TOPIC VI SHALLOW ARCH

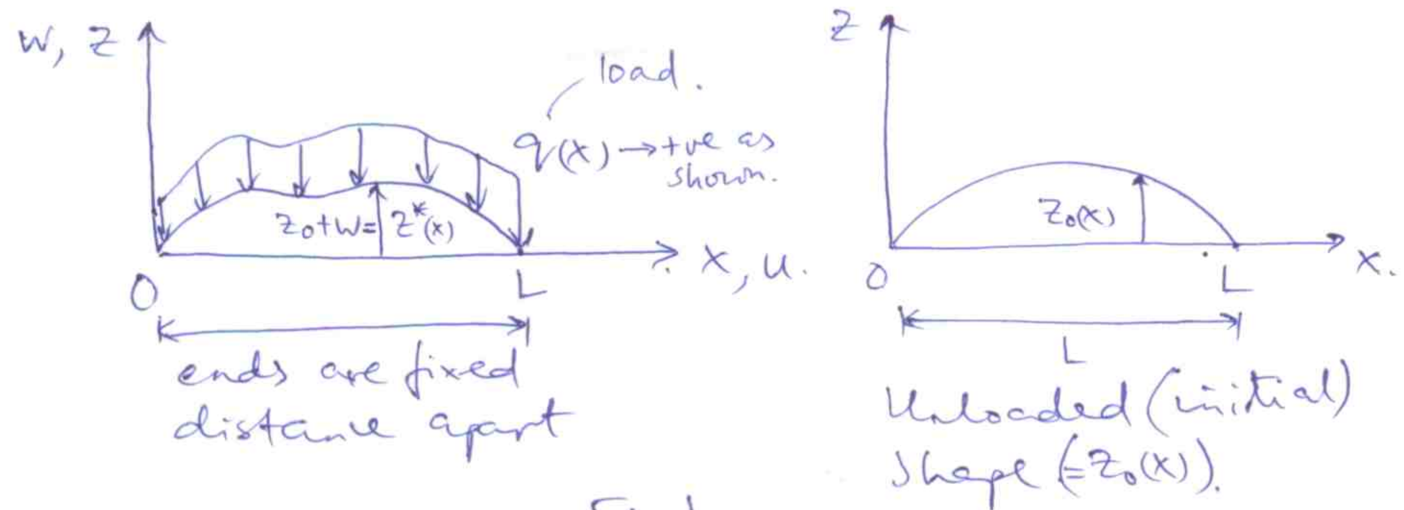


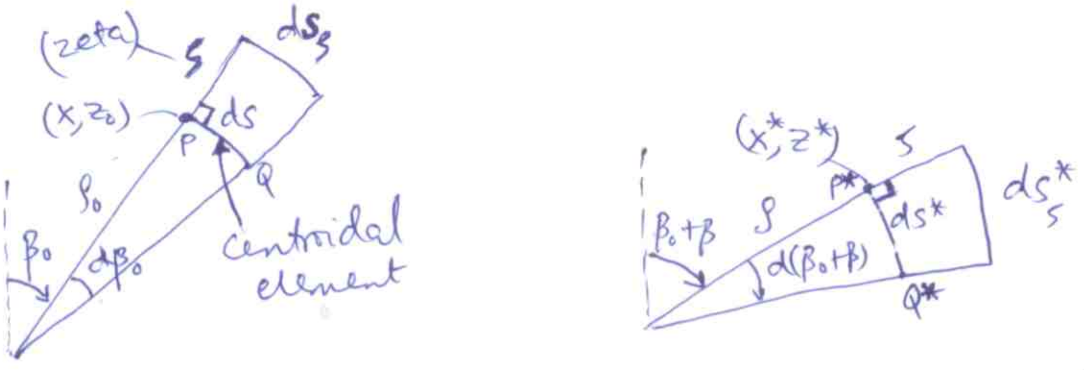
Fig 1

Kinematics

Assumptions same as for beam-column
 (i) Neglect transverse shear strains and transverse normal stresses.
 (ii) Plane sections remain plane.

Let $\bar{\sigma}$ and $\bar{e}$ denote tangential stress & strain.

$$\bar{\sigma} = E\bar{e} \quad \text{(Hooke's law)}$$ → these are the only nonzero stress and strain.



Initial position   Deflected position.

(Note the right angle and same ζ, in line with the assumptions).

ζ = thickness-wise coordinate measured from centroidal locus. $|\zeta| < h$, h = arch thickness.

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(x, z_0) displaces to (x^*, z^*),

$$x^* = x + u(x) \quad , \quad z^* = z_0(x) + w(x)$$

$u(x)$, $w(x)$ are displ components along x, z, respectively. The tangent and normal to centroidal locus rotate thru β.

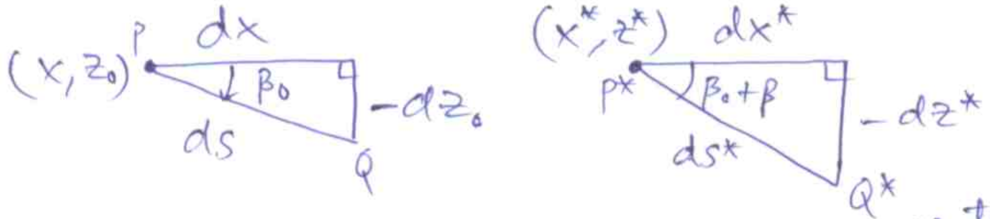

Fig. 2

Initial centroidal element of arc-length

Deflected centroidal element of arc-length.

$$\bar{e} = \frac{ds^*_\varsigma - ds_\varsigma}{ds_\varsigma}$$ = tangential strain of any parallel (non-centroidal) element (of arc-length).

$$= \frac{ds^* + \varsigma\, d(\beta_0+\beta) - ds - \varsigma\, d\beta_0}{ds + \varsigma\, d\beta_0} = \frac{ds^* - ds + \varsigma\, d\beta}{ds + \varsigma\, d\beta_0}$$

$$\bar{e} = \left(e + \varsigma \frac{d\beta}{ds}\right)\left(1 + \varsigma \frac{d\beta_0}{ds}\right)^{-1}$$

where $e = \dfrac{ds^* - ds}{ds}$ = tangential strain of centroidal element (of arc-length).

Curvatures of centroidal line are,

$$\begin{cases} K_0 = \dfrac{1}{\rho_0} = \dfrac{d\beta_0}{ds} & \text{(Before deformation)} \\ K = \dfrac{1}{\rho} = \dfrac{d(\beta+\beta_0)}{ds^*} & \text{(After deformation).} \end{cases}$$

$$\Rightarrow \left(\frac{d\beta}{ds} + \frac{d\beta_0}{ds}\right) \cancel{\frac{ds}{ds^*}}(1+e)^{-1} = K \quad \Rightarrow \frac{d\beta}{ds} = K(1+e) - K_0$$

$$\bar{e} = \left(e + \zeta\left[K(1+e) - K_0\right]\right)\left(1+\zeta K_0\right)^{-1}$$

For thin shallow arch, $|\zeta K_0| = \left|\dfrac{\zeta}{\rho_0}\right| \ll 1.$

$$\bar{e} = e - e\zeta K_0 + \zeta(K-K_0) - \zeta^2(K-K_0)K_0 + \zeta K e - \zeta^2 K K_0 e$$

Let strains be small, ie $\left(e, \dfrac{\zeta}{\rho_0}, \dfrac{\zeta}{\rho}\right) = O(\varepsilon)$

$$\boxed{\bar{e} = e + \zeta(K-K_0)} + O(\varepsilon^2)$$

neglect $\longrightarrow$ (1)

$$\boxed{K - K_0 = \frac{d\beta}{ds}} + O(\varepsilon^2)$$

neglect

Now we want to get e and $(K-K_0)$ in terms of $u(x)$, $w(x)$. From CE623,

$$2M = \frac{ds^{*2} - ds^2}{ds^2} = 2\underline{n}^T \underline{E}\,\underline{n} \quad , \quad \underline{n} = \begin{pmatrix} \cos\beta_0 \\ -\sin\beta_0 \\ 0 \end{pmatrix}$$

$$= \left(E_{11}\cos\beta_0 - E_{12}\sin\beta_0\right)\cos\beta_0 + \left(E_{12}\cos\beta_0 - E_{22}\sin\beta_0\right)\left(-\sin\beta_0\right)$$

$$= E_{11}\cos^2\beta_0 + E_{22}\sin^2\beta_0 - 2E_{12}\cos\beta_0\sin\beta_0$$

$$E_{11} = u_{,x} + \frac{u_{,x}^2}{2} + \frac{w_{,x}^2}{2}$$

$$E_{22} = w_{,z} + \frac{u_{,z}^2}{2} + \frac{w_{,z}^2}{2} = 0$$

$$2E_{12} = \underbrace{u_{,z}}_{0} + w_{,x} + u_{,x}\underbrace{u_{,z}}_{0} + w_{,x}\underbrace{w_{,z}}_{0} = w_{,x}$$

$$\cos\beta_0 = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2 + dz_0^2}} = \frac{1}{\sqrt{1+z_0'^2}}$$

$$\sin\beta_0 = \frac{-dz_0}{ds} = \frac{-z_0'}{\sqrt{1+z_0'^2}}$$

where $()' = d()/dx$

$$2M = \left(\frac{u' + \frac{u'^2}{2} + \frac{w'^2}{2} + w' z_0'}{1 + z_0'^2} \right) * 2$$

Now $e = \sqrt{1 + 2M} - 1 = \dfrac{ds^* - ds}{ds}$

$$= \cancel{1} + \frac{1}{2}(2M) - \frac{1}{8}(2M)^2 - \cancel{1} + O(M^3)$$

$$= \frac{u' + w' z_0' + \frac{u'^2}{2} + \frac{w'^2}{2}}{1 + z_0'^2} - \frac{1}{8} \frac{\left[4u'^2 + 4 w'^2 z_0'^2 + 8 u' w' z_0' \right]}{(1 + z_0'^2)^2} + \cancel{O(u'^3, w'^3)}$$

cubic in displ gradients

$$e = \frac{u' + z_0' w'}{1 + z_0'^2} + \frac{1}{2} \frac{(w' - z_0' u')^2}{(1 + z_0'^2)^2} \longrightarrow 2$$

For the rotations, from geometry,

$$\tan \beta_0 = -z_0' \;, \quad \tan(\beta + \beta_0) = -\frac{dz^*}{dx^*} = -\left(\frac{z_0' + w'}{1 + u'} \right)$$

For a shallow arch the angles β_0, $\beta + \beta_0$ are small (ie tangent = angle). For small strains, ie $(1+u')^{-1} \simeq 1 - u'$. Thus we get,

$$\beta_0 = -z_0', \quad \beta + \beta_0 = -(w' - z_0' u') \quad \Rightarrow \beta = -(w' - z_0' u')$$

where quadratic and higher terms in displ-gradients are neglected (ie $w'u' \simeq O(\varepsilon^3)$ neglected – note $u' = O(\varepsilon^2)$ $w' = O(\varepsilon)$ for "small strains & moderate rotations" theory).

Since $K = d(\beta + \beta_0)/ds^*$ we do,

$$\frac{d \tan(\beta + \beta_0)}{ds^*} = -\frac{d}{dx} \left(\frac{z_0' + w'}{1 + u'} \right) \frac{dx}{ds^*}$$

$$\frac{1}{\cos^2(\beta+\beta_0)}\frac{d(\beta+\beta_0)}{ds^*} = -\left(\frac{(1+u')(z_0''+w'') - u''(z_0'+w')}{(1+u')^2}\right)\frac{dx}{ds^*}$$

In the above use $(1+u')\dfrac{dx}{ds^*} = \dfrac{dx^*}{ds^*} = \cos(\beta+\beta_0)$ (ref Fig 2, p.2)

$$(z_0'+w')\frac{dx}{ds^*} = \frac{dz^*}{ds^*} = \sin(\beta+\beta_0).$$

We get,

$$\frac{d(\beta+\beta_0)}{ds^*} = K = -\left(\cos(\beta+\beta_0)(z_0''+w'') - u''\sin(\beta+\beta_0)\right)\left(\frac{dx}{ds^*}\right)^2$$

(use $ds^* = (1+e)ds$).

Also take $(\beta+\beta_0)$, β_0 as small angles.

$$K = -\left(z_0''+w'' - u''(\beta+\beta_0)\right)\underbrace{\cancel{\left(\frac{dx}{ds}\right)^2}}_{\cos^2\beta_0 \simeq 1}\underbrace{\cancel{\frac{1}{(1+e)^2}}}_{\simeq(1-2e)\simeq 1} \longrightarrow 3$$

$\because e \ll 1$.

For shallow shells $z_0' = -\beta_0 = O(\varepsilon)$

$$\Rightarrow (\beta+\beta_0) = O(\varepsilon)$$

$$\left.\begin{aligned} u''(\beta+\beta_0) &= O(\varepsilon^3)\\ (z_0'u')^2 &= O(\varepsilon^6)\\ z_0'u' &= O(\varepsilon^3)\\ (z_0')^2 &\ll 1 \end{aligned}\right\}$$

use these in 2, 3 keep upto $O(\varepsilon^2)$ in e, upto $O(\varepsilon)$ in K, and upto $O(\varepsilon)$ in β.

We get from 2, 3,

$$e = u' + z_0'w' + \frac{w'^2}{2} = O(\varepsilon^2) \text{ (each term of same order).}$$

$$K = -(z_0''+w'') \Rightarrow K - K_0 = -w''$$

(each term in K, $K-K_0$ is of $O(\varepsilon)$).

$$\beta = -w'$$

Thus we have a fully consistent set of kinematic relations (ie consistent with 'small strains, moderate rotation' theory) summarized below:

$$\boxed{\begin{aligned} e &= u' + z_0' w' + \frac{w'^2}{2} = O(\varepsilon^2) \\ K - K_0 &= -w'' = O(\varepsilon) \\ \beta &= -w' = O(\varepsilon) \\ \bar{e} &= e + \zeta(K - K_0) = O(\varepsilon^2) \end{aligned}}$$

→ For shallow thin arches.

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→ put $z_0 = K_0 = 0$ and you recover kinematics for straight column.

Strain Energy

$$U_T = U_i + U_p$$

$$U = \int_0^S \iint_A \frac{1}{2} E \bar{e}^2 \, dA \, ds = \int_0^S \iint_A \frac{1}{2} E \left(e + \zeta(K - K_0)\right)^2 dA \, ds$$

$$= \int_0^S \frac{1}{2} E \left(e^2 A + I (K - K_0)^2\right) \underbrace{ds}_{dx\sqrt{1+z_0'^2} \approx dx} \qquad \left(I = \iint_A \zeta^2 dA\right)$$

$$= \frac{1}{2} \int_0^L \left(EAe^2 + EI(K - K_0)^2\right) dx$$

$$\boxed{U = \frac{1}{2} \int_0^L \left(Ne + M(K - K_0)\right) dx}$$

where $\boxed{N = EAe, \quad M = EI(K - K_0)}$ → 5

$$\boxed{U_p = -\int_0^L -q w \, dx = \int_0^L q w \, dx} \qquad \boxed{(q \downarrow +ve)}$$

In terms of displacements,

$$U_T = \int_0^L \left(\frac{1}{2} EA \left[u' + z_0' w' + \frac{w'^2}{2} \right]^2 + \frac{1}{2} EI (-w'')^2 + q w \right) dx = \int_0^L F(u, u', w, w', w'') dx$$

$\delta U_T = 0$ for Equilibrium.

where $u = u_e + u_1$, $w = w_e + w_1$ (recall Topic II).

We directly use the Euler Lagrange Equations developed in Topic-II p. 15–16.

$$\frac{\partial F}{\partial u_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_e'}\right) = 0 \Rightarrow N_e' = 0 = \left(EA\left[u_e' + z_0' w_e' + \frac{w_e'^2}{2}\right]\right)'$$

$$\frac{\partial F}{\partial w_e} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_e'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_e''}\right) = 0 \Rightarrow q - \left[N_e (z_0' + w_e')\right]' - M_e'' = 0 = q - EA\left[u_e' + z_0' w_e' + \frac{w_e'^2}{2}\right][z_0' + w_e'] + \left[EI w_e''\right]''$$

Equilibrium Equations in Stress/Moment resultant form and displacement form. 6

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Put $z_0 = 0$ and you get equilibrium equations for straight column.

Boundary conditions:

Since we have no loading at boundary points, all the small 'f' terms in eqns on p. 15–16 T-II vanish. Thus,

$$\frac{\partial F}{\partial u_e'} = N_e = 0 \quad \text{or} \quad u = \bar{u}_0,\ u = \bar{u}_L \quad \underline{\text{at } x = 0, L.}$$

$$\frac{\partial F}{\partial w_e'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_e''}\right) = N_e (z_0' + w_e') - M_e' = 0 \quad \text{or} \quad w = \bar{w}_0,\ w = \bar{w}_L$$

$$-\frac{\partial F}{\partial w_e''} = M_e = 0 \quad \text{or} \quad w' = \bar{w}_0',\ w' = \bar{w}_L'.$$

$\delta(\delta^2 U_T) = 0$ for Stability

Take $\delta^2 U_T$ from p. 19, T-II. Keeping only relevant terms,

$$\delta^2 U_T = \int_0^L \left(\frac{\partial^2 F}{\partial u_e'^2} u_1'^2 + \frac{\partial^2 F}{\partial w_e'^2} w_1'^2 + \frac{\partial^2 F}{\partial w_e''^2} w_1''^2 + 2 \frac{\partial^2 F}{\partial u_e' \partial w_e'} u_1' w_1' \right) dx$$

$$= \int_0^L \Big[EA u_1'^2 + \frac{1}{2} EA \left(2 z_0'^2 + 3 w_e'^2 + 6 z_0' w_e' + 2 u_e' \right) w_1'^2 + EI w_1''^2 + 2 \frac{EA}{2} \left(2 z_0' + 2 w_e' \right) u_1' w_1' \Big] dx$$

For $\delta(\delta^2 U_T)$, instead of taking variation of the u_1, w_1 terms and integrating by parts, directly use Euler Lagrange Equations (T-II, p15, 16) with $F(u_1, u_1', w_1, w_1', w_1'')$ defined as

$$\delta^2 U_T = \int_0^L F(u_1, u_1', w_1, w_1', w_1'')\, dx$$ as above, and $u_e \to u_1$, $w_e \to w_1$ in the Euler Lagr Equns.

Thus, keeping only relevant terms,

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial u_1'} \right) = -\left[2EA \left(u_1' + (z_0' + w_e') w_1' \right) \right]' = -2N_1' = 0$$

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial w_1'} \right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_1''} \right) = -\Big[EA \left(2z_0'^2 + 3 w_e'^2 + 6 z_0' w_e' + 2 u_e' \right) w_1' + 2EA (z_0' + w_e') u_1' \Big]' + \left[2EI w_1'' \right]''$$

$$= -\Big[2EA \left(u_e' + z_0' w_e' + \frac{w_e'^2}{2} \right) w_1' + 2EA (z_0' + w_e') \left(u_1' + [z_0' + w_e'] w_1' \right) \Big]' + \left[2EI w_1'' \right]''$$

$$= -\left[\cancel{2} N_e w_1' + \cancel{2} N_1 (z_0' + w_e') \right]' + \left[\cancel{2} EI w_1'' \right]'' = 0$$

So stability Equations are, → 7

$$\boxed{\begin{aligned} & N_1' = 0 \\ & \left[N_e w_1' + N_1 (z_0' + w_e') \right]' + M_1'' = 0 \end{aligned}}$$

where,

$$\boxed{\begin{aligned} & N_1 = EA \left(u_1' + [z_0' + w_e'] w_1' \right) \\ & M_1 = -EI w_1'' \end{aligned}}$$

with boundary conditions,

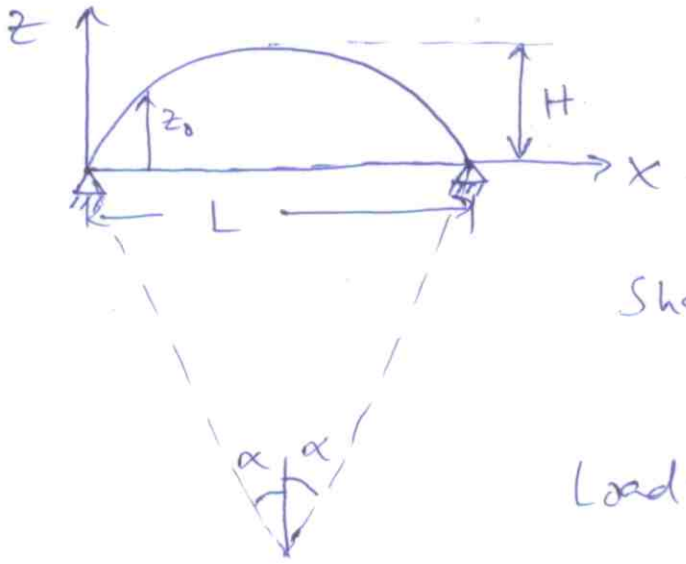
$$\frac{\partial F}{\partial u_1'} = N_1 = 0 \quad \text{or } u_1 = 0 \text{ at } x = 0, L$$

$$\frac{\partial F}{\partial w_1'} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1''}\right) = N_0 w_1' + N_1 (z_0' + w_e') + M_1' = 0 \quad \text{or } w_1 = 0 \text{ at } x = 0, L$$

$$-\frac{\partial F}{\partial w_1''} = M_1 = 0 \quad \text{or } w_1' = 0 \text{ at } x = 0, L$$

$\longrightarrow$ (7a)

SHALLOW SINUSOIDAL ARCH WITH SINUSOIDAL LOAD



$EI = \text{const}$

$z_0 = H \sin \frac{\pi x}{L}$

$\alpha = z_0'(0) = \frac{\pi H}{L}$

Shallow $\Rightarrow \frac{H}{L} \ll 1$, ie $\alpha \ll 1$ (small angle)

Load $= q(x) = q_0 \sin \frac{\pi x}{L}$

Equilibrium solution:

Drop 'e' subscript for convenience keeping in mind that we are dealing with equilibrium quantities.

$N' = 0 \Rightarrow N = \text{const} = C \longrightarrow$ (1)

$EI w^{IV} - N(z_0'' + w'') = -q \quad (N = \text{const used}) \longrightarrow$ (2)

Since $N = \text{const}$,

$$\int_0^L \underbrace{N}_{C} dx = EA \int_0^L \left(u' + z_0' w' + \frac{w'^2}{2}\right) dx = EA\,\underbrace{\cancel{(u(L) - u(0))}}_{0} + EA \int_0^L \left(z_0' w' + \frac{w'^2}{2}\right) dx$$

$$\Rightarrow \quad N = \frac{EA}{L}\int_0^L \left(z_0' w' + \frac{w'^2}{2} \right) dx \longrightarrow (3)$$

So although N is const, its value is not known a-priori since it depends on w (see integrand above). Same holds for column ($z_0 = 0$), but there P is applied and $N = P$.

BC's: $w(0) = w(L) = w''(0) = w''(L) = 0$.

Non-dimensionalization.

$$\boxed{x = L\bar{x}, \quad w = c_1 \bar{w}, \quad z_0 = c_2 \bar{z}_0, \quad q = c_3 \bar{q}} \longrightarrow (4)$$

(c_1, c_2 units of L, c_3 units of NL^{-1})

$$\frac{EIc_1}{L^4}\bar{w}^{IV} - N\left(\frac{c_2}{L^2}\bar{z}_0'' + \frac{c_1}{L^2}\bar{w}'' \right) = -c_3 \bar{q} \qquad \left(()' = \frac{d}{d\bar{x}} \right).$$

Define

$$\boxed{\begin{aligned} &\gamma^2 = -\frac{NL^4}{EIL^2} = -\frac{NL^2}{EI}, \quad c_1 = -2\sqrt{\frac{I}{A}} = -c_2 \\ &c_3 = -\frac{EI c_1}{L^4}\pi^4 = \frac{EI}{L^4}\sqrt{\frac{I}{A}}\, 2\pi^4 \end{aligned}} \longrightarrow (5)$$

$$\Rightarrow \bar{w}^{IV} + \gamma^2\left(-\bar{z}_0'' + \bar{w}'' \right) = \pi^4 \bar{q}$$

← (derivatives wrt $\bar{x}$)

Also from (3), $N = \frac{EA}{L}\int_0^1 \left(-4\frac{I}{A}\frac{1}{L^2} \right)\left(\bar{z}_0'\bar{w}' - \frac{1}{2}\bar{w}'^2 \right) L\, d\bar{x}$

ie, $$\boxed{\gamma^2 = 2\int_0^1 \left(2\bar{z}_0'\bar{w}' - \bar{w}'^2 \right) d\bar{x}} \longrightarrow (6)$$ (γ = Thrust 'N' parameter)

BC: $$\boxed{\bar{w}(0) = \bar{w}(1) = \bar{w}''(0) = \bar{w}''(1) = 0.} \longrightarrow (7)$$

λ = arch rise parameter (8)

$$z_0 = 2\sqrt{\frac{I}{A}}\,\bar{z}_0 = H \sin \pi\bar{x} \Rightarrow \boxed{\bar{z}_0 = \lambda \sin \pi\bar{x}, \quad \lambda = \frac{H}{2}\sqrt{\frac{A}{I}}}$$

$$\boxed{\bar{q} = \frac{q}{c_3} = \frac{L^4 q_0 \sin \pi\bar{x}}{2\pi^4 EI\sqrt{I/A}} = p \sin \pi\bar{x}, \quad p = \frac{q_0 L^4}{2\pi^4 EI\sqrt{I/A}}} \longrightarrow (9)$$

Thus,

$$\boxed{\bar{w}^{IV} + \gamma^2\left(\lambda\pi^2 \sin\pi\bar{x} + \bar{w}''\right) = \pi^4 p \sin\pi\bar{x}} \rightarrow (10)$$

$$\bar{w} = \bar{w}_H + \bar{w}_P = \underbrace{\left(A_1 \sin\gamma\bar{x} + A_2\cos\gamma\bar{x} + A_3\bar{x} + A_4\right)}_{\bar{w}_H} + \underbrace{\left(A_5 \sin\pi\bar{x}\right)}_{\bar{w}_P}$$

for $\gamma^2 > 0$ (ie $N < 0$, compressive). Here A_5 determined by putting $\bar{w}_P$ in (10), equating coeffs of $\sin\pi\bar{x}$ — you get A_5 in terms of γ^2, λ. Then use BC's to get $A_1 - A_4$. Then put this solution in (6) to get an integro-algebraic eqn for γ (not $\bar{w}$ will be in terms of γ, as seen above, when inserting in the integral in (6)). So this method is not possible analytically (ie in closed-form). In the case of eccentrically loaded columns we had a homogenous 4th order ODE so solution was possible by above approach. In case of column with imperfection we had a non-homogenous 2nd order ODE with $N = P$ applied, ie known. The main difficulty in arches is that $N =$ constant but not directly related to the applied load.

So we use Fourier series soln of (10). Let, solution be

$$\bar{w} = \sum_{n=1}^{\infty} b_n \sin n\pi\bar{x} \longrightarrow (11)$$

This satisfies primed BC's (which cosine series violates, hence excluded). Subst (11) in (10) to get,

$$\sum_{n=1}^{\infty}\left[b_n(n\pi)^4 \sin n\pi\bar{x} - b_n\gamma^2(n\pi)^2\sin n\pi\bar{x}\right] + \left(\gamma^2\lambda\pi^2 - \pi^4 p\right)\sin\pi\bar{x} = 0.$$

Equating coeffs of $\sin n\pi\bar{x}$

$$\boxed{\begin{aligned} & b_n\left(n^4 - \left(\frac{\gamma}{\pi}\right)^2 n^2\right) = 0, \quad n = 2, 3, \cdots \infty \\ & b_1\left(1 - \left(\frac{\gamma}{\pi}\right)^2\right) + \left(\frac{\gamma}{\pi}\right)^2\lambda - p = 0, \quad n = 1. \end{aligned}} \longrightarrow (12).$$

Subst (11) in (6) (along with (8)), get,

$$\gamma^2 = 2\int_0^1\left[2\lambda\pi\cos\pi\bar{x}\sum_{n=1}^{\infty} b_n(n\pi)\cos n\pi\bar{x} - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} b_n b_j (nj\pi^2)\cos n\pi\bar{x} * \cos j\pi\bar{x}\right]d\bar{x}$$

$$\gamma^2 = 2\left[2\lambda\pi^2 b_1\cdot\frac{1}{2} - \sum_{n=1}^{\infty} b_n^2 n^2\pi^2\cdot\frac{1}{2}\right]$$

$$\boxed{\left(\frac{\gamma}{\pi}\right)^2 = 2\lambda b_1 - b_1^2 - \sum_{n=2}^{\infty} b_n^2 n^2} \longrightarrow (13)$$

Look at (12). Its obvious that solutions can be only of two types, ie $b_1 \neq 0$ other b_n's $= 0$; and $b_1 \neq 0$, $b_m = 0$, $b_n = 0$, $n = 2, \cdots, n \neq m$.

Case I Solution.

$b_1 \neq 0$, $b_n = 0$, $n = 2, 3, \cdots \infty$

$$\boxed{\begin{aligned} & b_1 - \left(\frac{\gamma}{\pi}\right)^2(b_1 - \lambda) = p \longrightarrow 14(a) \\ & \left(\frac{\gamma}{\pi}\right)^2 = 2\lambda b_1 - b_1^2 \longrightarrow 14(b). \end{aligned}}$$

$\longrightarrow$ Can solve for b_1, then for γ^2. (but see alternative below to generate $p-\gamma$ and $p-b_1$ plots).

$\bar{w}\left(\frac{1}{2}\right) = b_1$

If p (load), λ (rise) such that soln is $b_1 = \lambda$, then

$$\bar{w}\left(\tfrac{1}{2}\right) = b_1 = \lambda = -\frac{w(L/2)}{2}\sqrt{\frac{A}{I}} = \frac{H}{2}\sqrt{\frac{A}{I}} \Rightarrow w\left(\frac{L}{2}\right) = -H.$$

i.e. mid-pt of arch displaces to ℄ of line joining supports.

Let $\frac{\gamma}{\pi} = f$ = another form of axial load (thrust) parameter

From 14(a), 14(b),

$b_1^2 - 2\lambda b_1 + f^2 = 0 \Rightarrow$ $\boxed{b_1 = \lambda \pm \sqrt{\lambda^2 - f^2}}$ → so real solns for $\lambda \geq f$.

$p - \lambda = (b_1 - \lambda)(1 - f^2) \Rightarrow$ $\boxed{p = \lambda \pm (1-f^2)\sqrt{\lambda^2 - f^2}}$ (15)

Plot p v/s f (load–thrust) and p v/s b_1 (load–midpoint deflection) for various λ (rise parameter) values.

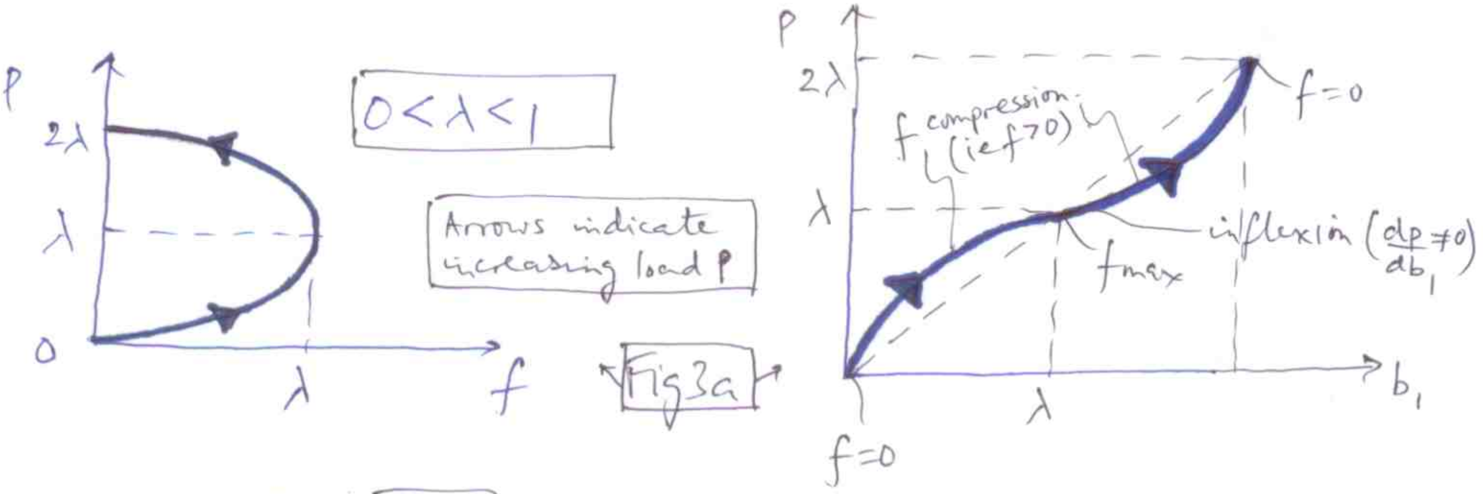


Fig3a

⟹ For $\boxed{\lambda = 1}$ plots look same as for $\lambda < 1$ except that inflexion occurs at zero slope $\left(\frac{dp}{db_1} = 0\right)$

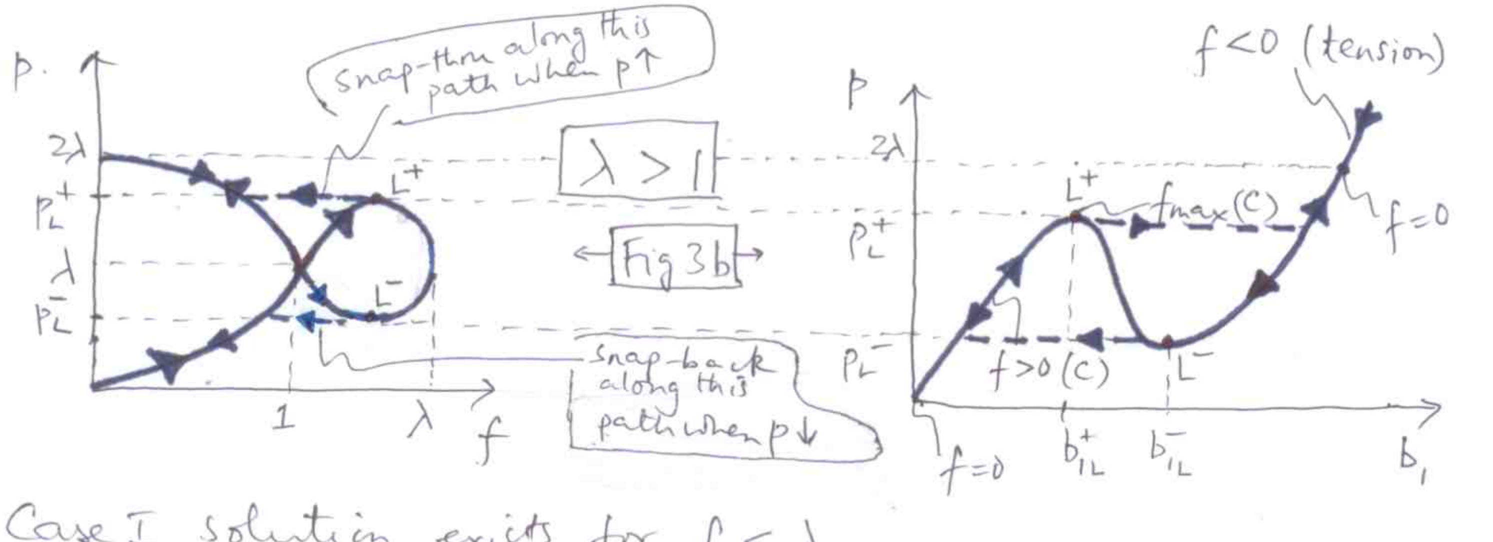


Fig 3b

Case I solution exists for $f \leq \lambda$.

To plot: (i) Take fixed value of λ (arch rise).

(ii) Take various f values: $f \leq \lambda$. Get b_1, p from (15) & plot.

$$\frac{dp}{db_1} = \frac{dp}{df}\,\frac{df}{db_1} = \frac{dp/df}{db_1/df}$$

$$\frac{dp}{df} = \pm\left\{-2f\sqrt{\lambda^2-f^2} - \frac{1}{2}\,2f\,\frac{(1-f^2)}{\sqrt{\lambda^2-f^2}}\right\} = \mp\frac{f}{\sqrt{\lambda^2-f^2}}\left\{2\lambda^2-3f^2+1\right\}$$

$$\frac{db_1}{df} = \mp\frac{1}{2}\,\frac{2f}{\sqrt{\lambda^2-f^2}}$$

$$\Rightarrow \frac{dp}{db_1} = 2\lambda^2-3f^2+1$$

$$= 0 \text{ for } f^2 = \frac{1+2\lambda^2}{3}$$

$\left(\frac{dp}{db_1}=0,\ \frac{dp}{df}=0\right.$ give same $\left.f^2\right)$.

For $dp/db_1 = 0$, ie the limit points, we have from (15),

$$b_1 = \lambda \pm \sqrt{\lambda^2 - \frac{1+2\lambda^2}{3}} = \lambda \pm \sqrt{\frac{\lambda^2-1}{3}}$$

$$p = \lambda \pm \left(1 - \frac{1+2\lambda^2}{3}\right)\sqrt{\lambda^2 - \frac{1+2\lambda^2}{3}} = \lambda \mp 2\left(\frac{\lambda^2-1}{3}\right)^{3/2}$$

$$\Rightarrow \boxed{\begin{array}{ll} p_L^+ = \lambda + 2\left(\frac{\lambda^2-1}{3}\right)^{3/2}, & p_L^- = \lambda - 2\left(\frac{\lambda^2-1}{3}\right)^{3/2} \\ b_{1L}^+ = \lambda - \sqrt{\frac{\lambda^2-1}{3}}, & b_{1L}^- = \lambda + \sqrt{\frac{\lambda^2-1}{3}} \end{array}} \rightarrow (16)$$

Case I Solution Summary:

(i) $\lambda \le 1$, no snap thru occurs. The load-deflection curve is monotonically increasing. At $p=\lambda$, $b_1=\lambda$, ie midpoint of arch lies on ℄ joining support, ie arch begins inverting without snap-thru (ie inversion takes place in a smooth continuous manner). After inversion, for $p > 2\lambda$, $f < 0$, ie arch under tension.

(ii) $\lambda > 1$, snap thru occurs. When $p\uparrow$, snap-thru occurs at p_L^+ (upper limit load) when arch snaps to higher value of b_1. When $p\downarrow$ it snaps back along reverse path as shown. Beyond $p = 2\lambda$ the inverted arch is in tension (ie $f<0$).

Case II solution.

b_1, b_m non-zero, other $b_i = 0$ $(i \neq 1, i\neq m)$.

From 12, 13,

$$\boxed{\begin{aligned} &\left(\frac{\gamma}{\pi}\right)^2 = f^2 = m^2 \\ &b_1 - m^2(b_1-\lambda) = p \\ &m^2 = 2\lambda b_1 - b_1^2 - m^2 b_m^2 \end{aligned}} \longrightarrow 17\,(a,b,c)$$

where $\bar{w} = b_1 \sin \pi \bar{x} + b_m \sin m\pi \bar{x}$

From 17(c),

$$m^2 b_m^2 = -b_1^2 + 2\lambda b_1 - m^2 = (b_1 - r_1)(r_2 - b_1)$$

where $r_1, r_2 = \lambda \mp \sqrt{\lambda^2 - m^2}$, $r_2 > r_1$

Hence Case II soln exists iff $\lambda \geq m$ and $r_1 \leq b_1 \leq r_2$

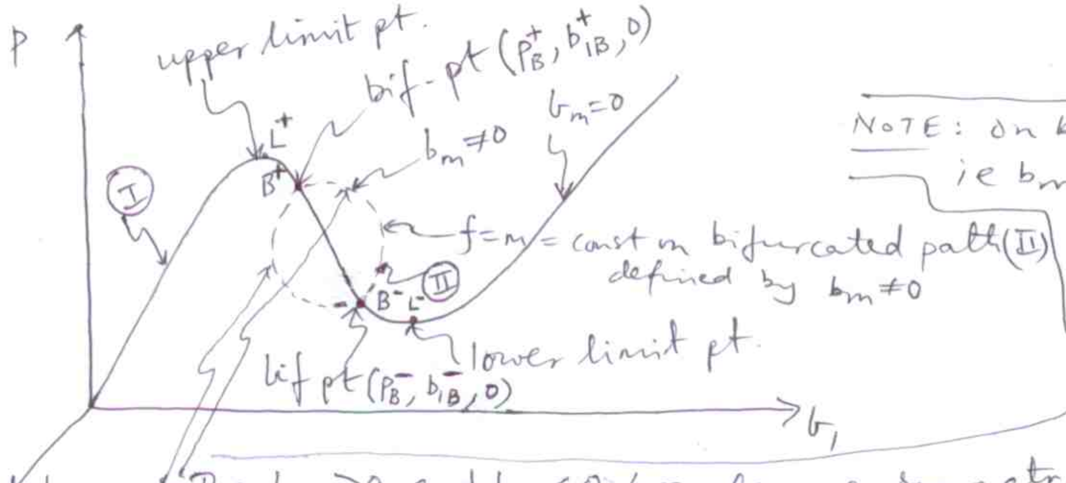
i.e,

$$\boxed{\begin{aligned} &\lambda - \sqrt{\lambda^2 - m^2} \leq b_1 \leq \lambda + \sqrt{\lambda^2-m^2} \\ \text{From } 17(b) \longrightarrow\ &\lambda - (m^2-1)\sqrt{\lambda^2-m^2} \leq p \leq \lambda + (m^2-1)\sqrt{\lambda^2-m^2} \end{aligned}} \longrightarrow 18$$

i.e, $b_{1B}^+ \leq b_1 \leq b_{1B}^-$, $p_B^- \leq p \leq p_B^+$

where p_B^+ corresponds to b_{1B}^+ and p_B^- corresponds to b_{1B}^-

In load-deflection plot, ie (p, b_1, b_m) 3-D plot, a bifurcation point exists if Case I & Case II solutions meet at that point. Thus $b_m = 0$ at bif. pt. From 15, 17a and 18 it is evident that $(p_B^+, b_{1B}^+, 0)$ and $(p_B^-, b_{1B}^-, 0)$ are bifurcation points, ie these points satisfy Case II solution eqs (Eq. 17 with $b_m=0$) as well as Case I sol eqns (Eq 14 (or Eq 15)).



NOTE: on bifurcated path, ie $b_m \neq 0$, $f = m$ = const, unlike on the primary equil (I) path, ie $b_m = 0$, $b_1 \neq 0$

The $b_m > 0$ and $b_m < 0$ branches are symmetric wrt $b_m = 0$ plane see (17C)

For bif pt. and limit pt. to coincide, from (16), (18),

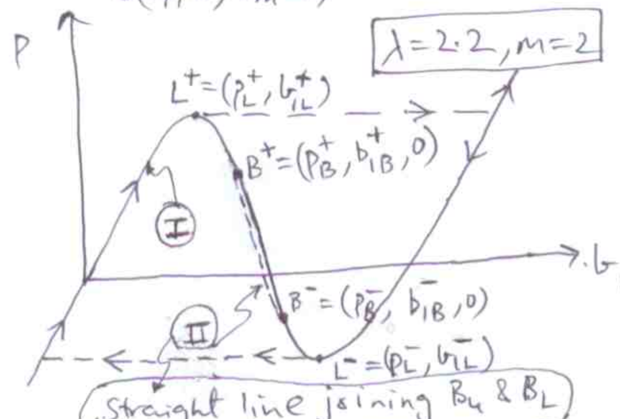
$p_L^+ = p_B^+$, $p_L^- = p_B^-$, $b_{1L}^+ = b_{1B}^+$, $b_{1L}^- = b_{1B}^-$ → any of these give same result,

or alternatively,

$$f^2 = \frac{1 + 2\lambda^2}{3} = m^2 \Rightarrow \lambda = \sqrt{\frac{3m^2 - 1}{2}}$$

So for $m = 2$, when $\lambda = \sqrt{5.5} = 2.345 = \lambda^*$ the bif pt and limit points coincide. For $\lambda < \lambda^*$ bif. pt comes after lim. pt and vice-versa for $\lambda > \lambda^*$. <u>Look at projected plots in p–b_1 plane</u>

I → Primary equilibrium $(b_1 \neq 0, b_m = 0)$ – solid, II → secondary equilibrium $(b_1 \neq 0, b_m \neq 0)$ – dotted (non-horizontal line).



NOTE: From 17(b) you see that for m = const, λ = const, b_1 v/s p is linear.

$m = 2$, $\underline{\lambda = 2.2}$ (ie $< \lambda^*$)

$p_B^{+,-} = 2.2 \pm (2^2 - 1)\sqrt{2.2^2 - 2^2}$

$p_B^+ = 4.95$, $p_B^- = -0.55$ (ie arch being pulled up).

$b_{1B}^{+,-} = 2.2 \mp \sqrt{2.2^2 - 2^2}$

$b_{1B}^- = 3.12$, $b_{1B}^+ = 1.28$

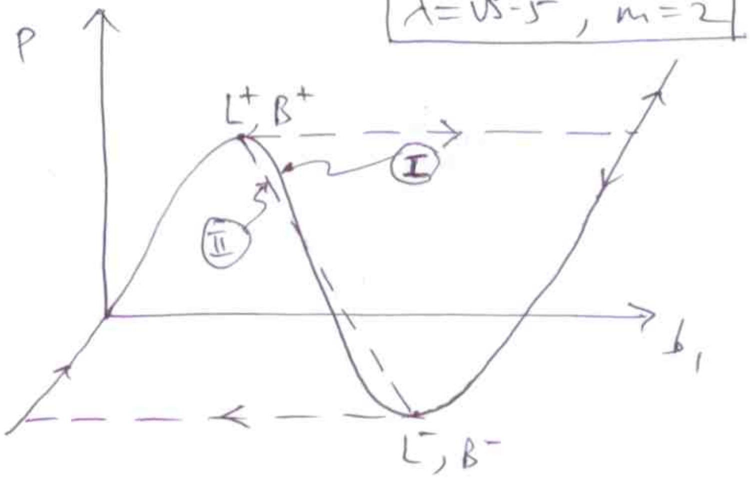
$p_L^{+,-} = 2.2 \pm 2\left(\frac{2.2^2 - 1}{3}\right)^{3/2}$

$p_L^+ = 5.1$, $p_L^- = -0.7$

$b_{1L}^{+,-} = 2.2 \mp \sqrt{\frac{2.2^2 - 1}{3}}$

$b_{1L}^+ = 1.07$, $b_{1L}^- = 3.33$.

$\boxed{\lambda = \sqrt{5.5}\,,\ m = 2}$



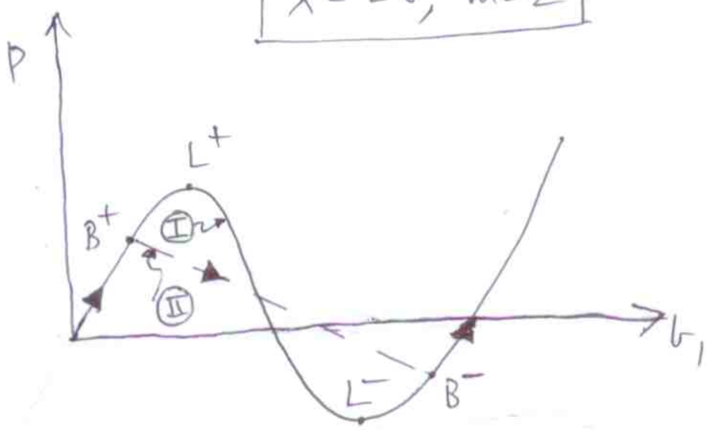
$$p_B^{+,-} = 2.345 \pm (2^2-1)\sqrt{2.345^2-2^2} = 6.019,\ -1.329$$

$$b_{1B}^{+,-} = 2.345 \mp \sqrt{2.345^2-2^2} = 1.12046,\ 3.5699$$

$$p_L^{+,-} = 2.345 \pm 2\left(\frac{2.345^2-1}{3}\right)^{3/2} = 6.019,\ -1.329$$

$$b_{1L}^{+,-} = 2.345 \mp \sqrt{\frac{2.345^2-1}{3}} = 1.12046,\ 3.5699$$

$\boxed{\lambda = 2.8,\ m = 2}$



$$p_B^{+,-} = 2.8 \pm (2^2-1)\sqrt{2.8^2-2^2} = 8.6787,\ -3.0787$$

$$b_{1B}^{+,-} = 2.8 \mp \sqrt{2.8^2-2^2} = 0.84,\ 4.7566$$

$$p_L^{+,-} = 2.8 \pm 2\left(\frac{2.8^2-1}{3}\right)^{3/2} = 9.685,\ -4.085$$

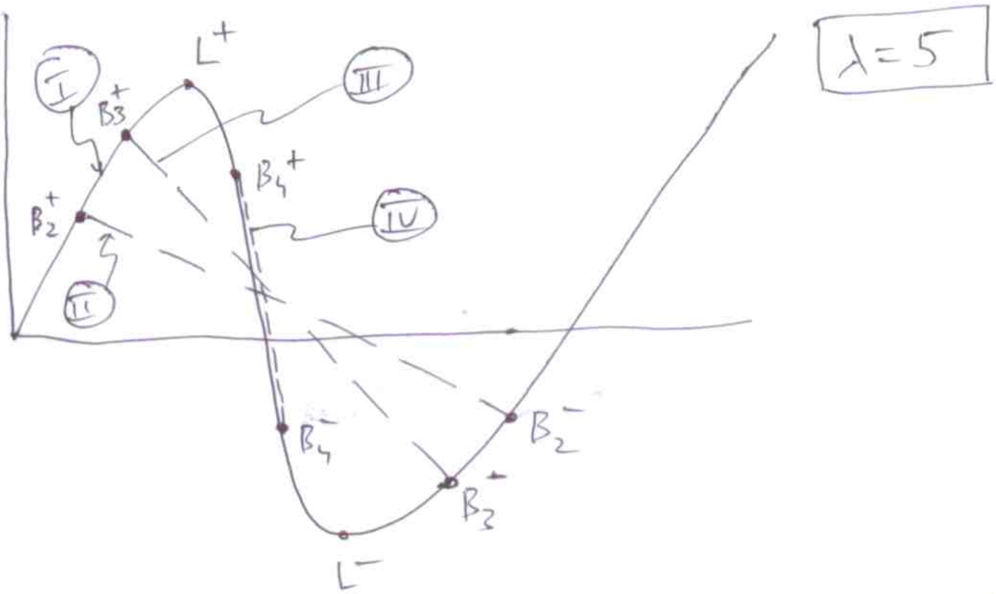
$$b_{1L}^{+,-} = 2.8 \mp \sqrt{\frac{2.8^2-1}{3}} = 1.29,\ 4.31$$

(a) For $m=2$, $\lambda = 2.2$ (ie $< \lambda^*$), as $p\uparrow$, snap-thru occurs at L^+ along horizontal dotted line, snap-back occurs (as $p\downarrow$) at L^- along horizontal dotted line. Here B^+ comes after L^+ and B^- before L^- as we traverse I.

(b) For $m=2$, $\lambda = \sqrt{5.5} = \lambda^*$, L^+, B^+ and L^-, B^- coincide. Again snap thru & snap back occurs along horizontal dotted lines, unless an $m=2$ disturbance is present, in which case it occurs along B^+B^- line. In both these cases we need $p<0$ (ie upward applied load) to effect snap-back.

(c) For $m=2$, $\lambda = 2.8$ (ie $> \lambda^*$), B^+ comes before L^+ and B^- after L^- as we traverse I. Thus if there is a disturbance proportional to $\sin 2\pi x$, the arch will snap-thru along path II (inclined dotted line) from B^+ to B^-. In most practical cases this is what will occur since small

disturbances would be present in every mode.
In general, looking at all possible equilibrium paths we have,



Here also arch will snap-thru from B_2^+ to B_2^- and snap back along this path in most practical cases, since $m=2$ disturbance will be present along with others.

Thus we see that although the sinusoidal arch and its sinusoidal loading and bc's are symmetric, we <u>must</u> consider the antisymmetric modes (ie $m=2,4,6$) in equilibrium solutions to get correct snap-thru behavior. Note that snap-thru occurs instantaneously (ie dynamic phenomenon) so the antisymm mode does not persist in the solution once we reach the opposite side (ie from B_2^+ to B_2^-). After reaching B_2^- we are back to the primary ($m=1$) path which is symmetric, in conformity with the symmetry of the problem. ~~The~~ antisymmetric mode only persists for the short duration of snap-thru.

STABILITY ANALYSIS.

Compare equil eqns (6) with stability eqns (7). They are distinct, unlike in case of columns / plates. Let us solve the stability equations corresponding to the primary equilibrium path ((I)), ie we wish to study the stability of the primary equilibrium solution

$$N_e = \text{const} \quad , \quad \overline{W}_e = b_1 \sin \pi \overline{x}$$

For the uniform pinned-pinned arch, the stability eqns and bc's become (ref. (7), (7a)),

$$\left.\begin{aligned} & N_1' = 0 \\ & EI\, w_1^{IV} - \left(N_e w_1' + N_1 (z_0' + w_e')\right)' = 0 \\ & u_1(0) = u_1(L) = 0 \\ & w_1(0) = w_1(L) = 0 \\ & M_1(0) = M_1(L) = 0 \Rightarrow w_1''(0) = w_1''(L) = 0 . \end{aligned}\right\} \rightarrow (19)$$

(19a) $\Rightarrow N_1 = \text{const} = EA\left(u_1' + (z_0' + w_e')\, w_1'\right)$

$$\int_0^L N_1\, dx = N_1 L = EA\Big(\underbrace{u_1(L)}_{0} - \underbrace{u_1(0)}_{0} + \int_0^L (z_0' + w_e') w_1'\, dx \Big)$$

$$N_1 = \frac{EA}{L} \int_0^L (z_0' + w_e') w_1'\, dx = \text{const.} \rightarrow (20)$$

Non-dimensionalization: Same as for equilibrium equation with additional non-dimensionalization

$$\overline{W}_1 = -\frac{w_1}{2}\sqrt{\frac{A}{I}} = \frac{w_1}{c_1} \quad , \quad \gamma_1^2 = -\frac{N_1 L^2}{EI} \quad (\text{ref. } (4), (5)).$$

Thus (19b), (20) become,

$$EI \frac{1}{L^4}\left(-2\overline{w}_1 \sqrt{\frac{I}{A}}\right)^{IV} - \left(-\frac{EI}{L^2}\gamma_e^2 \frac{1}{L^2}\left(-2\overline{w}_1\sqrt{\frac{I}{A}}\right)'' \right.$$
$$\left. - \frac{EI}{L^2}\gamma_1^2 \frac{1}{L^2}\left(2\overline{z}_0\sqrt{\frac{I}{A}} - 2\overline{W}_e\sqrt{\frac{I}{A}}\right)''\right) = 0$$

$$-\gamma_1^2 \frac{EI}{L^2} = \frac{EA}{L}\int_0^1 \frac{1}{L^2}\left(2\bar{z}_0\sqrt{\frac{I}{A}} - 2\bar{w}_e\sqrt{\frac{I}{A}}\right)'\left(-2\bar{w}_1\sqrt{\frac{I}{A}}\right)' L\, d\bar{x}$$

$$\Rightarrow \quad \left.\begin{aligned} &\bar{w}_1^{IV} + \gamma_e^2 \bar{w}_1'' - \gamma_1^2 \bar{z}_0'' + \gamma_1^2 \bar{w}_e'' = 0 \\ &\gamma_1^2 = 4\int_0^1 (\bar{z}_0 - \bar{w}_e)' \bar{w}_1' \, d\bar{x} \\ &\text{B.C's} \quad \bar{w}_1(0) = \bar{w}_1(1) = \bar{w}_1''(0) = \bar{w}_1''(1) = 0 \end{aligned}\right\} \rightarrow (21)$$

Eqn (21) is a linear system in $\bar{w}_1$, i.e the buckling mode. Compare with (10) which contains $\gamma_e^2 \bar{w}_e''$ which is a nonlinear term since γ_e^2 depends on $\bar{w}_e$. Here either $\bar{w}_1$ or γ_1^2 appear in each term, hence linear in $\bar{w}_1$ (see (21b) where γ_1^2 is linear in $\bar{w}_1$).

Solution is,

$$\bar{w}_1 = \sum_{n=1}^{\infty} c_n \sin n\pi\bar{x} \rightarrow (22) \quad \text{(similar to equil soln.)}$$

which satisfies the bc's above.

Substitute $\bar{z}_0 = \lambda \sin \pi\bar{x}$, $\bar{w}_e = b_1 \sin \pi\bar{x}$ and (22) in (21), (also use (14) for γ_e^2),

$$\gamma_1^2 = 4\int_0^1 \pi(\lambda \cos \pi\bar{x} - b_1 \cos \pi\bar{x}) \sum_{n=1} \pi n c_n \cos n\pi\bar{x}\, d\bar{x}$$

$$\boxed{\gamma_1^2 = 4\pi^2 c_1\left(\tfrac{1}{2}\right)(\lambda - b_1) = 2\pi^2 c_1(\lambda - b_1)} \rightarrow (23)$$

$$\sum_{n=1}^{\infty} \pi^4 n^4 c_n \sin n\pi\bar{x} + \gamma_e^2 \sum_{n=1}^{\infty} -\pi^2 n^2 c_n \sin n\pi\bar{x} + 2\pi^2 c_1(\lambda - b_1) * (-b_1\pi^2 + \lambda\pi^2)\sin \pi\bar{x} = 0$$

Equate coeffs of $\sin n\pi\bar{x}$,

$$\boxed{\begin{aligned} &\cancel{\pi^4}\left(1 - \frac{\gamma_e^2}{\pi^2} + 2(\lambda - b_1)^2\right) c_1 = 0 \\ &\cancel{n^2}\cancel{\pi^4}\left(n^2 - \frac{\gamma_e^2}{\pi^2}\right) c_n = 0 \quad , n = 2, 3, \cdots \infty \end{aligned}} \rightarrow (24).$$

Eq (24) is a set of homogeneous equations in $c_1, \ldots c_n$. For non-trivial solutions,

$$\det[\underline{\underline{C}}] = 0 \quad \text{where}$$

$$\underbrace{\underline{\underline{C}}}_{\underline{\underline{C}}}\begin{Bmatrix} c_1 \\ c_2 \\ \vdots \end{Bmatrix} = \begin{bmatrix} 1-\frac{\gamma_e^2}{\pi^2}+2(\lambda-b_1)^2 & & & \\ & 2^2-\left(\frac{\gamma_e}{\pi}\right)^2 & & \\ & & \ddots & \\ & & & m^2-\left(\frac{\gamma_e}{\pi}\right)^2 \\ & & & & \ddots \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \\ \vdots \end{Bmatrix} = 0$$

$$\Rightarrow \underbrace{1-\left(\frac{\gamma_e}{\pi}\right)^2+2(\lambda-b_1)^2=0}_{\text{"(a) solution"}} \quad \text{or} \quad \underbrace{m^2-\left(\frac{\gamma_e}{\pi}\right)^2=0}_{\text{"(b) solution"}}, \; m=2, \text{or } 3, \text{or} \ldots \infty$$

For "(a) solution", using (14),

$$1-2\lambda b_1+b_1^2+2\lambda^2+2b_1^2-4\lambda b_1=0$$

$$3(b_1^2-2\lambda b_1+\lambda^2)-\lambda^2+1=0$$

$$\boxed{b_1=\lambda\pm\sqrt{\frac{\lambda^2-1}{3}}} \longrightarrow \text{Buckling value of } b_1 \text{ (limit point)}$$

$$p=\lambda\pm\sqrt{\frac{\lambda^2-1}{3}}-\left(1+2\left(\frac{\lambda^2-1}{3}\right)\right)\left(\pm\sqrt{\frac{\lambda^2-1}{3}}\right)$$

$$\boxed{p=\lambda\mp 2\left(\frac{\lambda^2-1}{3}\right)^{3/2}} \longrightarrow \text{Buckling load } P. \text{ (limit point)}$$

These correspond to the values of b_1 and p at L^+ and L^- done by equilibrium solution approach. The buckling mode corresponds to $c_1 \neq 0$ other c_i's $=0$, ie, $c_1 \sin \pi \bar{x}$.

For "(b) solution", from (14),

$$\left(\frac{\delta e}{\pi}\right)^2 = m^2 = 2\lambda b_1 - b_1^2 \implies \boxed{b_1 = \lambda \pm \sqrt{\lambda^2 - m^2}}$$

Buck b_1 & p (bif. pt).

$$p = \lambda \pm \sqrt{\lambda^2 - m^2} - m^2\left(\pm\sqrt{\lambda^2-m^2}\right) = \boxed{\lambda \pm (1-m^2)\sqrt{\lambda^2-m^2} = p}$$

The corresponding buckling mode is for $c_m \neq 0$, $c_1 = 0$, other c_i's $= 0$, i.e. $c_m \sin m\pi\bar{x}$. This "b solution" exists for $\lambda \geq m$ only.

The critical load is the first buckling load encountered on the primary equilibrium path. For $\lambda \leq 1$, no critical load. For $1 \leq \lambda < \sqrt{5.5}$ we have limit point snap-thru buckling with buckl. mode $c_1 \sin \pi\bar{x}$. For $\lambda > \sqrt{5.5}$ we have bifurcation snap-thru along the $m=2$ path, i.e. buckling mode $c_2 \sin 2\pi\bar{x}$.