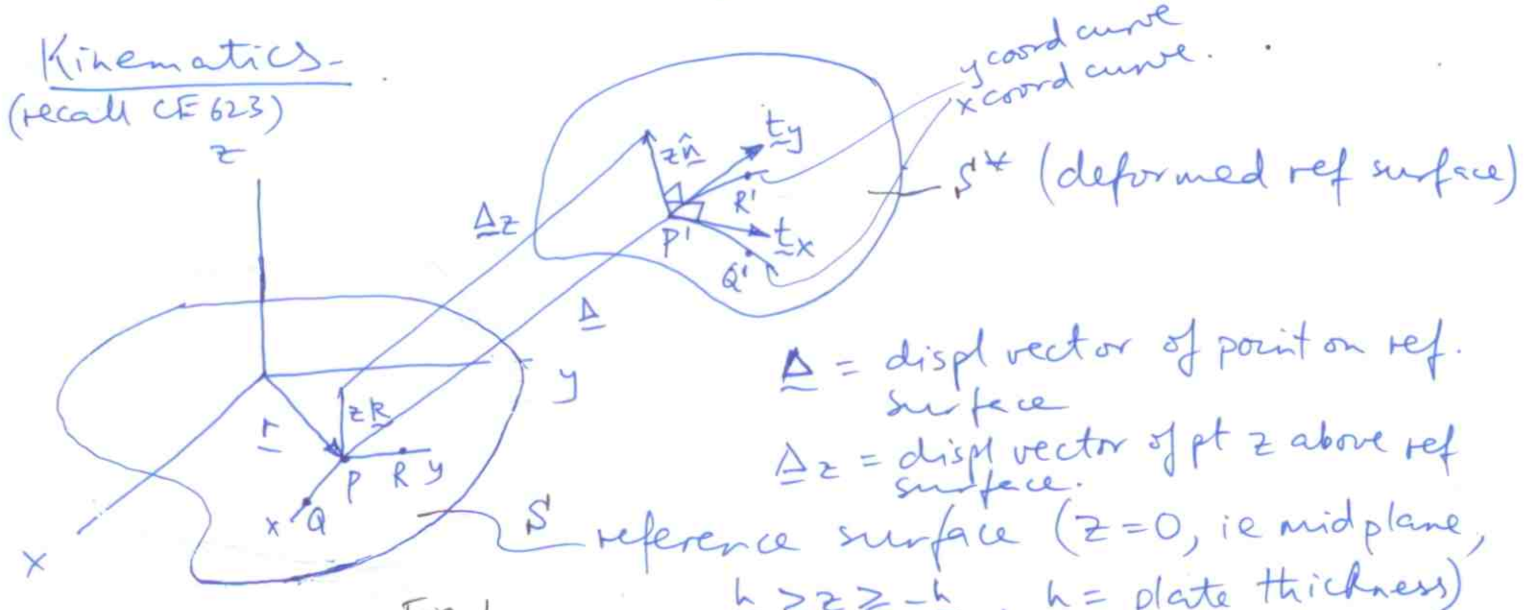
TOPIC V

PLATES.

We consider thin plates only ($L/h > 10$) where L is a characteristic length in the plan area.

Kinematics-
(recall CE 623)



$\underline{\Delta}$ = displ vector of point on ref. surface

$\underline{\Delta}z$ = displ vector of pt z above ref surface.

S reference surface ($z=0$, ie midplane, $\frac{h}{2} \geq z \geq -\frac{h}{2}$, h = plate thickness)

Fig 1

Point P (p.v. $\underline{r}$) moves to point P' (p.v. $\underline{r}^*$)

$$\underline{r}^*_{(x,y)} = \underline{r} + \underline{\Delta} = x\underline{i} + y\underline{j} + u(x,y)\,\underline{i} + v(x,y)\,\underline{j} + w(x,y)\,\underline{k}$$

Also shown are the displaced neighboring pts $R \to R'$ and $Q \to Q'$. Thus, in general

$$\underline{dr}^* = \frac{\partial \underline{r}^*}{\partial x} dx + \frac{\partial \underline{r}^*}{\partial y} dy .$$

For $PQ \to P'Q'$, $\underline{dr}^* = \frac{\partial \underline{r}^*}{\partial x} dx$ = tangential to x coordinate curve in S*

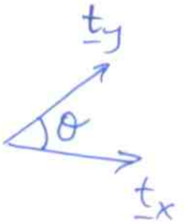
$PR \to P'R'$, $\underline{dr}^* = \frac{\partial \underline{r}^*}{\partial y} dy$ = tangential to y coord curve in S*

Engineering extensional strains are, (recall from CE 623).

$$\varepsilon_{Ex} = \varepsilon_x = \left(\frac{dr^* - dr}{dr}\right)_{PQ} = \frac{P'Q' - PQ}{PQ} = \sqrt{1 + 2E_{xx}} - 1 \quad , PQ = dx$$

$$\varepsilon_{Ey} = \varepsilon_y = \left(\frac{dr^* - dr}{dr}\right)_{PR} = \frac{P'R' - PR}{PR} = \sqrt{1 + 2E_{yy}} - 1 , \quad PR = dy$$

and engineering shear strain is,



$$\gamma_{xy} = \frac{\pi}{2} - \theta$$

$$\sin\gamma_{xy} = \frac{2E_{xy}}{\sqrt{1+2E_{xx}}\sqrt{1+2E_{yy}}} = \frac{2E_{xy}}{(1+\varepsilon_x)(1+\varepsilon_y)}$$

where $E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})$ (CE 623).

For small strains (displ. gradients) retaining only upto quadratic terms,

$$\varepsilon_x = \sqrt{1 + 2u_{,x} + u_{,x}^2 + v_{,x}^2 + w_{,x}^2} - 1$$

$$= 1 + \frac{1}{2}(2u_{,x} + u_{,x}^2 + v_{,x}^2 + w_{,x}^2) - \frac{1}{8}(2u_{,x} + u_{,x}^2 + v_{,x}^2 + w_{,x}^2)^2 - 1 + O(\text{cubic})$$

$$= u_{,x} + \frac{1}{2}(v_{,x}^2 + w_{,x}^2)$$

$$\therefore \varepsilon_y = v_{,y} + \frac{1}{2}(u_{,y}^2 + w_{,y}^2)$$

$$\sin\gamma_{xy} \simeq \gamma_{xy} = \frac{(u_{,y} + v_{,x} + u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y})}{(1 + u_{,x} + HOT)(1 + v_{,y} + HOT)}$$

$$\gamma_{xy} = u_{,y} + v_{,x} - u_{,x}v_{,x} - u_{,y}v_{,y} + w_{,x}w_{,y} + O(\text{cubic})$$

$$\gamma_{xy} = (1 - v_{,y})u_{,y} + (1 - u_{,x})v_{,x} + w_{,x}w_{,y} + O(\text{cubic}).$$

Rotations:

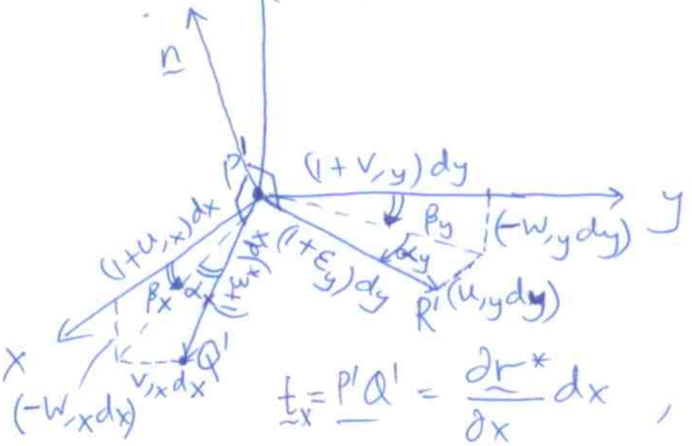


Fig. 2

$$\underset{\sim}{t}_x = \underline{P'Q'} = \frac{\partial \underset{\sim}{r}^*}{\partial x}dx \quad , \quad \underline{P'R'} = \frac{\partial r^*}{\partial y}dy = \underline{t}_y$$

out of plane rotations

$$\tan\beta_x = \frac{-w_{,x}}{1+u_{,x}} \simeq -w_{,x}$$

$$\tan\beta_y = \frac{-w_{,y}}{1+v_{,y}} \simeq -w_{,y}$$

(Neglected nonlinearities in $w_{,x}u_{,x}$ & $w_{,y}v_{,y}$). (See reason below).

Rotations about z axis, ie in-plane rotations

$$\tan\alpha_x = \frac{v_{,x}}{1+u_{,x}} \simeq v_{,x}(1-u_{,x})$$

$$\tan\alpha_y = \frac{u_{,y}}{1+v_{,y}} \simeq u_{,y}(1-v_{,y})$$

For moderate out-of-plane rotations, $(w_{,x}, w_{,y}) = O(\varepsilon)$

For small strains, $(u_{,x}, v_{,y}) = O(\varepsilon^2)$

For out-of-plane rot larger than in-plane rot, $(v_{,x}, u_{,y}) = O(\varepsilon^2)$

For small displ gradients, i.e, $\varepsilon \ll 1$, $\tan\beta_x = \beta_x$, $\tan\beta_y = \beta_y$.

Thus, neglect $w_{,x}u_{,x} = O(\varepsilon^3)$ compared to $w_{,x} = O(\varepsilon)$.

Thus, neglect $v_{,x}u_{,x} = O(\varepsilon^4)$ " " $v_{,x} = O(\varepsilon^2)$

Thus, neglect $v_{,x}^2 = O(\varepsilon^4)$ " " $w_{,x}^2 = O(\varepsilon^2)$.

So we get,

①

$$\boxed{\begin{aligned}
\varepsilon_x &= u_{,x} + \tfrac{1}{2}w_{,x}^2 + \overset{\text{neglect.}}{\cancel{O(\varepsilon^3)}} = u_{,x} + \tfrac{1}{2}\beta_x^2 + \cancel{O(\varepsilon^3)} \\
\varepsilon_y &= v_{,y} + \tfrac{1}{2}w_{,y}^2 + \cancel{O(\varepsilon^3)} = v_{,y} + \tfrac{1}{2}\beta_{,y}^2 + \cancel{O(\varepsilon^3)} \\
\gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}w_{,y} + \cancel{O(\varepsilon^3)} = u_{,y} + v_{,x} + \beta_x\beta_y + \cancel{O(\varepsilon^3)}
\end{aligned}}$$

For strains away from reference surface $(z=0)$ consider,

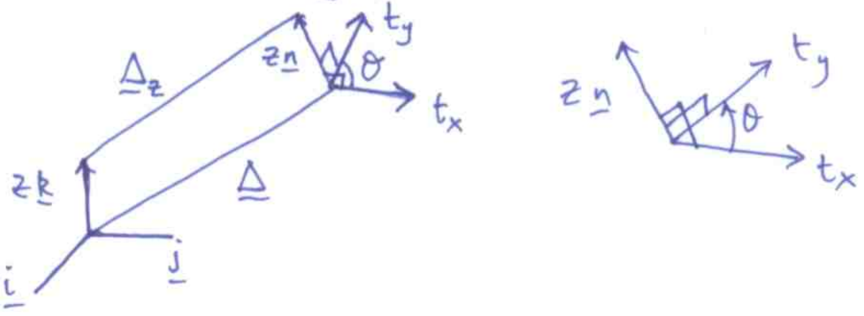


Fig. 3

(over → See p. 3a)

Kirchoff Assumptions.

(1) Normals to mid-surface remain normal and inextensible (ie $\gamma_{xz} = \gamma_{yz} = \varepsilon_z = 0$).

(2) Neglect σ_z in Hooke's law. This contradicts $\varepsilon_z = 0$ but we accept it.

$\underline{\Delta}_z$ = displ of pt z - above ref. surface

$\underset{\sim}{\Delta}_z = -z\underline{k} + \underline{\Delta} + z\underset{\sim}{n}$ (used inextensibility of element normal to ref. surface)

Now $\underline{t}_x \times \underline{t}_y = \underline{P'Q'} \times \underline{P'R'} = (P'Q')(P'R')\sin\theta\, \underset{\sim}{n}$

$$\Rightarrow \frac{\partial \underline{r}^*}{\partial x}\cancel{dx} \times \frac{\partial \underline{r}^*}{\partial y}\cancel{dy} = (1+\varepsilon_x)\cancel{dx}(1+\varepsilon_y)\cancel{dy}\sin\theta\, \underset{\sim}{n}$$

$$\frac{\left((1+u_{,x})\underline{i} + v_{,x}\underline{j} + w_{,x}\underline{k}\right) \times \left(u_{,y}\underline{i} + (1+v_{,y})\underline{j} + w_{,y}\underline{k}\right)}{(1+\varepsilon_x)(1+\varepsilon_y)\cos\gamma_{xy}} = \underline{n}$$

$$\underset{\sim}{n} = \frac{\begin{pmatrix}\underline{k}\left(1+\cancel{u_{,x}}+\cancel{v_{,y}}+\cancel{u_{,x}v_{,y}}-\cancel{v_{,x}u_{,y}}\right)+\underline{j}\left(-w_{,y}-\cancel{u_{,x}w_{,y}}+\cancel{u_{,y}w_{,x}}\right)\\ +\,\underline{i}\left(\cancel{v_{,x}w_{,y}}-w_{,x}-\cancel{w_{,x}v_{,y}}\right)\end{pmatrix}}{(1+\cancel{\varepsilon_x})(1+\cancel{\varepsilon_y})\overset{\nearrow\simeq 1}{\cancel{\cos\gamma_{xy}}}}$$

For small displ gradients → drop terms as indicated in $\underline{k}(\ )$ and denominator.

For small strains & inplane rotations and moderate out of plane rotations, drop terms indicated in $\underline{i}(\ )$, $\underline{j}(\ )$.

$$\Rightarrow \underset{\sim}{n} = -w_{,x}\underline{i} - w_{,y}\underline{j} + \underline{k}$$

$$\Rightarrow \underline{\Delta}_z = \underline{\Delta} + z(\underset{\sim}{n} - \underline{k}) = \bar{u}\underline{i} + \bar{v}\underline{j} + \bar{w}\underline{k}$$

$$\boxed{\begin{aligned}\bar{u}(x,y,z) &= u(x,y) - z\,w_{,x}(x,y)\\ \bar{v}(x,y,z) &= v(x,y) - z\,w_{,y}(x,y)\\ \bar{w}(x,y,z) &= w(x,y)\end{aligned}} \longrightarrow ②$$

NOTE: (we could have guessed this by considering rot of the normal, but this is the consistent way.)

Here $\bar{u}, \bar{v}, \bar{w}$ are displs of pt away from ~~mid~~ ref surface.

Strains for pt. away from ref surface can be derived exactly as for a pt on ref. surface. Only change would be overbars on the strains and displ quantities in Eq. ①. Then using ② in the resulting strains we get,

$$\overline{\varepsilon}_x = \overline{u}_{,x} + \frac{\overline{w}_{,x}^2}{2} = u_{,x} + \frac{w_{,x}^2}{2} + z\beta_{x,x}$$

$$\overline{\varepsilon}_y = v_{,y} + \frac{w_{,y}^2}{2} + z\beta_{y,y}$$

$$\overline{\gamma}_{xy} = \overline{u}_{,y} + \overline{v}_{,x} + \overline{w}_{,x}\overline{w}_{,y} = u_{,y} + v_{,x} + w_{,x}w_{,y} + z(\beta_{x,y} + \beta_{y,x})$$

$$\Rightarrow \boxed{\begin{aligned}
&\overline{\varepsilon}_x = u_{,x} + \frac{w_{,x}^2}{2} + z\kappa_x = \varepsilon_x + z\kappa_x \\
&\overline{\varepsilon}_y = v_{,y} + \frac{w_{,y}^2}{2} + z\kappa_y = \varepsilon_y + z\kappa_y \\
&\overline{\gamma}_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} + 2z\kappa_{xy} = \gamma_{xy} + 2z\kappa_{xy} \\
&\kappa_x = \beta_{x,x}, \quad \kappa_y = \beta_{y,y}, \quad \kappa_{xy} = \beta_{x,y} = \beta_{y,x} \\
&\beta_x = -w_{,x}, \quad \beta_{,y} = -w_{,y}
\end{aligned}} \rightarrow \textcircled{3}$$

Constitutive Equations.

Put $\sigma_z = 0$ in constitutive law. (NOTE: $\sigma_{zz} \to \sigma_z$, $\sigma_{xx} \to \sigma_x$, $\sigma_{yy} \to \sigma_y$ notation used).

$$\left.\begin{aligned}
\overline{\varepsilon}_x &= \frac{1}{E}(\overline{\sigma}_x - \nu\overline{\sigma}_y) \\
\overline{\varepsilon}_y &= \frac{1}{E}(\overline{\sigma}_y - \nu\overline{\sigma}_x) \\
\overline{\gamma}_{xy} &= \frac{2(1+\nu)}{E}\overline{\tau}_{xy}
\end{aligned}\right\} \Rightarrow
\begin{aligned}
\overline{\sigma}_x &= \frac{E}{1-\nu^2}(\overline{\varepsilon}_x + \nu\overline{\varepsilon}_y) \\
\overline{\sigma}_y &= \frac{E}{1-\nu^2}(\overline{\varepsilon}_y + \nu\overline{\varepsilon}_x) \\
\overline{\tau}_{xy} &= \frac{E}{2(1+\nu)}\overline{\gamma}_{xy}
\end{aligned}$$

Defined stress and moment resultants:

$$N_x \triangleq \int_h \overline{\sigma}_x\,dz = \int_h \frac{E}{1-\nu^2}(\overline{\varepsilon}_x + \nu\overline{\varepsilon}_y)\,dz = \int_{-h/2}^{h/2} \frac{E}{(1-\nu^2)}\left[\varepsilon_x + \nu\varepsilon_y + z(\kappa_x + \nu\kappa_y)\right]dz$$

$$N_x = \frac{Eh}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y)$$

$$N_y = \frac{Eh}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x) \text{ , similarly}$$

$$N_{xy} \triangleq \int_h \bar{\tau}_{xy}\,dz = \frac{Eh}{2(1+\nu)}\gamma_{xy}$$

(ie $(N_x, N_y, N_{xy}) \triangleq \int_h (\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy})\,dz$).

& $(M_x, M_y, M_{xy}) \triangleq \int_h (\bar{\sigma}_x, \bar{\sigma}_y, \tau_{xy})\,z\,dz$

$$M_x = \int_{-h/2}^{h/2} \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y + z(K_x + \nu K_y)\right] z\,dz$$

$$M_x = \frac{Eh^3}{12(1-\nu^2)}(K_x + \nu K_y)$$

$$M_y = \frac{Eh^3}{12(1-\nu^2)}(K_y + \nu K_x)$$, similarly.

$$M_{xy} = \frac{Eh^3}{12(1+\nu)}K_{xy}$$

Summary of stress, moment resultants:

④

$$C = Eh/(1-\nu^2)$$

$$D = Eh^3/12(1-\nu^2)$$

$$(N_x, N_y, N_{xy}) \triangleq \int_{-h/2}^{h/2} (\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy})\,dz$$

$$(M_x, M_y, M_{xy}) \triangleq \int_{-h/2}^{h/2} (\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy})\,z\,dz$$

$$(Q_x, Q_y) \triangleq \int_{-h/2}^{h/2} (\bar{\tau}_{xz}, \bar{\tau}_{yz})\,dz \rightarrow \text{transverse shear resultants.}$$

$$N_x = C(\varepsilon_x + \nu\varepsilon_y), \quad N_y = C(\varepsilon_y + \nu\varepsilon_x), \quad N_{xy} = \frac{C(1-\nu)}{2}\gamma_{xy}$$

$$M_x = D(K_x + \nu K_y), \quad M_y = D(K_y + \nu K_x), \quad M_{xy} = D(1-\nu)K_{xy}$$

$= M_{yx}$

<u>NOTE</u>: Although we assume normal remains normal, ie $(\gamma_{xz}, \gamma_{yz}) = 0$, we consider Q_x, Q_y in equilibrium

Equilibrium.

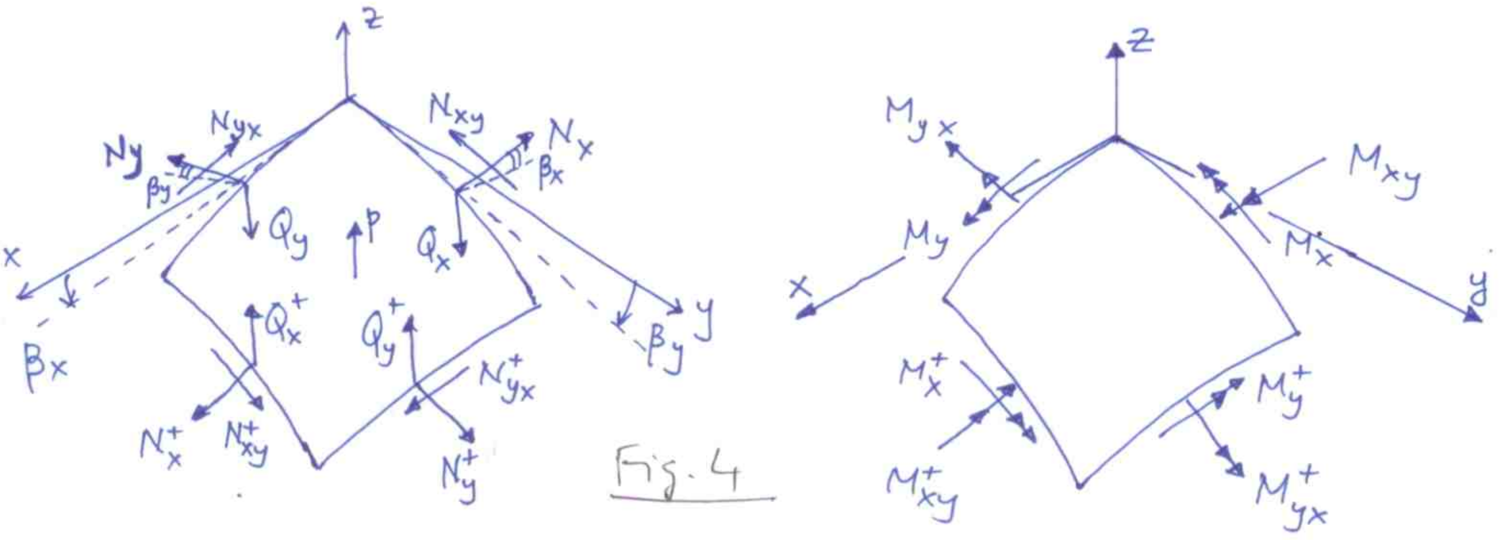


Fig. 4

β_x, β_y measured w.r.t. upper corner of plate element.

$$(\ )^+ = (\ ) + \frac{\partial(\ )}{\partial(x \text{ or } y)} d(x \text{ or } y)$$

$$\cos(\beta_x, \beta_y) = 1; \quad \sin(\beta_x, \beta_y) = (\beta_x, \beta_y); \quad \overbrace{(Q_x, Q_y) * (\beta_x, \beta_y) \simeq 0}^{\text{ie quadratic terms involving small } Q_x, Q_y \text{ \& rot's neglected}}$$

$$\sum F_x : (\cancel{N_x} + N_{x,x}dx)dy - \cancel{N_x dy} + (\cancel{N_{yx}} + N_{yx,y}dy)dx - \cancel{N_{yx}dx} \underbrace{+ (Q_x + Q_{x,x}dx)dy(\beta_x + \beta_{x,x}dx) - Q_x dy \beta_x}_{\text{drops out. see below}} = 0$$

$$\Rightarrow \quad N_{x,x} + N_{yx,y} = 0 \longrightarrow ⑤$$

Similarly $\sum F_y$: $\quad N_{xy,x} + N_{y,y} = 0 \longrightarrow$ ⑥; $N_{xy} = N_{yx}$ (recall).

$$\begin{aligned}\sum F_z : \quad & N_y dx \beta_y - (N_y + N_{y,y}dy)dx(\beta_y + \beta_{y,y}dy) \\ & + N_x dy \beta_x - (N_x + N_{x,x}dx)dy(\beta_x + \beta_{x,x}dx) \\ & + N_{xy}dy\beta_y - (N_{xy} + N_{xy,x}dx)dy(\beta_y + \beta_{y,x}dx) \\ & + N_{yx}dx\beta_x - (N_{yx} + N_{yx,y}dy)dx(\beta_x + \beta_{x,y}dy) \\ & + (\cancel{Q_y} + Q_{y,y}dy)dx - \cancel{Q_y dx} \\ & + (\cancel{Q_x} + Q_{x,x}dx)dy - \cancel{Q_x dy} \quad = -p(x,y)\end{aligned}$$

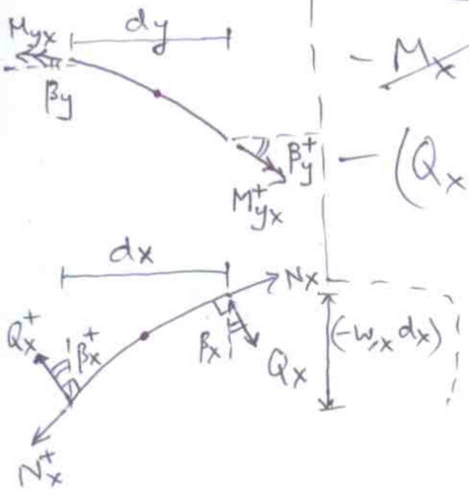
In the above we have dropped $Q_x\beta_x$, $Q_y\beta_y$, $Q_{x,x}\beta_x$, $Q_{x,x}\beta_{x,x}$ etc (ie all quadratics involving Q's & β's) and also used $\cos(\beta_x, \beta_y) = 1$, $\sin(\beta_x, \beta_y) = \beta_x, \beta_y$.

i.e, quadratics in the small transverse shear forces (Q's) and small rotations (β's) are small compared to in-plane forces (N's).

$\sum F_z$ gives,

$$\left(-N_y \beta_{y,y} - \cancel{N_{y,y}\beta_y} - N_x \beta_{x,x} - \cancel{N_{x,x}\beta_x} - N_{xy}\beta_{y,x} - \cancel{N_{xy,x}\beta_y} - N_{yx}\beta_{x,y} - \cancel{N_{yx,y}\beta_x} + Q_{y,y} + Q_{x,x}\right)\cancel{dx\,dy} = -p(x,y) \rightarrow ⑦$$

where we have neglected H.O.T's involving $dx\,dy^2$ or dx^2dy, and the cancellations indicated by arrows are due to Eqs ⑤, ⑥.



$$\sum M_y: \quad \left(M_{yx,y}\,dy + \cancel{M_{yx}}\right)dx - \cancel{M_{yx}\,dx} + \left(M_{x,x}\,dx + \cancel{M_x}\right)dy$$
$$- \cancel{M_x\,dy} - \left(Q_x + Q_x + \overset{\text{H.O.T.}}{\cancel{Q_{x,x}\,dx}}\right)dy\left(\frac{dx}{2}\right)$$
$$- \left(\underset{\text{H.O.T}}{\cancel{Q_x\,dy\,\beta_x}} + \underset{\text{H.O.T.}}{\cancel{\left[Q_x + Q_{x,x}\right]dy\left[\beta_x + \beta_{x,x}dx\right]}}\right)\left(\frac{\overset{\beta_x}{\cancel{-w_{,x}}}\,dx}{2}\right)$$
$$-\left(N_x + \overset{\text{H.O.T}}{\cancel{N_{x,x}\,dx}} + N_x\right)dy\left(\frac{-\overset{\beta_x}{\cancel{w_{,x}}}\,dx}{2}\right)$$
$$+\left(N_x\,dy\,\beta_x + \left(\underset{\text{H.O.T.}}{\cancel{N_{x,x}}}\,dx + N_x\right)dy\left(\underset{\text{H.O.T.}}{\cancel{\beta_{x,x}}}\,dx + \beta_x\right)\right)\frac{dx}{2}$$
$$= 0$$

$$\Rightarrow \left(M_{yx,y} + M_{x,x} - Q_x - \cancel{N_x\beta_x} + \cancel{N_x\beta_x}\right)\cancel{dx\,dy} = 0$$

$$\Rightarrow \quad M_{yx,y} + M_{x,x} - Q_x = 0. \longrightarrow ⑧$$

Similarly $\sum M_x$: $\quad M_{xy,x} + M_{y,y} - Q_y = 0 \rightarrow ⑨$

$$\sum M_z: \left[\left(\overset{\text{H.O.T}}{\cancel{N_{xy,x}}}\,dx + N_{xy}\right)dy + N_{xy}\,dy\right]\frac{dx}{2} - \left[\left(\overset{\text{H.O.T}}{\cancel{N_{yx,y}}}\,dy + N_{yx}\right)dx + N_{yx}\,dx\right]\frac{dy}{2}$$
$$\left(M_y + \underline{M_{y,y}\,dy}\right)dx\left(\underline{\beta_{x,y}dy} + \beta_x\right) - M_y\,dx\,\beta_x - \left(M_{yx} + \underline{M_{yx,y}}dy\right)dx\left(\beta_y + \underline{\beta_{y,y}dy}\right)$$
$$+ M_{yx}\,dx\,\beta_y - \left(M_x + \underline{M_{x,x}dx}\right)dy\left(\beta_y + \underline{\beta_{y,x}dx}\right) + M_x\,dy\,\beta_y + \left(M_{xy} + \underline{M_{xy,x}}dx\right)dy\left(\beta_x + \underline{\beta_{x,x}dx}\right)$$
$$- M_{xy}\,dy\,\beta_x = 0.$$

where all underlined terms are H.O.T. (higher order terms or higher order producing terms).

$$\Rightarrow (N_{xy} - N_{yx})\cancel{\underline{dx\,dy}} + \cancel{M_y dx}\,\beta_x - \cancel{M_y dx}\,\beta_x - \cancel{M_{yx} dx}\,\beta_y$$
$$+ \cancel{M_{yx} dx}\,\beta_y - \cancel{M_x dy}\,\beta_y + \cancel{M_x dy}\,\beta_y + \cancel{M_{xy} dy}\,\beta_x - \cancel{M_{xy} dy}\,\beta_x = 0$$

$\Rightarrow N_{xy} = N_{yx}$ → which is an identity from definition of N_{xy} & N_{yx} (see Eq ④).

Substitute ⑧, ⑨, in ⑦,

$$M_{yx,yx} + M_{x,xx} + M_{xy,xy} + M_{y,yy} - N_y \beta_{y,y} - N_x \beta_{x,x}$$
$$- 2N_{xy}\beta_{x,y} = -p.$$

Substitute ④, ③ (ie constitutive & strain displ relations),

$$-2D(1-\cancel{\nu})\,w_{,xyxy} - D\left(w_{,xxxx} + \cancel{\nu w_{,yyxx}} + w_{,yyyy} + \cancel{\nu w_{,xxyy}}\right)$$
$$+ N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} = -p(x,y)$$

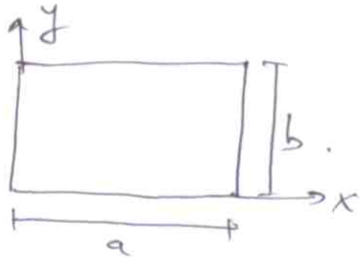
$$\Rightarrow \boxed{\begin{array}{l} D\nabla^4 w - N_x w_{,xx} - N_y w_{,yy} - 2N_{xy} w_{,xy} = p(x,y) \rightarrow ⑩ \\ \text{and} \quad N_{x,x} + N_{yx,y} = 0 \rightarrow ⑤ \\ \quad N_{xy,x} + N_{y,y} = 0 \rightarrow ⑥ \end{array}}$$

Uniform plate, ie D = const here.

If applied edge loads are constant (say $\bar{N}_x$, $\bar{N}_y$, $\bar{N}_{xy}$) then $N_x = \bar{N}_x$, $N_y = \bar{N}_y$, $N_{xy} = \bar{N}_{xy}$ are solutions of ⑤, ⑥ and we insert these solutions in ⑩ and solve ⑩ only. Otherwise, ⑤, ⑥, ⑩ have to be solved together.

<u>Boundary Conditions</u> (Rectangular Plates).

(i) <u>Simply supported</u> If edge x=0 is S.S,



$$\boxed{w(0,y)=0},\quad M_x(0,y) = -D\left(w_{,xx} + \nu \underset{\substack{=0 \\ \because\, w(0,y)=0.}}{\cancel{w_{,yy}}}\right)\Bigg|_{(0,y)} = 0$$

$$\boxed{w_{,xx}(0,y) = 0}$$

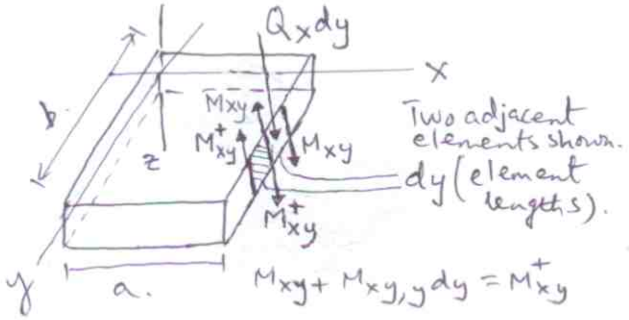
(ii) Fixed. If edge $y=b$ is fixed,

$$\boxed{w(x,b)=0 \,, \quad w_{,y}(x,b)=0}$$

(iii) Free. If edge $x=a$ is free,

$$M_x(a,y) = M_{xy}(a,y) = Q_x(a,y) = 0 \longrightarrow \text{for free edge, no resultant forces, moments act.}$$

However, the order of the differential equation is 4th order in space (see ⑩), hence only 2 BC's per edge can be prescribed. Thus we consider the following due to Kirchoff.

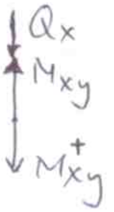


$$M_{xy} + M_{xy,y}\,dy = M^+_{xy}$$

All forces/equivalent forces are in ±z direction. Positive convention as per Fig 4, p.7

Represent $M_{xy}dy$ as a couple, ie $M_{xy}\uparrow$, $M_{xy}\downarrow$ seperated by dy, as shown.
In addition we also have $Q_x dy\downarrow$ on element.

Thus,



$$\equiv \;\downarrow Q_x dy + M_{xy,y}\,dy \equiv Q^*_x\,dy$$

Thus we define the equivalent Kirchoff Transverse shear as,

$$Q^*_x = Q_x + M_{xy,y}$$

Similarly $Q^*_y = Q_y + M_{xy,x}$

and apply the BC's on these.

Thus $M_x(a,y) = 0$, $Q^*_x(a,y) = 0$.

$$Q^*_x = M_{yx,y} + M_{x,x} + M_{xy,y} = M_{x,x} + 2M_{xy,y}$$

$$= -D\left(w_{,xxx} + \nu w_{,yyx}\right) - 2D(1-\nu)\left(w_{,xyy}\right)$$

$$\boxed{Q^*_x = -D\left(w_{,xxx} + (2-\nu)\,w_{,xyy}\right)}$$

Similarly

$$\boxed{Q^*_y = -D\left(w_{,yyy} + (2-\nu)\,w_{,xxy}\right)} \longrightarrow ⑪$$

So BC is,

$$\boxed{-\cancel{D}\left(w_{,xx}+\nu w_{,yy}\right)\Big|_{(a,y)}=0 \quad , \quad -\cancel{D}\left(w_{,xxx}+(2-\nu)w_{,xyy}\right)\Big|_{(a,y)}=0}$$

STRAIN ENERGY.

$$U=\frac{1}{2}\iiint_{Vol}\left(\bar{\sigma}_x\bar{\varepsilon}_x+\bar{\sigma}_y\bar{\varepsilon}_y+\underbrace{\bar{\sigma}_z\cancelto{0}{\bar{\varepsilon}_z}+\bar{\tau}_{yz}\cancelto{0}{\bar{\gamma}_{yz}}+\bar{\tau}_{xz}\cancelto{0}{\bar{\gamma}_{xz}}}_{\text{Kirchoff Assumption.}}+\bar{\tau}_{xy}\bar{\gamma}_{xy}\right)dV$$

Insert Eq ③ p. 5, and constitutive relations Eq ④ p. 6,

$$U=\frac{1}{2}\underset{\substack{S\\ \text{midsurface area}}}{\iint}\ \underset{\substack{h\\ \text{thickness}}}{\int}\left[\bar{\sigma}_x\left(\varepsilon_x+z\kappa_x\right)+\bar{\sigma}_y\left(\varepsilon_y+z\kappa_y\right)+\bar{\tau}_{xy}\left(\gamma_{xy}+2z\kappa_{xy}\right)\right]dz\,dS$$

$$\boxed{U=\underbrace{\frac{1}{2}\iint_S\left(N_x\varepsilon_x+N_y\varepsilon_y+N_{xy}\gamma_{xy}\right)dS}_{U_M=\text{Membrane strain energy}}+\underbrace{\frac{1}{2}\iint_S\left(M_x\kappa_x+M_y\kappa_y+2M_{xy}\kappa_{xy}\right)dS}_{U_b=\text{Bending Strain Energy}}} \to ⑫$$

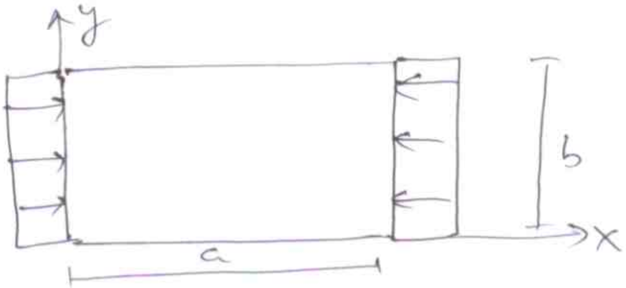
$$U=U_M+U_b.$$

U_M = due to work done by in-plane forces (N_x, N_y, N_{xy}) during overall deformation.

U_b = due to work done (during bending) by moment resultants (M_x, M_y, M_{xy}).

Exact Solutions.

Ex1 Simply supported rectangular plate with uniaxial uniform compression.



$$D\nabla^4 w + N_x \frac{\partial^2 w}{\partial x^2} = 0.$$

Nondimensionalize : $x = a\xi$, $y = b\eta$. , $p = a/b$ aspect ratio.

So we get,

$$D\left(\frac{\partial^4 w}{\partial\xi^4} + 2p^2\frac{\partial^4 w}{\partial\xi^2\partial\eta^2} + p^4\frac{\partial^4 w}{\partial\eta^4}\right) + a^2 N_x \frac{\partial^2 w}{\partial\xi^2} = 0 \longrightarrow ①$$

BC's : $w(0,\eta) = w(1,\eta) = w(\xi,0) = w(\xi,1) = 0$

$$M_x = \frac{\cancel{D}}{\cancel{a^2}}\left(\frac{\partial^2 w}{\partial\xi^2} + \nu p^2\frac{\partial^2 w}{\partial\eta^2}\right) = 0 \;,\; \xi = 0,1 \Rightarrow \frac{\partial^2 w}{\partial\xi^2} = 0 \;,\; \xi = 0,1$$

$$M_y = \frac{\cancel{D}}{\cancel{a^2}}\left(p^2\frac{\partial^2 w}{\partial\eta^2} + \nu\frac{\partial^2 w}{\partial\xi^2}\right) = 0 \;,\; \eta = 0,1 \Rightarrow \frac{\partial^2 w}{\partial\eta^2} = 0 \;,\; \eta = 0,1$$

Choose Double Fourier series solution satisfying BC's, ie,

$$w(\xi,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin m\pi\xi \sin n\pi\eta \, .$$

Insert in ①, equate coeffs of

$$\sum_{m,n=1}^{\infty}\sum \sin m\pi\xi \sin n\pi\eta \, A_{mn}\left[(m\pi)^4 + 2p^2(m\pi)^2(n\pi)^2 + p^4(n\pi)^4 - \frac{N_x a^2 (m\pi)^2}{D}\right] = 0$$

$\Rightarrow A_{mn}\left[\quad\right] = 0$, $A_{mn} \neq 0$ for buckling (ie non-trivial solution).

Define $k^2 = \dfrac{N_x}{D}$.

$$\Rightarrow \left[(m\pi)^2 + (pn\pi)^2\right]^2 - a^2k^2(m\pi)^2 = 0$$

$$N_x = \frac{D}{a^2}\pi^2\left[m + p^2\frac{n^2}{m}\right]^2 = \frac{D\pi^2}{b^2}\left[\frac{m}{p} + \frac{p}{m}n^2\right]^2$$

Its obvious that lowest N_x is for $N=1$.

$$\Rightarrow \boxed{N_x = \frac{D\pi^2}{b^2}\left[\frac{m}{p}+\frac{p}{m}\right]^2 = \lambda\frac{D\pi^2}{b^2}} \quad , \quad \boxed{\lambda = \left[\frac{m}{p}+\frac{p}{m}\right]^2}$$

From this we get:

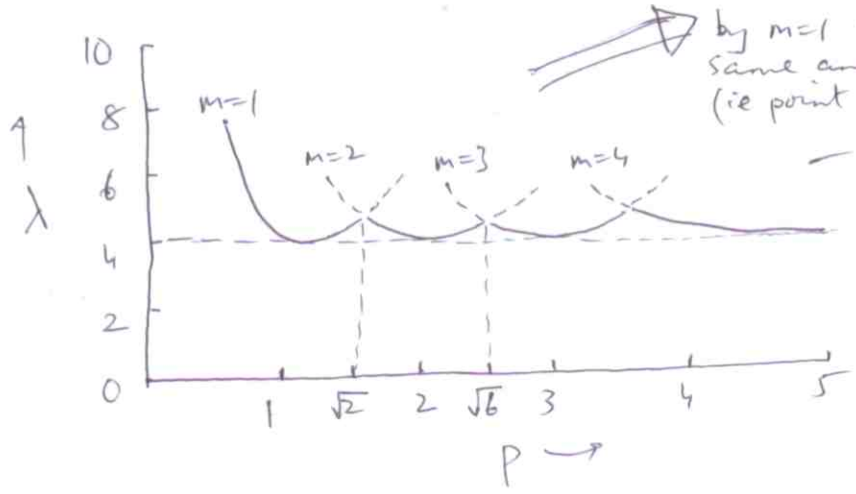
To find m for which N_x is lowest,

$$\frac{dN_x}{dm} = 0 = \frac{D\pi^2}{b^2}\left[\frac{m}{p}+\frac{p}{m}\right]\left[\frac{1}{p} - \frac{p}{m^2}\right]$$

$$\Rightarrow \boxed{m = p.} \longrightarrow$$ only for integer aspect ratios.

When $p \neq$ integer, we draw curves of λ v/s p for various m.

The $m \neq 1$ curves can be obtained by $m=1$ curve by keeping λ value same and multiplying p by m (ie point is $[\lambda, mp]$)



(solid line) lowermost envelope gives λ_{cr} v/s p.

The transition points from m to m+1 half sine waves are given by,

$$\left(\frac{m}{p}+\frac{p}{m}\right) = \left(\frac{m+1}{p} + \frac{p}{m+1}\right)$$

$$\Rightarrow \frac{1}{p} = p\left(\frac{1}{m(m+1)}\right) \Rightarrow p = \sqrt{m(m+1)}$$

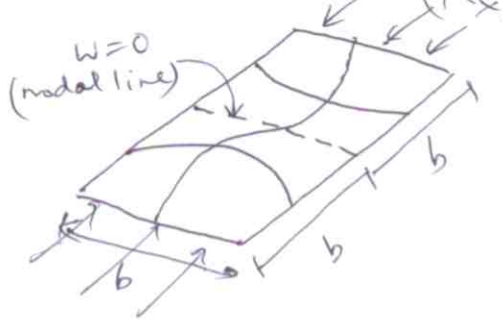
ie $p = \sqrt{2}$ for transition from $m=1$ to $m=2$
$= \sqrt{6}$ " " " $m=2$ " $m=3$

For $p > 4$, $\lambda = 4$ gives good enough approx of lowermost curve. [redacted]

Thus $\frac{N_x}{D} = k^2 = \frac{\lambda \pi^2}{b^2}$, ie $kb = \sqrt{\lambda}\,\pi$.

For p = integer the plate buckles with n = 1 (always) and m = p. That is it buckles in square sub-plates or buckles.

$(N_x)_{cr} = \frac{D\pi^2}{b^2}(4)$



a = 2b, p = 2

SQUARE BUCKLES.

For SS column, $P_{cr} = \frac{\pi^2 EI}{L^2}$ versus SS plate where

$$(N_x)_{cr} = \frac{\pi^2 D}{b^2}\lambda$$

So $(N_x)_{cr}$ inversely proportional to width of plate.

Ex 2 Rectangular plate under uniaxial compression (N_x), loaded edges simply-supported, unloaded edges elastically restrained (rotational Spring $\bar{\beta}$ N.m/m).

BC's: $w(0,\eta) = w(1,\eta) = 0$, $w_{,\xi\xi}(0,\eta) = w_{,\xi\xi}(1,\eta) = 0$

At $\eta = 0, 1$ (or $\eta = -\frac{1}{2}, \frac{1}{2}$ if we chose origin at mid-width) for symm BC's on unloaded edges.

(i) Clamped Clamped : $\bar{\beta} = \infty$ at $\eta = 0, 1$ (or $\eta = \pm\frac{1}{2}$)

(ii) Simply supported : $\bar{\beta} = 0$ at $\eta = 0, 1$. (or $\eta = \pm\frac{1}{2}$)

(iii) SS at $\eta = 0$, clamped at $\eta = 1$.

Choose $w(\xi,\eta) = F(\eta)\sum_{m=1}^{\infty}(\sin m\pi\xi)A_m$ → satisfies BC's at $\xi = 0, 1$.

Substitute in equilibrium eqn,

$$\sum_{m=1}^{\infty}\left[(m\pi)^4 F(\eta) - 2p^2(m\pi)^2 F'' + p^4 F^{IV} - a^2 k^2 m^2 \pi^2 F\right](\sin m\pi\xi)A_m = 0$$

$$\Rightarrow F^{IV} - \frac{2}{p^2}m^2\pi^2 F'' + \frac{m^4\pi^4}{p^4}F - \frac{a^2k^2m^2\pi^2 F}{p^4} = 0$$

put $f = e^{s\eta}$

$$\Rightarrow s^4 - \frac{2m^2\pi^2}{p^2} s^2 + \frac{m^4\pi^4}{p^4} - (ak)^2 \frac{m^2\pi^2}{p^4} = 0$$

$$\left(s^2 - \frac{m^2\pi^2}{p^2}\right)^2 = \left(\frac{akm\pi}{p^2}\right)^2$$

$$s^2 = \frac{m^2\pi^2}{p^2} \pm \frac{akm\pi}{p^2} \longrightarrow ①$$

For a wide-column (i.e, unloaded edges free), from column analogy, we have

$$\frac{N_x}{D} = k^2 = \frac{m^2\pi^2}{a^2}$$

Thus for plate with restraints on one or both of the unloaded edges we have

$$\frac{N_x}{D} > \frac{m^2\pi^2}{a^2}, \text{ ie } k > \frac{m\pi}{a}, \text{ ie } ak > m\pi$$

Hence s^2 in ① is < 0 when considering the $-ve$ sign.

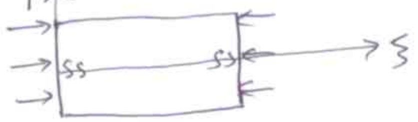
$$\Rightarrow s_1 = +\alpha, \quad s_2 = -\alpha, \quad s_3 = +i\beta, \quad s_4 = -i\beta$$

where,

$$\boxed{\begin{aligned} \alpha &= \left\{\frac{m^2\pi^2}{p^2} + \frac{akm\pi}{p^2}\right\}^{1/2} \\ \beta &= \left\{-\frac{m^2\pi^2}{p^2} + \frac{akm\pi}{p^2}\right\}^{1/2} \end{aligned}} \longrightarrow ①$$

$$\boxed{f(\eta) = A_1 \cosh\alpha\eta + A_2 \sinh\alpha\eta + A_3 \cos\beta\eta + A_4 \sin\beta\eta} \longrightarrow ②$$

<u>Case (i), (ii)</u> Here BC's on unloaded edges are symmetric, so easier (<u>not compulsory</u>) to keep origin as shown below

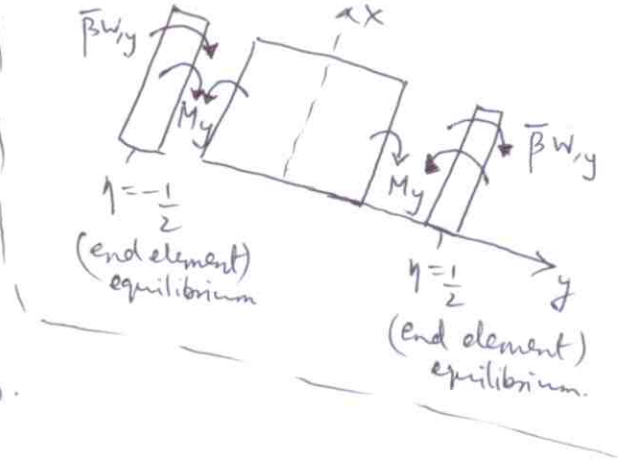


Due to symmetry we loose the $\sinh\alpha\eta$ and $\sin\alpha\eta$ terms. The BC's are,

$$W(\xi,\tfrac{1}{2}) = W(\xi,-\tfrac{1}{2}) = 0 \longrightarrow (a,b)$$

$$M_y = \frac{-D}{a^2}\Big(p^2 W_{,\eta\eta} + \nu \underset{\substack{=0\\ \text{along } \eta=\frac{1}{2}}}{\cancel{W_{,\xi\xi}}}\Big)\Big|_{\eta=\frac{1}{2}} = \frac{\bar{\beta}}{b} W_{,\eta}\Big|_{\eta=\frac{1}{2}} \quad \left(\text{from } M_y - \bar{\beta} W_{,y} = 0 \text{ at } \eta = \tfrac{1}{2}\right) \to (c)$$

$$M_y = \frac{-D}{a^2}\Big(p^2 W_{,\eta\eta} + \nu \underset{\substack{=0\\ \text{along } \eta=-\frac{1}{2}}}{\cancel{W_{,\xi\xi}}}\Big)\Big|_{\eta=-\frac{1}{2}} = -\frac{\bar{\beta}}{b} W_{,\eta}\Big|_{\eta=} \quad \left(\text{from } M_y + \bar{\beta} W_{,y} = 0 \text{ at } \eta = -\tfrac{1}{2}\right) \nearrow (d)$$



BC's (a,b) give, $A_1\cosh\frac{\alpha}{2} + A_3\cos\frac{\beta}{2} = 0$

(c,d) give,

$$-\beta_0\left(\alpha^2\cosh\frac{\alpha}{2}A_1 - \beta^2\cos\frac{\beta}{2}A_3\right) - \left(\alpha\sinh\frac{\alpha}{2}A_1 - \beta\sin\frac{\beta}{2}A_3\right) = 0.$$

where $\boxed{\beta_0 \triangleq \frac{D}{\bar{\beta} b}}$
(defined as)

Thus

$$\det\begin{bmatrix} \cosh\frac{\alpha}{2} & \cos\frac{\beta}{2} \\ -\beta_0\alpha^2\cosh\frac{\alpha}{2} - \alpha\sinh\frac{\alpha}{2} & \beta_0\beta^2\cos\frac{\beta}{2} + \beta\sin\frac{\beta}{2} \end{bmatrix} = 0$$

gives CE

Note that the CE contains the load N_x thru the parameter k which in turn appears in α and β. (i.e, the solution involves solution of a transcedental equation).

$$\left(\beta_0\beta^2\cos\frac{\beta}{2} + \beta\sin\frac{\beta}{2}\right)\cosh\frac{\alpha}{2} + \cos\frac{\beta}{2}\left(\beta_0\alpha^2\cosh\frac{\alpha}{2} + \alpha\sinh\frac{\alpha}{2}\right) = 0$$

$$\left(\beta_0\beta^2 + \beta\tan\frac{\beta}{2}\right) + \left(\beta_0\alpha^2 + \alpha\tanh\frac{\alpha}{2}\right) = 0 \quad \left(\because \cosh\frac{\alpha}{2} \neq 0,\ \cos\beta/2 \neq 0\right).$$

$$\boxed{\begin{aligned} &\beta_0\left(\frac{2ak\cancel{m\pi}}{p^{\cancel{2}}}\right) + \frac{m\cancel{\pi}}{\cancel{p}}\left(\frac{a}{m\pi}k-1\right)^{1/2}\tan\left[\frac{m\pi}{2p}\left(\frac{a}{m\pi}k-1\right)^{1/2}\right] \\ &\qquad + \frac{m\cancel{\pi}}{\cancel{p}}\left(\frac{a}{m\pi}k+1\right)^{1/2}\tanh\left[\frac{m\pi}{2p}\left(\frac{a}{m\pi}k+1\right)^{1/2}\right] = 0 \\ &\text{where } k^2 = \frac{N_x}{D} \end{aligned}} \rightarrow ③$$

$\hookrightarrow$ CE (transcedental eqn for k).

Case (i): Simply supported at $\eta =$ $\Rightarrow \bar{\beta} = 0 \Rightarrow \beta_0 \to \infty$

tanh function varies between -1 to $+1$.

$$\Rightarrow \left(\frac{a}{m\pi}k-1\right)^{1/2} \tan\left[\frac{m\pi}{2p}\left(\frac{a}{m\pi}k-1\right)^{1/2}\right] = -\infty.$$

finite so cancels out

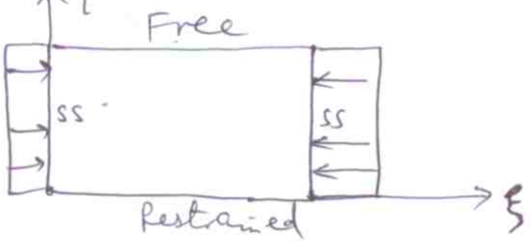
$$\Rightarrow \frac{m\pi}{2p}\left(\frac{a}{m\pi}k-1\right)^{1/2} = \frac{\pi}{2}(2n+1)$$ (take $n=0$ for k_{cr} smallest)

$$\frac{a}{m\pi}k - 1 = \frac{p^2}{m^2} \Rightarrow \frac{N_x}{D} = \left(1+\frac{p^2}{m^2}\right)^2 \frac{m^2\pi^2}{a^2}$$

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{m}{p}+\frac{p}{m}\right)^2 \longrightarrow \text{same result as before.}$$

Case (ii): Fixed at $\eta =$ $\Rightarrow \bar{\beta} = \infty \Rightarrow \beta_0 = 0 \rightarrow$ put in ③ and solve numerically

Ex 3. Rectangular Plate SS on loaded edges, elastically restrained on one edge, free on other edge, uniaxial compression.



origin fixed at lower left corner (∵ no symmetry in BC's).

BC's : $\eta = 0$: $w = 0$, $\beta_0 w_{,\eta\eta} = w_{,\eta}$

$$\eta = 1 : \begin{cases} p^2 w_{,\eta\eta} + \nu w_{,\xi\xi} = 0 \\ p^2 w_{,\eta\eta\eta} + (2-\nu) w_{,\eta\xi\xi} = 0 \end{cases}$$

$$w = \sum_{m=1}^{\infty} A_m \sin m\pi\xi \, f(\eta) .$$

$$w(\xi,0) = 0 \Rightarrow f(0) = 0 \Rightarrow A_1 + A_3 = 0$$

$$\beta_0 \, w_{,\eta\eta}(\xi,0) = w_{,\eta}(\xi,0) \Rightarrow \beta_0 f''(0) = f'(0) \Rightarrow \beta_0(\alpha^2 A_1 - \beta^2 A_3) = \alpha A_2 + \beta A_4$$

$$\left(p^2 w_{,\eta\eta} + \nu w_{,\xi\xi}\right)\Big|_{(\xi,1)} = 0 \Rightarrow p^2 f''(1) - \nu(m\pi)^2 f(1) = 0 .$$

$$\Rightarrow A_1\left(p^2\alpha^2 - \nu m^2\pi^2\right)\cosh\alpha + A_2\left(p^2\alpha^2 - \nu m^2\pi^2\right)\sinh\alpha$$
$$+ A_3\left(-p^2\beta^2 - \nu m^2\pi^2\right)\cos\beta + A_4\left(-p^2\beta^2 - \nu m^2\pi^2\right)\sin\beta$$

$$p^2\frac{\partial^3 w}{\partial\eta^3} + (2-\nu)\frac{\partial^3 w}{\partial\eta\,\partial\xi^2} = 0 \Rightarrow p^2 f'''(1) - (2-\nu)(m\pi)^2 f'(1) = 0$$

$$\Rightarrow A_1\left(p^2\alpha^3 - (2-\nu)m^2\pi^2\alpha\right)\sinh\alpha + A_2\left(p^2\alpha^3 - (2-\nu)m^2\pi^2\alpha\right)\cosh\alpha$$
$$+ A_3\left(p^2\beta^3 + (2-\nu)m^2\pi^2\beta\right)\sin\beta + A_4\left(-p^2\beta^3 - (2-\nu)m^2\pi^2\beta\right)\cos\beta = 0$$

$$\det\begin{bmatrix} 1 & 0 & 1 & 0 \\ \beta_0\alpha^2 & -\alpha & -\beta_0\beta^2 & -\beta \\ (p^2\alpha^2 - \nu m^2\pi^2)\cosh\alpha & (p^2\alpha^2 - \nu m^2\pi^2)\sinh\alpha & (-p^2\beta^2 - \nu m^2\pi^2)\cos\beta & (-p^2\beta^2 - \nu m^2\pi^2)\sin\beta \\ (p^2\alpha^3 - (2-\nu)m^2\pi^2\alpha)\sinh\alpha & (p^2\alpha^3 - (2-\nu)m^2\pi^2\alpha)\cosh\alpha & (p^2\beta^3 + (2-\nu)m^2\pi^2\beta)\sin\beta & (-p^2\beta^3 - (2-\nu)m^2\pi^2\beta) * \cos\beta \end{bmatrix} = 0 .$$

Now define $\tilde{\alpha} \triangleq p^2\alpha^2 - \nu m^2\pi^2 = p^2\beta^2 + (2-\nu)m^2\pi^2$

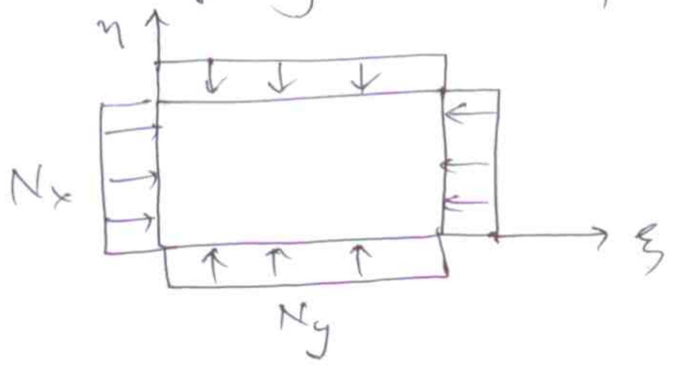
$\tilde{\beta} \triangleq p^2\beta^2 + \nu m^2\pi^2 = p^2\alpha^2 - (2-\nu)m^2\pi^2$

$$\Rightarrow -\alpha\left(\tilde{\beta}\tilde{\alpha}\beta\left[\cos^2\beta + \sin^2\beta\right]\right) + \beta_0\beta^2\left(-\tilde{\alpha}^2\beta\sinh\alpha\cos\beta + \tilde{\beta}^2\alpha\cosh\alpha\sin\beta\right)$$
$$-\beta\left(\beta\tilde{\alpha}^2\sinh\alpha\sin\beta + \alpha\tilde{\beta}^2\cosh\alpha\cos\beta\right)$$
$$+\beta_0\alpha^2\left(-\tilde{\alpha}^2\beta\sinh\alpha\cos\beta + \tilde{\beta}^2\alpha\cosh\alpha\sin\beta\right)$$
$$+\alpha\left(-\tilde{\alpha}^2\beta\cosh\alpha\cos\beta + \tilde{\beta}^2\alpha\sinh\alpha\sin\beta\right) - \beta\left(\tilde{\alpha}\tilde{\beta}\alpha\left[\cosh^2\alpha - \sinh^2\alpha\right]\right) = 0$$

$$\Rightarrow \boxed{\begin{aligned} &-2\alpha\beta\tilde{\alpha}\tilde{\beta} - \alpha\beta(\tilde{\alpha}^2 + \tilde{\beta}^2)\cosh\alpha\cos\beta - (\beta^2\tilde{\alpha}^2 - \alpha^2\tilde{\beta}^2)\sinh\alpha\sin\beta \\ &\beta_0(\alpha^2+\beta^2)\left(-\beta\tilde{\alpha}^2\sinh\alpha\cos\beta + \alpha\tilde{\beta}^2\cosh\alpha\sin\beta\right) = 0 . \end{aligned}}$$

↳CE. ↳ ①

Ex4 Uniformly bi-axially loaded S-S plate:



$$D\nabla^4 w + N_x w_{,xx} + N_y w_{,yy} = 0.$$

$$w = \sum_{m,n=1}^{\infty}\sum A_{mn} \sin m\pi\xi \sin n\pi\eta$$

(satisfies BC's)

$$D\left[\left(\frac{m\pi}{a}\right)^4 + \frac{2m^2n^2\pi^4}{a^2b^2} + \left(\frac{n\pi}{b}\right)^4\right] - N_x\left(\frac{m\pi}{a}\right)^2 - N_y\left(\frac{n\pi}{b}\right)^2 = 0$$

for $A_{mn} \neq 0$, ie buckling.

$$\Rightarrow D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - N_x\left(\frac{m\pi}{a}\right)^2 - N_y\left(\frac{n\pi}{b}\right)^2 = 0 \longrightarrow ①.$$

(i) If $N_x/N_y = r$

$$\Rightarrow N_y = \frac{D\pi^2\left[(m/a)^2 + (n/b)^2\right]^2}{\left[r(m/a)^2 + (n/b)^2\right]} = \frac{D\pi^2}{a^2}\frac{(m^2+p^2n^2)^2}{(rm^2+p^2n^2)}$$

For square plate with $N_x = N_y$, $p=1$, $r=1$,

$N_y = \frac{D\pi^2}{a^2}(m^2+n^2) \to$ lowest for $m=n=1$, $(N_x)_{cr} = 2\frac{D\pi^2}{a^2}$

For square plate, $p=1$,

$N_y = \frac{D\pi^2}{a^2}\frac{(m^2+n^2)^2}{(rm^2+n^2)} \to (m^2+n^2)^2$ grows faster than (rm^2+n^2) as either m or n increase

So min N_y for $m=n=1$

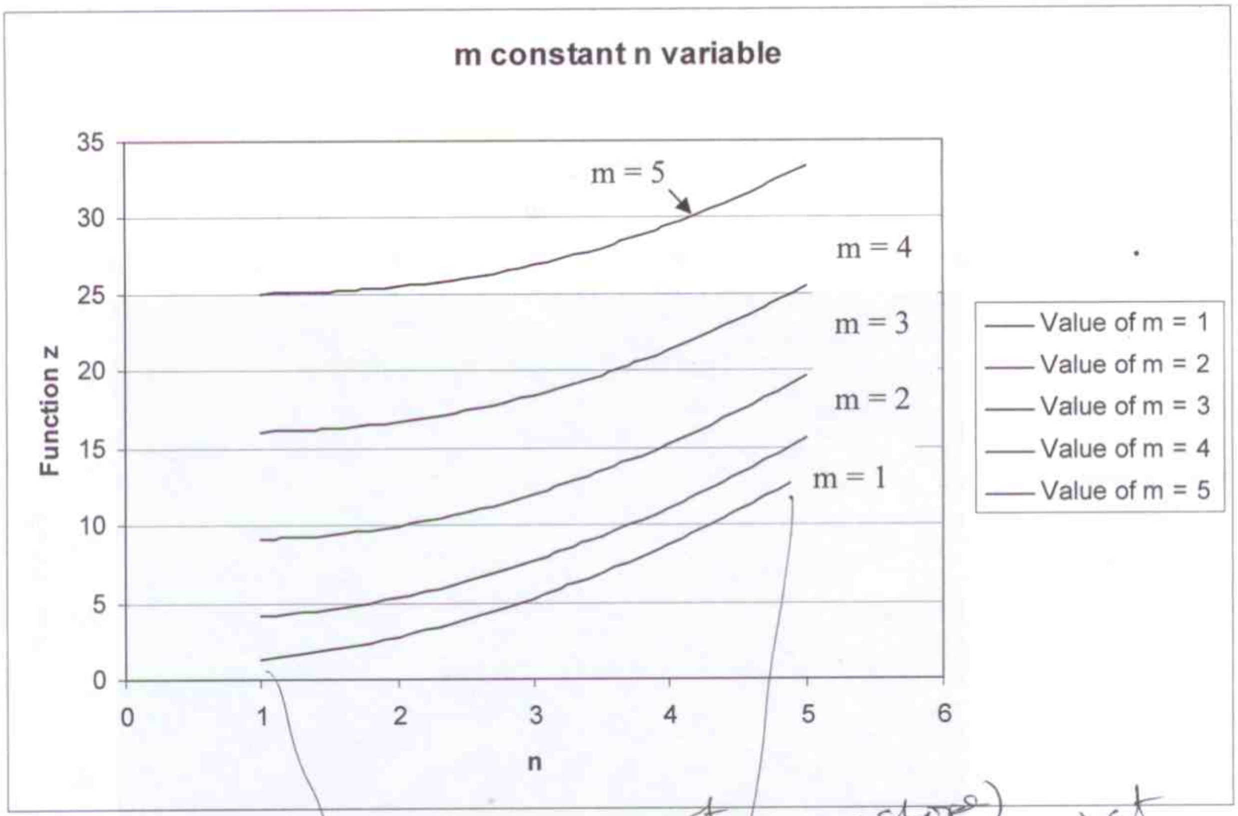
$(N_y)_{cr} = \frac{D\pi^2}{a^2}\frac{4}{(r+1)}$ (see graph p.19 a)

(ii) N_x = fixed, compute $(N_y)_{cr}$. → Use ①.

Note that even if $N_x < 0$ (or $N_y < 0$), ie one load tensile, we can still have buckling (ie if the other one comes out +ve according to ①, ie compressive). However, it is obvious that if, say N_x = fixed = tensile, then $(N_y)_{cr}$ increases (see ①), ie buckling load increases since the tensile load has a stabilizing effect.

TV – ⑲ⓐ

$r=2, \quad (m^2+n^2)^2/(m^2+rn^2)$



extremeties are not stationary values (i.e zero slope) so min occurs at lowermost point.

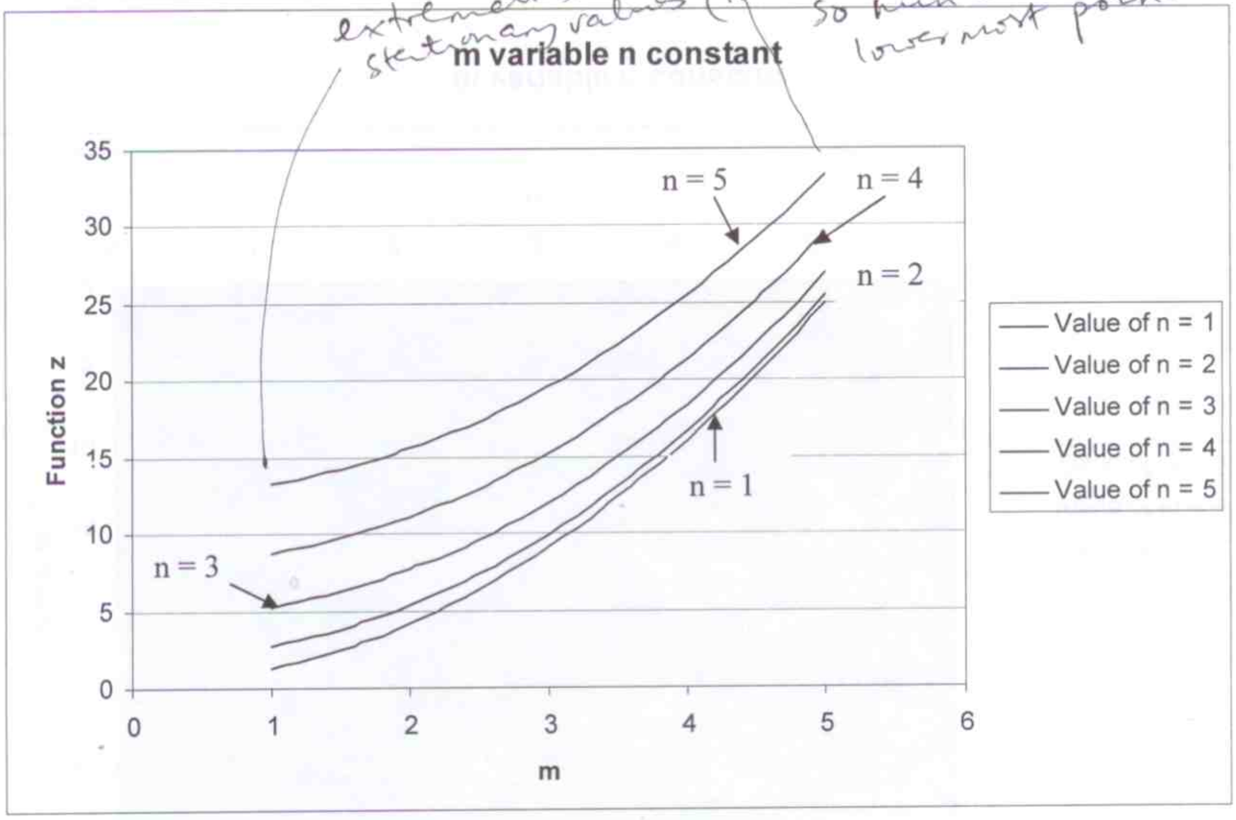


Fig shows lowermost envelope curves for various boundary conditions on unloaded edges (cases A – E as shown) for loaded edges clamped (dashed curve) & loaded edges simply supported (solid curve).

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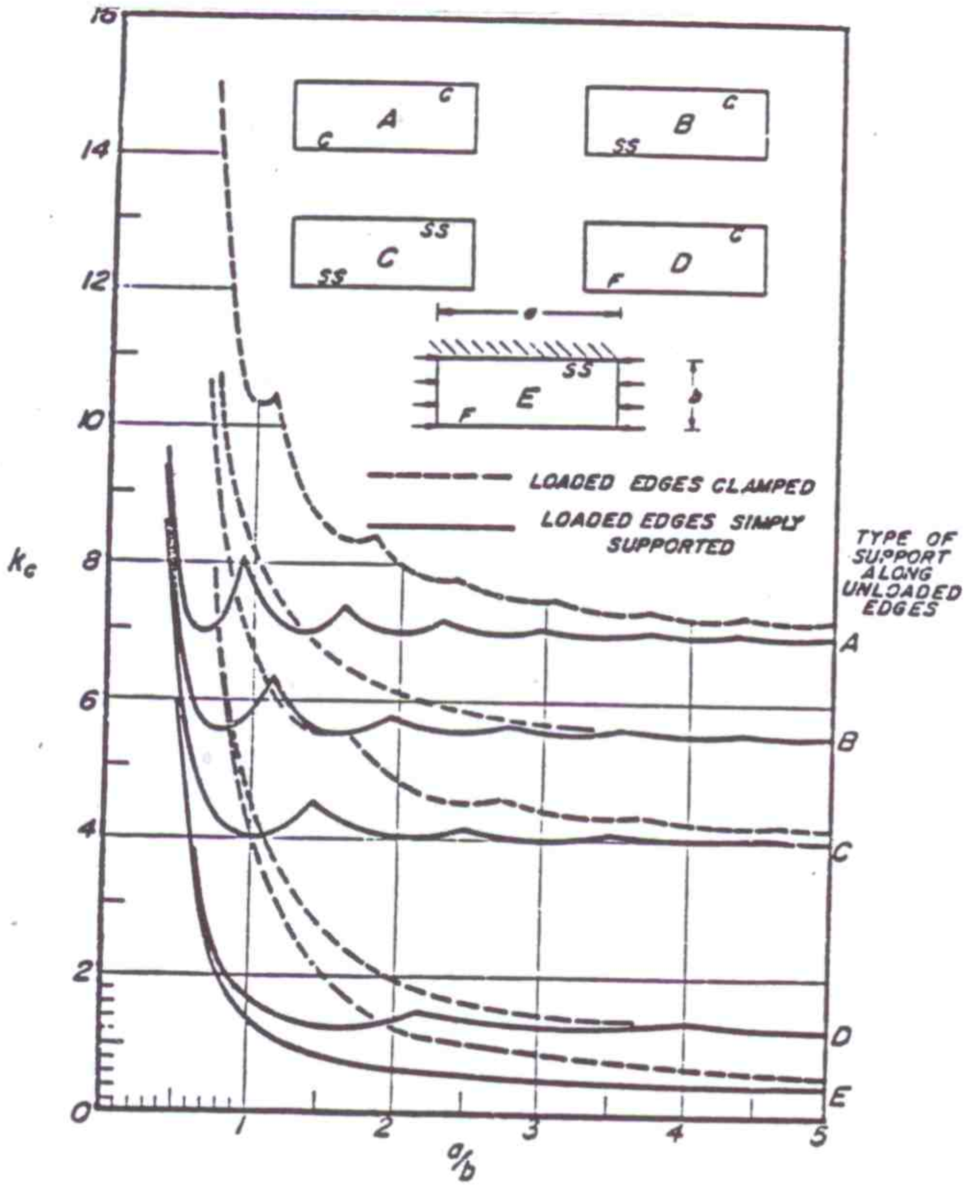
$P_{cr} \propto K_c$



ENERGY METHODS — APPROXIMATE SOLUTIONS

Ritz Method.

$$U_b = \frac{1}{2}\iint_S \left(M_x K_x + M_y K_y + M_{xy}\, 2K_{xy}\right) dS$$ (from Eq ⑫ p. 11).

S (area of undeformed mid surface)

$$U_b = \frac{1}{2}\iint_S \left[D\left(K_x^2 + K_y^2 + 2\nu K_x K_y\right) + 2D(1-\nu)K_{xy}^2\right] dS$$

$$= \frac{1}{2}\iint_S \left[D\left(K_x + K_y\right)^2 + 2D(1-\nu)\left(K_{xy}^2 - K_x K_y\right)\right] dS$$

if D=const (ie uniform plate) →

$$= \frac{D}{2}\iint_S \left[\left(w_{,xx} + w_{,yy}\right)^2 + \underbrace{2\,(1-\nu)\left(w_{,xy}^2 - w_{,xx}w_{,yy}\right)}_{\text{2nd Term.}}\right] dS.$$

Now if $w(x,y) = f(x)\,g(y)$ and plate is supported on all boundaries, ie $w=0$ on all boundaries, then

$$\int_0^b\!\!\int_0^a \left(w_{,xy}^2 - w_{,xx}w_{,yy}\right) dx\,dy = \int_0^b\!\!\int_0^a \left[(f'g')^2 - (f''g)(f\,g'')\right] dx\,dy$$

$$= \int_0^b\!\!\int_0^a (f'g')^2 dx\,dy - \int_0^a f''f\,dx * \int_0^b g''g\,dy$$

$$= \int_0^b\!\!\int_0^a (f'g')^2 dx\,dy - \Big(\underset{\substack{\swarrow \\ 0 \\ \text{at } 0,a}}{f'f}\Big|_0^a - \int_0^a f'f'\,dx\Big)\Big(\underset{\substack{\swarrow \\ =0 \text{ at } 0,b}}{g'g}\Big|_0^b - \int_0^b g'g'\,dy$$

$\because w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$

ie $f(0) = f(a) = g(0) = g(b) = 0$.

$$= \int_0^b\!\!\int_0^a \left((f'g')^2 - (f'g')^2\right) dx\,dy = 0$$

Note: D=const for 2nd term to vanish.
We have shown this for a rectangular plate but it can be shown for an arbitrary shaped plate (using Gauss Divergence theorem), so the conclusion is general:

ie if $w=0$ on boundaries & $w(x,y) = f(x)\,g(y)$, (ie seperable solution) then 2nd Term in U_b is zero.

$(W_p)_b$ = work done by N_x, N_y, N_{xy} during bending part of deformation (ie after inplane extension/compression has occurred).

(N_x, N_y, N_{xy}) ↗ p; bending ↖ b

$$(W_p)_b = \iint_S \left(N_x \frac{w_{,x}^2}{2} + N_y \frac{w_{,y}^2}{2} + N_{xy} w_{,x} w_{,y} \right) dS$$

(refer Eq ⑫ p.11, Eq ① p.3, & Note that N_x, N_y, N_{xy} are const over bending part of deformation)

So work done is not $\frac{1}{2} N_x \varepsilon_x$ but $N_x \varepsilon_x$, etc

$$\text{So, } \Pi = U_b - (W_p)_b = \frac{D}{2} \iint_S (w_{,xx} + w_{,yy})^2 dS - \frac{1}{2} \iint_S \left(N_x w_{,x}^2 + N_y w_{,y}^2 + 2 N_{xy} w_{,x} w_{,y} \right) dS$$

Π ↖ (not 'pi' = 22/7)

$$\boxed{\Pi = \frac{1}{pa^2} \frac{D}{2} \int_0^1 \int_0^1 \left(w_{,\xi\xi} + p^2 w_{,\eta\eta} \right)^2 d\xi\, d\eta - \frac{1}{2p} \int_0^1 \int_0^1 \left(N_x w_{,\xi}^2 + p^2 N_y w_{,\eta}^2 + p 2 N_{xy} w_{,\xi} w_{,\eta} \right) d\xi d\eta}$$

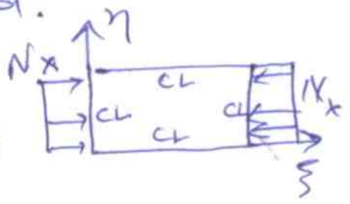
The procedure is same as for columns, ie

$$W_a = \sum_{m=1}^{r} \sum_{n=1}^{s} A_{mn} f(x)\, g(y)$$

$$\frac{\partial \Pi}{\partial A_{uv}} = 0 \longrightarrow \overset{u=1,\dots r,\ v=1,\dots s}{\text{gives equations from which}}$$

det (coeff matrix) = 0 gives CE

Ex 1 All edges clamped, only N_x applied.



BC's $w(0,\eta) = w(1,\eta) = w(\xi,0) = w(\xi,1) = 0$

$$\frac{\partial w}{\partial \xi}(0,\eta) = \frac{\partial w}{\partial \xi}(1,\eta) = \frac{\partial w}{\partial \eta}(\xi,0) = \frac{\partial w}{\partial \eta}(\xi,1) = 0$$

$$W_a = \sum_{\substack{m=1\cdots p \\ n=1,\cdots q}}\sum A_{mn}(1-\cos 2m\pi\xi)(1-\cos 2n\pi\eta)$$

Can also use $$w_a = \sum_{\substack{m=0,\cdots p \\ n=0,\cdots q}}\sum A_{mn}\xi^2(1-\xi)^2\eta^2(1-\eta)^2\xi^m\eta^n$$

Both these satisfy the geometric BC's.

We will use the trigonometric series. For a single-term series,

can call it A (∵ single term series considered)

$$w = w_a = \cancel{A_{mn}}(1-\cos 2m\pi\xi)(1-\cos 2n\pi\eta).$$

not 'pi'

$$\boxed{\Pi} = \frac{1}{2}\frac{D}{pa^2}\int_0^1\int_0^1\left(\frac{\partial^2 w}{\partial \xi^2} + p^2\frac{\partial^2 w}{\partial \eta^2}\right)^2 d\xi\, d\eta - \frac{1}{2}\int_0^1\int_0^1 \frac{N_x}{p}\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi\, d\eta$$

$$w_{,\xi} = A_{mn}\, 2m\pi \sin 2m\pi\xi(1-\cos 2n\pi\eta)$$

$$w_{,\xi\xi} = A_{mn}\, 4m^2\pi^2\cos 2m\pi\xi\,(1-\cos 2n\pi\eta)$$

$$w_{,\eta} = A_{mn}\, 2n\pi \sin 2n\pi\eta\,(1-\cos 2m\pi\xi)$$

$$w_{,\eta\eta} = A_{mn}\, 4n^2\pi^2\cos 2n\pi\eta(1-\cos 2m\pi\xi)$$

$$w_{,\xi\xi}^2 = A_{mn}^2\, 16m^4\pi^4\,\frac{(1+\cos 4m\pi\xi)}{2}\left(1 - 2\cos 2n\pi\eta + \left[\frac{1+\cos 4n\pi\eta}{2}\right]\right)$$

$$= A_{mn}^2\frac{16m^4\pi^4}{4}\Big[2 - 4\cos 2n\pi\eta + 1 + \cos 4n\pi\eta + 2\cos 4m\pi\xi - 4\cos 4m\pi\xi\cos 2n\pi\eta + \cos 4m\pi\xi + \cos 4m\pi\xi\cos 4n\pi\eta\Big]$$

$$w_{,\eta\eta}^2 = A_{mn}^2\frac{16n^4\pi^4}{4}\Big[3 - 4\cos 2m\pi\xi + \cos 4m\pi\xi + 3\cos 4n\pi\eta - 4\cos 4n\pi\eta\cos 2m\pi\xi + \cos 4m\pi\xi\cos 4n\pi\eta\Big].$$

$$w_{,\xi\xi}\, w_{,\eta\eta} = A_{mn}^2\frac{16m^2n^2\pi^4}{4}\Big[\cos 2m\pi\xi\cos 2n\pi\eta\,(1 - \cos 2n\pi\eta - \cos 2m\pi\xi + \cos 2m\pi\xi\cos 2n\pi\eta)\Big]$$

$$w_{,\xi\xi}\, w_{,\eta\eta} = \frac{A_{mn}^2\, 16 m^2 n^2 \pi^4}{4}\Big[\cos 2m\pi\xi \cos 2n\pi\eta - \cos 2m\pi\xi\Big\{\frac{1+\cos 4n\pi\eta}{2}\Big\}$$
$$- \cos 2n\pi\eta\Big\{\frac{1+\cos 4m\pi\xi}{2}\Big\}$$
$$+ \frac{1}{4}(1+\cos 4m\pi\xi)(1+\cos 4n\pi\eta)\Big]$$

$$w_{,\xi}^2 = A_{mn}^2\, 4\pi^2 m^2 \left(\frac{1-\cos 4m\pi\xi}{2}\right)\left(1 - 2\cos 2n\pi\eta + \Big\{\frac{1+\cos 4n\pi\eta}{2}\Big\}\right).$$

Now $\int_0^1 \cos 2k\pi\xi\, d\xi = \int_0^1 \cos 2k\pi\eta\, d\eta = 0.$, $\int_0^1\int_0^1 d\xi\, d\eta = 1$.

$$\Rightarrow \Pi = \frac{1}{2}\frac{D}{pa^2}\left[m^4\left(\frac{3}{4}\right) + p^4 n^4\left(\frac{3}{4}\right) + 2p^2 m^2 n^2\left(\frac{1}{4}\right)\right] 16\pi^4 A_{mn}^2$$
$$- \frac{1}{2}\frac{N_x}{p} 4m^2\pi^2 A_{mn}^2\left(\frac{3}{4}\right)$$

$$\frac{\partial \Pi}{\partial A_{mn}} = A_{mn}\left[\frac{16\pi^4 D}{pa^2}\left(\frac{3m^4 + 2p^2m^2n^2 + 3p^4n^4}{4}\right) - \frac{N_x}{p}\frac{12m^2\pi^2}{4}\right] = 0.$$

$$\because A_{mn} \neq 0,\ (N_x)_{cr} = \frac{4}{3}\frac{\pi^2 D}{a^2}\frac{1}{m^2}\left(3m^4 + 2p^2m^2n^2 + 3p^4n^4\right)$$

$(N_x)_{cr}$ min for $n=1$ (obvious).

$$(N_x)_{cr} = \frac{4}{3}\frac{\pi^2 D}{a^2}\frac{(3m^4 + 2p^2m^2 + 3p^4)}{m^2}$$

↓ Plot $(N_x)_{cr}$ versus p for various m and choose the lowermost envelope. (like in SS plate).

However, doing $\frac{d(N_x)_{cr}}{dm} = 0$ gives, $6m - \frac{(2)(3p^4)}{m^3} = 0$

$$\Rightarrow m = p.$$

So for integer p, this one term approx solution gives $(N_x)_{cr}$ for $m=p$.

For $p=1$ $(a=b)$, $(N_x)_{cr} = \frac{4}{3}\frac{\pi^2 D}{a^2}(3+2+3) = 10.67\frac{\pi^2 D}{a^2}$

Exact solution obtained by Levy is $(N_x)_{cr} = 10.07\frac{\pi^2 D}{a^2}$ for $a=b$. Thus the approx solution (one-term) is an upperbound with 6% error.

A two term approximation would be

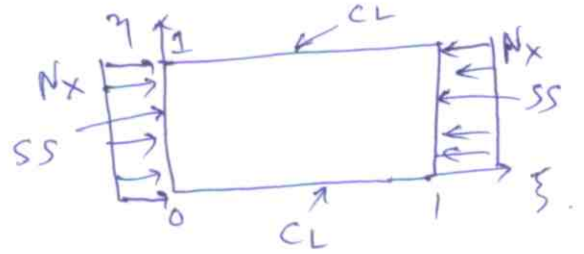
$$w_a = A(1-\cos 2m\pi\xi)(1-\cos 2\pi\eta) + B(1-\cos 4m\pi\xi)(1-\cos 2\pi\eta)$$

where we have used the intuition from the exact soln of SS plate & the Ritz solution of Clamped-Clamped plate that in η ($\overset{ie}{y}$) direction we will have only a single half-sine wave ($n=1$) for the buckled configuration when only N_x applied. We still leave 'm' as unknown and will plot $(N_x)_{cr}$ for various m and take the lowest envelope. In this case we will get $\frac{\partial \Pi}{\partial A} = 0$, $\frac{\partial \Pi}{\partial B} = 0$, $\because \begin{Bmatrix} A \\ B \end{Bmatrix} \neq \underset{\sim}{0}$ so determinant of coefficient matrix is zero (ie CE). This gives a quadratic in $(N_x)_{cr}$, plot both roots versus p for various m, take lowest envelope.

The problem with such approx solutions is that we dont know whether to take the lower few terms in the m-sequence (ie $m=1, 2, --$) since $(N_x)_{cr}$ depends on m and p. So the "complete" set of trial functions may not contain a high enough m-number for the domain (aspect ratio) of our interest. YET IT IS BEST TO TAKE A COMPLETE SET

STARTING FROM $m=1$ onwards (same for n also, in general).

Ex 2 N_x applied, loaded edges SS, other two clamped



$$\text{Choose} \quad W_a = A \sin m\pi\xi\,(1-\cos 2\overset{\text{can guess } n=1 \text{ (see below).}}{n}\pi\eta) \rightarrow \underline{\underline{\text{one-term approx.}}}$$

$\hookrightarrow$ satisfies kinematic BC's.

$$w_{,\xi} = Am\pi\cos m\pi\xi\,(1-\cos 2n\pi\eta)$$

$$w_{,\xi\xi} = -Am^2\pi^2 \sin m\pi\xi\,(1-\cos 2n\pi\eta)$$

$$w_{,\eta} = A\,2n\pi \sin m\pi\xi \sin 2n\pi\eta$$

$$w_{,\eta\eta} = A\,4n^2\pi^2 \sin m\pi\xi \cos 2n\pi\eta$$

$$\Pi = \frac{D\pi^4}{2pa^2} A^2\left[m^4\left(\frac{3}{4}\right) + p^4 16n^4\left(\frac{1}{4}\right) + 4m^2n^2\left(\frac{1}{4}\right)2p^2\right]$$

$$- \frac{N_x}{2p}\left(\frac{3}{4}\right)A^2 m^2\pi^2$$

$$\frac{d\Pi}{dA} = 0 = \cancel{2}A\left[\frac{D\pi^{\cancel{4}\,2}}{\cancel{2}pa^2}\left\{\frac{3m^4+16p^4n^4+8p^2m^2n^2}{\cancel{4}}\right\} - \frac{N_x}{\cancel{2}p}\frac{3}{\cancel{4}}m^2\cancel{\pi^2}\right]$$

$$N_x = \frac{D\pi^2}{a^2}\frac{1}{3m^2}\left\{3m^4+16p^4n^4+8p^2m^2n^2\right\}$$

$(N_x)_{cr}$ lowest for $n=1$. This was expected since intuitively & from exact soln of SS plate, $n=1$ gives critical load when only N_x applied.

$$\Rightarrow (N_x)_{cr} = \frac{D\pi^2}{a^2}\frac{1}{3}\left[3m^2 + \frac{16p^4}{m^2} + 8p^2\right]$$

$$\frac{d(N_x)_{cr}}{dm} = 0 = 6m - \frac{32p^4}{m^3} \Rightarrow m = \sqrt[4]{\frac{32}{6}}\,p \rightarrow \text{only when } p : m \text{ is integer.}$$

Otherwise plot $(N_x)_{cr}$ v/s p for various m and find envelope.

For $p=1$ $(a=b)$, we have

for $m=1 \rightarrow (N_x)_{cr} = 9\,\frac{D\pi^2}{a^2}$

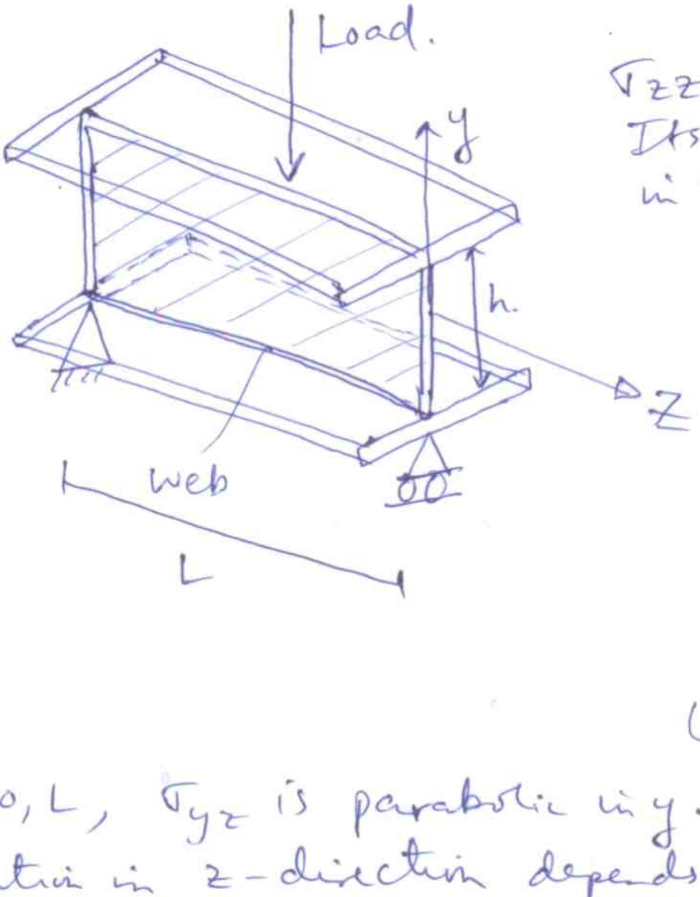
$m=2 \rightarrow (N_x)_{cr} = 8\,\frac{D\pi^2}{a^2} \longrightarrow$ critical

$m=3 \rightarrow (N_x)_{cr} = \left(\frac{35+16/9}{3}\right)\frac{D\pi^2}{a^2} > 8\,\frac{D\pi^2}{a^2}.$

$m>3 \rightarrow (N_x)_{cr} > 8\,\frac{D\pi^2}{a^2}$

Exact solution gives $(N_x)_{cr} = 7\cdot69\,\frac{D\pi^2}{a^2}$

Ex 3. Web Plate in Plate Girder.



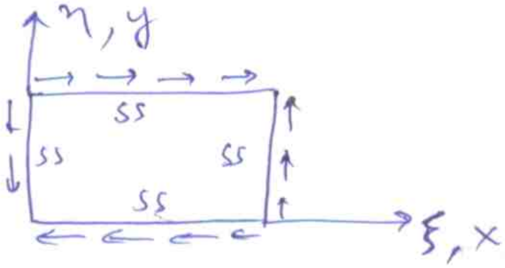
σ_{zz} varies along length. Its distribution is linear in y.

Force balance requires shear σ_{yz} here

Hence σ_{yz} acts on $y=\pm\frac{h}{2}$ edges of web, and also on $z = 0, L$ on web due to shear force.

On $z=0, L$, σ_{yz} is parabolic in y. On $y=\pm\frac{h}{2}$ σ_{yz} distribution in z-direction depends on $V(z)$ (ie shear force distribution), eg if $V(z)=$ const, ie $M(z)$ is linear, $d\sigma_{zz}/dz =$ const $\Rightarrow \sigma_{yz}$ is const in z-direction at $y=\pm h/2$.

For simplicity we assume some average σ_{yz} acts uniformly at $z=0, L$, $y=\pm\frac{h}{2}$. Also assume SS edges.



Governing eqn is,

0 in present problem:

$$\frac{\partial^4 w}{\partial \xi^4} + 2p^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + p^4 \frac{\partial^4 w}{\partial \eta^4} + \cancelto{0}{\frac{N_x}{D}} a^2 \frac{\partial^2 w}{\partial \xi^2} + \cancelto{0}{\frac{N_y}{D}} p^2 a^2 \frac{\partial^2 w}{\partial \eta^2}$$

$$- 2\frac{N_{xy}}{D} a^2 p \frac{\partial^2 w}{\partial \xi \partial \eta}$$

where N_x, N_y are +ve in compression and N_{xy} is positive as shown in Fig above. For exact solution, take

$$w = \sum_{n,m=1}^{\infty}\sum A_{mn} \sin m\pi\xi \sin n\pi\eta$$

get by subst in gov. eqn, ($= N_{xy}/D$)

$$\sum_{m,n=1}^{\infty}\sum A_{mn}\Big[(m^4\pi^4 + 2p^2 m^2 n^2 \pi^4 + p^4 n^4 \pi^4) \sin m\pi\xi \sin n\pi\eta$$
$$- k^2 a^2 p\, mn\pi^2 \cos m\pi\xi \cos n\pi\eta\Big] = 0.$$

So we see that solution not possible due to cosine terms from the mixed-second-partials, ie if you multiply & (with $\sin r\pi\xi \sin q\pi\eta$) integrate $\int_0^1\int_0^1 (\,)\,d\xi\, d\eta$ you get,

$$0 = A_{rq}(r^4\pi^4 + 2p^2 r^2 q^2 \pi^4 + p^4 q^4 \pi^4)\left(\tfrac{1}{4}\right)$$
$$- \sum_{m,n=1}^{\infty}\sum A_{mn} k^2 a^2 p\, mn\pi^2 \frac{1}{2\pi}\left[\frac{1}{r+m}\left((-1)^{r+m}-1\right) + \frac{1}{r-m}\left((-1)^{r-m}-1\right)\right] *$$
$$* \frac{1}{2\pi}\left[\frac{1}{q+n}\left((-1)^{q+n}-1\right) + \frac{1}{q-n}\left((-1)^{q-n}-1\right)\right]$$

$= 0$, $r \pm m$ even

$= \dfrac{2m}{r^2 - m^2}$, $r \pm m$ odd.

where we used the fact that

$$\int_0^1 \sin r\pi\xi \cos m\pi\xi \, d\xi = \frac{1}{2}\int_0^1 \left(\sin(r+m)\pi\xi + \sin(r-m)\pi\xi\right) d\xi$$

$$= \frac{1}{2\pi}\left[\frac{1}{r+m}\left((-1)^{r+m}-1\right) + \frac{1}{r-m}\left((-1)^{r-m}-1\right)\right]$$

$$\begin{cases} = 0, & r \pm m = \text{even} \\ = \dfrac{(-2)}{2\pi}\dfrac{2r}{(r^2-m^2)} = \dfrac{-1}{\pi}\dfrac{2r}{m^2-r^2}, & r \pm m = \text{odd} \end{cases}$$

So the exact analytical solution is not possible. We do approx soln by Ritz method.

$$W_a = \sum_{m=1}^{r}\sum_{n=1}^{s} A_{mn} \sin m\pi\xi \sin n\pi\eta .$$

$$\Pi = \underbrace{\frac{D}{2}\frac{1}{pa^2}\int_0^1\int_0^1\left(\frac{\partial^2 w}{\partial\xi^2} + p^2\frac{\partial^2 w}{\partial\eta^2}\right)^2 d\xi\, d\eta}_{U_b} - \underbrace{\int_0^1\int_0^1 N_{xy}\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\, d\xi\, d\eta}_{(\;\;)_b} .$$

work done by ext loads during bending only.

$$\left(\frac{\partial^2 w}{\partial\xi^2}\right)^2 = \sum_{m=1}^{r}\sum_{i=1}^{r}\sum_{n=1}^{s}\sum_{j=1}^{s}\left(A_{mn}A_{ij}\sin m\pi\xi \sin i\pi\xi * \sin n\pi\eta \sin j\pi\eta\right) m^2 i^2 \pi^4$$

$$\left(\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2}\right) = \sum\sum_{m,i=1}^{r}\sum\sum_{n,j=1}^{s} \pi^4 m^2 j^2 \left(A_{mn}A_{ij}\sin m\pi\xi \sin i\pi\xi \sin n\pi\eta \sin j\pi\eta\right)$$

$$U_b = \frac{D}{2}\frac{\pi^4}{pa^2}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\left(\frac{1}{4}\right)m^4 + \left(\frac{1}{4}\right)p^4 n^4 + \left(\frac{1}{4}\right)m^2 n^2 p^2\right] A_{mn}^2$$

$$= \frac{D}{8}\frac{\pi^4}{pa^2}\sum_{m=1}^{r}\sum_{n=1}^{s}\left(m^2 + p^2 n^2\right)^2 A_{mn}^2$$

$$(W_p)_b = N_{xy}\iint \sum\sum_{m,i=1}^{r}\sum\sum_{n,j=1}^{s} m j \pi^2 \left(\cos m\pi\xi \sin n\pi\eta \sin i\pi\xi \cos j\pi\eta\right) d\xi\, d\eta \; * A_{mn}A_{ij}$$

$$(W_p)_b = -N_{xy}\sum_{m,i=1}^{r}\sum\sum_{n,j=1}^{s}\sum\left(\frac{-2i}{m^2-i^2}\right)\left(\frac{-2n}{j^2-n^2}\right)\frac{1mj\pi^2}{\pi^2}\,\underbrace{*A_{mn}A_{ij}} = -4N_{xy}\sum_{m,i=1}^{r}\sum\sum_{n,j=1}^{s}\sum\frac{mnij \quad *A_{mn}A_{ij}}{(m^2-i^2)(j^2-n^2)}$$

where in $(W_p)_b$, m,n,i,j are such that $m\pm i$ & $n\pm j$ are odd only.

Restricting to 2 terms in each direction, ie $r=s=2$, the series in $(W_p)_b$ contains only the following combinations of $(m,n,i,j) = (1,1,2,2), (1,2,2,1), (2,1,1,2), (2,2,1,1)$

Thus,

$$\frac{\partial\Pi}{\partial A_{11}} = 0 = \frac{D\pi^4}{4pa^2}(1+p^2)^2A_{11} - 4N_x\left(\frac{4}{-9}\right)A_{22}*2$$

$$\frac{\partial\Pi}{\partial A_{12}} = 0 = \frac{D\pi^4}{4pa^2}(1+4p^2)^2A_{12} - 4N_x\left(\frac{4}{9}\right)A_{21}*2$$

$$\frac{\partial\Pi}{\partial A_{21}} = 0 = \frac{D\pi^4}{4pa^2}(4+p^2)^2A_{21} - 4N_x\left(\frac{4}{9}\right)A_{12}*2$$

$$\frac{\partial\Pi}{\partial A_{22}} = 0 = \frac{D\pi^4}{4pa^2}(4+4p^2)^2A_{22} - 4N_x\left(\frac{-4}{9}\right)A_{11}*2$$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{D\pi^4}{4pa^2}(1+p^2)^2 & 0 & 0 & \frac{32}{9}N_x \\ 0 & \frac{D\pi^4}{4pa^2}(1+4p^2)^2 & -\frac{32}{9}N_x & 0 \\ 0 & -\frac{32}{9}N_x & \frac{D\pi^4}{4pa^2}(4+p^2)^2 & 0 \\ \frac{32}{9}N_x & 0 & 0 & \frac{D\pi^4}{4pa^2}(4+4p^2)^2 \end{bmatrix}}_{\underline{C}} \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{Bmatrix} = 0$$

$\mathrm{Det}[\underline{C}] = 0$ gives N_x.

$$\left(\frac{D\pi^4}{4pa^2}\right)^2(1+p^2)^2(4+4p^2)^2\left[\left(\frac{D\pi^4}{4pa^2}\right)^2(1+4p^2)^2(4+p^2)^2-\left(\frac{32}{9}\right)^2N_x^2\right]$$

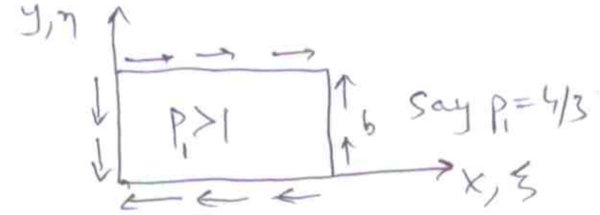
$$-\left(\frac{32}{9}N_x\right)^2\left[\qquad\right] = 0.$$

$\Rightarrow$ Let $\delta = D\pi^4/4pa^2$

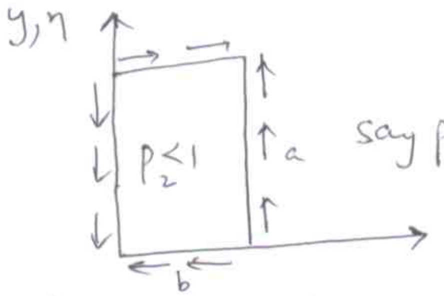
$$N_{xy} = \pm\frac{9}{32}\delta(1+p^2)(4+4p^2) \longrightarrow \text{lower root (obvious)} \longrightarrow \text{(**)}$$

or $$N_{xy} = \pm\frac{9}{32}\delta(1+4p^2)(4+p^2). \longrightarrow (*)$$

This shows that the direction (sign) of N_{xy} does not affect its critical value, ie,



say $p_1 = 4/3$



say $p_2 = \frac{3}{4}$

$$(N_{xy})_1 = \pm\frac{9}{32}\frac{D\pi^4}{4p_1a^2}4(1+p_1^2)^2$$

$$(N_{xy})_2 = \pm\frac{9}{32}\frac{D\pi^4}{4\frac{1}{p_1}b^2}4\left(1+\frac{1}{p_1^2}\right)^2$$

$$= \pm\frac{9}{32}\frac{D\pi^4}{4\frac{1}{p_1}b^2}\frac{4(p_1^2+1)^2}{p_1^4}$$

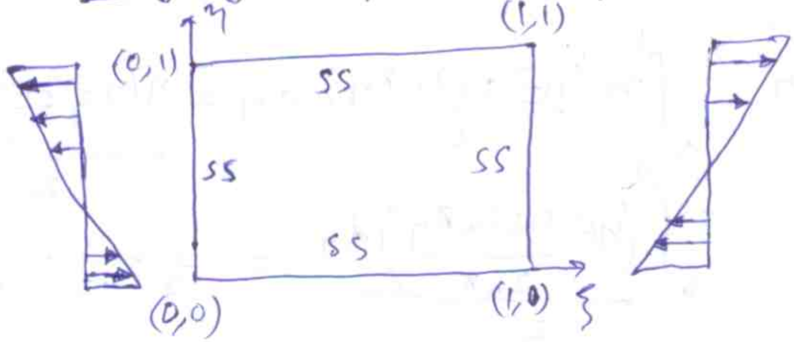
$\because p_1a^2 = p_1^3b^2$, $(N_{xy})_1 = (N_{xy})_2$ as expected.

This also holds true for N_{xy} given by (*) above, but that is not the lowest critical load.

For a square plate, $(N_{xy})_{cr} \xrightarrow{\text{from (**) above}} = 11.1\pi^2D^2/b^2$ which is around 16% higher than the exact solution, i.e.,

$$(N_{xy})_{cr} = 9.34\frac{\pi^2 D}{b^2}$$

Ex 4 Varying in-plane edge loads.



All edges SS.

Let $N_x = N_0(1-\alpha\eta)$ be variation of in-plane edge loads.

$$w_a = \sum_{m=1}^{r}\sum_{n=1}^{s} A_{mn} \sin m\pi\xi \sin n\pi\eta$$

$$U_b = \frac{D\pi^4}{8pa^2}\sum_{m=1}^{r}\sum_{n=1}^{s}(m^2+p^2n^2)^2 A_{mn}^2$$

(as in Ex 3, ie for all edges SS).

$$-(W_p)_b = -\frac{1}{2}\int_0^a\int_0^b N_x w_{,x}^2\,dx\,dy = -\frac{1}{2}\frac{1}{p}\int_0^1\int_0^1 N_0(1-\alpha\eta)\,w_{,\xi}^2\,d\xi\,d\eta$$

$$w_{,\xi}^2 = \sum\sum_{m,i=1}^{r}\sum\sum_{n,j=1}^{s} A_{mn}A_{ij}\,mi\pi^2 \cos m\pi\xi \cos i\pi\xi \sin n\pi\eta \sin j\pi\eta$$

We use the following results for integrals (details on reverse)

$$\int_0^1 \eta \sin n\pi\eta \sin j\pi\eta\,d\eta = \frac{1}{4}, \quad n=j$$

$$= 0, \quad n\pm j = \text{even}$$

$$= \frac{-4nj}{(n-j)^2(n+j)^2}\cdot\frac{1}{\pi^2}, \quad n\pm j = \text{odd}.$$

$$-(W_p)_b = -\frac{1}{2}\frac{N_0}{p}\left[\sum_{m=1}^{r}\sum_{n=1}^{s}\left\{A_{mn}^2\pi^2 m^2\left(\frac{1}{4}\right) - \alpha A_{mn}^2\pi^2 m^2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\right\}\right.$$

$$\left. - \sum_{m=1}^{r}\underbrace{\sum_{n=1}^{s}\sum_{j=1}^{s}}_{\text{Subject to } n\pm j = \text{odd}} \underbrace{\alpha A_{mn}A_{mj}\pi^2 m^2\left(\frac{1}{2}\right)\frac{1}{\pi^2}\left(\frac{-4nj}{(n-j)^2(n+j)^2}\right)}_{=F_{mnj}}\right]$$

$$\parallel$$

$$-\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{\substack{j=\text{only}\\ \text{those values}\\ \text{such that } n\pm j=\text{odd}}} 2F_{mnj} \leftarrow$$

just a simplification of the way to write the limits in the Σ.

Details of integrals used in Ex 4. :

$n=j$: $$\int_0^1 \eta \sin n\pi\eta \sin j\pi\eta \, d\eta = \int_0^1 \eta \left(\frac{1 \ominus \cos 2n\pi\eta}{2}\right) d\eta = \eta \frac{\left(\eta \ominus \frac{\sin 2n\pi\eta}{2n\pi}\right)}{2}\Bigg|_0^1$$

$$- \int_0^1 \frac{\left(\eta \ominus \frac{\sin 2n\pi\eta}{2n\pi}\right)}{2} d\eta = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

$n \neq j$: $$\int_0^1 \underbrace{\eta \sin n\pi\eta \sin j\pi\eta,}_{f(\eta)} d\eta = \int_0^1 \eta \frac{\left(\cos(n-j)\pi\eta - \cos(n+j)\pi\eta\right)}{2} d\eta$$

$$= \frac{1}{2}\eta \frac{1}{\pi}\left(\frac{1}{n-j}\sin(n-j)\pi\eta \ominus \frac{1}{n+j}\sin(n+j)\pi\eta\right)\Bigg|_0^1$$

$$- \frac{1}{2\pi}\int_0^1 \left(\frac{1}{n-j}\sin(n-j)\pi\eta \ominus \frac{1}{n+j}\sin(n+j)\pi\eta\right) d\eta$$

$$= \frac{1}{2\pi^2}\left(\frac{1}{(n-j)^2}\cos(n-j)\pi\eta \ominus \frac{1}{(n+j)^2}\cos(n+j)\pi\eta\right)\Bigg|_0^1$$

So for $n \pm j =$ even, $\int_0^1 f(\eta)\, d\eta = 0$.

$n \pm j =$ odd, $$\int_0^1 f(\eta)\, d\eta = +\frac{1}{\pi^2}\left[\frac{(-1)}{(n-j)^2} \ominus \frac{(-1)}{(n+j)^2}\right] = \frac{1}{\pi^2}\left[\frac{-4nj}{(n-j)^2(n+j)^2}\right]$$

For $\int_0^1 \eta \cos n\pi\eta \cos j\pi\eta \, d\eta$ only the "$-$" sign (circled) becomes $+$, so,

$n=j$: $\int_0^1 \eta \cos n\pi\eta \cos j\pi\eta \, d\eta = \frac{1}{4}$

$n \neq j$, $n \pm j$ even, $\int_0^1 \eta \cos n\pi\eta \cos j\pi\eta \, d\eta = 0$

$n \pm j$ odd, $$\int_0^1 \eta \cos n\pi\eta \cos j\pi\eta \, d\eta = -\frac{2}{\pi^2}\frac{(n^2+j^2)}{(n-j)^2(n+j)^2}$$

$$\Pi = U - (W_p)_b$$

$$\frac{\partial \Pi}{\partial A_{uv}} = \boxed{\begin{aligned} &\left[\frac{D\cancel{\pi}^{2\cancel{4}}}{4\cancel{b}a^2}(u^2+p^2v^2)^2 - \frac{1}{4}\frac{N_0}{\cancel{b}}\cancel{\pi}^2 u^2 + \alpha\frac{1}{8}\frac{N_0}{\cancel{b}}\cancel{\pi}^2 u^2\right] A_{uv} \\ &+ \alpha\, 2\frac{N_0}{\cancel{b}}\cancel{\pi}^2 u^2 \frac{1}{\cancel{\pi}^2} \sum_{\substack{j = \text{restricted} \\ \text{to only those} \\ \text{values so that} \\ v \pm j = \text{odd}}} A_{uj}\left(\frac{-vj}{(v-j)^2(v+j)^2}\right) = 0, \quad \begin{matrix} u = 1, \cdots r \\ v = 1, \cdots S. \end{matrix} \end{aligned}} \searrow ①$$

Written in matrix form

$$\underline{\underline{C}} = \begin{bmatrix} \underline{\underline{C}}_1 & \underline{\underline{O}} & \underline{\underline{O}} & \cdots & \underline{\underline{O}} \\ \underline{\underline{O}} & \underline{\underline{C}}_2 & \underline{\underline{O}} & \cdots & \underline{\underline{O}} \\ \underline{\underline{O}} & \underline{\underline{O}} & \underline{\underline{C}}_3 & \cdots & \underline{\underline{O}} \\ \vdots & \vdots & & \ddots & \underline{\underline{C}}_r \\ \underline{\underline{O}} & \underline{\underline{O}} & \cdots & \cdots & \end{bmatrix} \begin{Bmatrix} \underline{A}_1 \\ \underline{A}_2 \\ \vdots \\ \vdots \\ \underline{A}_r \end{Bmatrix} = 0$$

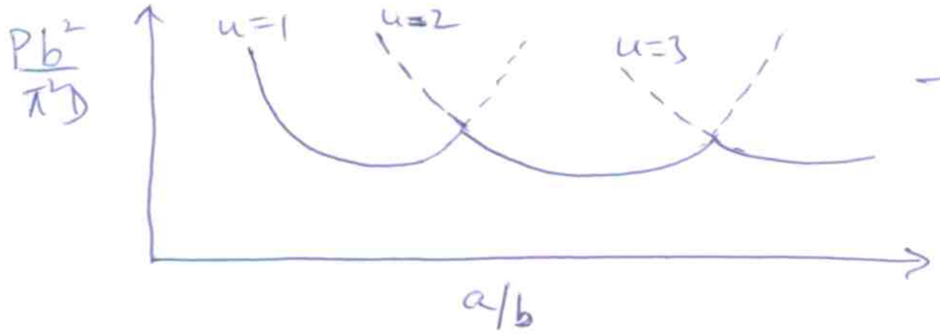
→ ~~For better understanding~~ Compare with Ex 3 where it is not block-diagonal!! The reason being that here u is the common index, ie A_{uv}, A_{uj}

where $\underline{\underline{C}}$ is a block-diagonal matrix, ie a matrix with blocks (or sub-matrices) of size s×s along its diagonals. Thus $\underline{\underline{C}}_i$ is a s×s submatrix corresponding to the equations for $u=i$ and $v=1, \cdots s$. Similarly $\underline{A}_i$ is a block of size s×1 (ie a sub-vector) corresponding to the equations for $u=i$, $v=1, \cdots s$. Thus $\det|\underline{\underline{C}}|=0$ if $\det|\underline{\underline{C}}_i|=0$ for any one (or more) $i=1,\cdots r$ ($\because \det|\underline{\underline{C}}| = \det|\underline{\underline{C}}_1| \det|\underline{\underline{C}}_2| \cdots \det|\underline{\underline{C}}_r|$).

Physically this represents finding the critical load for each sub-system, ie, $\underline{\underline{C}}_i \underline{A}_i = 0$ where each sub-system assumes the buckled shape to be i-half sine waves in x-direction.

So the procedure is to solve each subsystem, ie $\det|\underline{\underline{C}}_i|=0$ by increasing v from 1 to S until convergence in P_{cr} achieved for that subsystem. Then increase u, ie solve the next subsystem until convergence achieved.

This way we generate curves similar to the uniformly loaded simply supported plate, ie,



→ converged values (ie for convergence in y-direction, ie v=1,2,...) plotted for each subsystem (ie u)

The lowest envelope is the P_{cr} for corresponding a/b. As a/b increases we need higher wave number in x-direction (ie u) to get lowest P_{cr}. As α (ie non-uniformity of loading) increases we need higher wave number in y-direction (ie v) to get lowest P_{cr}.

(Next page)→

If we consider ① with $u=1$, $v=1,\dots s$ (ie by increasing s only until we get convergence) we have,

For $s=1$, (ie one-term approx) we have,

$$\left(\frac{D\pi^2}{4\,a^2}(1+p^2)^2 - \frac{1}{4}N_0 + \frac{\alpha}{8}N_0\right)A_{11} = 0.$$

$$(N_0)_{cr} = \frac{D\pi^2}{a^2}\frac{(1+p^2)^2}{1-\alpha/2}$$

For $S=2$ (ie two-term approx) we have,

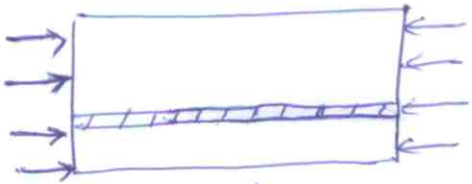
$$\left(\frac{D\pi^2}{4a^2}(1+p^2)^2 - \frac{1}{4}N_0 + \alpha\frac{1}{8}N_0\right)A_{11} - \alpha\frac{2N_0}{\pi^2}\left(\frac{-1*2}{(1-2)^2(1+2)^2}\right)A_{12} = 0$$

$$-\alpha\frac{2N_0}{\pi^2}\left(\frac{-2*1}{(2-1)^2(2+1)^2}\right)A_{11} + \left(\frac{D\pi^2}{4a^2}(1+4p^2)^2 - \frac{N_0}{4} + \alpha\frac{1}{8}N_0\right)A_{12} = 0$$

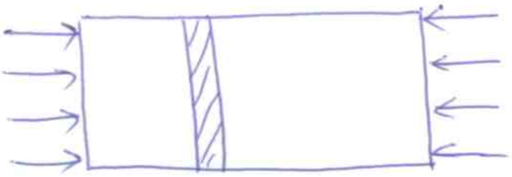
As α increases, ie loading gets more and more non-uniform (ie farther away from constant N_0 value) we need to go to higher value of s (ie more terms in the η-direction series) for convergence. As a/b increases we need to go to higher values of u (ie $u=2$, $u=3$ and solve those sub-systems until convergence in η-direction).

STIFFENED PLATES.

Used in Civil & Aerospace structures (slab over beams intersecting at right angles; stringers – in aircraft fuselage and wings – over which we place the thin sheets on either side to make up the structure which will not buckle). Note that buckling load is proportional to t^2/b^2 So Either increase t or decrease b_x to raise N_{cr}. Both these are impractical – former increases weight and latter alters design requirement/dimension. So we use stiffeners. Longitudinal stiffeners are more effective than transverse stiffeners in raising the buckling load. This is obvious, since the stiffeners deform due to direct action of the load when placed longitudinally.

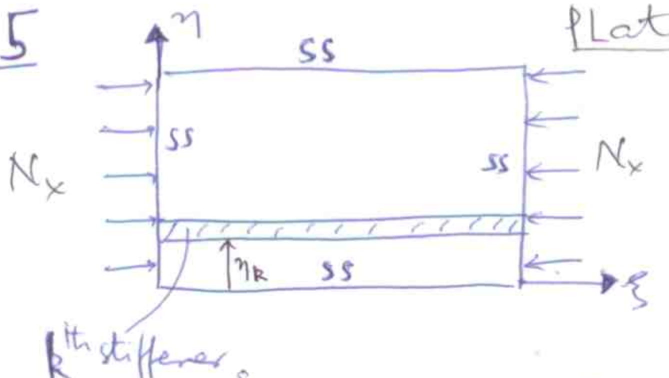


Longitudinal stiffener



Transverse stiffener.

Ex 5 Plate with Longitudinal Stiffeners



All sides SS.
Uniform load.
Use Ritz method.

$$W_a = \sum_{m=1}^{r}\sum_{n=1}^{s} A_{mn}\sin m\pi\xi \sin n\pi\eta .$$

Strain energy in bending for plate is, (Ex 3, 4),

$$\underset{\text{bending plate}}{(U)_{bp}} = \frac{D\pi^4}{8pa^2}\sum_{m=1}^{r}\sum_{n=1}^{s} A_{mn}^2(m^2+p^2n^2)^2$$

Bending strain energy for the stiffeners is,

$$\underset{\text{bending stiffener}}{(U)_{bs}} = \sum_{k=1}^{c}\int_0^1 \frac{1}{2}\frac{(EI_s)_k}{a^3}\, w_{,\xi\xi}^2\Big|_{\eta=\eta_k} d\xi$$

$$= \frac{1}{2a^3}\sum_{k=1}^{c}(EI_s)_k\int_0^1\Big(\sum_{m,i=1}^{r}\sum\ \sum_{n,j=1}^{s}\sum A_{mn}A_{ij}\, m^2 i^2\pi^4 \sin m\pi\xi \sin i\pi\xi \sin n\pi\eta_k \sin j\pi\eta_k\Big)d\xi$$

$$= \frac{\pi^4}{2a^3}\sum_{k=1}^{c}(EI_s)_k\sum_{m=1}^{r}\sum_{n,j=1}^{s}\sum A_{mn}A_{mj}\, m^4\left(\frac{1}{2}\right)\sin n\pi\eta_k \sin j\pi\eta_k$$

$$-(W_p)_{b,\,plate} = -\frac{1}{2p}\int_0^1\!\!\int_0^1 N_x w_{,\xi}^2\, d\eta\, d\xi = -\frac{N_x\pi^2}{2p}\sum_{m=1}^{r}\sum_{n=1}^{s} m^2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)A_{mn}^2$$

$$-(W_p)_{b,\,stiffeners} = -\frac{1}{2a}\sum_{k=1}^{c}\int_0^1 (P_s)_k\, w_{,\xi}^2\Big|_{\eta=\eta_k} d\xi$$

where $(P_s)_k$ is the axial load on the k^{th} stiffener, ie

$$\frac{(P_s)_k}{N_x b} = \frac{(a_s)_k}{bt}$$

(ie load is proportioned in ratio of areas of cross section of stiffener & plate, where $(a_s)_k$ = area of k^{th} stiffener, $t=h=$ plate thickness).

$$-(W_p)_{b,\,stiffener} = -\frac{N_x}{t}\frac{\pi^2}{2a}\sum_{k=1}^{c}(a_s)_k \sum_{m=1}^{r}\sum_{n,j=1}^{s}\sum A_{mn}A_{mj}\, m^2\left(\tfrac{1}{2}\right)\sin n\pi\eta_k \sin j\pi\eta_k$$

$$\Pi = (U_b)_p + (U_b)_s - (W_p)_{b,\,plate} - (W_p)_{b,\,stiffener}$$

$$\frac{\partial \Pi}{\partial A_{uv}} = \frac{D\pi^{\overset{2}{\cancel{4}}}}{4pa^2}(u^2+p^2v^2)^2 A_{uv} + \frac{\pi^{\overset{2}{\cancel{4}}}}{2\cancel{4}a^3}\sum_{k=1}^{c}(EI_s)_k\, u^4 \sin v\pi\eta_k \times \cancel{4}\sum_{j=1}^{s}A_{uj}\sin j\pi\eta_k$$

$$-\frac{N_x\cancel{\pi^2}}{4p}u^2A_{uv} - \frac{N_x\cancel{\pi^2}}{2\cancel{4}at}\sum_{k=1}^{c}(a_s)_k\, u^2 \sin v\pi\eta_k \times \cancel{4}\sum_{j=1}^{s}A_{uj}\sin j\pi\eta_k$$

$$= 0.$$

Defining the ratios,

$$\delta_k = \frac{(EI_s)_k}{Db}, \quad \gamma_k = \frac{(P_s)_k}{N_x b} = \frac{(a_s)_k}{bt},$$

and introducing in the above, get,

$$\boxed{\frac{D\pi^2}{p^2b^2}\left[(u^2+p^2v^2)^2A_{uv} + 2\sum_{k=1}^{c}\delta_k u^4 \sin v\pi\eta_k \sum_{j=1}^{s}A_{uj}\sin j\pi\eta_k\right] - N_x\left[u^2A_{uv} + 2\sum_{k=1}^{c}\gamma_k u^2 \sin v\pi\eta_k \sum_{j=1}^{s}A_{uj}\sin j\pi\eta_k\right] = 0}$$

← Here also we get block diagonal from since u appears in each term ie A_{uv}, A_{uj}.

$u = 1, \dots r$
$v = 1, \dots s$

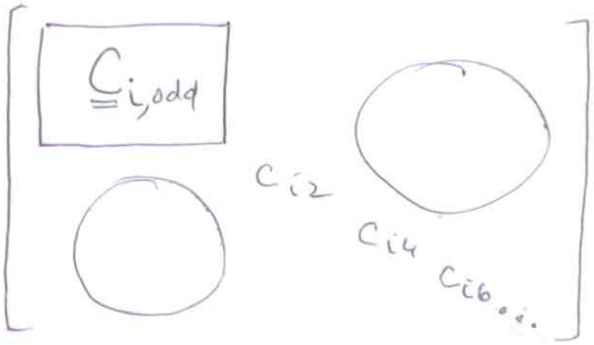
If we place a single stiffener at $\eta_1 = \frac{1}{2}$ (ie $y = b/2$), then the antisymmetric modes in y-direction (ie $v = 2, 4, 6, \dots$) imply that stiffener does not deform and hence it does not contribute to increase in buckling load. Thus for $v = 2, 4, 6, \dots$, the solution with or without stiffener will be same. [redacted]

This is evident from the equations for v = even, where $\sin v\pi\eta_1 = \sin v\pi\frac{1}{2} = 0$ ($\because$ v = even) so the stiffener contribution does not appear in those equations.

\# Moreover here also we have a block diagonal form, like in Ex 4, so we solve each subsystem, det $|\underline{\underline{C}}_i| = 0$, until it converges in the η-direction (ie $v = 1, \cdots s$, with some large enough s until convergence achieved). Then change u (ie u=2, u=3), solve subsystem, and choose lowest P_{cr} from lowest envelope as always.

Now for each subsystem $\underline{\underline{C}}_i$, we have, after re-arranging,

$\underline{\underline{C}}_i =$



→ Block diagonal.
We have moved the equations for v = odd to the top & the ones for v = even to the bottom.

$$\det\left[\underline{\underline{C}}_i\right] = \det\left[\underline{\underline{C}}_{i,odd}\right] * C_{i2} * C_{i4} * C_{i6} * \cdots\cdots = 0$$

So obtain roots of,

$$\det\left[C_{i,odd}\right] = 0 \longrightarrow$$

u=1, v=2 equation coeff of A_{12}

$$C_{i2} = \frac{D\pi^2}{p^2b^2}\left(i^2+4p^2\right)^2 - N_x\,i^2 = 0 \longrightarrow$$

u=1, v=4 equation coeff of A_{14}

$$C_{i4} = \frac{D\pi^2}{p^2b^2}\left(i^2+16p^2\right)^2 - N_x\,i^2 = 0 \longrightarrow$$

u=1, v=6 equation, coeff of A_{16}

$$C_{i6} = \frac{D\pi^2}{p^2b^2}\left(i^2+36p^2\right)^2 - N_x\,i^2 = 0 \longrightarrow$$

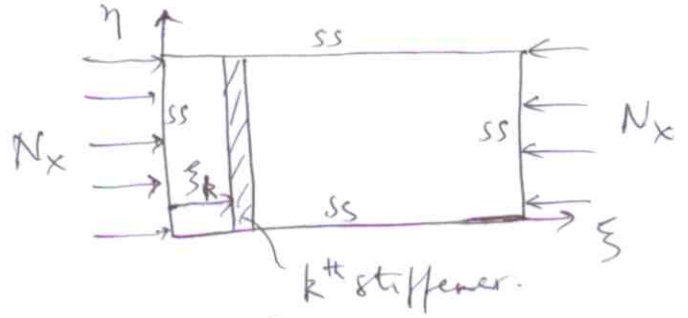
} choose lowest of these roots as converged value for that particular i. Then re-do by increasing i (ie u=i).

For example for u = i = 1, v = 1, … 6,

$$C_{i,odd} = \begin{array}{l} \\ \text{u=1, v=1 equation} \\ \text{u=1, v=3 equation} \\ \text{u=1, v=5 equation} \end{array} \begin{bmatrix} \frac{D\pi^2}{p^2b^2}\left[(1+p^2)^2+2\delta_1\right]-N_x\left[1+2\gamma_1\right], & \frac{D\pi^2}{p^2b^2}\left[-2\delta_1\right]-N_x\left[-2\gamma_1\right], & \frac{D\pi^2}{p^2b^2}\left[2\delta_1\right]-N_x\left[2\gamma_1\right] \\ \frac{D\pi^2}{p^2b^2}\left[-2\delta_1\right]-N_x\left[-2\gamma_1\right], & \frac{D\pi^2}{p^2b^2}\left[(1+9p^2)^2+2\delta_1\right]-N_x\left[1+2\gamma_1\right], & \frac{D\pi^2}{p^2b^2}\left[-2\delta_1\right]-N_x\left[-2\gamma_1\right] \\ \underbrace{\frac{D\pi^2}{p^2b^2}\left[2\delta_1\right]-N_x\left[2\gamma_1\right]}_{A_{11}\text{ coeff column}}, & \underbrace{\frac{D\pi^2}{p^2b^2}\left[-2\delta_1\right]-N_x\left[-2\gamma_1\right]}_{A_{13}\text{ coeff column}}, & \underbrace{\frac{D\pi^2}{p^2b^2}\left[(1+25p^2)^2+2\delta_1\right]-N_x\left[1+2\gamma_1\right]}_{A_{15}\text{ coeff column.}} \end{bmatrix}$$

Ex 6 Plate with Transverse Stiffeners.

The only change is that stiffeners dont contribute to work done by applied load.



$(U_b)_p$ and $(W_p)_{b,\text{plate}}$ remain same as in EX5.

$$(U_b)_s = \sum_{k=1}^{c}\int_0^1 \frac{1}{2}(EI_s)_k \frac{1}{b^3} w_{,\eta\eta}^2\Big|_{\xi=\xi_k} d\eta$$

$$= \frac{\pi^4}{2b^3}\sum_{k=1}^{c}(EI_s)_k \sum_{n=1}^{s}\sum_{m,i=1}^{r}\sum A_{mn}A_{in}\,n^4\left(\tfrac{1}{2}\right)\sin m\pi\xi_k \sin i\pi\xi_k$$

$$\frac{\partial\Pi}{\partial A_{uv}} = \frac{D\pi^4}{4pa^2}(u^2+p^2v^2)^2A_{uv} + \frac{\pi^4}{4\cdot b^3}\sum_{k=1}^{c}(EI_s)_k\, v^4 \sin u\pi\xi_k * 2\sum_{i=1}^{r}A_{iv}\sin i\pi\xi_k$$

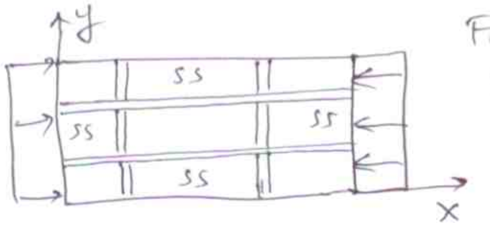
$$-\frac{N_x\pi^2u^2A_{uv}}{4p} = 0 \quad , \quad \begin{matrix}u=1,\cdots r\\ v=1,\cdots s\end{matrix}$$

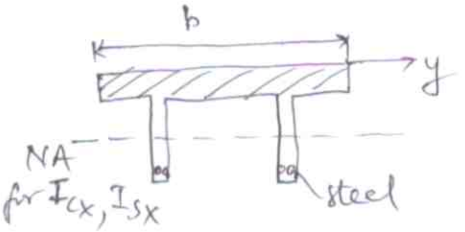
$$\Rightarrow \boxed{\begin{aligned}&\frac{D\pi^2}{b^2}\left[(u^2+p^2v^2)^2A_{uv} + 2\sum_{k=1}^{c}\delta_k p^3 v^4 \sin u\pi\xi_k \sum_{i=1}^{r}A_{iv}\sin i\pi\xi_k\right]\\ &-N_x p^2u^2A_{uv} = 0 \qquad\qquad u=1,\cdots r\\ &\qquad\qquad\qquad\qquad\qquad\quad v=1,\cdots s\end{aligned}}$$

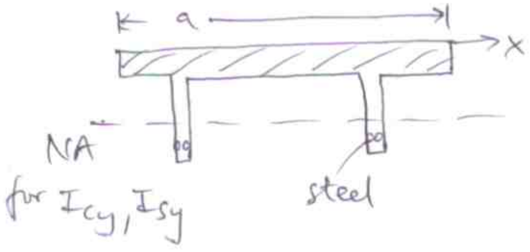
↳ Here also it is block diagonal. Each block is for a fixed value of v=j, ie $\underline{\underline{C}}_j$. For each block solve det $|\underline{\underline{C}}_j|=0$ by increasing terms in x-direction series, ie $u=1,\cdots r$. We get converged critical load, Then take lowest envelope of all such converged critical loads obtained from solution of all blocks det $|\underline{\underline{C}}_j|=0$, $j=1,\cdots s$. (ie Same procedure as before).

Ex 7 Orthotropic Plates. (Summary only).

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} = 0$$







For reinforced concrete slabs,

$$D_x = \frac{E_c}{1-\nu_c^2}\left[I_{cx} + (n-1) I_{sx}\right]$$

$$D_y = \frac{E_c}{1-\nu_c^2}\left[I_{cy} + (n-1) I_{sy}\right]$$

$$H = 2\sqrt{D_x D_y}$$

ν_c, E_c = Poissons ratio, Youngs' modulus of concrete

$n = E_s/E_c$ = modular ratio

I_{cx} = Moment of inertia of slab material taken about N.A of section defined by x = const

I_{sx} = MI of reinforcement about NA of section x = const.

I_{cy}, I_{sy} → likewise for section y = const.

For Fiber reinforced composite plates we also get orthotropic structure.

In general an orthotropic plate has different Youngs modulii and Poisson's ratio in the x, y directions.