TOPIC – IV

FRAMES.

Beam – column with end couples.

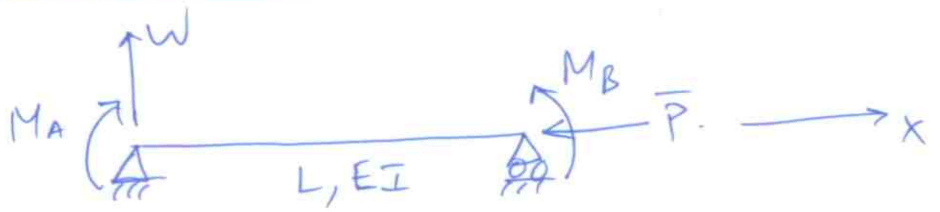


Fig. 0.

$$EI\,w^{IV} + \bar{P}\,w'' = 0 \quad \Rightarrow w = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4$$

BC's: $\quad w(0) = w(L) = 0$

$$EI\,w''(0) = M_A \,, \qquad EI\,w''(L) = M_B \,.$$

$$\Rightarrow A_2 + A_4 = 0 \,, \quad A_1 \sin kL + A_2 \cos kL + A_3 L + A_4 = 0$$

$$EI\left(-A_2 k^2\right) = M_A$$

$$EI\left(-A_1 k^2 \sin kL - A_2 k^2 \cos kL\right) = M_B$$

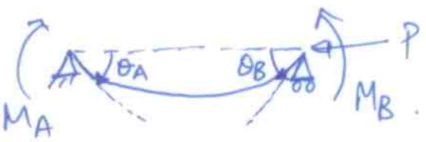
$$\Rightarrow A_2 = -M_A/\bar{P} = -A_4 \,, \qquad A_1 = \frac{M_A \cos kL - M_B}{\bar{P} \sin kL} \,,$$

$$A_3 = \frac{M_B - M_A}{\bar{P} L}$$

[NOTE: Response ∞ as $k \to n\pi$ so $\bar{P}_{cr} = \pi^2 EI/L^2$]

$$w = \frac{M_A}{\bar{P}}\left[\frac{\cos kL \sin kx}{\sin kL} - \cos kx - \frac{x}{L} + 1\right] + \frac{M_B}{\bar{P}}\left[-\frac{\sin kx}{\sin kL} + \frac{x}{L}\right]$$

$$\theta_A \triangleq -w'(0) \,, \qquad \theta_B \triangleq w'(L)$$



↳ defined for directions in Fig → (ie, they are magnitudes of rotations).

$$\Rightarrow \theta_A = -\frac{M_A}{\bar{P}}\left[\frac{k \cos kL}{\sin kL} - \frac{1}{L}\right] - \frac{M_B}{\bar{P}}\left[-\frac{k}{\sin kL} + \frac{1}{L}\right]$$

$$= \frac{M_A L}{EI}\left[\frac{1}{k^2 L^2} - \frac{k}{k^2 L}\frac{1}{\tan kL}\right] + \frac{M_B L}{EI}\left[\frac{k}{k^2 L}\frac{1}{\sin kL} - \frac{1}{k^2 L^2}\right]$$

$$= \frac{M_A L}{EI}\left[\frac{1}{(2u)^2} - \frac{1}{2u \tan 2u}\right] + \frac{M_B L}{EI}\left[\frac{1}{2u \sin 2u} - \frac{1}{(2u)^2}\right]$$

$$\theta_A = \frac{M_A L}{3EI}\psi(u) + \frac{M_B L}{6EI}\phi(u)$$

similarly $$\theta_B = \frac{M_B L}{3EI}\psi(u) + \frac{M_A L}{6EI}\phi(u)$$ → (1)

→ slope defl. equations with effect of axial load.

where $\boxed{u = kL/2}$ → (2)

$$\psi(u) = \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right)$$

$$\phi(u) = \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right)$$ → (3)

→ These represent the influence of axial load P.

$$\lim_{u\to 0}\phi(u) = \lim_{u\to 0}\frac{3(2u - \sin 2u)}{2u^2 \sin 2u} = \frac{0}{0}$$

$$= \lim_{u\to 0}\frac{3(2 - 2\cos 2u)}{4u\sin 2u + 4u^2\cos 2u} = \frac{0}{0}$$

$$= \lim_{u\to 0}\frac{3(4\sin 2u)}{4\sin 2u + 8u\cos 2u + 8u\cos 2u - 16u^2\sin 2u} = \frac{0}{0}$$

$$= \lim_{u\to 0}\frac{3(8\cos 2u)}{(8\cos 2u + 8\cos 2u - 16u\sin 2u + 8\cos 2u - 16u\sin 2u - 32u\sin 2u - 32u^2\cos 2u)}$$

$$= \frac{24}{24} = 1$$

similarly $\lim_{u\to 0}\psi(u) = 1$, i.e. $\boxed{\phi(0) = \psi(0) = 1}$

(The factors of 3, 6 in denominator of θ_A, θ_B, corresponding to 3 in numerator of $\psi(u)$, $\phi(u)$ have been put so as to make the about limits → 1 as $u \to 0$, i.e. as $\bar{P} \to 0$).

So for no axial load ($\bar{P} = 0$) θ_A, θ_B given by above with

$$\psi(u) = \phi(u) = 1.$$

[NOTE: Although we saw that $w(x) \to \infty$ as $\bar{P} \to \bar{P}_{cr} = \pi^2 EI/L^2$, the idea of this section is to develop Eq. (1).]

Rotationally Restrained Columns.

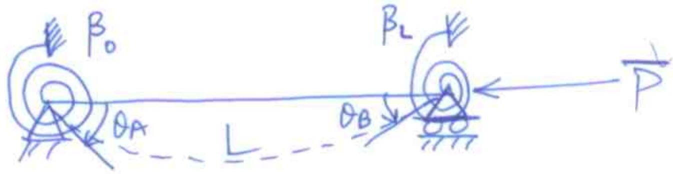


Fig 1: S.S. Column with Rot. Restraints.

Fig 2: S.S. Col. with end couples.

From Fig 1, using sign conventions as in Fig. 2,

$$M_A = -\beta_0 \theta_A \quad , \quad M_B = -\beta_L \theta_B \longrightarrow (4)$$

Use Eq (1), p. (2),

$$\left.\begin{aligned} M_A\left(1 + \beta_0 \frac{L\psi}{3EI}\right) + M_B\left(\beta_0 \frac{L\phi}{6EI}\right) &= 0 \\ M_A\left(\beta_L \frac{L\phi}{6EI}\right) + M_B\left(1 + \beta_L \frac{L\psi}{3EI}\right) &= 0 \end{aligned}\right\} \rightarrow (5)$$

For non-trivial (M_A, M_B), ie buckling, characteristic eqn is,

$$\left(\frac{1}{\beta_0} + \frac{L\psi}{3EI}\right)\left(\frac{1}{\beta_L} + \frac{L\psi}{3EI}\right) - \left(\frac{L\phi}{6EI}\right)^2 = 0 \rightarrow (6)$$

For case when $\beta_0 = \beta_L = \beta$ (equal springs), CE becomes

$$\frac{1}{\beta} + \frac{L\psi}{3EI} \pm \frac{L\phi}{6EI} = 0 \longrightarrow (7).$$

With '+' sign, (7) in (5a) gives $M_A = M_B \Rightarrow$ Symmetric buckling

" '−' " " " " " $M_A = -M_B \Rightarrow$ Antisymmetric buckling.

Symmetric buckling:

(3) (p.2) in (7) (with '+' sign) gives,

$$0 = \frac{1}{\beta} + \frac{3}{u}\frac{L}{3EI}\left(\cancel{\frac{1}{4u}} - \frac{1}{2\tan 2u} + \frac{1}{2\sin 2u} - \cancel{\frac{1}{4u}}\right) = \frac{1}{\beta} + \frac{1}{u}\frac{L}{EI}\left(\frac{1-\cos 2u}{2\sin 2u}\right)$$

$$\Rightarrow \frac{1}{\beta} + \frac{L}{2EI}\frac{\tan u}{u} = 0 \Rightarrow \tan u = -\frac{1}{\beta}\frac{2EI}{L}u \rightarrow (8).$$

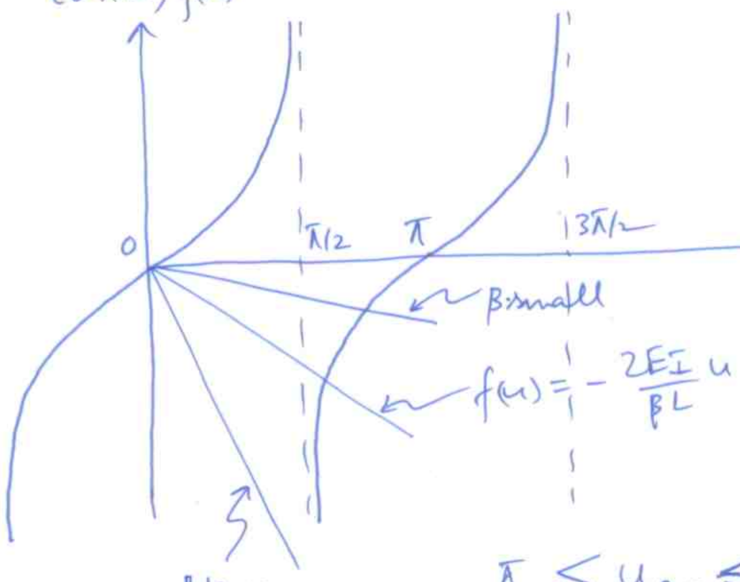
Antisymmetric buckling:

3 (p.2) in 7 with '−' sign gives,

$$0 = \frac{1}{\beta} + \frac{3}{u}\frac{L}{3EI}\left(\frac{1}{4u} - \frac{1}{2\tan 2u} - \frac{1}{2\sin 2u} + \frac{1}{4u}\right) = \frac{1}{\beta} + \frac{1}{u}\frac{L}{EI}\left(\frac{1}{2u} - \frac{[1+\cos 2u]}{2\sin 2u}\right)$$

$$\Rightarrow \frac{1}{\beta} + \frac{L}{2EI}\frac{1}{u}\left(\frac{1}{u} - \frac{1}{\tan u}\right) = 0 \Rightarrow \tan u = \frac{1}{\frac{2EI}{L\beta}u + \frac{1}{u}}$$

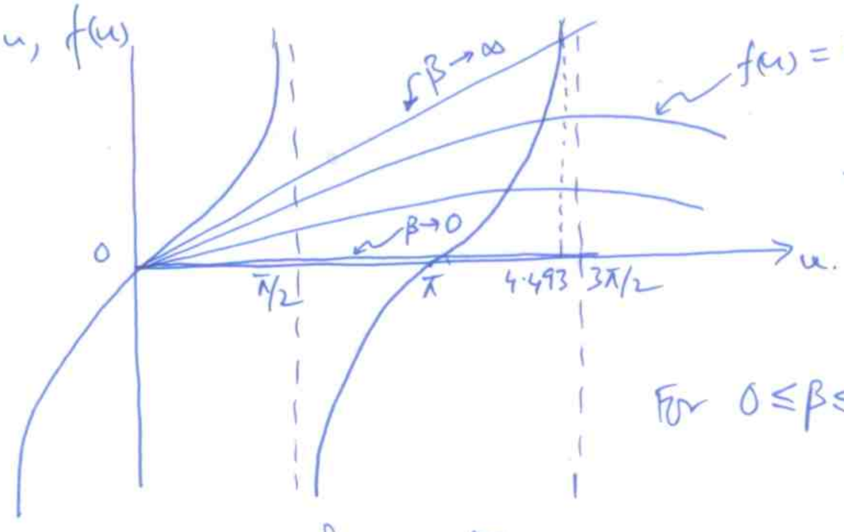
$$\Rightarrow \tan u = \frac{u}{1 + \frac{2EI}{L\beta}u^2} \longrightarrow 9 \text{ (call both as 9)}.$$



Plot for Eq 8

Symmetric buckl.

$$\frac{\pi}{2} \leq u_{cr} \leq \pi \quad \text{for } 0 \leq \beta \leq \infty$$



$f(u) = u\left[1 + \frac{2EI}{L\beta}u^2\right]^{-1}$

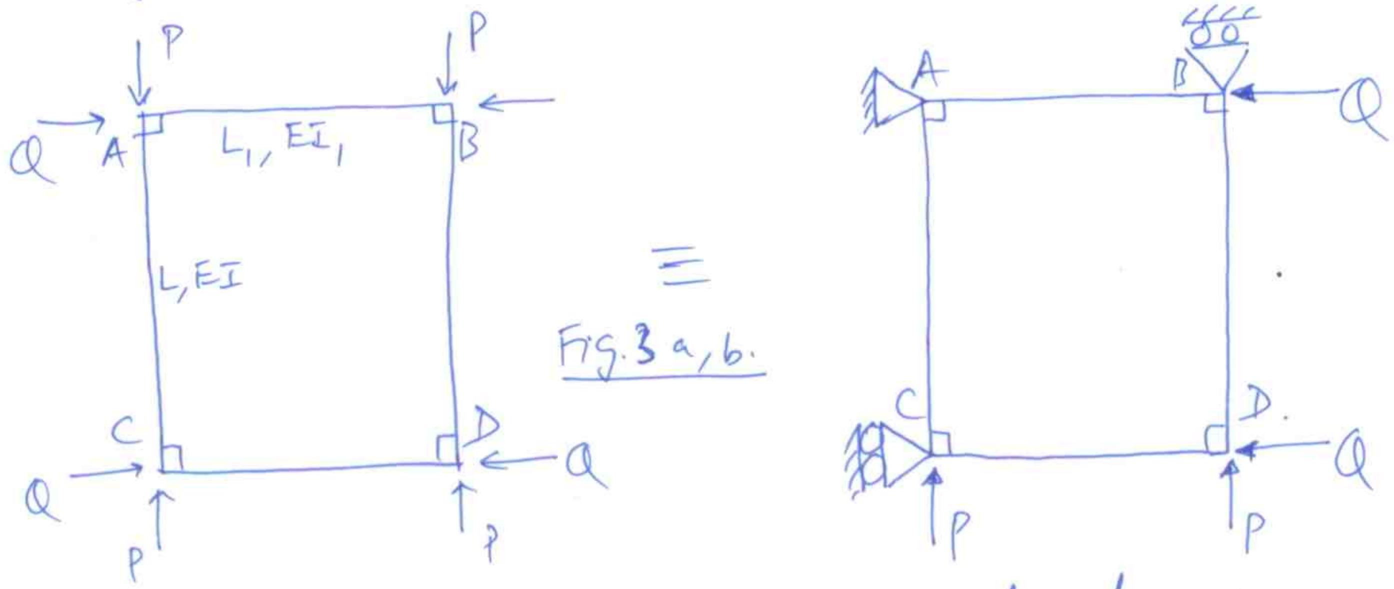
Plot for Eq. 9

Antisymm buckl.

For $0 \leq \beta \leq \infty$, we have $\frac{\pi}{2} \leq u_{cr} \leq 4.493$

So the symm & antisymm u_{cr} ranges are exclusive of each other (ie no overlap) $\Longrightarrow$ Symm buckling occurs

Rigid Frames (Rectangular)



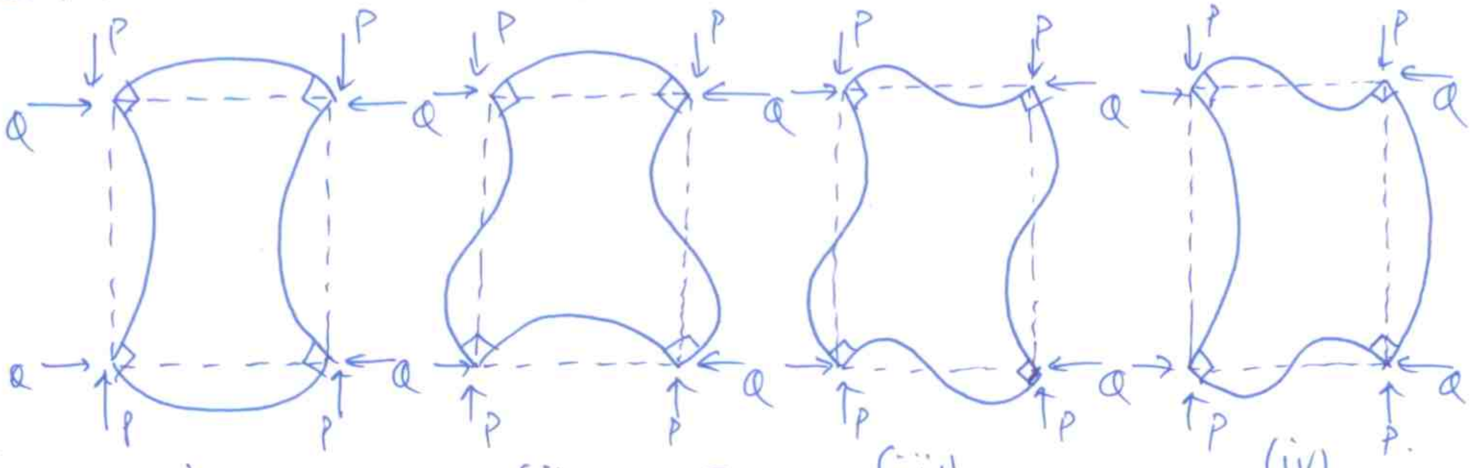
$\equiv$

Fig. 3 a, b.

Assume equal beams, equal columns.

We want to find critical (P, Q) combination.

The 4 possible buckling modes are,



(i) SYMMETRIC (ii) MIXED (iii) ANTISYMM. (iv) MIXED

Fig 4

In each case (i)-(iv) the rotational restraint (spring const) at each (corner) end of the beams/columns are identical since equal rotations are present due to equal-beams equal-col's assumption and the fact that bending moments at ends of columns are equal and equal to the BM at ends of beams (from equilibrium).

Now from results of rotationally restrained beam-cols, we saw that symmetric mode is prefered (ie lowest u_{cr} for symm mode).

So comparing (i) with (ii), since the equivalent rotational springs are equal* (*see below) the columns would buckle in symmetric mode so (ii) is discarded.

Similarly, (iv) discarded in favor of (iii).

The equivalent spring (β) for beams and columns in the symm and antisymm mode are obtained from 1, 3, p. 2. or from 8, 9 using convention of Fig 1, Fig. 2 & 4, p. 3–p4.

Symm mode : $M_A = M_B = M$ and use 1, 3,

or From 8, using Fig 1, Fig 3 convention & 4,

$$\frac{1}{\beta} = -\frac{L}{2EI}\frac{\tan u}{u} \longrightarrow (a)$$

$$\frac{1}{\beta} = +\frac{L_1}{2EI_1}\frac{\tan u_1}{u_1} \longrightarrow (b)$$

(signs are chosen since directions of M, θ in Fig 5 for columns are same as those in Fig. 2 so Eq 8 with $\beta \to \beta$ for vertical columns, while the sign of θ is opposite for horizontal beams when comparing Fig. 2 and Fig 5 so $\beta \to -\beta$ in 8). Else do by using 1, 3.

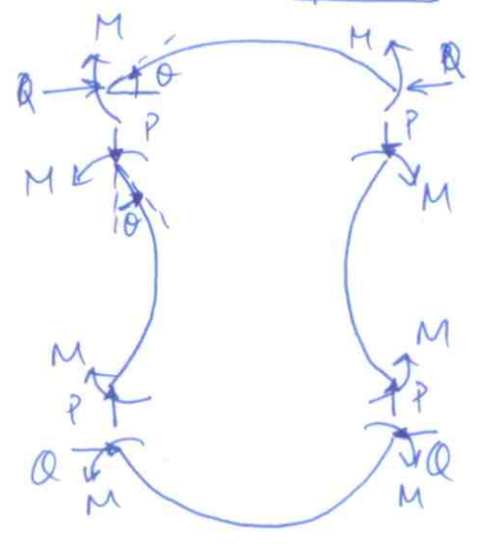


Fig. 5

Here $\boxed{u = \frac{kl}{2},\ u_1 = \frac{k_1 l_1}{2}}$. $k^2 = \frac{P}{EI}$, $k_1^2 = \frac{Q}{EI_1}$

(a), (b) $\Rightarrow$ $$\boxed{\frac{L}{EI}\frac{\tan u}{u} = -\frac{L_1}{EI_1}\frac{\tan u_1}{u_1}} \longrightarrow 10.$$

Antisymmetric mode :

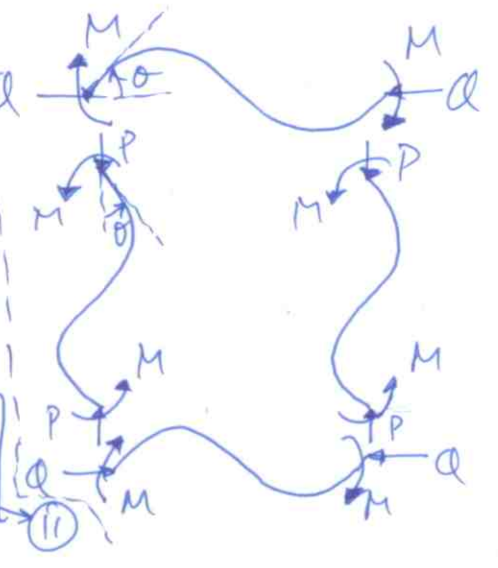
$M_A = -M_B$ in 1, 3 or

from 9,

$$\frac{1}{\beta} = -\frac{L}{2EI}\frac{1}{u}\left(\frac{1}{u} - \frac{1}{\tan u}\right) \to (c)$$

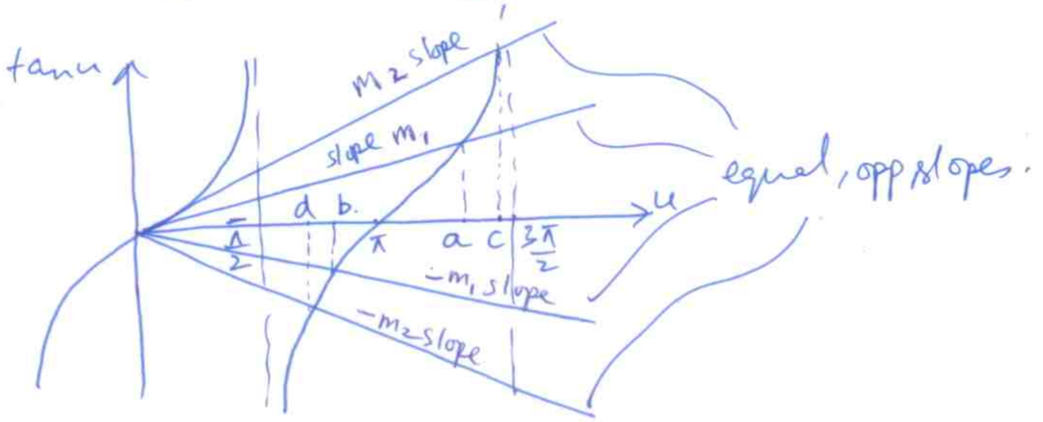
$$\frac{1}{\beta} = \frac{L_1}{2EI_1}\frac{1}{u_1}\left(\frac{1}{u_1} - \frac{1}{\tan u_1}\right) \to (d)$$

(c),(d) $\Rightarrow$ $$\boxed{\frac{L}{EI}\frac{1}{u}\left(\frac{1}{u} - \frac{1}{\tan u}\right) = -\frac{L_1}{EI_1}\frac{1}{u_1}\left(\frac{1}{u_1} - \frac{1}{\tan u_1}\right)} \to 11$$

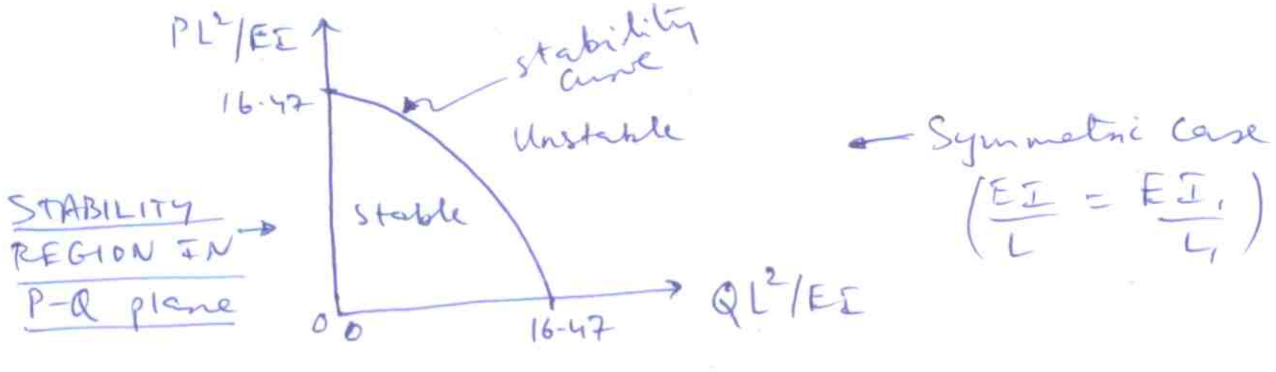


If columns = beams, (ie $\frac{L}{EI} = \frac{L_1}{EI_1}$),

(10) $\Rightarrow$ $\frac{\tan u}{u} = -\frac{\tan u_1}{u_1}$ $\longrightarrow$ symmetric buckl.



$\Rightarrow$ For various slope magnitudes, e.g. m_1, etc, find the intersection pts of tan u with $m_1 u$ and tan u with $-m_1 u$, say a, b, respectively. Solve for P, Q from $a = \frac{kL}{2} = \frac{\sqrt{P/EI}\cdot L}{2}$, $b = \frac{k_1 L_1}{2} = \frac{\sqrt{Q/EI_1}\cdot L_1}{2}$. Repeat for different slopes, m_2, m_3, etc. Then for each slope plot the solution pairs (P, Q). It looks like,



For special case, $Q = 0$, $\lim_{u_1 \to 0} \frac{\tan u_1}{u_1} = 1$

$$\Rightarrow \frac{\tan u}{u} = -\frac{EI}{L}\frac{L_1}{EI_1} = -1 \longrightarrow \text{smallest root is } u_{cr} = 2.029$$

$$\Rightarrow P_{cr} = 16.47\, EI/L^2$$

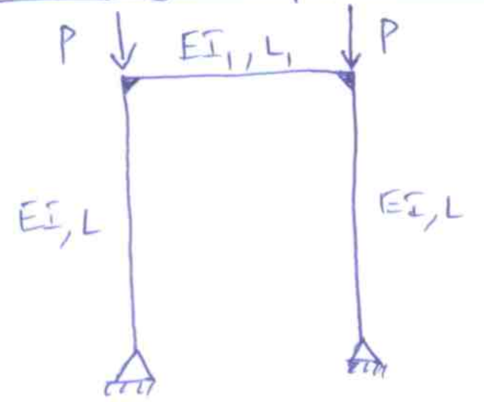
Similarly you can get the stability curve in (P, Q) plane for antisymmetric case. For $L = L_1$, $EI = EI_1$, the stability curve is higher in the +ve (P, Q) plane (1st quadrant). Hence we conclude that the frame buckles in symmetric mode first.

For $Q=0$, $EI/L = EI_1/L_1$, using $\psi(0)=1$, get

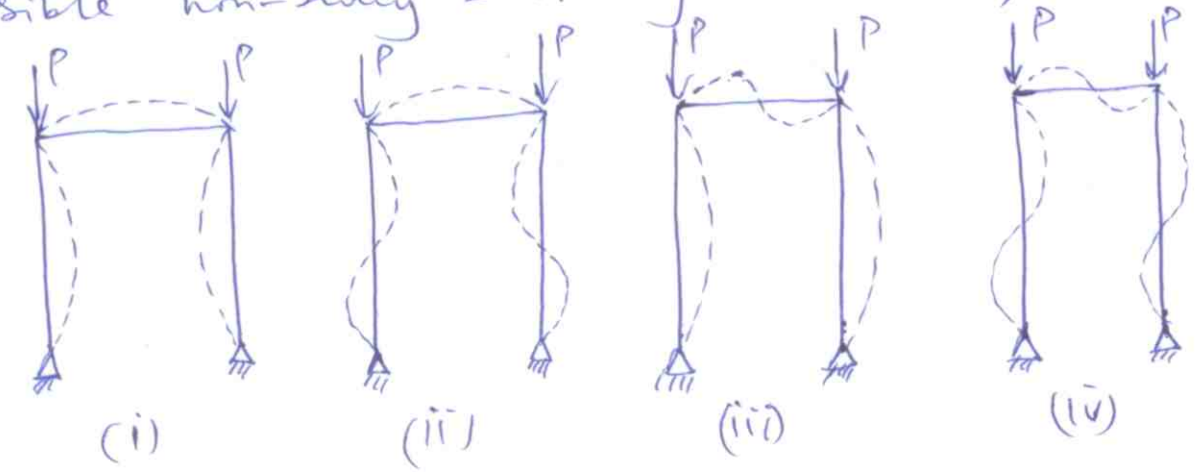
$$\frac{1}{u}\left(\frac{1}{u}-\cot u\right) = -\frac{1}{3} \longrightarrow \text{smallest root } u_{cr} > \pi, \text{ so}$$

symmetric mode buckling occurs first.

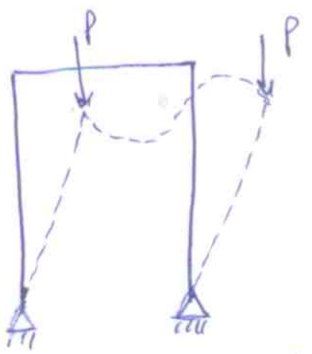
Simply Supported Portal frame.



Possible non-sway buckling modes are,



Sway mode is (only one possibility)



For non-sway modes, comparing (i) with (ii) and (iii) with (iv), we see that (i) preferred over (ii) & (iii) preferred over (iv), since within each comparison the beam mode remains same and column would first buckle in the lower (ie symmetric) mode. So analyze only (i), (iii).

(i) → symmetric beam mode → put $Q=0$, ie $\psi(u)=\phi(u)=1$ and $M_A = M_B$ in ① p.2,

$$\frac{\theta_A}{M_A} = \frac{\theta_B}{M_B} = \frac{1}{\beta} = \frac{L_1}{2EI_1}$$

(ii) → antisymm beam mode → $Q=0$, $\psi(u)=\phi(u)=1$, $M_B=-M_A$,

$$\frac{\theta_A}{M_A} = \frac{\theta_B}{M_B} = \frac{1}{\beta} = \frac{L_1}{6EI_1}$$

Thus the CE is obtained by putting $\beta_0=\beta$, $\beta_L=0$ (actually $\beta_L \to 0$ in practice) in ⑥ p.3,

We get, $$\frac{1}{\beta} + \frac{L\psi}{3EI} = 0$$

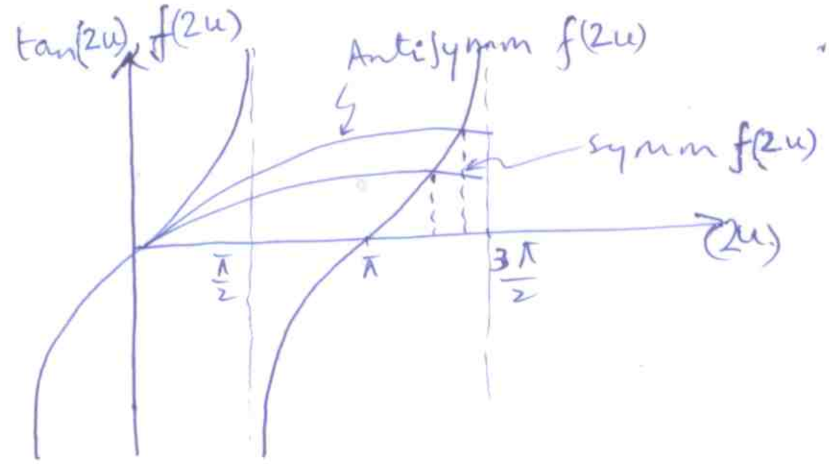
(i) Symmetric beam $\Rightarrow \dfrac{L_1}{2EI_1} + \dfrac{L}{3EI}\dfrac{3}{2u}\left(\dfrac{1}{2u} - \dfrac{1}{\tan 2u}\right) = 0$

$$\Rightarrow \cot 2u = \frac{1}{2u} + \frac{EI}{EI_1}\frac{L_1}{L}u$$

(ii) Antisymm beam $\Rightarrow \cot 2u = \dfrac{1}{2u} + \dfrac{EI}{3EI_1}\dfrac{L_1}{L}u$

Thus $$\tan 2u = \frac{2u}{1+c(2u)^2} = f(2u)$$

where $c = \dfrac{1}{2}\dfrac{EI}{EI_1}\dfrac{L_1}{L}$ for symm & $\dfrac{1}{6}\dfrac{EI}{EI_1}\dfrac{L_1}{L}$ for anti-symm beam mode.



$\because (C)_{\text{Antisymm}} < (C)_{\text{symm}}$ The AS curve for $f(u)$ will be higher, hence <u>symmetric mode is buckling critical.</u>

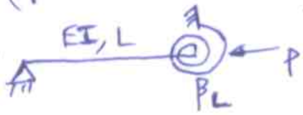
For case when horizontal beam is rigid, put $EI_1 \to \infty$,

$\Rightarrow \tan 2u = 2u$ (both SYM & ASYM) $\Rightarrow 2u = 4.493$

$\Rightarrow P_{cr} = 20.19\, EI/L^2$ → matches with pinned-fixed column.

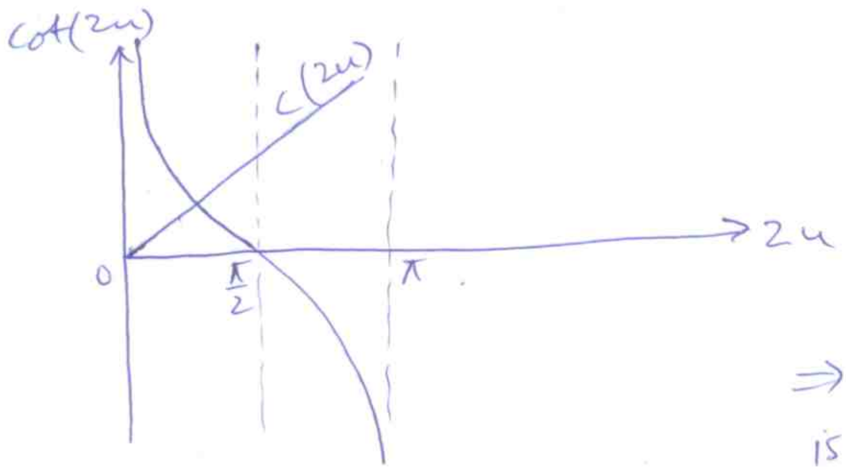
For sway buckling, we cannot use a specialization of (6) p. 3, since that assumes ends have no transverse translation. Treat this as a specialization of elastically supported columns (ie 6th order CE in u on p. 34, Topic-II). Rotational spring at top end is $\bar{\beta}_L = \frac{6EI_1}{L_1}$ (as for ASYM horz beam mode). So put $\alpha_0 = \infty$, $\alpha_L = 0$, $\beta_0 = 0$, $\beta_L = \frac{1}{EI}\left(\frac{6EI_1}{L_1}\right)$ in 6th order CE .. we get, (put $u \to 2u$),

Equivalent Column:



$$-\frac{(2u)^6}{L^6}\sin 2u + \left(\frac{6EI_1}{L_1}\right)\frac{1}{EI}\frac{(2u)^5}{L^5}\cos 2u = 0$$

$$\tan 2u = \frac{(6EI_1L/EIL_1)}{2u} \Rightarrow \cot 2u = 2cu, \quad c = \frac{1}{\left(\frac{6EI_1L}{EIL_1}\right)}$$



So P_{cr} for $2u_{cr} < \frac{\pi}{2}$

Compare with non-sway results for which $2u_{cr} > \pi$

$\Rightarrow$ Simply supported portal is sway-buckling critical.

For $\frac{EI_1}{L_1} = \frac{EI}{L}$, $\tan 2u = \frac{6}{2u}$

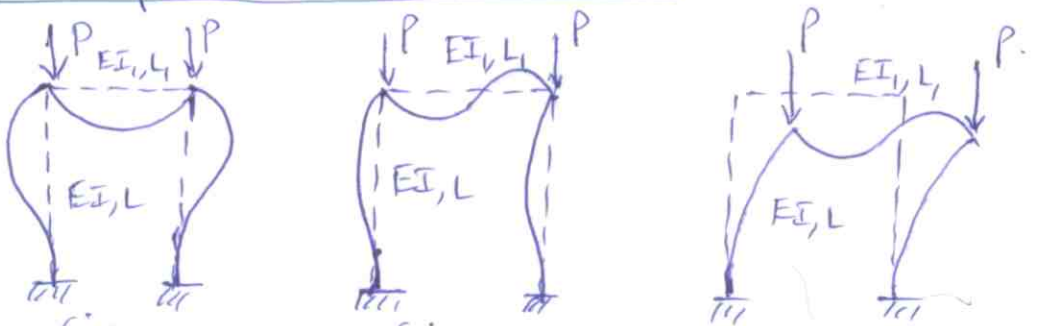
$$\Rightarrow 2u_{cr} = 1.350, \quad P_{cr} = 1.821\frac{EI}{L^2}$$

NOTE: We can use the approach of 6th order polynomial for the non-sway cases also, for which,

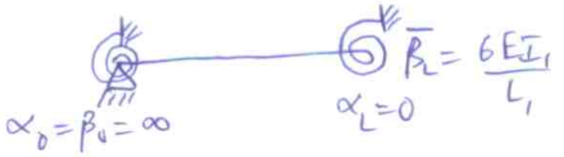
(i) $\rightarrow$ symmetric beam $\rightarrow$ $\alpha_0 = \alpha_L = \infty$, $\beta_0 = 0$, $\beta_L = 2EI_1/L_1$

(ii) $\rightarrow$ asymmetric beam $\rightarrow$ $\alpha_0 = \alpha_L = \infty$, $\beta_0 = 0$, $\beta_L = 6EI_1/L_1$

Clamped Portal Frame



Here it is easily shown that sway mode is critical. So we analyze only that. Equivalent column is,



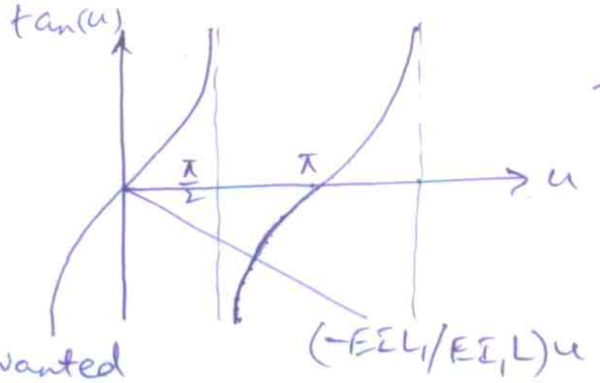
$\alpha_0 = \beta_0 = \infty$, $\alpha_L = 0$, $\beta_L = \frac{1}{EI}\frac{6EI_1}{L_1}$

CE is, (6th order CE, p34, Topic-II)

$$\frac{6EI_1}{EI\,L_1}\frac{u^4}{L^4}\sin u + \frac{u^5}{L^5}\cos u = 0$$

$$\tan u = -\frac{EI\,L_1}{6EI_1 L}u \;,\quad u = kl,\; k^2 = P/EI.$$

Not reqd due to



$\frac{\pi}{2} \le u_{cr} \le \pi$

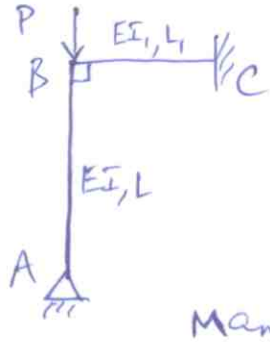
If $\frac{EI_1}{L_1} = \frac{EI}{L}$, $\tan u = -\frac{u}{6}$

$\Rightarrow u_{cr} = 2.716$

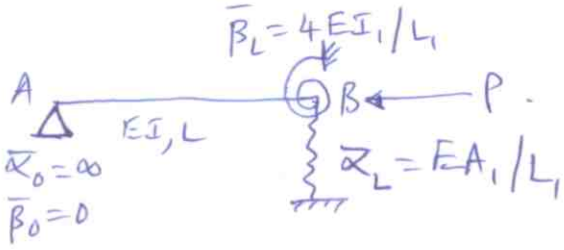
$P_{cr} = 7.379\frac{EI}{L^2}$

If we wanted to analyze mode(i) $\rightarrow$ $\alpha_L = \infty$, $\beta_L = 2EI_1/L_1$, to analyze mode(ii), $\alpha_L = \infty$, $\beta_L = 6EI_1/L_1$

Partial Frames.



equivalent column



Many times $\bar{\alpha}_L$ is large and taken as ∞ (ie only no-sway case practically realized).

So for $\alpha_0 = \alpha_L = \infty$, $\beta_0 = 0$, $\beta_L = \frac{4EI_1}{L_1}\frac{1}{EI}$, 6th order CE yields, (p.34, Topic II)

$$\left(\frac{Lu^4}{L^4} + \frac{4EI_1}{L_1}\frac{1}{EI}\frac{u^2}{L^2}\right)\sin u + \left(-\frac{4EI_1}{L_1}\frac{1}{EI}\frac{Lu^3}{L^3}\right)\cos u = 0.$$

$$\tan u = \frac{u}{\left(\frac{EIL_1}{4EI_1L}\right)u^2 + 1}, \quad (u = kL \text{ here}).$$

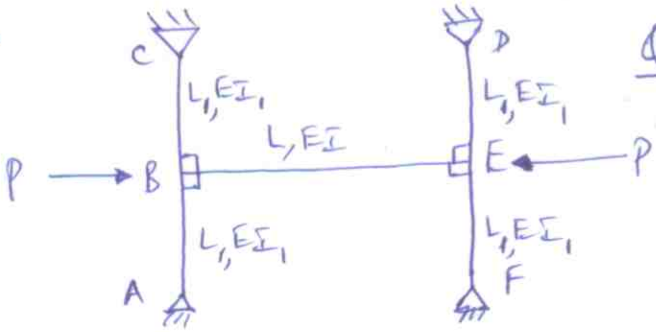
If end A is fixed, $\beta_0 = \infty$. Then,

$$\left(1 - \frac{4EI_1}{L_1}\frac{1}{EI}L\right)\frac{u^2}{L^2}\sin u - \frac{Lu^3}{L^3}\cos u + 2\cdot\frac{4EI_1}{L_1}\frac{1}{EI}\frac{u}{L} = 0.$$

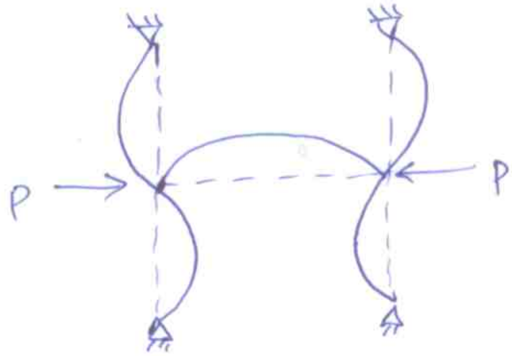
$(u = kL$ here.)

Problems from Simitses and Hodges book.

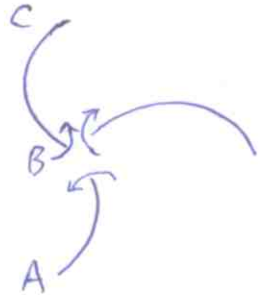
P. 4.1



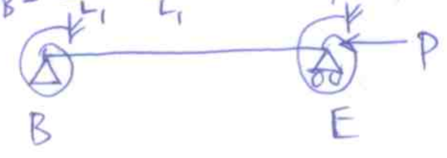
Q: Column BE rigidly attached to AC and DF. Find P_{cr} for in-plane buckling.



Symmetric column mode (1st mode) yields lowest critical load, so we analyze only that.



$\beta_B = \frac{3EI_1}{L_1} + \frac{3EI_1}{L_1}$ (from BC, from BA) — use $\phi(0) = \psi(0) = 1$, $M_C = M_A = 0$ in slope defl eqns ① p. 2.



Equivalent Column

$\beta_E = \beta_B$

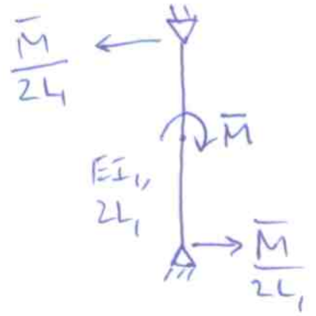
ie $\beta_B = \frac{6EI_1}{L_1}$

Use symmetric buckling eqn 8 p. 3,

$$\tan u = -\frac{1}{\beta}\frac{2EI}{L}u = -\frac{L_1}{6EI_1}\frac{2EI}{L}u \quad , \quad u = \frac{kL}{2}$$

i.e. $\frac{\pi}{2} \le u_{cr} \le \pi$.

Aside: verify β_B is correct by considering as single rod AC. Use Castigliano's theorem.



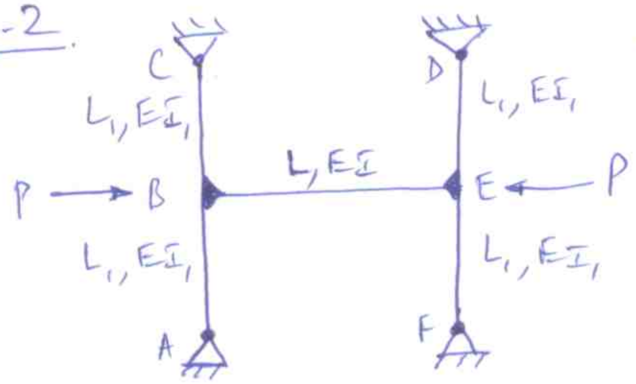
$$U = \frac{1}{2EI_1}\int_0^{2L_1} M(x)^2\,dx = \frac{1}{2EI_1}\left[\int_0^{L_1}\frac{\bar{M}^2}{4L_1^2}x^2\,dx + \int_{L_1}^{2L_1}\left(\frac{\bar{M}}{2L_1}x - \bar{M}\right)^2 dx\right]$$

$$= \frac{1}{2EI_1}\left[\frac{\bar{M}^2}{4L_1^2}\frac{L_1^3}{3} + \left(\frac{\bar{M}^2}{4L_1^2}\frac{x^3}{3} + \bar{M}^2 x - \frac{\bar{M}^2}{L_1}\frac{x^2}{2}\right)\Bigg|_{L_1}^{2L_1}\right]$$

$$= \frac{1}{2EI_1}\bar{M}^2 L_1\left[\frac{1}{12} + \frac{7}{12} + 1 - \frac{3}{2}\right] = \frac{\bar{M}^2 L_1}{12EI_1}$$

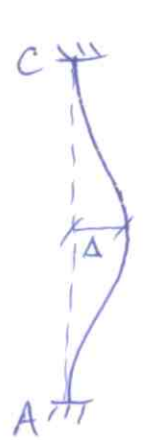
$$\theta(L_1) = \frac{\partial U}{\partial \bar{M}} = \frac{\bar{M}L_1}{6EI_1} \longrightarrow \text{verified}$$

P4-2

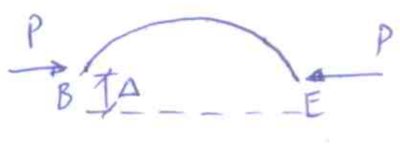


Q: Flexural rigidities EI, EI_1 are for out-of-plane bending. By raising EI_1 we can increase P_{cr} for out-of-plane buckling. Show that increase in P_{cr} is possible only until EI_1 reaches value $\pi^2 L_1^3 EI/12L^3$. Neglect torsional rigidity of AC & FD.

Case I

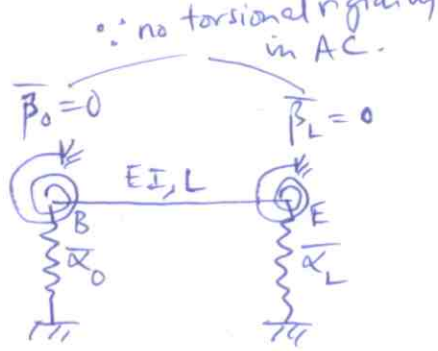


SIDE VIEW



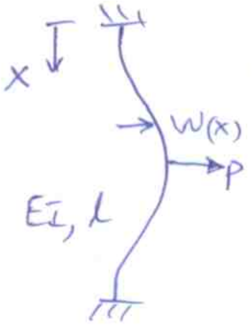
TOP VIEW.

Equt Column $\longrightarrow$



$\alpha_0 = \alpha_L$

For $\bar{\alpha}_0$ refer Solid Mech text (eg Popov) (or derive it).



Result : $w(x) = \dfrac{Px^3}{12EI} - \dfrac{PLx^2}{16EI} \Rightarrow w\left(\dfrac{l}{2}\right) = \dfrac{Pl^3}{192EI}$

Put $l = 2L_1$, $EI = EI_1 \Rightarrow w(L_1) = \dfrac{PL_1^3}{24EI_1}$.

$\Rightarrow \bar{\alpha}_0 = \bar{\alpha}_L = 24EI_1/L^3$

Put $\alpha_0 = \alpha_L = \dfrac{24EI_1}{EIL_1^3}$, $\beta_0 = \beta_L = 0$ in 6th order CE (p-34 - Topic II). We get,
$= \alpha$

$$\left(-2\alpha \frac{u^6}{L^6} + \alpha^2 L \frac{u^4}{L^4}\right)\sin u = 0 \quad , \quad u = kL$$

$$\Rightarrow \sin u = 0 \quad \text{or} \quad \frac{u^2}{L^2} = \frac{\alpha L}{2}$$

$k_1 = k = \dfrac{n\pi}{L}$ or $k^2 = \dfrac{\alpha L}{2} = \dfrac{24EI_1}{EIL_1^3}\dfrac{L}{2} = \dfrac{12EI_1L}{EIL_1^3} = k_2^2$

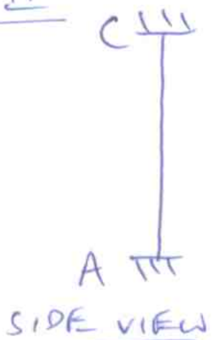
$n=1$
critical.

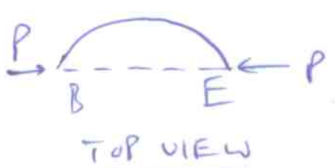
So k_2 is critical (ie decides buckling load) and increases as EI_1 is increased until $k_2 = k_1 = \pi/L$, ie,

$$\frac{12EI_1L}{EIL_1^3} = \frac{\pi^2}{L^2} \Rightarrow (EI_1)_{critical} = \frac{\pi^2 EIL_1^3}{12L^3}$$

beyond this value of EI_1, the column BE buckles at the Euler buckling load irrespective of how much more you increase EI_1.

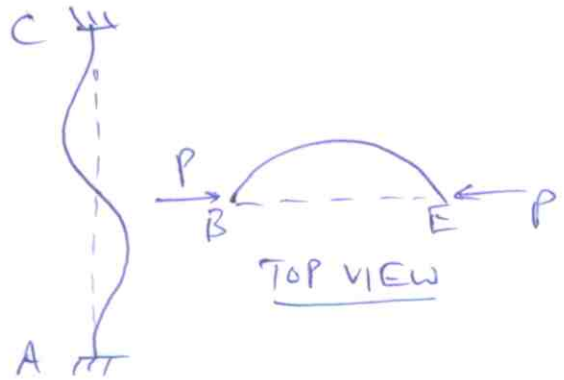
Case II





i.e, AC remains straight ($w(x) = \Delta = 0$ in Case I).
Since AC has no torsional rigidity, BE will buckle at Euler load P_E.
So Case II is not the critical one, since Case I gives lower P_{cr}.

Case III

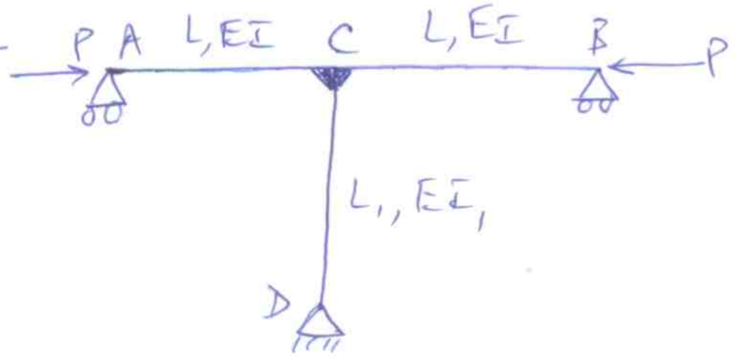


SIDE VIEW

Again BE will buckle at P_{Euler}, so Case III not critical.

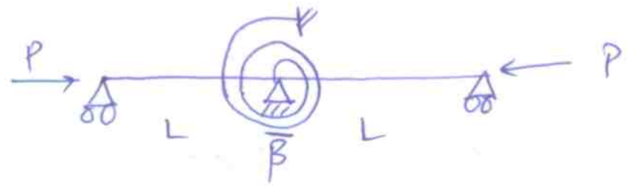
So we need only consider Case I.

P4.3

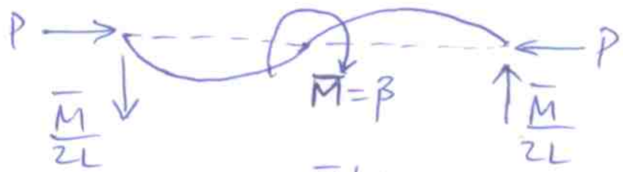


Q: AB and CD are rigidly attached. Derive CE and specialize when $EI = EI_1$, $2L_1 = L$.

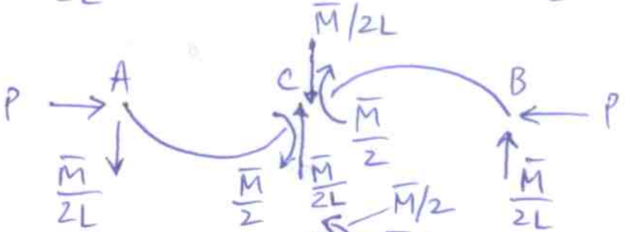
Equivalent column is:



For CD, $M_D = 0$, $\phi = \psi = 1$ in ① p-2 gives $\dfrac{M_C}{\theta_C} = \dfrac{3EI_1}{L_1} = \bar\beta$



⇐ FBD in deformed configuration



⇐ Internal forces on AC, CB

⇐ Internal forces at C.

$$\frac{M_C}{\theta_C} = \frac{\bar M}{\theta_C} = \bar\beta \Rightarrow \frac{\bar M/2}{\theta_C} = \bar\beta/2$$

So problem reduces to

$\bar{\beta} = 3EI_1/L_1$

Analogous to

Put $\alpha_0 = \alpha_L = \infty$, $\beta_0 = 0$, $\beta_L = \dfrac{3}{2}\dfrac{EI_1/L_1}{EI}$ in 6^{th} order CE (p.3.4, Topic II)

$$\left(L\frac{u^4}{L^4} + \beta_L\frac{u^2}{L^2}\right)\sin u - L\beta_L\frac{u^3}{L^3}\cos u = 0$$

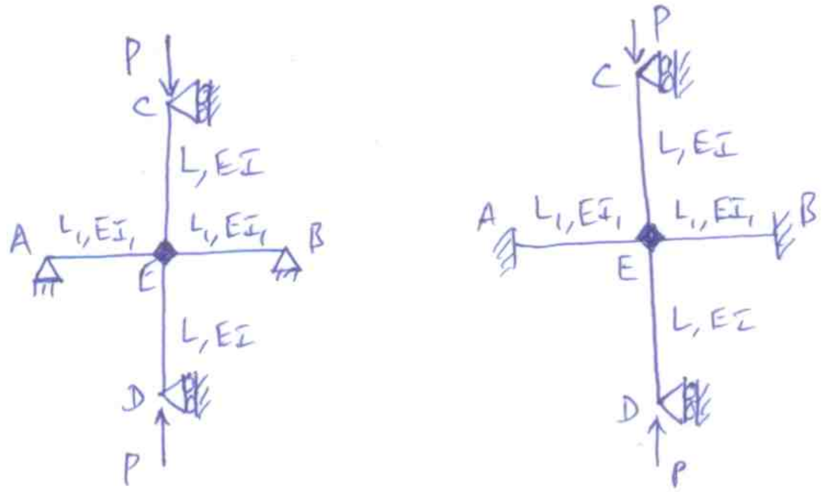
$$\tan u = \frac{u}{\frac{1}{\beta_L L}u^2 + 1} = \frac{u}{\frac{2EIL_1}{3EI_1L}u^2+1}$$

For $EI = EI_1$, $2L_1 = L$, $\tan u = \dfrac{u}{\frac{u^2}{3}+1}$

Will be done later by Matrix Stiffness Method also.

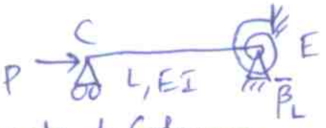
P.4.4

Q: Derive CE's.

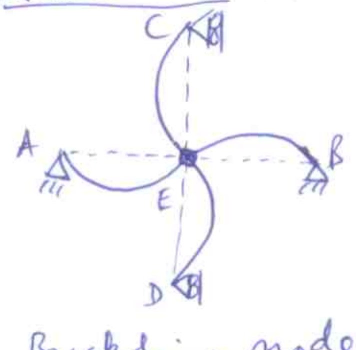


(a) (b)

(a)



Equivalent Column.



Buckling mode

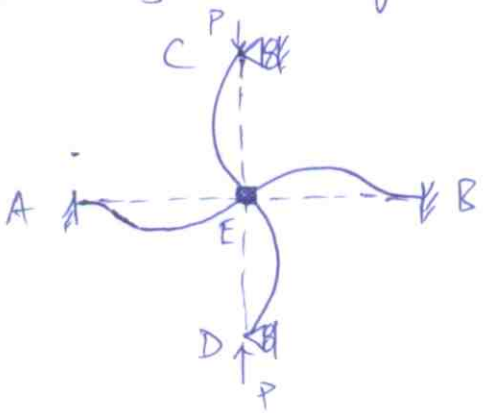
$M_A = 0$, $\phi = \psi = 1$, Slope-defl eq (1), p 2,

$\dfrac{M_E}{\theta_E} = \bar{\beta}_L = \dfrac{3EI_1}{L_1} \Rightarrow \beta_L = \dfrac{3EI_1}{EIL_1}$

Rest is same as in P.4.3, so result is

$\tan u = u\Big/\left[\dfrac{EIL_1}{3EI_1L}u^2 + 1\right]$

(b) Only β_L changes. Equivalent column remains same.

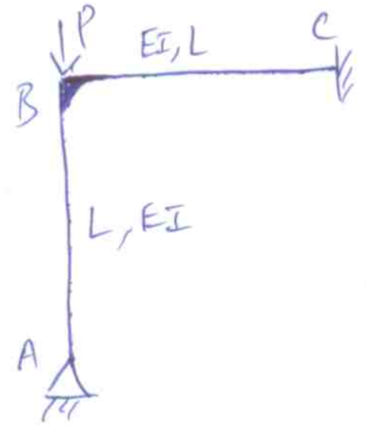


$\theta_A=0,\ \phi=\psi=1$ in slope-defl eq ① p2,

$$\frac{M_E}{\theta_E}=\bar{\beta}_L=\frac{4EI_1}{L_1} \Rightarrow \beta_L=\frac{4EI_1}{EIL_1}$$

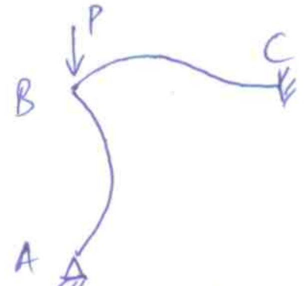
$$\Rightarrow \tan u = \frac{u}{\left(\frac{EIL_1}{4EI_1L}u^2+1\right)}$$

P4-6

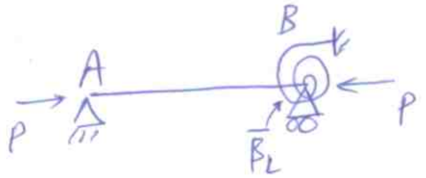


Q: Will it buckle in-plane or out of plane? Cross section circular, $R_0 \ll L$.

In-plane buckling



Buckled mode.



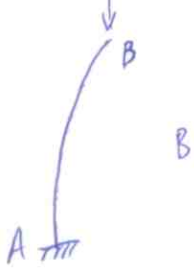
Eqvt column

$$\bar{\beta}_L=\frac{4EI}{L},\ \beta_L=\frac{4}{L}$$

$$\alpha_0=\alpha_L=\infty,\ \beta_0=0$$

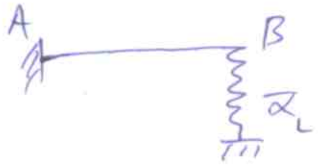
CE (6th order) yields, $\tan u = \dfrac{u}{(\beta_L L)^{-1}u^2+1} = \dfrac{u}{\frac{u^2}{4}+1}$

Out of plane buckling



SIDE VIEW

TOP VIEW



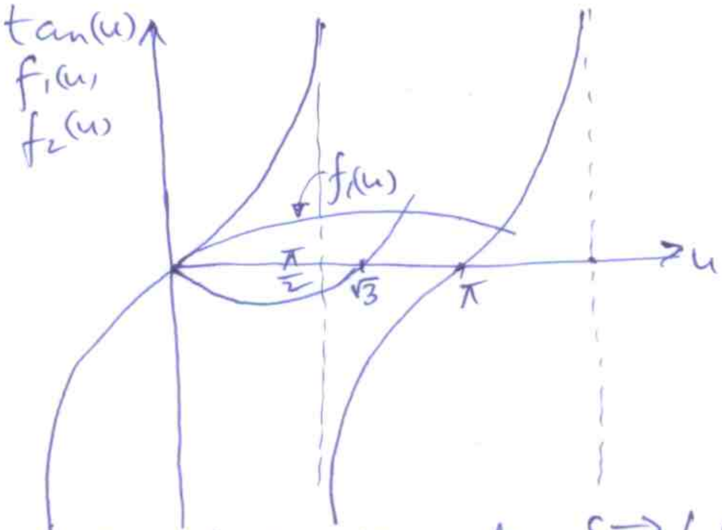
$$\alpha_0=\beta_0=\infty,\ \beta_L=0$$

$$\bar{\alpha}_L=3EI/L^3$$

$$\alpha_L=3/L^3$$

CE (6th order, p.34, Topic II) $\longrightarrow \alpha_L\frac{u^2}{L^2}\sin u+\left(\frac{u^5}{L^5}-\alpha_L L\frac{u^3}{L^3}\right)\cos u=0$

$$\tan u = \frac{u^3}{\alpha_L L^3} - u = \frac{u^3}{3} - u$$



Qualitative reasoning. (can do numerical plot instead)

$$f_1(u) = \frac{u^2}{4} + 1$$

$$f_2(u) = \frac{u^3}{3} - u$$

$f_2(u) < 0$ for small u

$f_2(u) = 0$ for $u = \sqrt{3}$

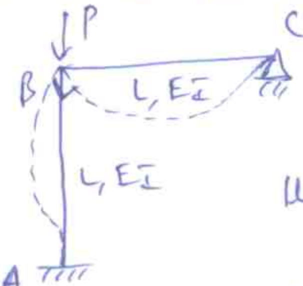
$f_1(u) > 0 \quad \forall u$

$$\left.\frac{d f_2(u)}{du}\right|_{u=\pi} > \left.\frac{d(\tan u)}{du}\right|_{u=\pi}$$

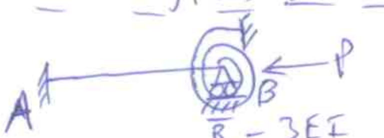
► It will buckle in inplane mode $\Leftarrow \Big\{ \Rightarrow f_2(u)$ intersects $\tan u$ at $u \to$
$\Rightarrow f_1(u)$ intersects $\tan u$ for $\pi \le u \le 3\pi/2$

P. 4.7 (a)

$\boxed{\text{Q = Analyze for inplane buckling.}}$



No-sway case (it is critical one)



Eqvt Column

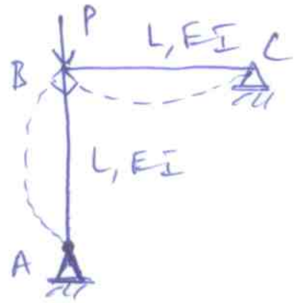
$\bar{\beta}_L = \frac{3EI}{L}$, $\beta_L = 3/L$, $\alpha_0 = \beta_0 = \infty$, $\alpha_L = \infty$

Use 6th order CE.
see top of p.12. So the result is,

$$(1-4)\frac{u^2}{L^2}\sin u - \frac{u^3}{L^2}\cos u + 2 \times \frac{3}{L^2} u = 0$$

$$3u \sin u + u^2 \cos u = 6$$

(b)



No sway case (its the critical one)

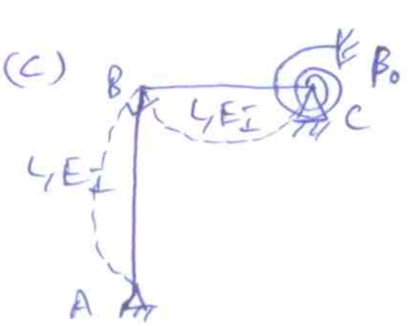
Eqvt Column

$\bar{\beta}_L = \frac{3EI}{L}$, $\beta_L = 3/L$, $\alpha_0 = \alpha_L = \infty$, $\beta_0 = 0$

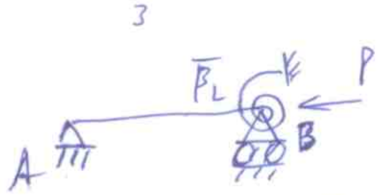
So solution identical to P(4-4)(a) with $EI = EI_1$, $L = L_1$, hence

$$\tan u = \frac{u}{\frac{u^2}{3} + 1}$$

(c)



No sway case (critical one)



Eqvt Column.

In slope-defl eqn ① p.2, put $M_C = -\beta_0 \theta_C$, $\phi = \psi = 1$, gives

$$\theta_B = \frac{M_B L}{3EI} - \frac{\beta_0 \theta_C L}{6EI}, \quad \theta_C = -\frac{\beta_0 \theta_C L}{3EI} + \frac{M_B L}{6EI}$$

solution is, $\theta_B = M_B \frac{L}{EI}\left(\frac{1}{3} - \frac{\beta_0 L}{6EI}\left[\frac{1}{6}\Big/\left\{1+\beta_0/3EI\right\}\right]\right)$

$$1/\bar{\beta}_L = \frac{\theta_B}{M_B} = \frac{L}{EI}\left(\frac{1}{3} - \frac{\beta_0 L}{\cancel{6EI}_{2}}\,\frac{1}{6}\,\frac{\cancel{3EI}}{3EI+\beta_0 L}\right)$$

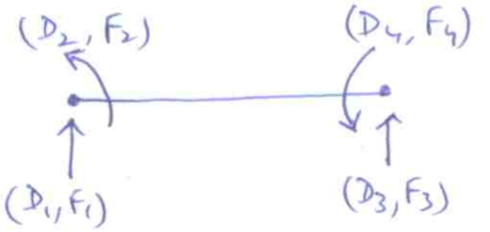
$$= \frac{L}{EI}\left(\frac{12EI + 3\beta_0 L}{36EI + 12\beta_0 L}\right)$$

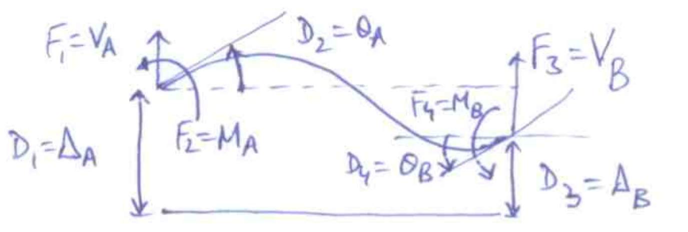
$$\Rightarrow \bar{\beta}_L = \frac{EI}{L}\left(\frac{12EI/\beta_0 L + 4}{4EI/\beta_0 L + 1}\right), \quad \beta_L = \frac{\bar{\beta}_L}{EI}, \quad \alpha_0 = \alpha_L = \infty, \quad \beta_0 = 0$$

6th order CE gives,

$$\tan u = \frac{u}{\left(\frac{4EI/\beta_0 L + 1}{12EI/\beta_0 L + 4}\right)u^2 + 1}$$

MATRIX (STIFFNESS) METHOD





→ replace notation $V_A \to Q_A$, $V_B \to -Q_B$ for transverse shear forces — standard notation.

Fig: Degrees of Freedom for Beam Element.

Concept of stability. (* Axial compressive/tensile).

*Axial load P has the effect of lowering/increasing the stiffness (rotational or shear) of the element and structure as a whole. We will incorporate this effect into our existing knowledge of stiffness matrix method.

For example consider the rigid joint of a structure, with load M applied.

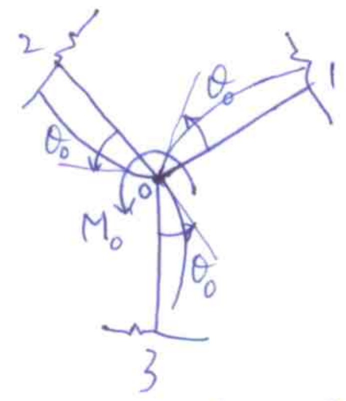


Fig: Moment applied at rigid joint.

Joint stiffness $= k_{01} + k_{02} + k_{03} = k_0$

$$M_0 = k_0 \theta_0$$

When $k_0 \to 0$, $M_0/k_0 \to \infty$ ie $\theta_0 \to \infty$ ie unstable. As axial compressive load (P) in members increases, k_0 decreases until $k_0 = 0$ is the buckling condition.

In general, for a structure we have,

$$\underline{F} = \underline{\underline{K}}\, \underline{D}$$

$\underline{F}$ = load vector (force, moments at joints)

$\underline{D}$ = displ vector (rotations, translations at joints).

$\underline{\underline{K}}$ = stiffness matrix defined by

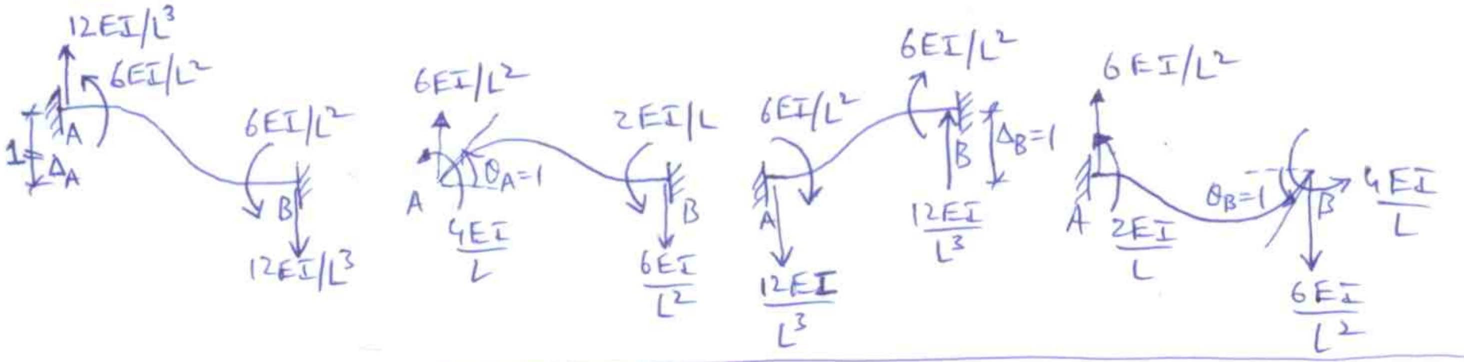
K_{ij} = F_i required to produce $D_j = 1$, $D_k = 0$, $k \neq j$, $(i, k) = (1, \ldots N)$.

Thus when $\det[\underline{\underline{K}}] = 0$, we have ∞ $\underline{D}$ for $\underline{F} \neq 0$ or non-trivial $\underline{D}$ for $\underline{F} = 0$, ie, condition of instability.

Thus joint stiffness = 0 or $\det[\underline{\underline{K}}] = 0$ yield the characteristic equation for P_{cr}. As P increases, overall resistance of structure to random disturbance decreases. At P_{cr} the structure offers no resistance to disturbance loading, ie $\underline{D}$ can increase without bound with $\underline{F}$ held constant, and $\underline{D}$ (representing configuration of structure) is non-unique for the beam-column theory considered in *CE619 (recall initial classes where columns were stated as neutral in post-buckling, ie, w-displ or end shortening was a flat line beyond P_{cr}).

(* Moderate rotations, small strains.)

STIFFNESS MATRIX without P effect.



$$\underset{=}{K} = \frac{EI}{L}\begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix}, \quad \underline{D} = \begin{Bmatrix} \Delta_A \\ \theta_A \\ \Delta_B \\ \theta_B \end{Bmatrix}, \quad \underline{F} = \begin{Bmatrix} Q_A \\ M_A \\ Q_B \\ M_B \end{Bmatrix}$$

or $$\underset{=}{K} = \frac{EI}{L}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}, \quad \underline{D} = \begin{Bmatrix} \Delta_A/L \\ \theta_A \\ \Delta_B/L \\ \theta_B \end{Bmatrix}, \quad \underline{F} = \begin{Bmatrix} Q_A L \\ M_A \\ Q_B L \\ M_B \end{Bmatrix}$$

→ 12

STABILITY FUNCTIONS. → We wish to find effect of P on K_{ij}'s.

(I) → Unit Rotation at A.

Fig: Beam-Col with unit Rotation applied.

$$w^{IV} + k^2 w'' = 0$$

$$w(x) = \bar{A}\sin kx + \bar{B}\cos kx + \bar{C}x + \bar{D}$$

(as before)

B.C's: $w(0) = w(L) = w'(L) = 0, \quad w'(0) = 1$

Apply BC's and solve for $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ (details omitted, straightforward, not required).

You get, $\bar{A} = \dfrac{1 - uS - C}{k(2 - 2C - uS)}$, $\bar{D} = -\bar{B}$, $\bar{C} = L(1 - \bar{A}k)$.

$\bar{B} = \dfrac{S - uC}{k(2 - 2C - uS)}$, $\boxed{u = kL,\ S = \sin u,\ C = \cos u}$

capital C

$$M_A = -EIw''(0) = r(EI/L)$$
$$M_B = EIw''(L) = r\underset{\text{(small c)}}{c}(EI/L)$$
$\rightarrow$ (13a) (Convention used is of Fig on p. 19).

$$\boxed{r = r(u) = \frac{u(S-uC)}{(2-2C-uS)}, \quad rc = rc(u) = \frac{u(u-S)}{(2-2C-uS)} \\ u = kL = \pi\sqrt{P/P_e} = \pi\sqrt{\rho}, \quad S = \sin u, \quad C = \cos u} \rightarrow (13)$$

$$\boxed{c = \frac{rc}{r} = \frac{(u-S)}{(S-uC)}}$$

small c represents carry-over, ie $c = M_B/M_A$

NOTE: Dont confuse the two c's.

From $Q_A = EIw'''(0)$

or from external equilibrium,

ie $M_A + M_B - Q_A L = 0$

we get,

$$Q_A = q\left(\frac{EI}{L^2}\right) \rightarrow (14a)$$

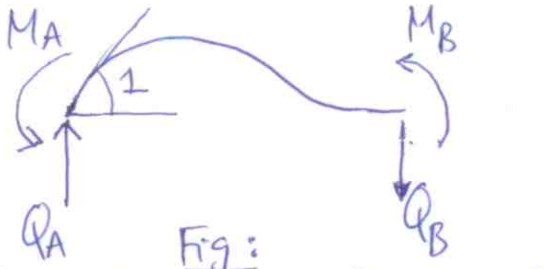


Fig: Positive convention used here for M_A, M_B, Q_A, Q_B

$$\boxed{q = \frac{u^2(1-C)}{(2-2C-uS)}} \rightarrow (14)$$

Thus, $\boxed{k_{12} = q\left(\frac{EI}{L^2}\right), \; k_{22} = r\left(\frac{EI}{L}\right), \; k_{32} = -q\left(\frac{EI}{L^2}\right), \; k_{42} = rc\left(\frac{EI}{L}\right)}$ $\rightarrow$ (14b)

If θ_A applied at left end (instead of unit rotation),

$M_A = r(EI/L)\theta_A$, $M_B = rc(EI/L)\theta_A$, $Q_A = q(EI/L^2)\theta_A$

Now if end B is pinned and unit rotation applied at A,

BC's: $w(0) = w(L) = 0$, $w'(0) = 1$, $w''(L) = 0$

Can solve again for A, B, C, D. But an easier way is to use previous solution (for $\theta_A = 1$, B fixed) and apply opposite moment at B (keeping A fixed), ie $M_B = -rc(EI/L)$ to make total $M_B = 0$. Then $\boxed{M_A = \left(\frac{EI}{L}\right)(r - (rc)c) = (EI/L)\,r(1-c^2) = (EI/L)\,r'; \; k_{22} = (EI/L)\,r', \; r' = r(1-c^2)}$ → (15)

$\theta_A = 1$ applied, A, B pinned = $\theta_A = 1$ applied, B fixed, A pinned + $M_B = -rc(EI/L)$ applied, B pinned, A fixed

So If θ_A applied at A, B pinned, $M_A = \left(\frac{EI}{L}\right) r'\,\theta_A$. → (a)

Note that this case (θ_A applied, B pinned) yields, from slope-defl equations (eq ①, p 2, with $M_B = 0$)

$$\theta_A = \frac{M_A L}{3EI}\psi(u) \Rightarrow M_A = (EI/L)\frac{3}{\psi}\theta_A \longrightarrow (b)$$

Comparing (a), (b), we expect $\boxed{r' = \frac{3}{\psi}}$ → (16). Lets verify this.

Verification: replace u by $2u$ in r' since $u = kL/2$ in ψ definition.

$$r' = r(1-c^2) = \frac{2u(\sin 2u - 2u\cos 2u)}{(2 - 2\cos 2u - 2u\sin 2u)} * \left[1 - \frac{(2u - \sin 2u)^2}{(\sin 2u - 2u\cos 2u)^2}\right]$$

$$= \frac{2u}{(2 - 2\cos 2u - 2u\sin 2u)}\left[\frac{(\sin 2u - 2u\cos 2u)^2 - (2u - \sin 2u)^2}{(\sin 2u - 2u\cos 2u)}\right]$$

$$= \frac{2u(4u^2\cos^2 2u - 4u\sin 2u\cos 2u - 4u^2 + 4u\sin 2u)}{(2 - 2\cos 2u - 2u\sin 2u)(\sin 2u - 2u\cos 2u)}$$

$$= \frac{(2u)(2u)\cancel{(-2u\sin 2u - 2\cos 2u + 2)}(\sin 2u)}{\cancel{(2 - 2\cos 2u - 2u\sin 2u)}(\sin 2u - 2u\cos 2u)} = \text{LHS}.$$

$$RHS = \frac{3}{\psi(u)} = \frac{2u(2u\tan 2u)}{(\tan 2u - 2u)} = \frac{(2u)(2u)\sin 2u}{\sin 2u - 2u\cos 2u}$$

LHS = RHS → hence verified.

Coefficients r, rc, c, r', q are tabulated in given handout (Appendix A.1) for various $\rho \left(= \sqrt{P/P_E} = u^2/\pi\right)$.
When we obtain the CE's, you solve them either numerically or by using these tables and hit-n-trial with interpolation.

For $P = 0$ (no axial load), ie $\rho = 0$, refer $\underline{K}$ p. 21,

$M_A = 4(EI/L)$, $Q_A = 6(EI/L^2)$, $M_B = 2EI/L$

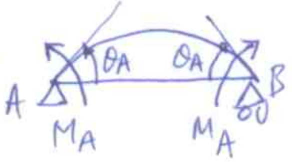
ie $\boxed{r(0) = 4,\ q(0) = 6,\ rc(0) = 2}$ for $P = 0$ (when comparing with M_A, M_B, Q_A, eqs 13a, 14a, p. 22)

Table also verifies this.

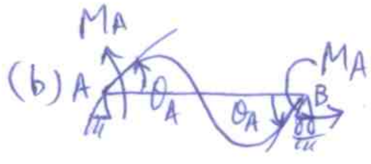
Note: From (13), (14) p. 22 you can use L'Hopitals rule for $u \to 0$ and also verify the same.

Similarly from slope-defl eq (1), p. 2, with $M_B = 0$, $\psi = \phi = 1$,
get $M_A = 3(EI/L) \Rightarrow \boxed{r'(0) = 3.}$ → (Table also verifies it)

For $P = 0$, the following cases (well known from before) are also useful:

(a) 

ie symmetric bending with equal but* opposite applied end moments. In slope defl eqn (1) p 2, put $\phi = \psi = 1$, $M_A = M_B$, get $\theta_A = \theta_B$, $\boxed{M_A/\theta_A = 2EI/L}$

(b) 

ie antisymm bending with equal applied end moments. In slope defl eqn (1) p 2, put $M_B = -M_A$, ($\phi = \psi = 1$)
get $\theta_B = -\theta_A$, $\boxed{M_A/\theta_A = 6EI/L}$

(* opposite wrt convention of Fig on P. 19) (Note: Convention in Figs on p. 183 is not same).

(II) Unit Displacement at A

(ie equivalent to unit relative displ between ends A & B).

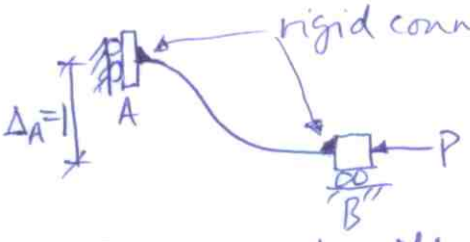


Fig: Beam-Col with unit Displ applied

BC's: $W'(0) = W'(L) = W(L) = 0$

$W(0) = 1$

Solve for $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, get,

$$\bar{A} = -\frac{S}{(2-2C-uS)}, \quad \bar{B} = -\frac{(C-1)}{(2-2C-uS)}$$

$$\bar{C} = -\frac{uS}{(2-2C-uS)}, \quad \bar{D} = \frac{1-C-uS}{(2-2C-uS)}$$

$$\left.\begin{aligned} M_A &= -EIW''(0) = q\,(EI/L^2) \\ M_B &= EIW''(L) = q\,(EI/L^2) \end{aligned}\right\} \rightarrow (17a)$$

(Convention used is of Fig. on p-19).

$$\left.\begin{aligned} Q_A &= EIW'''(0) = s\,(EI/L^3) \\ Q_B &= EIW'''(L) = s\,(EI/L^3) \end{aligned}\right\} \rightarrow (17b)$$

(convention used is of Fig on p. 22) or p-19

(small 's')

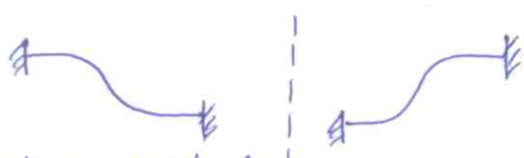
$$\boxed{s = \frac{u^3 S}{(2-2C-uS)}} \rightarrow (17)$$

Thus, $\boxed{k_{11} = s(EI/L^3),\ k_{21} = q(EI/L^2),\ k_{31} = -s(EI/L^3),\ k_{41} = q(EI/L^2)} \rightarrow (17c)$

STIFFNESS MATRIX WITH P EFFECT

Thus we can now write

$$\underline{K} = \frac{EI}{L}\begin{bmatrix} s & q & -s & q \\ q & r & -q & rc \\ -s & -q & s & -q \\ q & rc & -q & r \end{bmatrix}$$



To get 3rd col: Take 1st column in $\underline{K}$, interchange 1st & 3rd, 2nd & 4th entries, then change the sign on resulting 2nd & 4th entries which represent moments.

↳ To get 4th col of $\underline{\underline{K}}$, take 2nd col, interchange 1st & 3rd, 2nd & 4th entries, then change sign on the resulting 1st & 3rd entries which represent shear forces.

The corresponding $\underline{D}$ & $\underline{F}$ are defined in Eq (12)$_2$, p. 21

For $P=0$, $\underline{\underline{K}}$ reduces to that in Eq (12)$_2$

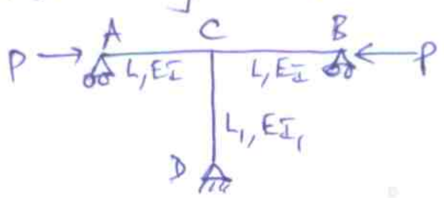
For tensile P the d.e. is $w^{IV} - k^2 w'' = 0$. Thus if we replace $k \to ik$, $u \to iu$, in Eqs (13), (14), (17) on p. 22, 25, we get the stability functions. Now note that $\sin iu = i\sinh u$, $\cos = \cosh u$. Thus stability functions become

$$r = \frac{u(uC-S)}{(2-2C+uS)}, \quad rc = \frac{u(S-u)}{(2-2C+uS)}, \quad q = \frac{u^2(C-1)}{(2-2C+uS)}$$

$$s = \frac{u^3 S}{(2-2C+uS)}, \quad S = \sinh u, \quad C = \cosh u$$

(Ex1) Rework P4.3 of Simitses and Hodges, p. 15.

(i) Using Joint Stiffness.

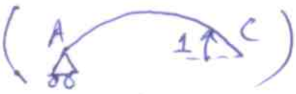


$k_C = k_{CA} + k_{CB} + k_{CD}$

$k_{CA} = k_{CB} = \dfrac{3}{\psi}\dfrac{EI}{L}$ (from slope-defl Eq (1), p. 2, with $M_B = 0$, find M_A/θ_A).

$k_{CD} = \dfrac{3EI_1}{L_1}$ (ie $\psi = 1$ in k_{CA}; $EI \to EI_1$, $L \to L_1$)

$$k_C = \frac{6}{\psi}\frac{EI}{L} + 3\frac{EI_1}{L_1} \longrightarrow (a)$$

Alternatively, $k_{CA} = k_{CB} = r'\dfrac{EI}{L}$ ()

$$k_{CD} = r'\big|_{P=0}\frac{EI_1}{L_1} = \frac{3EI_1}{L_1}$$

Again $k_C = 2r'\dfrac{EI}{L} + \dfrac{3EI_1}{L_1} \longrightarrow (b)$

· From Eq (16) p. 23, (a), (b) are the same. The CE is $\boxed{k_C = 0 = \dfrac{6}{\psi}\dfrac{EI}{L} + \dfrac{3EI_1}{L_1}}$

This CE matches the one obtained on p.16 for this problem.
To see this, subst Eq ③ p. 2 with $2u \to u$.

$$\frac{6}{\psi}\frac{EI}{L} + \frac{3EI_1}{L_1} = 0 = \frac{6u^2 \tan u}{3(\tan u - u)}\frac{EI}{L} + \frac{3EI_1}{L_1}$$

$$\Rightarrow \tan u = \left(\frac{3}{2}\frac{EI_1}{L_1}\frac{L}{EI}\,u\right)\Big/\left(u^2 + \frac{3}{2}\frac{EI_1}{L_1}\frac{L}{EI}\right) \to \text{verified}.$$

(ii) Using the Stiffness Matrix
Four d.o.f's, $\{\theta_C, \theta_A, \theta_B, \theta_D\}^T$ with corresponding $\{M_C, M_A, M_B, M_D\}^T$

$$\underline{\underline{K}} = \frac{EI}{L}\begin{bmatrix} r + r + \frac{L}{EI}\frac{4EI_1}{L_1} & rc & rc & \frac{2EI_1}{L_1}\frac{L}{EI} \\ rc & r & 0 & 0 \\ rc & 0 & r & 0 \\ 2\frac{EI_1}{L_1}\frac{L}{EI} & 0 & 0 & \frac{L}{EI}\frac{4EI_1}{L_1} \end{bmatrix}$$

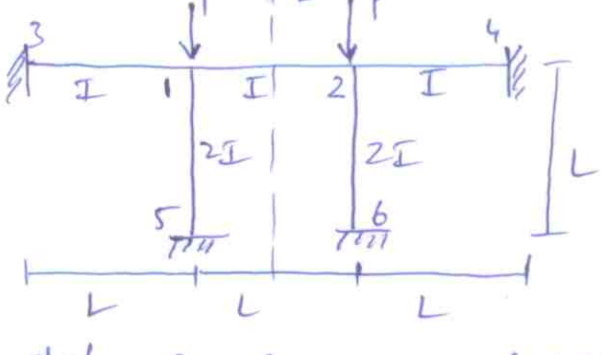
Now $M_A = M_B = M_D = 0$ (let $C \to 1$, $A \to 2$, $B \to 3$, $D \to 4$) jt numbering.

$$\Rightarrow \theta_4 = -\theta_1/2, \quad \theta_3 = \theta_2 = -\frac{rc}{r}\theta_1 = -c\theta_1$$

$$\Rightarrow \left(r + r + \frac{L}{EI}\frac{4EI_1}{L_1} - rc^2 - rc^2 - \frac{1}{2}\frac{2EI_1}{L_1}\frac{L}{EI}\right)\theta_1 = \left(\cancel{M_1}\right)_{\text{applied}} \to 0$$

$$\Rightarrow 2r(1-c^2) + 3\frac{EI_1}{L_1}\frac{L}{EI} = 0 = \boxed{2r' + 3\frac{EI_1}{L_1}\frac{L}{EI}} \to \text{same CE.}$$

(Ex 2)



$$\underline{K} = \begin{bmatrix} \frac{4EI}{L} + \frac{4EI}{L} + r\frac{E(2I)}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} + \frac{4EI}{L} + r\frac{E2I}{L} \end{bmatrix}$$

2-dof, θ_1, θ_2.

$\det(\underline{K}) = 0 \to$ CE.

Q: Find P_{cr}

$$\Rightarrow \left(8\frac{EI}{L} + r\,2\frac{EI}{L}\right)^2 - \left(2\frac{EI}{L}\right)^2 = 0$$

$$\boxed{(6+2r)(10+2r) = 0}$$

$\rho = P/P_E$ is $\simeq 2.74$ and 3.00 for $r = -3$ & -5, respectively (from Tables, Appendix A.1 of Gambhir).

$\Rightarrow P_{cr} \simeq (2.74)\, \cancel{P_E} \nearrow \frac{\pi^2 E(2I)}{L^2}$

Alternatively, by joint stiffness method,

$k_1 = k_{12} + k_{13} + k_{15} = \frac{2EI}{L} + \frac{4EI}{L} + r\frac{E(2I)}{L}$.

$k_1 = 0$ is CE $\Rightarrow \boxed{2r + 6 = 0}$ → same result.

Note: You can use the joint stiffness method only since we know that member 1-2 will deform symmetrically wrt ℄, ie equal rotations ($\theta_1 = -\theta_2$ in p.19 Fig convention or $\theta_1 = \theta_2$ in p.1 Fig convention).

If lower ends are pinned, the joint stiffness method gives,

$k_1 = \frac{2EI}{L} + \frac{4EI}{L} + r'\frac{E(2I)}{L} = 0 \Rightarrow \boxed{6 + 2r' = 0}$

Alternatively, doing by $\underline{K}$ matrix method,

$\underline{D} = \{\theta_1 \ \theta_2 \ \theta_5 \ \theta_6\}^T$

$$\underline{K} = \frac{EI}{L}\begin{bmatrix} 4+4+2r & 2 & 2rc & 0 \\ 2 & 4+4+2r & 0 & 2rc \\ 2rc & 0 & 2r & 0 \\ 0 & 2rc & 0 & 2r \end{bmatrix}$$

(The 2 in 2r is due to 2I).

Use $M_5 = M_6 = 0$ to eliminate θ_5, θ_6 dof. Get,

$\theta_5 = -c\theta_1$, $\theta_6 = -c\theta_1$

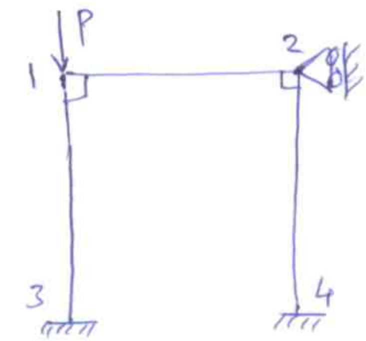
$$\Rightarrow \underline{K} = \begin{bmatrix} 4+4+2r' & 2 \\ 2 & 4+4+2r' \end{bmatrix}\frac{EI}{L}, \quad \underline{D} = \{\theta_1 \ \theta_2\}^T$$

$|\underline{K}| = 0 = \boxed{(6+2r')(10+2r') = 0}$ → partly same as before.

$(6+2r')$ gives lower ρ from Table → $\rho \simeq 1.4$, $P_{cr} = 1.4\pi^2\frac{2EI}{L^2}$

→ same as by joint stiffness method.

(Ex 3.). (i)



L, EI, all members.

$$\underline{D} = \{\theta_1 \;\; \theta_2\}^T$$

$$\underline{\underline{K}} = EI \begin{bmatrix} r+4 & 2 \\ 2 & 4+4 \end{bmatrix}$$

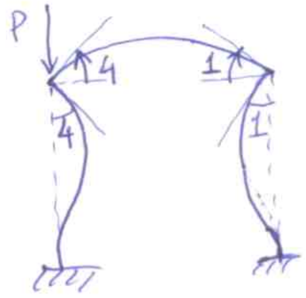
Q: Find P_{cr}, modeshape.

$$\det[\underline{\underline{K}}] = 0 \Rightarrow \boxed{8r + 28 = 0}$$

For $r = -3.5$, $\rho \simeq 2.81$, $P_{cr} = 2.81 \frac{\pi^2 EI}{L^2}$

To find buckling mode solve for $\{\theta_1, \theta_2\}^T$ (eigenvector) corresponding to $r = -3.5$, ie solve

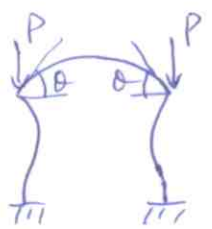
$$EI \begin{bmatrix} -3.5+4 & 2 \\ 2 & 8 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \Rightarrow \theta_1 = -4\theta_2 \rightarrow \text{mode shape}$$



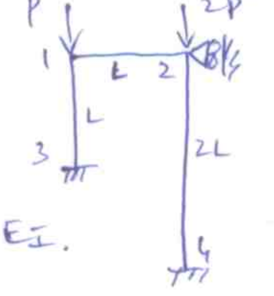
(ii) In addition vertical P (downward) applied at joint 2.
Thus symmetric buckling will occur. So use method of joint stiffness instead (shortcut).

$$k_1 = k_{13} + k_{12} = (r+2)\frac{EI}{L} = 0 \Rightarrow \boxed{r = -2}$$

$\rho = 2.545$, $P_{cr} = 2.545\pi^2 EI/L^2$, $\theta_1 = -\theta_2$



(iii) Instead downward 2P at joint 2. Also member 2-4 has length 2L.



All EI.

Stability ~~coeffs~~ functions of member 1-3 are different from 2-4 since k's and u's are different, ie,

for 1-3 : $k_1^2 = P/EI$, $u_1 = k_1 L$

for 2-4 : $k_2^2 = 2P/EI$, $u_2 = k_2 2L$

So use subscripts 1, 2 for the respective stability functions.

$$\underline{\underline{K}} = \frac{EI}{L}\begin{bmatrix} r_1+4 & 2 \\ 2 & r_2/2+4 \end{bmatrix} \Rightarrow (r_1+4)(0.5r_2+4)-4=0$$

$$\rho_1 = PL^2/\pi^2 EI\ , \quad \rho_2 = \frac{2P(4L^2)}{\pi^2 EI}\ , \quad \frac{\rho_2}{\rho_1} = 8.$$

Choose ρ → read r from table → take ρ_2 as $8 \times$(chosen ρ_1) → read corresponding r_2 → check if CE satisfied → iterate (hit-n-trial), <u>or solve</u> (<u>computationally</u>)

$$0 = (r_1+4)(0.5r_2+4)-4 = \left(\frac{\sqrt{\frac{P}{EI}}\,L\left(\sin\sqrt{\frac{P}{EI}}L - \sqrt{\frac{P}{EI}}L\cos\sqrt{\frac{P}{EI}}L\right)}{2-2\cos\sqrt{\frac{P}{EI}}L - \sqrt{\frac{P}{EI}}L\,\sin\sqrt{\frac{P}{EI}}L}+4\right) *$$

$$\text{Solve for } P \longleftarrow \left(\frac{1}{2} * \frac{\sqrt{\frac{2P}{EI}}\,2L\left(\sin\sqrt{\frac{2P}{EI}}2L - \sqrt{\frac{2P}{EI}}2L\cos\sqrt{\frac{2P}{EI}}2L\right)}{\left[2-2\cos\sqrt{\frac{2P}{EI}}2L - \sqrt{\frac{2P}{EI}}2L\,\sin\sqrt{\frac{2P}{EI}}2L\right]}+4\right)$$

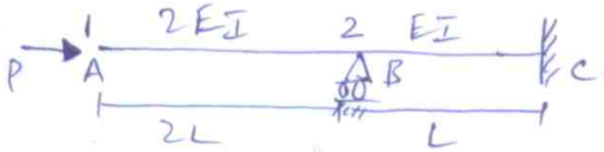
$$-4=0$$

ie
$$\left[\frac{u(\sin u - u\cos u)}{(2-2\cos u - u\sin u)}+4\right]\left[\frac{2\sqrt{2}u(\sin 2\sqrt{2}u - 2\sqrt{2}u\cos u)}{(2-2\cos 2\sqrt{2}u - 2\sqrt{2}u\sin 2\sqrt{2}u)}+4\right]-4=0$$

or solve for u. → then get P from $u = \sqrt{\frac{P}{EI}}L$

After solving r_1, r_2, get eigenvector for mode shape by solving $\underline{\underline{K}}\,\underline{D} = \underline{0}$.

<u>(Ex 4)</u>



Continuous beam.

Q: Solve P_{cr}.

$$\underline{D} = \left\{\frac{\Delta_1}{L}, \theta_1, \theta_2\right\} \qquad \because (\Delta_2, \Delta_3, \theta_3) = (0,0,0) \text{ knock out } 3^{rd}, 5^{th}, 6^{th} \text{ rows and columns.}$$

After assembly of individual $\underline{K}$'s of both elements we get,

$$\underline{\underline{K}} = \frac{EI}{L}\begin{bmatrix} s_1/2^3 & q_1/2^2 & -s_1/2^3 & q_1/2^2 & 0 & 0 \\ q_1/2^2 & r_1/2 & -q_1/2^2 & r_1c_1/2 & 0 & 0 \\ \cancel{-s_1/2^3} & \cancel{-q_1/2^2} & \cancel{s_1/2^3+s_2} & \cancel{-q_1/2^2+q_2} & \cancel{-s_2} & \cancel{q_2} \\ q_1/2^2 & r_1c_1/2 & -q_1/2^2+q_2 & r_1/2+r_2 & -q_2 & r_2c_2 \\ \cancel{0} & \cancel{0} & \cancel{-s_2} & \cancel{-q_2} & \cancel{s_2} & \cancel{-q_2} \\ \cancel{0} & \cancel{0} & \cancel{q_2} & \cancel{r_2c_2} & \cancel{-q_2} & \cancel{r_2} \end{bmatrix}$$

$$\underline{\underline{K}} = \frac{EI}{L}\begin{bmatrix} s_1/8 & q_1/4 & q_1/4 \\ q_1/4 & r_1/2 & r_1c_1/2 \\ q_1/4 & r_1c_1/2 & r_1/2 + r_2 \end{bmatrix}$$

$$\frac{\rho_1}{\rho_2} = \frac{P4L^2}{\pi^2 2EI}\,\frac{\pi^2 EI}{PL^2} = 2$$

Do $|\underline{\underline{K}}| = 0$, solve by procedure in (Ex 3) either by hit-n-trial from Tables or by getting CE in terms of ρ or u.

If end C is hinged instead

$$\underline{\underline{K}} = \frac{EI}{L}\left[\begin{array}{c|c} \frac{L}{EI}\underline{\underline{K}}_{\substack{\text{previous}\\3\times3}} & \begin{matrix}0\\0\\-q_2\end{matrix} \\ \hline \begin{matrix}0 & 0 & -q_2\end{matrix} & s_2 \end{array}\right]$$

NOTE: For the original 6×6 $\underline{\underline{K}}$, $\underline{D} = \left\{\frac{\Delta_1}{L}\ \theta_1\ \frac{\Delta_2}{L}\ \theta_2\ \frac{\Delta_3}{L}\ \theta_3\right\}^T$

$\underline{F} = \{Q_1L\ \ M_1\ \ Q_2L\ \ M_2\ \ Q_3L\ \ M_3\}^T$, hence the division factors $/2$, $/2^2$, $/2^3$ arise $\because L_1 = 2L$, $L_2 = L$.

Appendix A

Stability Functions

A.1 Stability Functions for Compression Members

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
0.00	4.0000	4.0000	0.5000	3.0000	6.0000	12.0000	1.0000	1.0000	−1.0000
0.02	3.9736	4.0265	0.5050	2.9603	5.9802	11.7631	1.0168	0.9333	−1.0337
0.04	3.9471	4.0535	0.5101	2.9201	5.9604	11.5260	1.0343	0.8648	−1.0690
0.06	3.9204	4.0808	0.5153	2.8795	5.9405	11.2889	1.0525	0.7943	−1.1060
0.08	3.8936	4.1086	0.5206	2.8384	5.9206	11.0516	1.0714	0.7218	−1.1448
0.10	3.8667	4.1369	0.5260	2.7968	5.9006	10.8142	1.0913	0.6471	−1.1856
0.12	3.8396	4.1656	0.5316	2.7547	5.8805	10.5767	1.1120	0.5701	−1.2285
0.14	3.8123	4.1947	0.5372	2.7120	5.8604	10.3391	1.1336	0.4905	−1.2737
0.16	3.7849	4.2244	0.5430	2.6688	5.8403	10.1014	1.1563	0.4083	−1.3213
0.18	3.7574	4.2545	0.5490	2.6251	5.8200	9.8636	1.1801	0.3233	−1.3715
0.20	3.7297	4.2851	0.5550	2.5808	5.7998	9.6256	1.2051	0.2351	−1.4245
0.22	3.7019	4.3162	0.5612	2.5359	5.7794	9.3875	1.2313	0.1438	−1.4805
0.24	3.6739	4.3479	0.5676	2.4904	5.7590	9.1493	1.2589	0.0489	−1.5398
0.26	3.6457	4.3801	0.5741	2.4443	5.7385	8.9110	1.2880	−0.0498	−1.6027
0.28	3.6174	4.4128	0.5807	2.3975	5.7180	8.6726	1.3186	−0.1527	−1.6694
0.30	3.5889	4.4460	0.5875	2.3500	5.6974	8.4340	1.3511	−0.2599	−1.7402
0.32	3.5602	4.4799	0.5945	2.3019	5.6768	8.1953	1.3854	−0.3720	−1.8157
0.34	3.5314	4.5143	0.6017	2.2531	5.6561	7.9565	1.4218	−0.4894	−1.8961
0.36	3.5024	4.5493	0.6090	2.2035	5.6353	7.7176	1.4604	−0.6125	−1.9820
0.38	3.4732	4.5849	0.6165	2.1532	5.6145	7.4785	1.5015	−0.7418	−2.0738
0.40	3.4439	4.6211	0.6242	2.1021	5.5936	7.2393	1.5453	−0.8781	−2.1723
0.42	3.4144	4.6580	0.6321	2.0502	5.5726	7.0000	1.5922	−1.0219	−2.2781
0.44	3.3847	4.6955	0.6402	1.9974	5.5516	6.7605	1.6423	−1.1741	−2.3919
0.46	3.3548	4.7337	0.6485	1.9438	5.5305	6.5210	1.6962	−1.3357	−2.5148
0.48	3.3247	4.7725	0.6571	1.8893	5.5093	6.2813	1.7542	−1.5076	−2.6477
0.50	3.2945	4.8121	0.6659	1.8338	5.4881	6.0414	1.8168	−1.6910	−2.7918
0.52	3.2640	4.8524	0.6749	1.7774	5.4668	5.8015	1.8846	−1.8875	−2.9487
0.54	3.2334	4.8934	0.6841	1.7200	5.4455	5.5614	1.9583	−2.0986	−3.1199
0.56	3.2025	4.9351	0.6937	1.6615	5.4240	5.3211	2.0387	−2.3264	−3.3075
0.58	3.1715	4.9776	0.7035	1.6020	5.4026	5.0807	2.1267	−2.5733	−3.5137
0.60	3.1403	5.0210	0.7136	1.5414	5.3810	4.8402	2.2234	−2.8419	−3.7414
0.62	3.1088	5.0651	0.7239	1.4795	5.3594	4.5996	2.3304	−3.1359	−3.9941
0.64	3.0771	5.1100	0.7346	1.4165	5.3377	4.3588	2.4491	−3.4592	−4.2758
0.66	3.0453	5.1558	0.7456	1.3522	5.3159	4.1179	2.5819	−3.8172	−4.5918
0.68	3.0132	5.2025	0.7570	1.2866	5.2941	3.8768	2.7311	−4.2162	−4.9485
0.70	2.9809	5.2500	0.7687	1.2197	5.2722	3.6356	2.9003	−4.6645	−5.3541

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
0.72	2.9484	5.2985	0.7807	1.1512	5.2502	3.3943	3.0935	−5.1725	−5.8190
0.74	2.9156	5.3479	0.7932	1.0814	5.2282	3.1528	3.3165	−5.7540	−6.3571
0.76	2.8826	5.3983	0.8060	1.0099	5.2060	2.9112	3.5766	−6.4273	−6.9865
0.78	2.8494	5.4497	0.8193	0.9368	5.1838	2.6694	3.8839	−7.2174	−7.7323
0.80	2.8159	5.5020	0.8330	0.8621	5.1616	2.4275	4.2526	−8.1592	−8.6295
0.82	2.7822	5.5555	0.8472	0.7855	5.1392	2.1854	4.7032	−9.3032	−9.7285
0.84	2.7483	5.6100	0.8618	0.7071	5.1168	1.9432	5.2664	−10.7253	−11.1051
0.86	2.7141	5.6655	0.8770	0.6267	5.0943	1.7008	5.9904	−12.5445	−12.8784
0.88	2.6797	5.7223	0.8927	0.5442	5.0718	1.4583	6.9556	−14.9591	−15.2467
0.90	2.6450	5.7801	0.9090	0.4596	5.0491	1.2157	8.3069	−18.3264	−18.5672
0.92	2.6100	5.8392	0.9258	0.3727	5.0264	0.9728	10.3336	−23.3606	−23.5541
0.94	2.5748	5.8995	0.9433	0.2835	5.0036	0.7299	13.7113	−31.7284	−31.8742
0.96	2.5392	5.9611	0.9615	0.1917	4.9808	0.4867	20.4663	−48.4298	−48.5275
0.98	2.5035	6.0239	0.9804	0.0972	4.9578	0.2434	40.7308	−98.4647	−98.5138
1.00	2.4674	6.0881	1.0000	0.0000	4.9348	0.0000	∞	∞	∞
1.02	2.4311	6.1536	1.0204	−0.1002	4.9117	−0.2436	−40.3255	101.4645	101.5141
1.04	2.3944	6.2206	1.0416	−0.2035	4.8885	−0.4874	−20.0610	51.4286	51.5283
1.06	2.3575	6.2889	1.0638	−0.3102	4.8652	−0.7313	−13.3060	34.7259	34.8762
1.08	2.3202	6.3588	1.0868	−0.4204	4.8419	−0.9754	−9.9283	26.3561	26.5576
1.10	2.2827	6.4302	1.1109	−0.5343	4.8185	−1.2196	−7.9016	21.3194	21.5725
1.12	2.2448	6.5032	1.1360	−0.6522	4.7950	−1.4640	−6.5503	17.9491	18.2544
1.14	2.2066	6.5778	1.1623	−0.7743	4.7714	−1.7086	−5.5851	15.5308	15.8889
1.16	2.1681	6.6541	1.1898	−0.9009	4.7477	−1.9534	−4.8610	13.7074	14.1189
1.18	2.1293	6.7321	1.2185	−1.0324	4.7239	−2.1983	−4.2978	12.2806	12.7459
1.20	2.0901	6.8119	1.2487	−1.1690	4.7001	−2.4434	−3.8472	11.1312	11.6510
1.22	2.0506	6.8935	1.2804	−1.3112	4.6761	−2.6886	−3.4785	10.1835	10.7584
1.24	2.0107	6.9770	1.3137	−1.4592	4.6521	−2.9341	−3.1711	9.3869	10.0176
1.26	1.9705	7.0625	1.3487	−1.6137	4.6280	−3.1797	−2.9110	8.7066	9.3936
1.28	1.9299	7.1499	1.3855	−1.7750	4.6038	−3.4255	−2.6880	8.1174	8.8615
1.30	1.8889	7.2394	1.4244	−1.9437	4.5795	−3.6714	−2.4947	7.6012	8.4029
1.32	1.8476	7.3311	1.4655	−2.1204	4.5552	−3.9176	−2.3255	7.1441	8.0041
1.34	1.8058	7.4249	1.5089	−2.3058	4.5307	−4.1639	−2.1762	6.7357	7.6547
1.36	1.7637	7.5210	1.5549	−2.5006	4.5061	−4.4104	−2.0434	6.3677	7.3464
1.38	1.7212	7.6195	1.6038	−2.7058	4.4815	−4.6571	−1.9246	6.0337	7.0729
1.40	1.6782	7.7203	1.6557	−2.9221	4.4568	−4.9039	−1.8176	5.7286	6.8289
1.42	1.6348	7.8237	1.7109	−3.1507	4.4319	−5.1510	−1.7208	5.4481	6.6103
1.44	1.5910	7.9296	1.7699	−3.3929	4.4070	−5.3982	−1.6328	5.1888	6.4138
1.46	1.5468	8.0383	1.8329	−3.6499	4.3820	−5.6457	−1.5523	4.9480	6.2363
1.48	1.5021	8.1496	1.9005	−3.9233	4.3569	−5.8933	−1.4786	4.7231	6.0758
1.50	1.4570	8.2638	1.9731	−4.2150	4.3317	−6.1411	−1.4107	4.5123	5.9301
1.52	1.4114	8.3810	2.0512	−4.5269	4.3064	−6.3891	−1.3480	4.3139	5.7975
1.54	1.3653	8.5012	2.1356	−4.8616	4.2809	−6.6373	−1.2900	4.1264	5.6768
1.56	1.3187	8.6246	2.2271	−5.2217	4.2554	−6.8857	−1.2360	3.9486	5.5667
1.58	1.2716	8.7512	2.3264	−5.6105	4.2298	−7.1343	−1.1858	3.7794	5.4661
1.60	1.2240	8.8813	2.4348	−6.0320	4.2041	−7.3831	−1.1389	3.6179	5.3741
1.62	1.1759	9.0148	2.5534	−6.4906	4.1783	−7.6321	−1.0949	3.4634	5.2900
1.64	1.1272	9.1519	2.6838	−6.9919	4.1524	−7.8813	−1.0537	3.3150	5.2130
1.66	1.0780	9.2928	2.8278	−7.5424	4.1264	−8.1307	−1.0150	3.1722	5.1426
1.68	1.0282	9.4376	2.9877	−8.1502	4.1003	−8.3803	−0.9786	3.0344	5.0782
1.70	0.9779	9.5864	3.1662	−8.8253	4.0741	−8.6302	−0.9442	2.9012	5.0195
1.72	0.9270	9.7394	3.3667	−9.5800	4.0478	−8.8802	−0.9116	2.7720	4.9659
1.74	0.8754	9.8968	3.5936	−10.4299	4.0213	−9.1305	−0.8809	2.6465	4.9170
1.76	0.8233	10.0587	3.8524	−11.3949	3.9948	−9.3809	−0.8517	2.5244	4.8727
1.78	0.7705	10.2252	4.1504	−12.5011	3.9681	−9.6316	−0.8240	2.4053	4.8325
1.80	0.7170	10.3966	4.4969	−13.7828	3.9414	−9.8825	−0.7977	2.2889	4.7963
1.82	0.6629	10.5731	4.9051	−15.2868	3.9145	−10.1336	−0.7726	2.1750	4.7638
1.84	0.6081	10.7548	5.3929	−17.0776	3.8876	−10.3850	−0.7487	2.0634	4.7347
1.86	0.5526	10.9419	5.9859	−19.2479	3.8605	−10.6365	−0.7259	1.9537	4.7090

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
1.88	0.4964	11.1347	6.7223	−21.9352	3.8333	−10.8883	−0.7041	1.8459	4.6864
1.90	0.4394	11.3335	7.6612	−25.3521	3.8059	−11.1404	−0.6833	1.7397	4.6668
1.92	0.3817	11.5383	8.8990	−29.8466	3.7785	−11.3926	−0.6633	1.6349	4.6500
1.94	0.3232	11.7496	10.6056	−36.0301	3.7510	−11.6451	−0.6442	1.5314	4.6360
1.96	0.2639	11.9675	13.1087	−45.0844	3.7233	−11.8978	−0.6259	1.4291	4.6246
1.98	0.2038	12.1923	17.1355	−59.6290	3.6955	−12.1508	−0.6083	1.3277	4.6157
2.00	0.1428	12.4244	24.6841	−86.8644	3.6676	−12.4040	−0.5914	1.2272	4.6093
2.02	0.0809	12.6640	43.9616	−156.3627	3.6396	−12.6574	−0.5751	1.1275	4.6052
2.04	0.0182	12.9114	197.3863	−709.2395	3.6115	−12.9111	−0.5594	1.0284	4.6034
2.06	−0.0455	13.1671	−79.8138	289.5707	3.5832	−13.1650	−0.5443	0.9298	4.6039
2.08	−0.1101	13.4313	−33.2921	121.9015	3.5548	−13.4192	−0.5298	0.8316	4.6066
2.10	−0.1757	13.7045	−21.0722	77.8328	3.5263	−13.6736	−0.5158	0.7337	4.6113
2.12	−0.2423	13.9870	−15.4361	57.4874	3.4976	−13.9283	−0.5022	0.6360	4.6182
2.14	−0.3099	14.2793	−12.1925	45.7629	3.4689	−14.1832	−0.4892	0.5385	4.6272
2.16	−0.3786	14.5818	−10.0850	38.1320	3.4400	−14.4384	−0.4765	0.4409	4.6382
2.18	−0.4485	14.8950	−8.6059	32.7650	3.4109	−14.6939	−0.4643	0.3433	4.6512
2.20	−0.5194	15.2194	−7.5107	28.7813	3.3818	−14.9496	−0.4524	0.2456	4.6662
2.22	−0.5916	15.5555	−6.6673	25.7044	3.3525	−15.2055	−0.4410	0.1476	4.6832
2.24	−0.6649	15.9039	−5.9978	23.2542	3.3231	−15.4618	−0.4298	0.0493	4.7022
2.26	−0.7395	16.2652	−5.4537	21.2552	3.2935	−15.7183	−0.4191	−0.0494	4.7231
2.28	−0.8154	16.6400	−5.0027	19.5917	3.2638	−15.9751	−0.4086	−0.1486	4.7460
2.30	−0.8926	17.0289	−4.6230	18.1845	3.2340	−16.2321	−0.3985	−0.2483	4.7709
2.32	−0.9713	17.4328	−4.2988	16.9775	3.2040	−16.4895	−0.3886	−0.3487	4.7978
2.34	−1.0513	17.8523	−4.0190	15.9298	3.1739	−16.7471	−0.3790	−0.4498	4.8267
2.36	−1.1328	18.2883	−3.7750	15.0109	3.1436	−17.0050	−0.3697	−0.5517	4.8576
2.38	−1.2159	18.7416	−3.5604	14.1977	3.1132	−17.2632	−0.3607	−0.6545	4.8906
2.40	−1.3006	19.2131	−3.3703	13.4723	3.0827	−17.5216	−0.3519	−0.7582	4.9256
2.42	−1.3869	19.7038	−3.2006	12.8204	3.0520	−17.7804	−0.3433	−0.8630	4.9628
2.44	−1.4749	20.2148	−3.0484	12.2310	3.0212	−18.0395	−0.3350	−0.9689	5.0021
2.46	−1.5647	20.7470	−2.9111	11.6949	2.9902	−18.2988	−0.3268	−1.0761	5.0435
2.48	−1.6563	21.3018	−2.7865	11.2047	2.9591	−18.5585	−0.3189	−1.1845	5.0872
2.50	−1.7499	21.8804	−2.6732	10.7543	2.9278	−18.8184	−0.3112	−1.2943	5.1332
2.52	−1.8454	22.4841	−2.5695	10.3386	2.8964	−19.0787	−0.3036	−1.4057	5.1814
2.54	−1.9430	23.1144	−2.4744	9.9534	2.8648	−19.3393	−0.2963	−1.5186	5.2321
2.56	−2.0427	23.7728	−2.3869	9.5952	2.8330	−19.6001	−0.2891	−1.6332	5.2852
2.58	−2.1447	24.4610	−2.3061	9.2607	2.8011	−19.8613	−0.2821	−1.7496	5.3409
2.60	−2.2490	25.1808	−2.2312	8.9475	2.7691	−20.1229	−0.2752	−1.8679	5.3991
2.62	−2.3557	25.9341	−2.1618	8.6533	2.7368	−20.3847	−0.2685	−1.9883	5.4600
2.64	−2.4650	26.7230	−2.0971	8.3761	2.7044	−20.6469	−0.2620	−2.1107	5.5237
2.66	−2.5769	27.5497	−2.0369	8.1142	2.6719	−20.9094	−0.2556	−2.2355	5.5902
2.68	−2.6915	28.4165	−1.9805	7.8662	2.6392	−21.1722	−0.2493	−2.3626	5.6597
2.70	−2.8091	29.3262	−1.9278	7.6308	2.6063	−21.4353	−0.2432	−2.4922	5.7323
2.72	−2.9296	30.2815	−1.8784	7.4067	2.5732	−21.6988	−0.2372	−2.6245	5.8080
2.74	−3.0533	31.2854	−1.8319	7.1931	2.5400	−21.9627	−0.2313	−2.7596	5.8871
2.76	−3.1803	32.3413	−1.7882	6.9889	2.5066	−22.2269	−0.2255	−2.8976	5.9696
2.78	−3.3108	33.4526	−1.7470	6.7934	2.4730	−22.4914	−0.2199	−3.0389	6.0558
2.80	−3.4449	34.6234	−1.7081	6.6059	2.4393	−22.7563	−0.2144	−3.1834	6.1456
2.82	−3.5828	35.8577	−1.6714	6.4257	2.4054	−23.0215	−0.2090	−3.3314	6.2394
2.84	−3.7246	37.1601	−1.6366	6.2522	2.3713	−23.2871	−0.2037	−3.4832	6.3374
2.86	−3.8707	38.5358	−1.6038	6.0849	2.3370	−23.5531	−0.1984	−3.6389	6.4396
2.88	−4.0213	39.9901	−1.5726	5.9234	2.3025	−23.8195	−0.1933	−3.7987	6.5463
2.90	−4.1765	41.5290	−1.5430	5.7671	2.2678	−24.0862	−0.1883	−3.9629	6.6578
2.92	−4.3366	43.1591	−1.5149	5.6158	2.2330	−24.3533	−0.1834	−4.1318	6.7743
2.94	−4.5019	44.8876	−1.4882	5.4690	2.1979	−24.6207	−0.1785	−4.3057	6.8960
2.96	−4.6727	46.7223	−1.4628	5.3264	2.1627	−24.8886	−0.1738	−4.4847	7.0233
2.98	−4.8492	48.6720	−1.4387	5.1878	2.1273	−25.1568	−0.1691	−4.6694	7.1564
3.00	−5.0320	50.7463	−1.4157	5.0528	2.0917	−25.4255	−0.1645	−4.8599	7.2957

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
3.02	−5.2212	52.9557	−1.3937	4.9212	2.0558	−25.6945	−0.1600	−5.0567	7.4416
3.04	−5.4174	55.3121	−1.3728	4.7927	2.0198	−25.9640	−0.1556	−5.2603	7.5943
3.06	−5.6209	57.8285	−1.3529	4.6672	1.9836	−26.2338	−0.1512	−5.4709	7.7545
3.08	−5.8323	60.5193	−1.3339	4.5444	1.9472	−26.5041	−0.1469	−5.6892	7.9225
3.10	−6.0519	63.4008	−1.3157	4.4242	1.9105	−26.7747	−0.1427	−5.9156	8.0988
3.12	−6.2805	66.4910	−1.2983	4.3063	1.8737	−27.0458	−0.1386	−6.1507	8.2840
3.14	−6.5186	69.8101	−1.2817	4.1907	1.8366	−27.3174	−0.1345	−6.3952	8.4787
3.16	−6.7669	73.3808	−1.2659	4.0771	1.7993	−27.5893	−0.1304	−6.6496	8.6836
3.18	−7.0262	77.2287	−1.2508	3.9654	1.7618	−27.8617	−0.1265	−6.9148	8.8994
3.20	−7.2971	81.3826	−1.2363	3.8556	1.7241	−28.1345	−0.1226	−7.1915	9.1269
3.22	−7.5807	85.8752	−1.2224	3.7474	1.6862	−28.4078	−0.1187	−7.4806	9.3670
3.24	−7.8779	90.7436	−1.2092	3.6407	1.6480	−28.6815	−0.1149	−7.7833	9.6206
3.26	−8.1899	96.0299	−1.1965	3.5355	1.6096	−28.9557	−0.1112	−8.1004	9.8890
3.28	−8.5178	101.7826	−1.1844	3.4317	1.5710	−29.2304	−0.1075	−8.4333	10.1732
3.30	−8.8629	108.0569	−1.1729	3.3291	1.5321	−29.5055	−0.1039	−8.7834	10.4746
3.32	−9.2269	114.9167	−1.1618	3.2276	1.4930	−29.7810	−0.1003	−9.1520	10.7948
3.34	−9.6114	122.4357	−1.1512	3.1272	1.4537	−30.0571	−0.0967	−9.5411	11.1354
3.36	−10.0183	130.6995	−1.1412	3.0278	1.4141	−30.3336	−0.0932	−9.9524	11.4983
3.38	−10.4497	139.8079	−1.1315	2.9293	1.3743	−30.6107	−0.0898	−10.3880	11.8857
3.40	−10.9082	149.8780	−1.1223	2.8316	1.3342	−30.8882	−0.0864	−10.8506	12.3001
3.42	−11.3965	161.0474	−1.1135	2.7347	1.2939	−31.1662	−0.0830	−11.3428	12.7442
3.44	−11.9178	173.4792	−1.1052	2.6385	1.2533	−31.4448	−0.0797	−11.8679	13.2211
3.46	−12.4757	187.3678	−1.0972	2.5428	1.2125	−31.7238	−0.0764	−12.4294	13.7346
3.48	−13.0745	202.9458	−1.0896	2.4478	1.1714	−32.0034	−0.0732	−13.0316	14.2888
3.50	−13.7190	220.4940	−1.0824	2.3532	1.1301	−32.2835	−0.0700	−13.6794	14.8886
3.52	−14.4149	240.3536	−1.0755	2.2591	1.0884	−32.5641	−0.0668	−14.3785	15.5397
3.54	−15.1689	262.9423	−1.0690	2.1653	1.0465	−32.8453	−0.0637	−15.1356	16.2488
3.56	−15.9890	288.7758	−1.0628	2.0719	1.0044	−33.1270	−0.0606	−15.9586	17.0239
3.58	−16.8845	318.4967	−1.0570	1.9787	0.9619	−33.4093	−0.0576	−16.8568	17.8742
3.60	−17.8668	352.9140	−1.0514	1.8857	0.9192	−33.6921	−0.0546	−17.8417	18.8111
3.62	−18.9494	393.0567	−1.0462	1.7930	0.8762	−33.9755	−0.0516	−18.9268	19.8483
3.64	−20.1492	440.2504	−1.0413	1.7003	0.8329	−34.2595	−0.0486	−20.1290	21.0024
3.66	−21.4868	496.2245	−1.0367	1.6077	0.7893	−34.5441	−0.0457	−21.4687	22.2941
3.68	−22.9879	563.2705	−1.0324	1.5151	0.7455	−34.8292	−0.0428	−22.9719	23.7493
3.70	−24.6852	644.4737	−1.0284	1.4225	0.7013	−35.1149	−0.0399	−24.6712	25.4005
3.72	−26.6208	744.0683	−1.0247	1.3298	0.6568	−35.4013	−0.0371	−26.6086	27.2898
3.74	−28.8496	867.9886	−1.0212	1.2370	0.6120	−35.6883	−0.0343	−28.8391	29.4721
3.76	−31.4449	1024.7578	−1.0180	1.1441	0.5669	−35.9758	−0.0315	−31.4360	32.0208
3.78	−34.5066	1226.9671	−1.0151	1.0509	0.5215	−36.2641	−0.0288	−34.4991	35.0356
3.80	−38.1745	1493.8426	−1.0125	0.9575	0.4758	−36.5529	−0.0260	−38.1683	38.6565
3.82	−42.6506	1855.9180	−1.0101	0.8638	0.4297	−36.8424	−0.0233	−42.6456	43.0854
3.84	−48.2381	2364.0441	−1.0079	0.7698	0.3834	−37.1326	−0.0206	−48.2341	48.6254
3.86	−55.4130	3108.0245	−1.0061	0.6754	0.3367	−37.4234	−0.0180	−55.4100	55.7527
3.88	−64.9691	4258.6957	−1.0045	0.5805	0.2896	−37.7149	−0.0154	−64.9669	65.2609
3.90	−78.3349	6174.3571	−1.0031	0.4852	0.2422	−38.0070	−0.0127	−78.3333	78.5786
3.92	−98.3675	9714.4644	−1.0020	0.3894	0.1945	−38.2999	−0.0102	−98.3665	98.5630
3.94	−131.7337	17392.3511	−1.0011	0.2930	0.1464	−38.5934	−0.0076	−131.7331	131.8806
3.96	−198.4334	39414.6837	−1.0005	0.1960	0.0980	−38.8877	−0.0050	−198.4331	198.5316
3.98	−398.4666	158814.7958	−1.0001	0.0983	0.0492	−39.1827	−0.0025	−398.4665	398.5158
4.00	0.0000	0.0000	−1.0000	0.0000	0.0000	−39.4784	0.0000	0.0000	0.0000
4.50	16.5908	322.3967	−1.0823	−2.8415	−1.3646	−47.1425	0.0579	16.6303	−17.9159
5.00	7.4983	111.7156	−1.4096	−7.4004	−3.0712	−55.4905	0.1107	7.6683	−10.3996
5.50	3.4609	76.8742	−2.5334	−18.7516	−5.3069	−64.8967	0.1636	3.8948	−8.3338
6.00	0.2974	76.1659	−29.3489	−255.8395	−8.4299	−76.0775	0.2216	1.2315	−7.7932
6.50	−3.1859	100.7494	3.1505	28.4371	−13.2233	−90.5991	0.2919	−1.2559	−8.1074
7.00	−8.3115	181.7543	1.6220	13.5564	−21.7931	−112.6735	0.3868	−4.0963	−9.2664
7.50	−19.3323	532.5795	1.1937	8.2164	−42.4100	−158.8420	0.5340	−8.0091	−11.7544
8.00	−85.6372	7760.1380	1.0287	4.9793	−173.7288	−426.4145	0.8148	−14.8570	−17.3114
8.50	54.6147	2850.6399	0.9776	2.4191	108.0060	132.1204	1.6350	−33.6783	−34.9016
9.00	19.2650	371.1392	1.0000	0.0000	48.0000	7.1736	−6.5491	145.4326	145.4326