.c's for clamped-clamped beam since for $i=0$ this is the soln. for CL-CL beam with udl. Equivalently you can also see that w_i of Note (4) above also satisfies all b.c's for CL-CL beam.

(6) <u>Polynomial trial functions</u>: Choose w_1 as polynomial of order m where m are number of kinematic b.c's to be satisfied. Solve for m constants by applying b.c's.

<u>Eg.</u> <u>CL-SS</u> : $W(0) = W'(0) = W(L) = 0$.

$$w_1 = \sum_{i=0}^{3} a_i x^i \quad \rightarrow m+1 = 4 = \text{no. of constants in polynomial}$$

$$w(0) = a_0 = 0, \quad W'(0) = a_1 = 0, \quad W(L) = a_2 L^2 + a_3 L^3 = 0$$

$$w_1 = a_3(-Lx^2 + x^3)$$

Additionally, if you want to satisfy $M(L) = EI\,w''(L) = 0$ choose $w_1 = \sum_{i=0}^{4} a_i x_i$

$$\left.\begin{aligned} 0 &= w(L) = a_2 L^2 + a_3 L^3 + a_4 L^4 \\ 0 &= w''(L) = 2a_2 + 6a_3 L + 12 a_4 L^2 \end{aligned}\right\} \Rightarrow \begin{aligned} a_2 &= \tfrac{3}{2} a_4 L^2 \\ a_3 &= -\tfrac{5}{2} a_4 L \end{aligned}$$

$$w_1 = a_4\left(\frac{3L^2}{2}x^2 - \frac{5}{2}Lx^3 + x^4\right) \longrightarrow \text{CL-SS}.$$

Then $\boxed{w_i = x^i w_1, \quad w_a = \sum_{i=1}^{N} a_i x^i w_1}$

CL-Free: $w_1 = x^2$ (satisfies kin. bc's) ; $w_1 = 3Lx^2 - x^3$ (satisfies kin + zero moment at $x=L$)

SS-SS: $w_1 = x(L-x) \rightarrow$ (satisfies kin. bc's) ; $w_1 = (L^3x - 2Lx^3 + x^4)$ → satisfies kin + natural bc's

CL-CL: $w_1 = x^2(L-x)^2 \rightarrow$ satisfies kinematic + natural bc's

Fourier trial functions.

Can choose buckling modes. They are superior to the polynomial functions that satisfy only kinematic b.c's.

SS-SS: $w_i = \sin\frac{i\pi x}{L} \rightarrow i^{th}$ buckling mode

CL-CL: $w_1 = \left(\cos\frac{2\pi x}{L} - 1\right)$ (Note: higher modes come from CE $\frac{kL}{2} = \tan\frac{kL}{2}$).

CL-Free: $w_i = \cos\left[\left(\frac{2i+1}{2}\right)\frac{\pi x}{L}\right] - 1 \rightarrow i^{th}$ buckling mode.

Some results for uniform columns ($EI = \text{const.}$) using one-term approximate solution (i.e. Rayleigh Quotient = P_{cr}) are as follows:

CL-Free: $w_1 = a_1 x^2(3L-x)$ gives $R[w_1] = P_{cr} = 2.5\,EI/L^2$ — only 1.32% higher than exact solution.
(satisfies kin + zero moment bc)

$w_1 = \left(\cos\frac{\pi x}{2L} - 1\right) \rightarrow R[w_1] = P_{cr} = \frac{\pi^2 EI}{4L^2} \rightarrow$ exact soln $\because w_1 =$ exact 1st buckling mode.

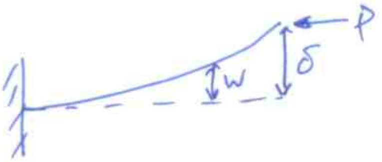
$w_1 = x^2 \rightarrow R(w_1) = P_{cr} = \frac{3EI}{L^2} \rightarrow$ 21.3% higher than exact sol, $\because$ only kin bc's satisfied.
(satisfies kin bc's)

SS-SS: $w_1 = (L-x)x \rightarrow R(w_1) = P_{cr} = \frac{12\,EI}{L^2} \rightarrow$ 21.3% higher than exact sol, $\because$ only kin bc's satisfied
satisfies kin bc's

$w_1 = (L^3x - 2Lx^3 + x^4) \rightarrow R(w_1) = P_{cr} = 9.88\,EI/L^2 \rightarrow$ 0.13% higher than exact sol.
(all bc's satisfied)

Better forms of Rayleigh Quotient, for use in Cantilever & Simply supported columns are given as follows:

(i) Cantilever Column (CL – Free).



$$M(x) = P(\delta - w)$$

$$EI(w'')^2 = \frac{(EIw'')^2}{EI} = \frac{M^2}{EI} = \frac{P^2(\delta-w)^2}{EI}$$

$$P = \int_0^L \frac{P^2(\delta-w)^2}{EI}dx \Big/ \int_0^L (w')^2 dx$$

$$\Rightarrow \boxed{P = \int_0^L (w')^2 dx \Big/ \int_0^L \frac{(\delta - w)^2}{EI} dx} \rightarrow \text{TIMOSHENKO COEFF for CL-Free}$$

Gives more accurate results than Rayleigh quotient since double-differentiation (which magnifies errors) avoided. Using $w_1 = x^2$ gives $P_{cr} = 2.5EI/L^2$ which is same as what we had using w_1 which satisfied kint zero-moment bc's in the Rayleigh Quotient. Using $w_1 = x^2(3L-x)$ in Timoshenko coeff gives $P_{cr} = 2.47\frac{EI}{L^2}$ which is even close to exact sol than we got from Rayleigh coeff using same trial function.

(ii) Simply supported Column (SS – SS).

$$EIw'' + Pw = 0 \Rightarrow w'' = -Pw/EI$$

$$P = \int_0^L P^2w^2/EI\, dx \Big/ \int_0^L (w')^2 dx$$

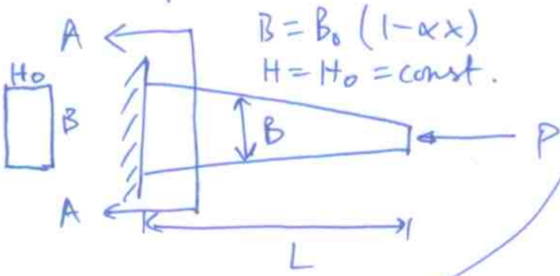
$$\boxed{P = \int_0^L (w')^2 dx \Big/ \int_0^L (w^2/EI)\, dx} \rightarrow \text{TIMOSHENKO COEFF for SS-SS.}$$

Elastically supported Columns.

$$\boxed{R(w) = \frac{\int_0^L EI(w'')^2 dx + \bar{\alpha}_0 w^2(0) + \bar{\alpha}_L w^2(L) + \bar{\beta}_0 (w'(0))^2 + \bar{\beta}_L (w'(L))^2}{\int_0^L w'^2 dx}}$$

↳ we merely added the additional energy arising due to elastic supports in the bending strain energy (ie due to $w(x)$).

Example



Plan view → bending (W) out-of-plane of paper.

$\Rightarrow EI = EI_0(1-\alpha x)$, $0 \le \alpha \le 1$

$$\left\{\begin{array}{l} w_1 = 3Lx^2 - x^3 \\ w_2 = x(3Lx^2 - x^3) \end{array}\right\} \text{ or } \left\{\begin{array}{l} w_1 = x^2 \\ w_2 = x^3 \end{array}\right\}$$

will give better results ∵ moment zero at free end is also satisfied.

Work with these in the following.

$w_1' = 6Lx - 3x^2$, $w_1'' = 6L - 6x$, $w_2' = 3Lx^2 - x^3 + 6Lx^2 - 3x^3 = 9Lx^2 - 4x^3$

$w_2'' = 18Lx - 12x^2$

$$N = \int_0^L EI\left[(w_1'')^2 a_1^2 + 2w_1'' w_2'' a_1 a_2 + (w_2'')^2 a_2^2\right] dx$$

$$D = \int_0^L \left(w_1'^2 a_1^2 + 2w_1' w_2' a_1 a_2 + w_2'^2 a_2^2\right) dx.$$

$$n_{11} = \left.\frac{\partial N}{\partial a_1}\right|_{\substack{a_1=1\\a_2=0}} = \int_0^L EI_0(1-\alpha x)\left[2\cdot[6(L-x)]^2\right] dx$$

$$n_{12} = \left.\frac{\partial N}{\partial a_1}\right|_{\substack{a_1=0\\a_2=1}} = \int_0^L EI_0(1-\alpha x)\left[2[6(L-x)][18Lx - 12x^2]\right] dx = n_{21} = \left.\frac{\partial N}{\partial a_2}\right|_{\substack{a_2=0\\a_1=1}}$$

$$n_{22} = \left.\frac{\partial N}{\partial a_2}\right|_{\substack{a_2=1\\a_1=0}} = \int_0^L EI_0(1-\alpha x)\left[2\cdot[18Lx - 12x^2]^2\right] dx$$

$$d_{11} = \left.\frac{\partial D}{\partial a_1}\right|_{\substack{a_1=1\\a_2=0}} = \int_0^L 2\,(6Lx-3x^2)^2\,dx \;;\; d_{22} = \left.\frac{\partial D}{\partial a_2}\right|_{\substack{a_2=1\\a_1=0}} = \int_0^L 2\,(9Lx^2-4x^3)^2\,dx$$

$$d_{12} = d_{21} = \left.\frac{\partial D}{\partial a_1}\right|_{\substack{a_2=1\\a_1=0}} = \left.\frac{\partial D}{\partial a_2}\right|_{\substack{a_2=0\\a_1=1}} = \int_0^L 2\,(6Lx-3x^2)(9Lx^2-4x^3)\,dx$$

Solve the eigenvalue problem

$$\begin{bmatrix} n_{11}-Pd_{11} & n_{12}-Pd_{12} \\ n_{12}-Pd_{12} & n_{22}-Pd_{22} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0$$

$$\Rightarrow (n_{11}-Pd_{11})(n_{22}-Pd_{22}) - (n_{12}-Pd_{12})^2 = 0$$

↓ solve quadratic in P to get $(P_{cr})_1$, $(P_{cr})_2$.

✗ NOTE: The Ritz method always gives symmetric matrices n_{ij}, d_{ij}.

Its a lot easier if you work with strictly admissible functions (ie only kinematic bc's satisfied) and take more terms in the series. For 2-term series, $w_1 = x^2$, $w_2 = x^3$

$$w_1' = 2x,\quad w_1'' = 2,\quad w_2' = 3x^2,\quad w_2'' = 6x$$

$$n_{11} = \int_0^L EI_0(1-\alpha x)\,2.(2)^2\,dx = EI_0[8L - 4\alpha L^2], \quad n_{22} = \int_0^L EI_0(1-\alpha x)\,2(36x^2)\,dx = EI_0[24L^3 - 18\alpha L^4]$$

$$n_{12} = \int_0^L EI_0(1-\alpha x)\,2(12x)\,dx = EI_0[12L^2 - 8\alpha L^3], \quad d_{11} = \int_0^L 2(4x^2)\,dx = \frac{8}{3}L^3, \quad d_{22} = \int_0^L 2(9x^4)\,dx = \frac{18}{5}L^5$$

$$d_{12} = \int_0^L 2.(6x^3)\,dx = 3L^4$$

It appears that working with admissible functions is somewhat easier, especially if we want to get general expressions for n_{ij}, d_{ij} for programming. In that case, $w_i = x^{i+1}$,

$$w_i' = (i+1)\,x^{i}, \quad w_i'' = (i+1)(i)\,x^{i-1}$$

$$N = \int_0^L \sum_{i,j=1}^{N}\sum EI_0 (1-\alpha x)(i+1)(j+1)(i)(j)\, x^{i-1+j-1} a_i a_j \, dx$$

$$D = \int_0^L \sum_{i,j=1}^{N}\sum (i+1)(j+1)\, x^{i+j} a_i a_j \, dx$$

$$\frac{\partial N}{\partial a_m} = \sum_{k=1}^{N} 2EI_0 (m+1)(m)(k+1)(k) \int_0^L x^{m+k-2}(1-\alpha x)\,dx \; a_k \;, \quad m = 1, \cdots N$$

$$\frac{\partial D}{\partial a_m} = \sum_{k=1}^{N} 2(m+1)(k+1) \int_0^L x^{m+k}\,dx \; a_k \;, \quad m = 1, \cdots N$$

So $n_{mk} = 2EI_0 (m+1)(m)(k+1)(k) \int_0^L x^{m+k-2}(1-\alpha x)\,dx$

$$= 2EI(m+1)(m)(k+1)(k)\left[\frac{L^{m+k-1}}{m+k-1} - \alpha \frac{L^{m+k}}{m+k}\right], \quad m,k = 1, \cdots N$$

$$d_{mk} = 2(m+1)(k+1)\left[\frac{L^{m+k+1}}{m+k+1}\right], \quad m,k = 1, \cdots N$$

$\left[n_{mk} - P d_{mk}\right]\{a_k\} = 0 \rightarrow$ solve EVP.

$\rightarrow$ Ready for programming $\rightarrow$ use N = some big number.

So if we want to work with admissible functions it becomes more amenable to programming, and then we can take more terms in the series to get results that are as good as the approach which uses trial functions satisfying some or all of the natural b.c's also (in addition to kinematic bc's) where we take fewer terms in the series. Usually the former approach (using strictly admissible functions and not bothering about other bc's) is done in practice.

(II) RAYLEIGH-RITZ METHOD.

We begin with total potential energy ($U = U_i + U_p$) (see p. 13, T-II). So in the present version, applicability for conservative systems only. However, there are "generalized potential energies" defined for non-conservative systems also (not done in CE619). For beam-columns,

$$U = \frac{1}{2}\int_0^L \left[EA\left(u' + \frac{w'^2}{2}\right)^2 + EI(w'')^2\right]dx - \int_0^L (q^* w - tu)\,dx$$
$$+ \bar{P}u\Big|_0^L - \bar{M}w'\Big|_0^L - \bar{R}w\Big|_0^L$$

Put $w_a = \sum_{i=1}^{N} a_i w_i$, $u_a = \sum_{j=1}^{M} b_j u_j$ $\longrightarrow$ w_i, u_j must be admissible. Other guidelines for their choice is as for Rayleigh quotient method.

Thus,

$U = U(a_1, \dots a_N, b_1, \dots b_M; P)$ $\longrightarrow$ So the ∞ degree of freedom system has been discretized.

Equilibrium $\Longrightarrow$ $\dfrac{\partial U}{\partial a_i} = 0$, $\dfrac{\partial U}{\partial b_j} = 0$, $i = 1, \dots N$, $j = 1, \dots M$.

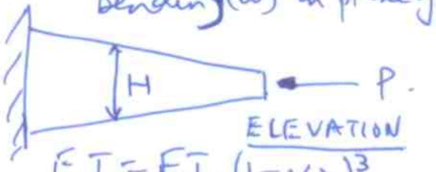
Gives $(M+N)$ equations to solve for $(M+N)$ unknowns $(a_1, \dots a_N, b_1, \dots b_M)$ in terms of load $\bar{P}$.

These equations are usually non-linear in a_i, b_j, since U is not quadratic (see the $\frac{w'^2}{2}$ term).

Stability $\Longrightarrow$ $\underline{\underline{C}} = \left[\dfrac{\partial^2 U}{\partial q_i \partial q_j}\Bigg|_{q\,\text{equil}}\right]$ is pos. def. where $\underline{q}^T = [a_1, \dots a_N, b_1, \dots b_M]$

Example

Bending (w) in plane of paper.



ELEVATION

$H = H_0(1 - \alpha x)$

$B = \text{const} = B_0$

$A = H.B.$

$EI = EI_0(1 - \alpha x)^3$

$EA = EA_0(1 - \alpha x)$

$w_a = \sum_{i=1}^{N} a_i x^{i+1}$

$u_a = \sum_{i=1}^{M} b_i x^i$

$$2U = \int_0^L \Big\{ EA_0 (1-\alpha x) \Big[\sum_{i,j=1}^{M}\sum b_i b_j (i)(j)\, x^{i-1} x^{j-1} + \sum_{\substack{i=1,\dots,M \\ j,k=1,\dots N}}\sum\sum b_i a_j a_k (i)(j+1)(k+1)\, x^{i-1} x^j x^k + \frac{1}{4}\sum\sum_{i,j,k,l=1}^{N}\sum\sum a_i a_j a_k a_l (i+1)(j+1)(k+1)(l+1)\, x^i x^j x^k x^l \Big]$$

$$+ EI_0 (1-\alpha x)^3 \Big[\sum_{i,j=1}^{N}\sum a_i a_j \;(i+1)(i)(j+1)(j)\, x^{i-1} x^{j-1} \Big] \Big\} dx + 2\bar{P} \sum_{i=1}^{M} b_i L^i$$

$$2U = EA_0 \Bigg\{ \sum_{i,j=1}^{M}\sum \underbrace{(i)(j)\left(\frac{L^{i+j-1}}{i+j-1} - \alpha \frac{L^{i+j}}{i+j}\right)}_{\beta_{ij}} b_i b_j + \sum_{\substack{i=1,\dots M \\ j,k=1\dots N}}\sum\sum \underbrace{(i)(j+1)(k+1)\left(\frac{L^{i+j+k}}{i+j+k} - \alpha \frac{L^{i+j+k+1}}{i+j+k+1}\right)}_{\gamma_{ijk}} b_i a_j a_k$$

$$+ \frac{1}{4} \sum_{i,j,k,l=1\dots N}\sum\sum\sum \underbrace{(i+1)(j+1)(k+1)(l+1)\left(\frac{L^{i+j+k+l+1}}{i+j+k+l+1} - \alpha \frac{L^{i+j+k+l+2}}{i+j+k+l+2}\right)}_{4\theta_{ijkl}} * a_i a_j a_k a_l \Bigg\}$$

$$+ EI_0 \sum_{i,j=1}^{N}\sum \underbrace{(i+1)(i)(j+1)(j)\left(\frac{L^{i+j-1}}{i+j-1} - 3\alpha \frac{L^{i+j}}{i+j} + 3\alpha^2 \frac{L^{i+j+1}}{i+j+1} - \alpha^3 \frac{L^{i+j+2}}{i+j+2}\right)}_{\xi_{ij}} a_i a_j$$

$$+ 2\bar{P} \sum_{i=1}^{M} b_i L^i$$

$$2U = EA_0 \Big\{ \sum_{i,j=1}^{M}\sum \beta_{ij} b_i b_j + \sum_{\substack{i=1,\dots M \\ j,k=1,\dots N}}\sum\sum \gamma_{ijk} b_i a_j a_k + \sum_{i,j,k,l=1\dots N}\sum\sum\sum \theta_{ijkl} a_i a_j a_k a_l \Big\}$$
$$+ EI_0 \sum_{i,j=1,\dots N}\sum \xi_{ij} a_i a_j + 2\bar{P} \sum_{i=1,\dots M} b_i L^i$$

$$\frac{\partial(2U)}{\partial a_p} = EA_0 \Big\{ \sum_{\substack{i=1,\dots M \\ j=1,\dots N}}\sum 2\gamma_{ijp} b_i a_j + \sum_{i,j,k=1,\dots N}\sum\sum 4\theta_{ijkp} a_i a_j a_k \Big\}$$
$$+ EI_0 \sum_{i=1,\dots N} 2\xi_{ip} a_i = 0. \longrightarrow (1), \quad \underline{p=1,\dots N.}$$

$$\frac{\partial(2U)}{\partial b_p} = EA_0\left\{\sum_{i=1,\dots M} 2\beta_{ip} b_i + \sum_{j,k=1,\dots N}\sum \gamma_{pjk} a_j a_k\right\} + 2\bar{P}L^P = 0 \rightarrow (2)\ \underline{p=1,\dots M}$$

where we have used the fact that $\beta_{ij} = \beta_{ji}$, $\gamma_{ijk} = \gamma_{ikj}$, θ_{ijkl} is symmetric in all its indices, $\xi_{ij} = \xi_{ji}$.

So (1) & (2) represent (N+M) equations to solve for (N+M) unknowns a_i, b_j, $i = 1,\dots N$, $j = 1,\dots M$, in terms of $\bar{P}$, after we solve for $\bar{P}$ (ie to solve for mode shapes). These are nonlinear in the a_i's. One possible solution is $a_i = 0$, $i = 1,\dots N$ and $\boxed{EA_0\sum_{i=1}^{M} 2\beta_{ip} b_i + 2\bar{P}L^P = 0} \rightarrow (3)$

$p = 1,\dots M$, to solve for the b_i's, ie, these are linear equations relating b_i's to $\bar{P}$. This represents the unbuckled column ($a_i = 0$) with end-shortening ($b_i \neq 0$). Lets analyze the stability of this configuration since it is the case of practical interest

$$\frac{\partial^2(2U)}{\partial a_p \partial a_q} = EA_0\left\{\sum_{i=1\dots M} 2\gamma_{ipq} b_i + \sum_{i,j=1\dots N}\sum 12\,\theta_{ijpq}\overset{*}{}a_i a_j\right\} + EI_0(2\xi_{pq}), \quad p, q, = 1,\dots N$$

symmetric in $p \gtrless q$

$$\frac{\partial^2(2U)}{\partial a_p \partial b_q} = EA_0\left\{\sum_{i=1,\dots N} 2\underbrace{\gamma_{qip}}_{=\gamma_{qpi}} a_i\right\}, \quad p = 1,\dots N,\ q = 1,\dots M$$

$$\frac{\partial^2(2U)}{\partial b_p \partial b_q} = EA_0\left\{2\beta_{pq}\right\}, \quad p, q, = 1\dots M$$

$\rightarrow$ symmetric in $p \gtrless q$

$$\frac{\partial^2(2U)}{\partial b_p \partial a_q} = EA_0\left\{\sum_{i=1,\dots N} 2\gamma_{pqi} a_i\right\}, \quad p = 1,\dots M,\ q = 1,\dots N$$

$$\underline{\underline{C}} = \left[\begin{array}{c|c} \dfrac{\partial^2(2U)}{\partial a_p \partial a_q} & \dfrac{\partial^2(2U)}{\partial a_p \partial b_q} \\ (N\times N) & (N\times M) \\ \hline \dfrac{\partial^2(2U)}{\partial b_p \partial a_q} & \dfrac{\partial^2(2U)}{\partial b_p \partial b_q} \\ (M\times N) & (M\times M) \end{array}\right]_{(\underset{\sim}{a}_e = \underset{\sim}{0},\ \underline{b}_e)} = \underline{\underline{C}}^T$$

Note symmetry of $\underline{\underline{C}}$ (as expected)

$\because a_i = 0$, upper-right & lower-left partitions of $\underline{\underline{C}}$ are zero $\left(\text{ie, } \left.\frac{\partial^2(2U)}{\partial a_p \partial b_q}\right|_{\underset{\sim}{a}=0} = \left.\frac{\partial^2(2U)}{\partial b_p \partial a_q}\right|_{\underset{\sim}{a}=0} = 0\right)$

So $\det|\underline{\underline{C}}| = \det\left[\frac{\partial^2(2U)}{\partial a_p \partial a_q}\right]_{(\underset{\sim}{a}=0,\underset{\sim}{b}_e)} * \det\left[\frac{\partial^2(2U)}{\partial b_p \partial b_q}\right]_{(\underset{\sim}{a}=0,\underset{\sim}{b}_e)}$.

$$\boxed{\det|\underline{\underline{C}}| = \det\Big[\underbrace{EA_0\sum_{i=1}^{M} 2\gamma_{ipq}\, b_i + 2EI_0\,\xi_{pq}}_{N\times N \text{ matrix, } p,q=1,\cdots N}\Big] * \det\Big[\underbrace{2EA_0\,\beta_{pq}}_{M\times M \text{ matrix, } p,q=1,\cdots M.}\Big] = 0}$$

where b_i are solutions of (3) p12.

Note that b_i will be proportional to $\bar{P}$ (ie $b_i = \text{const} \times \bar{P}$)

So $\det|\underline{\underline{C}}| = 0$ gives an <u>N^{th} order</u> polynomial equation in $\bar{P}$ to be solved for $\bar{P}_{cr}$. Since $\det[2EA_0\beta_{pq}] \neq 0$ and independent of $\bar{P}$, the CE reduces to,

$$\boxed{\det\left[EA_0 \sum_{i=1}^{M} 2\gamma_{ipq}\, b_i + 2EI_0\,\xi_{pq}\right] = 0} \rightarrow \text{CE}.$$

METHOD OF WEIGHTED RESIDUALS.

$$W_a = \sum_{i=1}^{N} a_i W_i(x) \longrightarrow \text{approx solution.} \longrightarrow (1)$$

$W_i \rightarrow$ comparison functions, ie, they satisfy all BC's (not just the kinematic ones).

Let $\mathcal{L}$ be the differential operator representing the differential equation (DE) to be approximately solved. For our case,

$$(EIw'')'' + Pw'' = 0 \Rightarrow \mathcal{L} = (EI(\ )'')'' + P(\ )''.$$

Define error (residual) of not satisfying DE as,

$$e(x) = \mathcal{L}(W_a - W) = \mathcal{L}W_a \quad (\because \mathcal{L}(w) = (EIw'')'' + Pw'' = 0).$$

Let $v_j(x)$, $j = 1, \dots N$ representing weighting functions. Then in weighted residuals method we make $e(x)$ 'orthogonal' to $v_j(x)$, ie

$$\int_0^L e(x)\, v_j(x)\, dx = 0, \ j = 1, \dots N \longrightarrow (2a)$$

$$\Rightarrow \int_0^L \left(\mathcal{L} \sum_{i=1}^{N} a_i W_i(x)\right) v_j(x)\, dx = 0 \longrightarrow (2b)$$

Choose $v(x)$ $(= 1, \dots N)$ as independent, for example $v_1(x) = x$, $v_2(x) = x^2$, $\dots$ $v_N(x) = x^N$ (or similarly sine or cosine terms in a series). So the greater N is, it means that the error $e(x)$ is orthogonal to a large number of independent functions ($x, x^2, \dots x^N$, for example). This is like driving the error to zero.

In the context of an $N \times 1$ column vector, think of the error representing an arbitrary $N \times 1$ column vector. If you make this orthogonal to the $N \times 1$

unit vectors $\{1, 0, \cdots\}^T$, $\{0, 1, 0, \cdots\}^T \cdots\cdots \{0, 0, \cdots 0, 1, 0\}^T$, $\{0, 0, \cdots, 0, 1\}^T$, it means that the $N \times 1$ error vector is being driven to zero. This is the basic idea of weighted residuals.

15 T-III

(1) <u>Collocation Method.</u>

Choose weighting functions as Dirac-Delta functions, $v_i = \delta(x - x_j)$, $j = 1, \cdots N$.

(2a) $\Rightarrow$ $e(x_j) = 0$, $j = 1, \cdots N$.

i.e, the error at N discrete points $x = x_j$, $j = 1, \cdots N$ is driven to zero, ie,

$$\boxed{\left[\left(EI \sum_{i=1}^{N} a_i w_i''\right)'' + P \sum_{i=1}^{N} a_i w_i''\right]_{x = x_j} = 0 , \quad j = 1, \cdots N \quad 0 \le x_j \le L} \leftarrow (3)$$

So if N is increased, say to ∞, then the DE is being exactly satisfied at ∞ points in $0 < x < L$, ie increasing N makes $w_a \to w$ (exact soln).

From (3),

$$\underbrace{\begin{bmatrix} \left[(EIw_1'')'' + Pw_1''\right]_{x=x_1} & \left[(EIw_2'')'' + Pw_2''\right]_{x=x_1} & \cdots & \left[(EIw_N'')'' + Pw_N''\right]_{x=x_1} \\ \vdots & \vdots & & \vdots \\ \left[(EIw_1'')'' + Pw_1''\right]_{x=x_N} & \left[(EIw_2'')'' + Pw_2''\right]_{x=x_N} & \cdots & \left[(EIw_N'')'' + Pw_N''\right]_{x=x_N} \end{bmatrix}}_{\underline{\underline{C}}} \begin{Bmatrix} a_1 \\ \vdots \\ a_N \end{Bmatrix} = \underset{\sim}{0}.$$

$\Rightarrow \underline{\underline{C}}\, \underline{a} = 0$

For $\underline{a} \neq \underline{0}$, $\det[\underline{\underline{C}}] = 0$ → gives N^{th} order polynomial eqn in P → ie C.E.

However the Collocation method yields an unsymmetric $\underline{\underline{C}}$ and requires relatively larger number of approximating terms (ie 'N' trial functions) compared to other methods like the Galerkin method.

(2) Galerkin method

Choose $V_i(x) = W_i(x)$

ie weighting functions are chosen same as trial functions.

Then, from (2b),

$$\boxed{\sum_{i=1}^{N} a_i \int_0^L \left[\left(EI(x) W_i''(x) \right)'' + P W_i''(x) \right] W_j(x)\, dx = 0, \quad j = 1, \cdots N.} \quad (4)$$

$\hookrightarrow$ GALERKIN EQUATIONS

i.e,

$$\underline{\underline{C}} = \begin{bmatrix} \int_0^L \left[(EI(x) W_1''(x))'' + P W_1''(x) \right] W_1(x)\, dx & \cdots\cdots & \int_0^L \left[(EI W_N'')'' + P W_N'' \right] W_1\, dx \\ \vdots & & \vdots \\ \int \left[(EI W_1'')'' + P W_1'' \right] W_N\, dx & \cdots\cdots & \int \left[(EI W_N'')'' + P W_N'' \right] W_N\, dx \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_N \end{Bmatrix} = 0.$$

ie $\underline{\underline{C}}\, \underline{a} = 0$, $\det |\underline{\underline{C}}| = 0$ gives CE in P, ie N^{th} order polynomial eqn in P. Choose lowest value as P_{cr}.

Choice of W_i (trial functions) follows the guidelines below:

(1) W_i must satisfy all BC's. However we will see when this can be relaxed to only kinematic BC's

(2) Choose complete set of w_i's (eg. $\sin \frac{m\pi x}{L}$, $m = 1, \cdots N$, for SS BC's,

without missing any integer in between $1 \cdots N$). Choosing a complete set enhances convergence.

For a column with surface traction $t=0$, equilibrium equation is (repeated from Topic-II p. 8)

$$(EIw'')'' + (Pw')' = q + \sum_{p=1}^{n} Q_p \delta[x-x_p] - \sum_{l=1}^{m} C_l \eta[x-x_l]$$

Galerkin Equations are,

$$\sum_{i=1}^{N} a_i \int_0^L \left[(EI(x) W_i''(x))'' + P W_i''(x) \right] W_j(x)\,dx = \int_0^L \left[q + \sum_{p=1}^{n} Q_p \delta[x-x_p] - \sum_{l=1}^{m} C_l \eta[x-x_l] \right] * W_j(x)\,dx \quad (5)$$

$j = 1, \cdots N.$

ie N equations for N unknowns $a_1 \cdots a_N$.

Lets look at Modified Galerkins Method. Go one step backward to variations $\delta U_i + \delta U_p$, p. 5, 7, Topic II, with $-P' = t = 0$ and $P = +\bar{P}$ at $x = 0, L$,

$$\delta U_i + \delta U_p = \int_0^L \Big(+ \overbrace{P'}^{=0} \delta u + (Pw')'\delta w + (EIw'')''\delta w - q\delta w + \overbrace{\cancel{t}}^{=0}\delta u - \sum_{p=1}^{n} Q_p \delta[x-x_p]\delta w + \sum_{l=1}^{m} C_l \eta[x-x_l]\delta w \Big) dx$$

boundary terms

$$+ \left[\begin{array}{l} \cancel{-P\delta u\Big|_0^L} - Pw'\delta w\Big|_0^L + EIw''\delta w'\Big|_0^L - (EIw'')'\delta w\Big|_0^L \\ + \bar{M}_0 \delta w_0' - \bar{M}_L \delta w_L' + \bar{R}_0 \delta w_0 - \bar{R}_L \delta w_L - \cancel{\bar{P}(\delta u_0 - \delta u_L)} \end{array} \right] = 0 \quad (6)$$

$(\because (P - \bar{P}) = 0)$

Now we do the discretization at this stage itself, ie in the variational principle, ie put $w_a = \sum_{i=1}^{N} a_i W_i(x)$ and $\delta w = \delta a_j W_j(x)$. Note that if $W_i(x)$ satisfy all b.c's then all boundary terms in (6) are zero and we get same result as (5) for the Galerkin Equations. When W_i are admissible only (ie satisfy Kinematic BC's),

$\delta U_i + \delta U_p = 0$

$$\Rightarrow \left\{ \sum_{i=1}^{N} a_i \int_0^L [P w_i'' + (EI w_i'')''] w_j \, dx - \int_0^L \left(q w_j + \sum_{p=1}^{n} Q_p \delta[x-x_p] w_j - \sum_{l=1}^{m} C_l \eta[x-x_l] w_j \right) dx \right. \rightarrow (7)$$

Boundary terms

$$\left. + \left[-P \sum_{i=1}^{N} a_i w_i' w_j \Big|_0^L + EI \sum_{i=1}^{N} a_i w_i'' w_j' \Big|_0^L - \left(EI \sum_{i=1}^{N} a_i w_i'' \right)' w_j \Big|_0^L + \bar{M}_0 w_j'(0) - \bar{M}_L w_j'(L) + \bar{R}_0 w_j(0) - \bar{R}_L w_j(L) \right] \right\}$$

$* \delta a_j = 0$.
$j = 1, \ldots N$.

$\because \delta w$ is arbitrary $\Rightarrow \delta a_j$ is arbitrary

$$\Rightarrow \{ \quad \} = 0 \rightarrow \text{Galerkin Equations}, \quad j = 1, \ldots N \rightarrow (7a)$$

Note that if w_i satisfy all BC's (ie comparison functions) then the boundary term in (7) is zero and (7a) reduces to (5). However, if w_i is chosen as admissible (ie satisfie only kinematic BC's) then boundary terms contribute to the Galerkin equations and hence the coefficient matrix $\underline{\underline{C}}$. That is the coefficient matrix represents the effect of driving to zero the error of not satisfying the differential equation as well as the natural (ie force, moment) boundary conditions. This is the main advantage of the Modified Galerkin method, ie w_i need be admissible only if we use this modified method.

(Ex) Apply modified Galerkin Method to a Clamped-Free Column. Only axial compression applied.

$$W_a = \sum_{i=1}^{N} a_i x^{i+1} \longrightarrow \text{admissible trial functions chosen}$$

$$w_j = x^{j+1}$$

$$\sum_{i=1}^{N} a_i \int_0^L \left[(i+1)(i)\, P x^{i-1} + EI (i+1)(i)(i-1)(i-2)\, x^{i-3} \Delta_{i3} \right] x^{j+1}\, dx$$

$$- P \sum_{i=1}^{N} a_i (i+1) x^i x^{j+1} \Big|_0^L + EI \sum_{i=1}^{N} a_i (i+1)(i)\, x^{i-1} (j+1) x^j \Big|_0^L$$

$$- EI \sum_{i=1}^{N} a_i (i+1)(i)(i-1)\, x^{i-2} x^{j+1} \Delta_{i2} \Big|_0^L = 0, \quad j = 1, \cdots N$$

$$\Rightarrow \sum_{i=1}^{N} a_i \left[(i)(i+1) P \frac{L^{i+j+1}}{i+j+1} + EI (i+1)(i)(i-1)(i-2) \frac{L^{i+j-1}}{i+j-1} \right]$$

$$- P \sum_{i=1}^{N} a_i (i+1) L^{i+j+1} + EI \sum_{i=1}^{N} a_i (i+1)(i)(j+1) L^{i+j-1}$$

$$- EI \sum a_i (i+1)(i)(i-1) L^{i+j-1} = 0.$$

For $N = 3$,

$$\begin{bmatrix} \left(\frac{2}{3} PL^3 - 2PL^3 + 4EIL\right) & \left(\frac{3}{2} PL^4 - 3PL^4 + 12EIL^2 - 6EIL^2\right) & \left(\frac{12}{5} PL^5 + 8EIL^3 - 4PL^5 + 24EIL^3 - 24EIL^3\right) \\ \left(\frac{1}{2} PL^4 - 2PL^4 + 6EIL^2\right) & \left(\frac{6}{5} PL^5 - 3PL^5 + 18EIL^3 - 6EIL^3\right) & \left(2PL^6 + 6EIL^4 - 4PL^6 + 36EIL^4 - 24EIL^4\right) \\ \left(\frac{2}{5} PL^5 - 2PL^5 + 8EIL^3\right) & \left(PL^6 - 3PL^6 + 24EIL^4 - 6EIL^4\right) & \left(\frac{12}{7} PL^7 + \frac{24}{5} EIL^5 - 4PL^7 + 48EIL^5 - 24EIL^5\right) \end{bmatrix}$$

$$* \{a_1 \ a_2 \ a_3\}^T = 0.$$

Put $\boxed{\lambda = PL^2 / EI}$,

$$\underbrace{\begin{bmatrix} 4\left(1-\frac{\lambda}{3}\right) & 3L\left(2-\frac{\lambda}{2}\right) & 8L^2\left(1-\frac{\lambda}{5}\right) \\ 3\left(2-\frac{\lambda}{2}\right) & 3L\left(4-\frac{3\lambda}{5}\right) & 2L^2\left(9-\lambda\right) \\ 8\left(1-\frac{\lambda}{5}\right) & 2L\left(9-\lambda\right) & 16L^2\left(\frac{9}{5}-\frac{\lambda}{7}\right) \end{bmatrix}}_{\underline{\underline{C}}} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = 0.$$

$\det |\underline{\underline{C}}| = 0 \rightarrow$ solve for λ from cubic polynomial equation.

$\lambda_{cr} = 2.4677$.

$(\lambda_{cr})_{exact} = (\pi/2)^2$