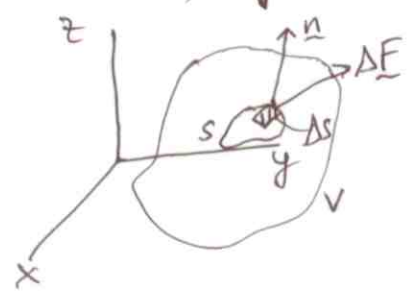# ELASTICITY FUNDAMENTALS:

(I) Analysis of Stress.

Surface and body forces/moments.

Surface forces act across a surface (external or internal) of the body. These occur due to direct contact.

$S \to$ close surface embedded in V.
To study interaction of material inside & outside S, we consider small surface element ΔS oriented at $\underline{n}$. $\Delta \underline{F}$ is force exerted by outside material on inside material, & acts across ΔS, & depends on position $(\underline{r})$ and $\Delta \underline{S}$.
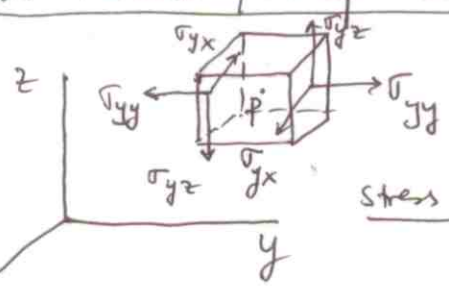
Define $\boxed{\underline{\sigma} = \lim_{\Delta S \to 0} \dfrac{\Delta \underline{F}}{\Delta S}}$ = stress vector ($\underline{t}$ in some books).

As $\Delta S \to 0$, moments due to $\Delta \underline{F}$ about any pt on DS vanishes $\Rightarrow$ surface moments don't exist.

Body forces/moments act at a distance (ie not due to direct contact) due to external field (gravity, magnetism) and are defined as:
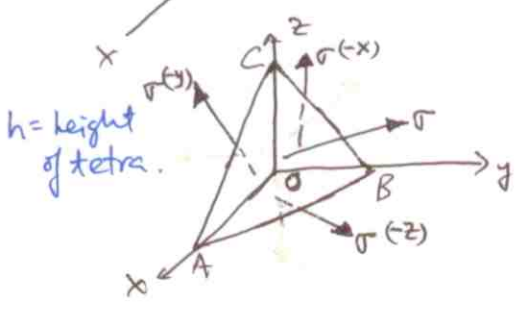
$$\underline{f} = \lim_{\Delta m \to 0} \frac{\Delta \underline{F}}{\Delta m} = \frac{d\underline{F}}{\int dv} = \frac{\underline{f}}{\int}$$

Stress Tensor / components of stress. and its relation with stress vector

stress tensor at P.

$$\underline{\underline{T}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

h = height of tetra.

$$\underline{\sigma}^{(-x)} = -\sigma_{xx}\,\underline{i} - \sigma_{xy}\,\underline{j} - \sigma_{xz}\,\underline{k}$$
$$\underline{\sigma}^{(-y)} = -\sigma_{yx}\,\underline{i} - \sigma_{yy}\,\underline{j} - \sigma_{yz}\,\underline{k}$$
$$\underline{\sigma}^{(-z)} = -\sigma_{zx}\,\underline{i} - \sigma_{zy}\,\underline{j} - \sigma_{zz}\,\underline{k}$$

$$l = \frac{\Delta OCB}{\Delta ABC}, \quad m = \frac{\Delta(OAC)}{\Delta(ABC)}, \quad n = \frac{\Delta(OAB)}{\Delta(ABC)}$$
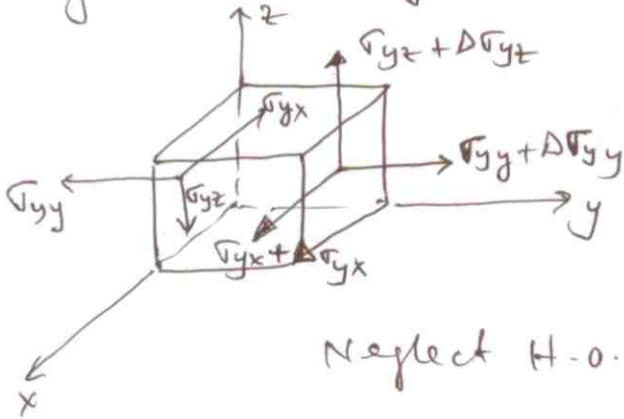
Equilibrium of forces $\Rightarrow$
(incl. body forces,)
(put lim $h \to 0$)
Cauchy rel. $\longrightarrow$

$$\underline{\sigma} = (l\,\sigma_{xx} + m\,\sigma_{yx} + n\,\sigma_{zx})\,\underline{i}$$
$$\underset{T_y}{} + (l\,\sigma_{xy} + m\,\sigma_{yy} + n\,\sigma_{yz})\,\underline{j} \qquad = \sigma_{ji}\,n_j\,\underline{e_i}$$
$$\underset{T_z}{} + (l\,\sigma_{zx} + m\,\sigma_{zy} + n\,\sigma_{zz})\,\underline{k} \qquad \begin{array}{l}\text{in tensorial}\\ \text{index}\\ \text{notation}\end{array}$$

where $\underline{n}$ = normal to ABC plane = $l\,\underline{i} + m\,\underline{j} + n\,\underline{k}$

Absence of body moments, when considered with moment equilibrium of an element yields,

$$\underline{k}: (\sigma_{yx} + \Delta\sigma_{yx} + \sigma_{yx})\,\frac{\Delta y}{2}^{\times \Delta x \Delta z} - (\sigma_{xy} + \Delta\sigma_{xy} + \sigma_{xy})\frac{\Delta x}{2}$$
$$\times \Delta y \Delta z$$
$$= \eta'_{hz}$$
$$= 0 \quad (\because \text{ no body moment})$$

Neglect H.o.T's, $\Rightarrow \boxed{\sigma_{yx} = \sigma_{xy}}$ etc.

Normal & Shearing stresses on arbitrary plane.

$$N = \sigma_N = \underline{\sigma} \cdot \underline{n} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2lm\,\sigma_{xy} + 2ln\,\sigma_{xz}$$
$$+ 2mn\,\sigma_{yz} \longrightarrow \text{①}$$

$$S = \sigma_S = \sqrt{|\underline{\sigma}|^2 - \sigma_N^2} \longrightarrow \text{②}$$

σ_N and σ_S are important in failure/design criteria.

Stress transformation.

l_1, m_1, n_1 denote dir. cosines between X and x, y, z, resply.
l_2, m_2, n_2 " " " Y " " "
l_3, m_3, n_3 " " " Z " " "

From ①,
$$\sigma_{XX} = l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + n_1^2\sigma_{zz} + 2l_1 m_1\,\sigma_{xy} + 2l_1 n_1\,\sigma_{xz} + 2m_1 n_1\,\sigma_{yz}$$

|||^{ally} for σ_{YY}, σ_{ZZ}

Let $\underline{\sigma}^{(X)}$ = stress vector on plane having X-axis as its normal.

Also $\underline{I} = l_2\,\underline{i} + m_2\,\underline{j} + n_2\,\underline{k}$

So, $\sigma_{XY} = \underline{\sigma}^{(X)} \cdot \underline{J} = \underline{\sigma}^{(Y)} \cdot \underline{I}$

$$= l_1 l_2 \sigma_{xx} + m_1 m_2 \sigma_{yy} + n_1 n_2 \sigma_{zz} + (m_1 l_2 + l_1 m_2) \sigma_{xy}$$
$$+ (n_1 l_2 + l_1 n_2) \sigma_{xz} + (m_1 n_2 + n_1 m_2) \sigma_{yz}$$

$|||^{arly}$ for σ_{XZ}, σ_{YZ}

In compact form, using index notation,

$\sigma'_{ij} = a_{im} a_{jn} \sigma_{mn}$ } where

or $\underline{\sigma}' = \underline{a}\, \underline{\sigma}\, \underline{a}^T$ } → ③

	x	y	z
X	a_{11}	a_{12}	a_{13}
Y	a_{21}	a_{22}	a_{23}
Z	a_{31}	a_{32}	a_{33}

We see that $a a^T = \underline{I}$

$$= \begin{pmatrix} X \cdot X & X \cdot Y & X \cdot Z \\ Y \cdot X & Y \cdot Y & Y \cdot Z \\ Z \cdot X & Z \cdot Y & Z \cdot Z \end{pmatrix}$$

Principal stresses and axes:

There exist 3 mutually $\perp^{ar}$ planes at any point for which σ_S vanishes on each of these planes, ie $\underline{\sigma} = \sigma_N \underline{n}$ on each of these planes. Further, σ_N also has stationary values ($i e |\underline{\sigma}| = \sigma_N$) (it may be shown (not here)) on each of these planes. Thus

$$\underline{\sigma} = \sigma_{ji} \, n_j \, \underline{e}_i = (l \sigma_{xx} + m \sigma_{yx} + n \sigma_{zx}) \underline{i} + (l \sigma_{xy} + m \sigma_{yy} + n \sigma_{yz}) \underline{j}$$
$$+ (l \sigma_{zx} + m \sigma_{zy} + n \sigma_{zz}) \underline{k}$$
$$= \sigma_N (l \underline{i} + m \underline{j} + n \underline{k}) \quad \text{since } (l, m, n) \text{ is principal plane.}$$

compare coeffs, get 3 linear eqns in l, m, n, then note non-triviality of solution

$$\Rightarrow \begin{vmatrix} \sigma_{xx} - \sigma_N & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_N & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_N \end{vmatrix} = 0 \longrightarrow \text{solve for } \sigma_N.$$

rename as σ.

Then solve for (l, m, n) from two indep. or one indep. simultaneous equations and $l^2 + m^2 + n^2 = 1$.

The cubic is, (rename $\sigma_N \to \sigma$)

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0.$$

$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$, $I_3 = \det$ (stress tensor)

$I_2 = \sigma_{xx} \sigma_{yy} + \sigma_{xx} \sigma_{zz} + \sigma_{yy} \sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$ } → ④

P-stresses & axes depend only on stress tensor at the point, ie on state of

stress at the point which in turn depends on applied loads on the body. Here I_1, I_2, I_3 are invariants.

If $\sigma_1 = \sigma_2$ (ie 2 equal P-stress) then ∞ set of two $\perp^{ar}$ axes contained in plane normal to third p-axes are the other two p-axes.

If $\sigma_1 = \sigma_2 = \sigma_3$, then ∞ set of three $\perp^{ar}$ axes are p-axes.

Octahedral stress

(i.e it coincides with p-axes at a particular pt P)

If (x^*, y^*, z^*) is a p-axes coord system, then referring to this system the eight planes satisfying $l^2 = m^2 = n^2 = \frac{1}{3}$ are octahedral planes.

$$(\sigma_N)_{oct} = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \frac{I_1}{3} \overset{\frown}{=} \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$9(\sigma_s)^2_{oct} = 9\tau^2_{oct} = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2I_1^2 - 6I_2$$

$$\overset{\frown}{=} (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2$$
$$+ 6\sigma_{xy}^2 + 6\sigma_{xz}^2 + 6\sigma_{yz}^2$$

⑤

Since I_1, I_2, I_3 are invariant, we substitute back their general-coord-system forms.

Octahedral stresses are required in failure criteria.

Pure shear state of stress.

If $\exists$ system x, y, z for which $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ vanish at a point, then the state of stress at that point is a Pure Shear state. Necessary + suff condt for Pure Shear state of stress to exist is
$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0.$$

Deviatoric stress.

deviatoric part. hydrostatic or spherical part

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ symm & & \sigma_{zz} \end{bmatrix} = \underset{\hat{\sigma}''}{\underline{I}_d} + \underset{\sigma_m}{\underline{I}_m}, \quad \text{where } \underline{I}_m = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$\underline{I} = \underline{\sigma} \Rightarrow \sigma_{ij} = \hat{\sigma}_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$$

where $\sigma_m = \dfrac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \dfrac{I_1}{3}$.

Note that $\underline{I}_d$ represents a pure shear state of stress.

Expts indicate that yielding & plastic deformation depend on $\underline{I}_d$ and independent of $\underline{I}_m$

Now evp for $\underline{T}$ in index notation is $\quad \underline{T} = \sigma n_i \underline{e}_i = \sigma_{ij} n_j \underline{e}_i$

$$\left(\sigma_{ij} - \sigma \delta_{ij}\right) n_j = 0. \qquad , \quad \text{where} \quad n_1 = l, \; n_2 = m, \; n_3 = l .$$

So evp for $\underline{\underline{T}}d$ is

$$\left(\hat{\sigma}_{ij} - \hat{\sigma} \delta_{ij}\right) n_j = 0 \;\Rightarrow\; \left(\sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} - \hat{\sigma}\,\delta_{ij}\right) n_j = 0$$

$$\Rightarrow \left(\sigma_{ij} - \underbrace{\left[\hat{\sigma} + \tfrac{1}{3}\sigma_{kk}\right]}_{= \,\sigma}\delta_{ij}\right) n_j = 0.$$

$$\Rightarrow \hat{\sigma} = \sigma - \tfrac{1}{3}\sigma_{kk} = \sigma - \sigma_m$$

Plane stress.

$$\sigma_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos2\theta + \sigma_{xy}\sin2\theta$$

$$\sigma_{yy} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta = \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) - \tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos2\theta - \sigma_{xy}\sin2\theta$$

$$\sigma_{xy} = -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) = -\tfrac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin2\theta + \sigma_{xy}\cos2\theta$$

$$\Rightarrow \left[\sigma_{xx} - \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})\right]^2 + \sigma_{xy}^2 = \tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xy}^2$$

Principal stresses occur for $\quad \tan2\theta = \dfrac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}}$

and are equal to $\quad \sigma_{1,2} = \dfrac{\sigma_{xx}+\sigma_{yy}}{2} \pm \tfrac{1}{4}\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xy}^2}$ $\qquad \Big\} \longrightarrow ⑥$

Maximum shearing stress.

$$(\sigma_s)_{max} = \max\left(\left|\tfrac{(\sigma_1-\sigma_2)}{2}\right|, \; \left|\tfrac{(\sigma_1-\sigma_3)}{2}\right|, \; \left|\tfrac{(\sigma_2-\sigma_3)}{2}\right|\right) \longrightarrow ⑦$$

and acts on $\pm45°$ planes (in the p-axes system) in the 1-2, 1-3, 2-3 plane, respectively.

MAXIMUM SHEAR STRESS.

We wish to find $\max(S)$. Refer to p-coord system :

$$\underline{\tau} = (\sigma_{(1)} n_1, \sigma_{(2)} n_2, \sigma_{(3)} n_3) \qquad \Big| \qquad S^2 = |\underline{\tau}|^2 - N^2$$
$$N = \sigma_{(1)} n_1^2 + \sigma_{(2)} n_2^2 + \sigma_{(3)} n_3^2$$

where $\underline{n} = (n_1, n_2, n_3)$ is plane containing $\max(S)$. Note that n_1, n_2, n_3 are w.r.t. p-coord system. Thus,

$$n_1^2 + n_2^2 + n_3^2 = 1 \longrightarrow ①$$

$$S^2 = \sigma^2_{(1)} n_1^2 + \sigma^2_{(2)} n_2^2 + \sigma^2_{(3)} n_3^2 \longrightarrow ⊛$$
$$- \left[\sigma_{(1)} n_1^2 + \sigma_{(2)} n_2^2 + \sigma_{(3)} n_3^2 \right]^2$$

Doing $\dfrac{\partial S^2}{\partial n_1^2} = \dfrac{\partial S^2}{\partial n_2^2} = \dfrac{\partial S^2}{\partial n_3^2} = 0$ to get n_1, n_2, n_3

wont work due to constrain ①. Instead you can use ① to eliminate (say) n_3^2 from ⊛ by putting $n_3^2 = 1 - n_1^2 - n_2^2$ in ⊛, and then do $\dfrac{\partial S^2}{\partial n_1} = \dfrac{\partial S^2}{\partial n_2} = 0$, since now n_1, n_2 are independent.

However, this is somewhat messy. So use the Lagrange Multiplier approach for constrained optimization. Form augmented function G :

$$G = S^2 + \lambda (n_1^2 + n_2^2 + n_3^2 - 1)$$
$$\qquad\qquad\quad \hookrightarrow \text{Lagrange multiplier.}$$

Now n_1, n_2, n_3, λ can be treated as independent. Note that constraint is included in G, ie., if constraint satisfied then $G = S^2$.

Do $\dfrac{\partial G}{\partial \lambda} = \dfrac{\partial G}{\partial n_1} = \dfrac{\partial G}{\partial n_2} = \dfrac{\partial G}{\partial n_3} = 0$

$$\frac{\partial G}{\partial \lambda} = 0 = n_1^2 + n_2^2 + n_3^2 - 1 \longrightarrow \text{①} \quad \left(\begin{array}{l}\text{retrieved} \\ \text{again!}\end{array}\right).$$

$$\frac{\partial G}{\partial n_1} = 0 = n_1 \left\{ \sigma^2(1) - 2\left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2 \right] \sigma(1) + \lambda \right\} \to \text{②}$$

$$\frac{\partial G}{\partial n_2} = 0 = n_2 \left\{ \sigma^2(2) - 2\left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2 \right] \sigma(2) + \lambda \right\} \to \text{③}$$

$$\frac{\partial G}{\partial n_3} = 0 = n_3 \left\{ \sigma^2(3) - 2\left[\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2 \right] \sigma(3) + \lambda \right\} \to \text{④}$$

Solutions:

(i) Case I : only one component of $\underline{n}$ is non-zero.

$$n_1 = 1, \; n_2 = n_3 = 0 \implies \lambda = \sigma^2(1), \quad S^2 = 0$$

or, $n_2 = 1, \; n_1 = n_3 = 0 \implies \lambda = \sigma^2(2), \quad S^2 = 0$

or, $n_3 = 1, \; n_1 = n_2 = 0 \implies \lambda = \sigma^2(3), \quad S^2 = 0$

This is the well-known solution that $S = 0$ on principal planes. Here $S = 0$ is a local extremum value but not the max(S) that we seek — so NOT INTERESTING

(ii) Case II : only one component of $\underline{n}$ is zero.

Say $n_1 = 0, \; n_2 \neq 0, \; n_3 \neq 0$

①, ③, ④ $\implies$

$$\sigma^2(2) - 2\left[\sigma(2) n_2^2 + \sigma(3) - \sigma(3) n_2^2 \right] \sigma(2) + \lambda = 0$$

$$\sigma^2(3) - 2\left[\sigma(2) n_2^2 + \sigma(3) - \sigma(3) n_2^2 \right] \sigma(3) + \lambda = 0$$

$$\left[\sigma(2) - \sigma(3) \right] \left[\sigma(2) + \sigma(3) - 2\left\{ \sigma(2) n_2^2 + \sigma(3) - \sigma(3) n_2^2 \right\} \right] = 0$$

$\implies$ If $\sigma(2) = \sigma(3) \implies \lambda = \sigma^2(2), \quad S^2 = 0, \; n_2 = \text{arb}, \; n_3 = \text{arb}$

If $\sigma(2) \neq \sigma(3) \implies n_2 = \pm \frac{1}{\sqrt{2}}, \; n_3 = \pm \frac{1}{\sqrt{2}}, \; \lambda = \sigma(2)\sigma(3),$

$$S^2 = \left(\frac{\sigma(2) - \sigma(3)}{2} \right)^2 \implies S = \left| \frac{\sigma(2) - \sigma(3)}{2} \right|$$

Thus possible solutions are,

$$n_1 = 0, \quad n_2 = \pm\sqrt{\frac{1}{2}}, \quad n_3 = \pm\sqrt{\frac{1}{2}} \qquad n_2 = \text{arb}, \; n_3 = \text{arb}, \quad \lambda = \sigma(2)\sigma(3),$$

$$\underbrace{\phantom{n_1 = 0, \quad n_2 = \pm\sqrt{\frac{1}{2}}, \quad n_3 = \pm\sqrt{\frac{1}{2}}}}_{\sigma(2) \neq \sigma(3)} \qquad \underbrace{\phantom{n_2 = \text{arb}, \; n_3 = \text{arb}}}_{\sigma(2) = \sigma(3)} \qquad S = \frac{1}{2}|\sigma(2) - \sigma(3)|$$

or

$$n_2 = 0, \quad n_1 = \pm\sqrt{\frac{1}{2}}, \quad n_3 = \pm\sqrt{\frac{1}{2}}, \qquad n_1 = \text{arb}, \; n_3 = \text{arb}, \quad \lambda = \sigma(1)\sigma(3),$$

$$\underbrace{\phantom{n_2 = 0, \quad n_1 = \pm\sqrt{\frac{1}{2}}, \quad n_3 = \pm\sqrt{\frac{1}{2}}}}_{\sigma(1) \neq \sigma(3)} \qquad \underbrace{\phantom{n_1 = \text{arb}, \; n_3 = \text{arb}}}_{\sigma(1) = \sigma(3)} \qquad S = \left|\frac{\sigma(1) - \sigma(3)}{2}\right|$$

or

$$n_3 = 0, \quad n_2 = \pm\sqrt{\frac{1}{2}}, \quad n_1 = \pm\sqrt{\frac{1}{2}}, \qquad n_1 = \text{arb}, \; n_2 = \text{arb}, \quad \lambda = \sigma(1)\sigma(2),$$

$$\underbrace{\phantom{n_3 = 0, \quad n_2 = \pm\sqrt{\frac{1}{2}}, \quad n_1 = \pm\sqrt{\frac{1}{2}}}}_{\sigma(1) \neq \sigma(2)} \qquad \underbrace{\phantom{n_1 = \text{arb}, \; n_2 = \text{arb}}}_{\sigma(1) = \sigma(2)} \qquad S = \left|\frac{\sigma(1) - \sigma(2)}{2}\right|$$

- So if all p-stresses are distinct then max(S) acts on a plane containing one p-axis & equally inclined (at ±45°) to other two p-axes.

- If any two p-stresses are same, say $\sigma(1) = \sigma(2)$, then $S_{max} = \frac{|\sigma(1) - \sigma(3)|}{2} = \frac{|\sigma(2) - \sigma(3)|}{2}$, acts on plane with $n_3 = \pm\sqrt{\frac{1}{2}}$ and n_1, n_2 arbitrary.

Case III : All components of $\underline{n}$ are non-zero.

Here ②,③,④ give an inconsistent system of equations if two or more p-stresses are distinct, i.e., determinant of coefficient matrix of $[n_1^2 \; n_2^2 \; n_3^2]^T$ vanishes. Hence solution is only possible if $\sigma(1) = \sigma(2) = \sigma(3)$ which yields $S = 0$ on all planes. Thus, this value of S is not an extremum — so NOT INTERESTING.

EQUILIBRIUM EQUATIONS.
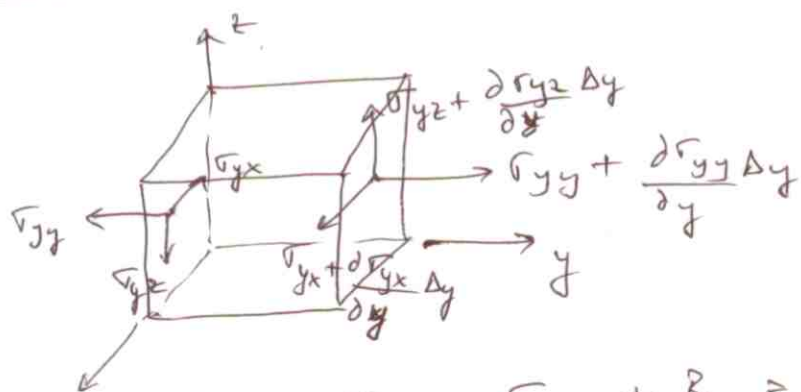
$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + B_x = 0$$

$$\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{yz,z} + B_y = 0$$

$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + B_z = 0$$

$$\sigma_{ij,j} + B_i = 0. \longrightarrow \boxed{8}$$

In general curvilinear orthogonal coordinates we have,

$$\frac{\partial(\beta\gamma\sigma_{xx})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{xy})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{xz})}{\partial z} + \gamma\sigma_{xy}\frac{\partial\alpha}{\partial y}$$

$$+ \beta\sigma_{xz}\frac{\partial\alpha}{\partial z} - \gamma\sigma_{yy}\frac{\partial\beta}{\partial x} - \beta\sigma_{zz}\frac{\partial\gamma}{\partial x} + \alpha\beta\gamma B_x = 0$$

$$\frac{\partial(\beta\gamma\sigma_{xy})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yy})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{yz})}{\partial z}$$

$$+ \gamma\frac{\partial\beta}{\partial x}\sigma_{xy} + \alpha\frac{\partial\beta}{\partial z}\sigma_{yz} - \gamma\frac{\partial\alpha}{\partial y}\sigma_{xx} - \alpha\frac{\partial\gamma}{\partial y}\sigma_{zz} + \alpha\beta\gamma B_y = 0$$

$$\frac{\partial(\beta\gamma\sigma_{xz})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yz})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{zz})}{\partial z}$$

$$+ \beta\frac{\partial\gamma}{\partial x}\sigma_{xz} + \alpha\frac{\partial\gamma}{\partial y}\sigma_{yz} - \alpha\frac{\partial\beta}{\partial z}\sigma_{yy} - \beta\frac{\partial\alpha}{\partial z}\sigma_{xx} + \alpha\beta\gamma B_z = 0$$

$$\longrightarrow \boxed{9}$$

Here (α, β, γ) are metric coefficients defined by

$$ds^2 = \alpha^2 dx^2 + \beta^2 dy^2 + \gamma^2 dz^2 = (\text{diagonal of element})^2$$

Cartesian — $\alpha = \beta = \gamma = 1$ ∴ $ds^2 = dx^2 + dy^2 + dz^2$

Polar/cylindrical $ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \Rightarrow \alpha = 1, \beta = r, \gamma = 1$

Spherical — $ds^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2 \Rightarrow \alpha = 1, \beta = r,$
$\gamma = r\sin\theta$

Cylindrical coords:

$$\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + B_r = 0$$

$$\sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \sigma_{\theta z,z} + \frac{2\sigma_{r\theta}}{r} + B_\theta = 0 \quad \longrightarrow ⑩$$

$$\sigma_{rz,r} + \frac{\sigma_{\theta z,\theta}}{r} + \sigma_{zz,z} + \frac{\sigma_{rz}}{r} + B_z = 0$$
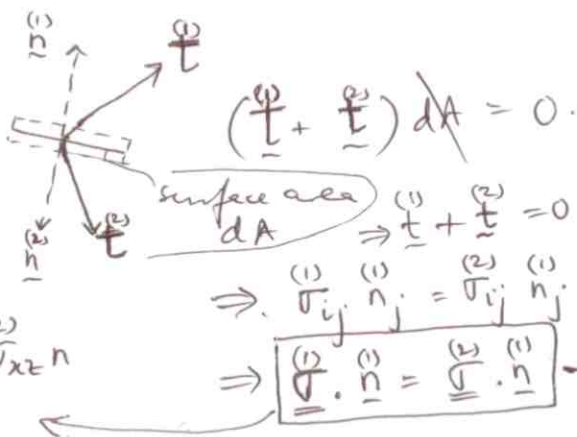
Spherical coords:

$$\sigma_{rr,r} + \frac{\sigma_{r\theta,\theta}}{r} + \frac{\sigma_{r\phi,\phi}}{r\sin\theta} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta\right) + B_r = 0$$

$$⑪ \quad\left\{\sigma_{r\theta,r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \frac{\sigma_{\theta\phi,\phi}}{r\sin\theta} + \frac{1}{r}\left[(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}\right] + B_\theta = 0\right.$$

$$\sigma_{r\phi,r} + \frac{\sigma_{\theta\phi,\theta}}{r} + \frac{\sigma_{\phi\phi,\phi}}{r\sin\theta} + \frac{1}{r}\left(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta\right) + B_\phi = 0$$

Stress Boundary Conditions.

$$\left(\underset{\sim}{t}^{(1)} + \underset{\sim}{t}^{(2)}\right) dA = 0.$$

surface area dA

$$\Rightarrow \underset{\sim}{t}^{(1)} + \underset{\sim}{t}^{(2)} = 0$$

$$\Rightarrow \sigma_{ij}^{(1)} n_j^{(1)} = \sigma_{ij}^{(2)} n_j^{(1)}$$

$$\Rightarrow \boxed{\underset{\approx}{\sigma}^{(1)} \cdot \underset{\sim}{n}^{(1)} = \underset{\approx}{\sigma}^{(2)} \cdot \underset{\sim}{n}^{(1)}} \longrightarrow ⑫$$

$$\left\{\sigma_{xx}^{(1)} \ell + \sigma_{xy}^{(1)} m + \sigma_{xz}^{(1)} n = \sigma_{xx}^{(2)} \ell + \sigma_{xy}^{(2)} m + \sigma_{xz}^{(2)} n\right.$$
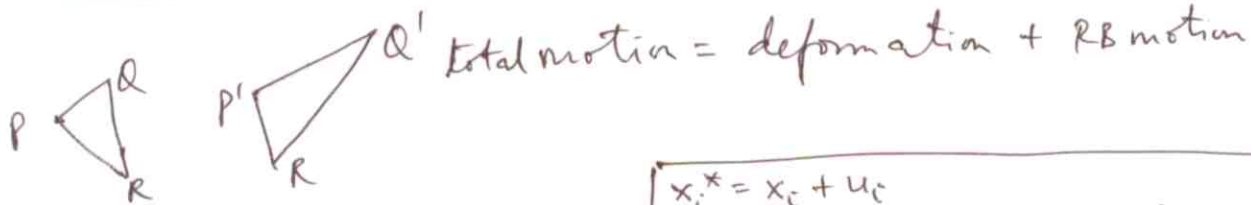$$\vdots \dots$$
$$\vdots \dots$$

(eg) $\underline{n} = (0,0,1) \Rightarrow \sigma_{xz}^{(1)} = \sigma_{xz}^{(2)}, \quad \sigma_{yz}^{(1)} = \sigma_{yz}^{(2)}, \quad \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)}$

No b.c's involving other stress components. Further, other stress components (ie those not involving 'z' are not necessarily continuous across boundary).

(2) <u>Deformation & Strain Analysis.</u> ⑧

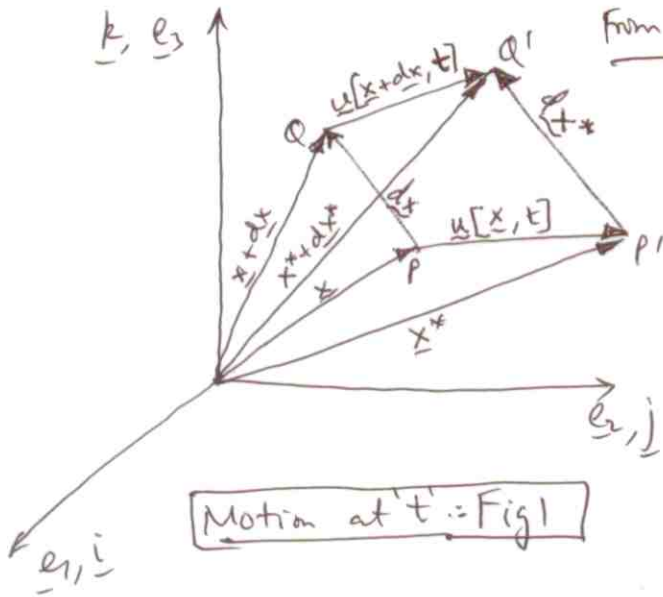Total motion = deformation + RB motion

$$x_i^* = x_i + u_i$$
$$x_i^* = x_i^*[x_1, x_2, x_3, t], \quad u_i = u_i[x_1, x_2, x_3, t]$$

From Fig 1 → $x_i^* + dx_i^* = x_i + dx_i + u_i[\underline{x} + d\underline{x}, t]$

$$du_i = u_i[Q] - u_i[P] = u_i[\underline{x} + d\underline{x}, t] - u_i[\underline{x}, t]$$

Note that du_i is evaluated at 't', ie it is a snapshot at 't'.

$$\Rightarrow \boxed{du_i = \frac{\partial u_i}{\partial x_j}\Big|_{P, t} dx_j}$$ (ie no change in 't' while finding du_i, ie $\Delta t = 0$). ①

$\overline{PQ} = |d\underline{x}| = dx_i \cdot dx_i = (dx)^2$, $\overline{P'Q'} = dx_i^* dx_i^* = (dx^*)^2$

$$\boxed{dx_i^* = \frac{\partial x_i^*}{\partial x_j} dx_j}$$ at given instant 't'. ②

Stretch ratio $\triangleq (dx^*)^2 - (dx)^2 = dx_i^* dx_i^* - dx_i \cdot dx_i = \left[\frac{\partial x_r^*}{\partial x_i}\frac{\partial x_r^*}{\partial x_j} - \delta_{ij}\right] dx_i \cdot dx_j$

$$= \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_r}{\partial x_i}\frac{\partial u_r}{\partial x_j}\right] dx_i \cdot dx_j$$

$$= 2\varepsilon_{ij} dx_i \cdot dx_j$$

$\Rightarrow M = \frac{1}{2}\left[\frac{(dx^*)^2 - (dx)^2}{(dx)^2}\right] = \frac{1}{2}\left[\left(\frac{dx^*}{dx}\right)^2 - 1\right] = \frac{1}{2}\left[(1 + \epsilon_E)^2 - 1\right] = \epsilon_E + \frac{\epsilon_E^2}{2}$

= magnification factor

$\hookrightarrow = \varepsilon_{ij} n_i \cdot n_j \longrightarrow$

Summary → $\boxed{\begin{array}{l} M \triangleq \frac{1}{2}\left[\dfrac{(dx^*)^2 - (dx)^2}{(dx)^2}\right] = \epsilon_E + \dfrac{\epsilon_E^2}{2} = \varepsilon_{ij} n_i \cdot n_j \quad ③ \\[4mm] \text{where } \epsilon_E \triangleq \dfrac{dx^* - dx}{dx}, \quad \varepsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{r,i} u_{r,j}\right] \quad ④ \\ \text{Ext Engg strain} \end{array}}$

Notes: (1) ε_{ij} is a measure of straining. If $dx \neq da$ then $\varepsilon_{ij} \neq 0$ (ie all components cannot be zero

(2) $\mathcal{E}_{ij} = \mathcal{E}_{ji}$

(3) R.B. motion does not affect M.

(4) M is analogous to N (normal stress on a plane) → compare their expressions.

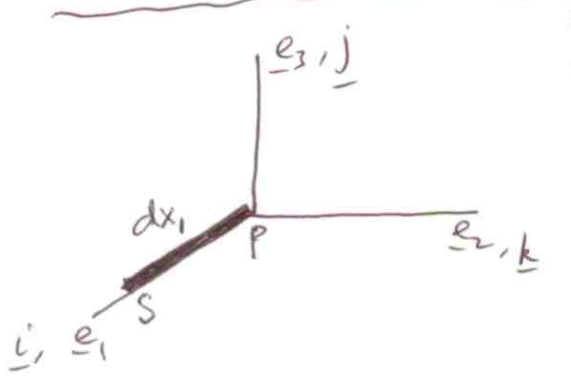Geometric (Physical) interpretation of $\mathcal{E}_{ij}$ (nonlinear).
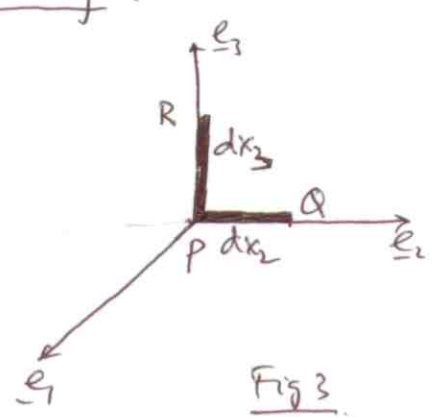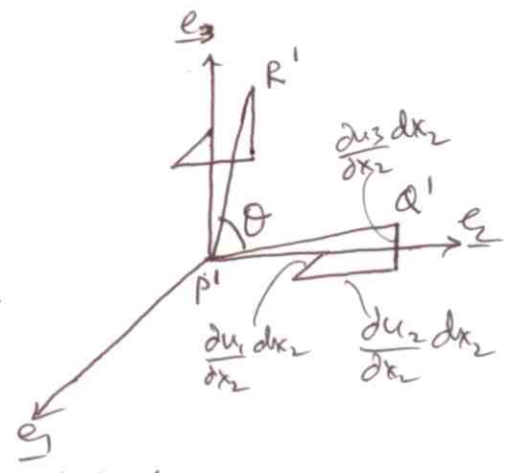

Fig 2.

Fig 3.

P,Q,R displace to P',Q',R'
(Note P's displ is not necessarily horizontal as is shown here).

Normal strains :

Ref. Fig 2. Engg. strain for element PS, having $n_i = (1,0,0)$, is

$$\varepsilon_{E1} = \frac{dx^* - dx}{dx} = \frac{(dx^*)^2 - (dx)^2}{(dx)^2 (\varepsilon_E + 2)} = \frac{2E_{11}}{2 + \varepsilon_{E1}}$$

$$\Rightarrow \boxed{\varepsilon_{E1} = \sqrt{1 + 2E_{11}} - 1} \quad , \quad \text{||}^{ally} \text{ for } E_{22}, E_{33}. \longrightarrow ⑤$$

$$\boxed{M_1 = \varepsilon_{E1} + \frac{\varepsilon_{E1}^2}{2} = E_{11}} \quad , \quad \text{||}^{ally} \text{ for } M_2, M_3. \longrightarrow ⑥$$

Shear strains

Ref Fig 3.

$$\overline{P'Q'} = dx^*\Big|_{PQ} = \sqrt{1 + 2E_{22}}\ dx_2$$

$$\overline{P'R'} = dx^*\Big|_{PR} = \sqrt{1 + 2E_{33}}\ dx_3$$

$$\underline{P'Q'} \cdot \underline{P'R'} = \cos\theta \sqrt{1 + 2E_{22}} \sqrt{1 + 2E_{33}}\ dx_2\ dx_3 \longrightarrow ⑦$$

Also $\underline{P'Q'} \cdot \underline{P'R'} = \underline{dx^*}\Big|_{PQ} \cdot \underline{dx^*}\Big|_{PR} = dx_i^*\Big|_{PQ} dx_i^*\Big|_{PR}$

$$= \left(\frac{\partial x_i^*}{\partial x_j}\right)_P dx_j\Big|_{PQ} \left(\frac{\partial x_i^*}{\partial x_R}\right)_P dx_R\Big|_{PR} = \left(\frac{\partial x_i^*}{\partial x_2} \frac{\partial x_i^*}{\partial x_3}\right)_P dx_2\ dx_3$$
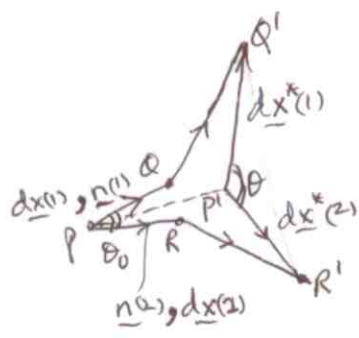
$$= 2E_{23}\ dx_2\ dx_3. \longrightarrow ⑧$$

From ⑦, ⑧ p.9,

$$2\epsilon_{23} = \sqrt{1+2\epsilon_{22}}\ \sqrt{1+2\epsilon_{33}}\ \cos\theta = \left(1+\epsilon_{E2}\right)\left(1+\epsilon_{E3}\right)\cos\theta.$$

$$⑨$$

Generalization of Shear Strain interpretation

Consider two line elements with directions $\underline{n}^{(1)}$, $\underline{n}^{(2)}$, resply initially being θ_0 apart, as shown below.

Now,

$$d\underline{x}^*_{(1)} \cdot d\underline{x}^*_{(2)} = dx^*_{i(1)}\, dx^*_{i(2)} = \sqrt{dx^*_{i(1)} dx^*_{i(1)}}\ \sqrt{dx^*_{j(1)} dx^*_{j(1)}}\ \cos\theta$$

$$\underset{LHS}{\downarrow} \qquad\qquad \underset{RHS}{\downarrow}$$

$$LHS = \left(\frac{\partial x^*_i}{\partial x_j} dx_j\right)_{(1)} \left(\frac{\partial x^*_i}{\partial x_k} dx_k\right)_{(2)} = \left(dx_i + \frac{\partial u_i}{\partial x_j} dx_j\right)_{(1)} \left(dx_i + \frac{\partial u_i}{\partial x_k} dx_k\right)_{(2)}$$

$$= dx_{i(1)}\, dx_{i(2)} + dx_{i(1)}\, dx_{j(2)}\left[u_{i,j} + u_{j,i} + u_{m,i}\,u_{m,j}\right]_P$$

Using ③, p.8,

$$RHS = \sqrt{(dx^*_{(1)})^2}\ \sqrt{(dx^*_{(2)})^2}\ \cos\theta = \sqrt{dx_{i(1)} dx_{i(1)} + 2\epsilon_{ij}\, dx_{i(1)} dx_{j(1)}}\ *$$

$$\sqrt{dx_{p(2)} dx_{p(2)} + 2\epsilon_{pq}\, dx_{p(2)} dx_{q(2)}}\ *\ \cos\theta$$

Divide LHS & RHS by $dx_{(1)} dx_{(2)}$, note that $n_i = dx_i/dx$, and equate
$(LHS = RHS)$
and note that $n_{i(1)} n_{i(1)} = n_i{}_{(2)} n_i{}_{(2)} = 1$, $n_{i(1)} n_{i(2)} = \cos\theta_0$, you get,

$$\cos\theta = \frac{\cos\theta_0 + 2 n_{i(1)}\, n_{j(2)}\, \epsilon_{ij}}{\sqrt{1 + 2 n_{i(1)} n_{j(1)} \epsilon_{ij}}\ \sqrt{1 + 2 n_{p(2)} n_{q(2)} \epsilon_{pq}}} \longrightarrow ⑩$$

Thus we have angle between two line elements after straining, in terms of ϵ_{ij} and angle before straining.

Strain Transformations.
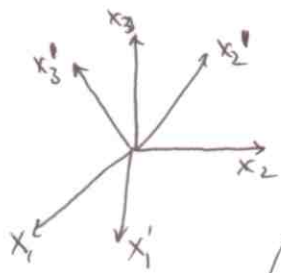
Let the element lie along x_1' axis. Then from
③ & ⑥ we have

$$M_1' = \epsilon_{11}' \quad \text{from} ⑥$$

& $$M_1' = \epsilon_{ij}\, n_i\, n_j \quad (\text{from} ③) = \epsilon_{ij}\, a_{1i}\, a_{1j}$$

$$(\because n_i = \text{unit vector along } x_1' \text{ axis}$$
$$= (a_{11}, a_{12}, a_{13}))$$

$$\Rightarrow \epsilon_{11}' = \epsilon_{ij}\, a_{1i}\, a_{1j} \quad (\text{similarly for } \epsilon_{22}',\, \epsilon_{33}') \rightarrow ⑪a$$

For transf. of shear strains, use ⑩ for two line elements originally along x'_2 & x'_3 axes, respectively; with **unit** vectors denoted as $\underline{n}(2)$, $\underline{n}(3)$, resply. From ⑨ and ⑩ we have,

from ⑨ → in $\underline{x}'$ system → $\dfrac{2\epsilon'_{23}}{\sqrt{1+2\epsilon'_{22}}\ \sqrt{1+2\epsilon'_{33}}} = \cos\theta \longrightarrow$ ⊛

from ⑩ → in $\underline{x}$ system →
$$\cos\theta = \left[\dfrac{\overbrace{\cos\phi_0}^{90°} + 2\epsilon_{ij}\,\overset{a_{2i}}{\overbrace{n_i(2)}}\,\overset{a_{3j}}{\overbrace{n_j(3)}}}{\underbrace{\sqrt{1+2\,\overset{a_{2i}}{n_i(2)}\,\overset{a_{2j}}{n_j(2)}\,\epsilon_{ij}}}_{2\epsilon'_{22}}\ \underbrace{\sqrt{1+2\,\overset{a_{3p}}{n_p(3)}\,\overset{a_{3q}}{n_q(3)}\,\epsilon_{pq}}}_{2\epsilon'_{33}}}\right]$$

using transf law ⑪ⓐ for normal strains

$\Rightarrow \cos\theta = \dfrac{2\epsilon_{ij}\,a_{2i}\,a_{3j}}{\sqrt{1+2\epsilon'_{22}}\ \sqrt{1+2\epsilon'_{33}}} \longrightarrow$ ⊛⊛

From ⊛ & ⊛⊛ → $\epsilon'_{23} = \epsilon_{ij}\,a_{2i}\,a_{3j}$ (similarly for $\epsilon'_{12}, \epsilon'_{13}$). → ⑪ⓑ

In compact form, ⑪(a, b) read,

$$\epsilon'_{ij} = a_{im}\,a_{jn}\,\epsilon_{mn} \longrightarrow ⑪$$

This is same as stress transf. law.

Principal strains & directions.

Seek mutually $\perp$ directions $\underline{n}(1)$, $\underline{n}(2)$, $\underline{n}(3)$, ie a coordinate transf from $\underline{x}$ to $\underline{x}'$ system, such that ϵ_{ij} gets diagonalized, ie $\epsilon'_{ij}=0$ for $i\neq j$. We know, this exists, since transf ⑪ is same as transf for stresses (ie, ③, p.3). Thus $\underline{n}(1) \equiv x'_1$ axis and so on. So we solve the evalue problem

$$(\epsilon_{ij} - \lambda\,\delta_{ij})\,n_j = 0. \longrightarrow ⑫$$

$M(1), M(2), M(3)$ due to analogy with $N(1), N(2), N(3)$.

Again, from analogy with stress(ref ①, p.2 & ③, p.8), we know that $M(N)$ is stationary at the p-axes of strain (stress). Further, since $\epsilon_{ij}=0$ for $i\neq j$, the p-axes retain their orthogonality during deformation. So from analogy between M & N, we get p-strains

as $M_{(1)}$, $M_{(2)}$, $M_{(3)}$ in book of Borési — but not adopted here due to conflict with Eq (H a) (12)

as M_1, M_2, M_3 where principal extensional engg strains

are $\epsilon'_{E1} + \frac{1}{2} \epsilon'^2_{E1} = M_{(1)}$, etc (for $\epsilon'_{E2}, \epsilon'_{E3}$). Now from ③,

$$dM = d\epsilon_E (1 + \epsilon_E) \implies d\epsilon_E = 0 \text{ if } dM = 0 \quad \because (1 + \epsilon_E) > 0 \text{ by}$$

$$\text{definition of } \epsilon_E \left(= \frac{dx^*}{dx} - 1\right).$$

Thus $\epsilon'_{E1}, \epsilon'_{E2}, \epsilon'_{E3}$ are stationary values of ext engg strains.

Analogous to stress, we have,

$$M^3 - J_1 M^2 + J_2 M - J_3 = 0 \longrightarrow \boxed{13}.$$

$$J_1 = \epsilon_{ii}, \quad J_2 = \sum \det(\text{cofactors of diag's of } [\epsilon_{ij}]).$$

$$J_3 = \det(\epsilon_{ij})$$

$$\implies J_1 = M_{(1)} + M_{(2)} + M_{(3)}, \qquad J_2 = M_{(1)} M_{(2)} + M_{(2)} M_{(3)} + M_{(3)} M_{(1)}$$

$$J_3 = M_{(1)} M_{(2)} M_{(3)}$$

So we solve $M_{(1)}, M_{(2)}, M_{(3)}$ from ⑬, then $\underline{n}_{(1)}, \underline{n}_{(2)}, \underline{n}_{(3)}$ from

⑫ subject to $|\underline{n}| = 1$.

(Next: see p.13 for linear p-strain theory — a different presentation.)

<u>Small displacement gradient theory — Linearization</u>

Assume $\partial u_i / \partial x_j \ll 1$.

$$\implies \epsilon_{ij} \to \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \xrightarrow{\text{small (infinitesimal)}} \boxed{14}$$

Thus $du_i = u_{i,j} dx_j$ is small $\implies dx + dx^* \simeq 2dx$

$$\implies M = \frac{1}{2} \frac{(dx^*)^2 - (dx)^2}{(dx)^2} = \frac{(dx^* - dx)\overbrace{(dx + dx^*)}^{\simeq 2dx}}{2(dx)^2} \simeq \frac{dx^* - dx}{dx} = \epsilon_E$$

ie $M \simeq \epsilon_E$ (can see from putting $\epsilon_E \ll 1$ in ③ as well).

Now $M = \varepsilon_{ij} n_i n_j \simeq \epsilon_E \longrightarrow \boxed{15}$

<u>Geometric interpretations</u>

Put $\epsilon_{ij} \ll 1$, ie infinitesimal theromt (retain only linear terms in ϵ_{ij}).

From ⑤,⑥ $\to M_1 = \epsilon_{E1} \simeq \varepsilon_{11}$, $\|^{arly}$ for $\varepsilon_{22}, \varepsilon_{33}$ · $\longrightarrow$ $\Big\}$ ⑯

From ⑨ (put $\epsilon_{ij} \ll 1$ or $\epsilon_E \ll 1$) $\to 2\varepsilon_{23} \simeq \cos\theta$, $\|^{arly}$ for $\varepsilon_{12}, \varepsilon_{13}$. $\to$

<u>Note</u>: that $u_{i,j} \ll 1 \implies \varepsilon_{ij} \ll 1 \implies M \ll 1 \implies \epsilon_E \ll 1$

Note on Engg shear strains:

$$\gamma_{ij} \triangleq 2\epsilon_{ij} \text{ or } 2\mathcal{E}_{ij} \text{ for } i \neq j.$$

This is done to effect tensor transformations of tensorial shear strains (ie ϵ_{ij} or $\mathcal{E}_{ij}$). For linear theory (ie $u_{i,j} \ll 1$),

for eg, $\gamma_{23} = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} - \theta = \alpha$, $\text{also for } \gamma_{12}, \gamma_{13}$

since $\frac{\pi}{2} - \theta$ is small due to $u_{i,j} \ll 1$.

$\boxed{\alpha_1 + \alpha_2 = \alpha.}$
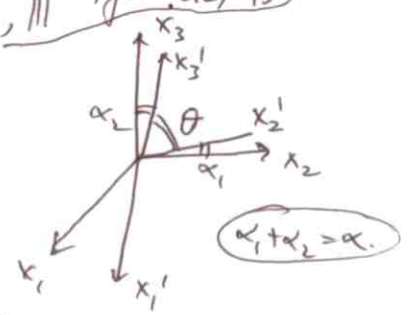
Other analogies with stress tensor (Nonlinear theory).

(i) Deviatoric strain: $\widehat{\mathcal{E}_{ij}} = \mathcal{E}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{E}_{mm}$
    deviatoric   total   spherical.

(ii) Pure shear state of strain: $\exists$ iff $\mathcal{E}_{ii} = 0$.
Physically it means that there exists three mutually $\perp^{ar}$ directions $\underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)}$ along which $\mathcal{E}_{E1}, \mathcal{E}_{E2}, \mathcal{E}_{E3}$ (ie engg ext strains) are zero, resply.

Infinitesimal Rotation & Analysis of Relative displ / Linear P-strain theory.
(to do after gneral nonlinear p-strain theory as a different treatment).

For $u_{i,j} \ll 1$ we have,

$$du_i = u_{i,j}\,dx_j = \frac{1}{2}\left[(u_{i,j} + u_{j,i}) + (u_{i,j} - u_{j,i})\right]dx_j = \left[\mathcal{E}_{ij} + \tilde{w}_{ij}\right]dx_j$$

$\underbrace{\phantom{(u_{i,j}+u_{j,i})}}_{\mathcal{E}_{ij}}$ $\underbrace{\phantom{(u_{i,j}-u_{j,i})}}_{\tilde{w}_{ij} \text{ (rot tensor)}}^{\text{linear}}$ $\boxed{17}$

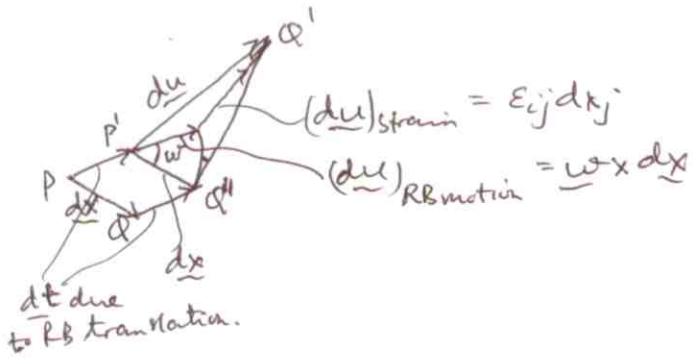RB motion $\Rightarrow \mathcal{E}_{ij} = 0 \Rightarrow du_i = \tilde{w}_{ij}\,dx_j$.

Note that by defn $\tilde{w}_{ij} = -\tilde{w}_{ji}$ (antisymm).

Now $du_i = \begin{bmatrix} 0 & \tilde{w}_{12} & \tilde{w}_{13} \\ -\tilde{w}_{12} & 0 & \tilde{w}_{23} \\ -\tilde{w}_{13} & -\tilde{w}_{23} & 0 \end{bmatrix}\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 & -\tilde{w}_{21} & \tilde{w}_{13} \\ \tilde{w}_{21} & 0 & -\tilde{w}_{32} \\ -\tilde{w}_{13} & \tilde{w}_{32} & 0 \end{bmatrix}\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$

$= \left[(-\tilde{w}_{21}dx_2 + \tilde{w}_{13}dx_3), (\tilde{w}_{21}dx_1 - \tilde{w}_{32}dx_3), (-\tilde{w}_{13}dx_1 + \tilde{w}_{32}dx_2)\right]^T$

$\longleftarrow$ $= \tilde{w} \times dx$ where $\tilde{w} \triangleq [\tilde{w}_{32}, \tilde{w}_{13}, \tilde{w}_{21}]^T = $ linear rot vector

Thus $w_{ij} dx_j = \underset{\sim}{\omega} \times \underline{dx}$ $\left(= \epsilon_{ijk} \omega_j dx_j \right)$ ← permutation symbol ⟶ not deform class

Thus $\underset{\sim}{\omega} \times \underline{dx}$ is the rotation component of du_i, the rel displ.

$(d\underline{u})_{strain} = \epsilon_{ij} dx_j$

$(d\underline{u})_{RB motion} = \underline{\omega} \times \underline{dx}$

dt' due to RB translation.

⟶ So relative motion = RBM + STRAINING.

So we can study their effects separately & superpose..

<u>Another explanation of p-strains for infinitesimal theory</u>

Consider du_i due to straining only.

⑲ ← $\mathcal{E}_i \triangleq \dfrac{du_i}{dx} = \dfrac{\epsilon_{ij} dx_j}{dx} = \epsilon_{ij} n_j =$ rel displ per unit original length.

In general $\mathcal{E}_i$ is neither directed along PQ or P'Q' (see above fig). In fact $\mathcal{E}_i$ is along the $(d\underline{u})_{strain}$ vector.

{ Now seek orientation n_i of element PQ for which the element stays perpendicular, to plane of particles (after deformation) to which it was originally $\perp^{a}$ before deformation. Since RBM has been excluded, this means that $\mathcal{E}_i$ directed along PQ, ie, along n_i. This does not mean that direction of PQ is invariant, since in general RBM is present. All it means is that filtering out the RBM you get direction of PQ as invariant. Thus,

another explanation

$$\mathcal{E}_i = \mathcal{E}^* n_i = \mathcal{E}^* \delta_{ij} n_j \longrightarrow ⑳$$
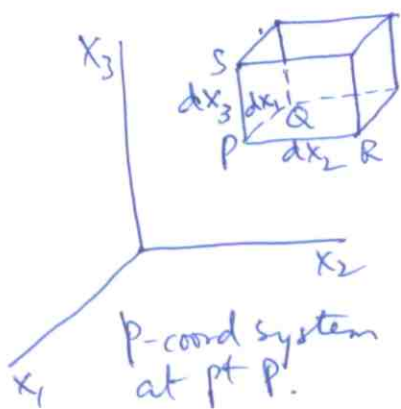
From ⑲ & ⑳,

$$(\mathcal{E}_{ij} - \mathcal{E}^* \delta_{ij}) n_j = 0 \longrightarrow \text{p-strain/axes evp.}$$

Volumetric Strain (Cubical Dilatation).

Its defined as

$$e \ (\text{or } \bar{\delta}) \triangleq \frac{dV^* - dV}{dV} \longrightarrow \boxed{21}$$

p-coord system at pt P.

Consider a rect. parallelopiped & p-coord system at pt. P. Thus sides stay $\parallel$ after deformation (since they mutually must remain $\perp$ as they lie along p-axes).

Thus,

$$dV^* = (dx_1^*)(dx_2^*)(dx_3^*) = (1 + \epsilon_{E_1})(1 + \epsilon_{E_2})(1 + \epsilon_{E_3})$$

$$\Rightarrow e(=\bar{\delta}) = \epsilon_{E_1} + \epsilon_{E_2} + \epsilon_{E_3} + \epsilon_{E_1}\epsilon_{E_2} + \epsilon_{E_2}\epsilon_{E_3}$$
$$+ \epsilon_{E_3}\epsilon_{E_1} + \epsilon_{E_1}\epsilon_{E_2}\epsilon_{E_3}$$

$$\boxed{\begin{array}{l} = \epsilon_{E_1} + \epsilon_{E_2} + \epsilon_{E_3} \quad (\text{linearized}) \\ = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii} \quad \}(\text{linearized}) \\ = \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \end{array}}$$

(only this for linear case)

An analogous magnification factor is defined as,

$$M_v = \frac{1}{2}\left(\left(\frac{dV^*}{dV}\right)^2 - 1\right) = e + \frac{e^2}{2} \longrightarrow \boxed{23} \longrightarrow M_v \simeq e \ (\text{linearized})$$

Also $M_1 = \frac{1}{2}\left(\frac{(dx_1^*)^2}{(dx_1)^2} - 1\right)$, $M_2 = \frac{1}{2}\left(\frac{(dx_2^*)^2}{(dx_2)^2} - 1\right)$, $M_3 = \frac{1}{2}\left(\frac{(dx_3^*)^2}{(dx_3)^2} - 1\right)$

where M_1, M_2, M_3 are principal strains (nonlinear case).

$$\Rightarrow M_v = \frac{1}{2}\left[(2M_1 + 1)(2M_2 + 1)(2M_3 + 1) - 1\right] = 4M_1 M_2 M_3 + 2(M_1 M_2 + M_2 M_3 + M_3 M_1) + M_1 + M_2 + M_3$$

$$= 4J_3 + 2J_2 + J_1 \longrightarrow \boxed{24}$$

Conservation of mass. (Lagrangian continuity eqn)

Consider above figure but x_i are not p-axes. Then deformed element is skewed, differential parallelopiped.

$$dV^* = dx_{PQ}^* \cdot (dx_{PR}^* \times dx_{PS}^*) = \text{vol of skewed parallelopiped}$$

(Triple product for vol of skewed ∥piped) (Since we can easily see that opp. faces and sides remain ∥, parallel.)

$$\Rightarrow dV^* = \left(\frac{\partial \underline{x}^*}{\partial x_1} dx_1\right)_{P'Q'} \cdot \left\{\left(\frac{\partial \underline{x}^*}{\partial x_2} dx_2\right)_{P'R'} \times \left(\frac{\partial \underline{x}^*}{\partial x_3} dx_3\right)_{P's'}\right\}$$

$$= \left(\frac{\partial \underline{x}^*}{\partial x_1}\right)_P \cdot \left(\frac{\partial \underline{x}^*}{\partial x_2} \times \frac{\partial \underline{x}^*}{\partial x_3}\right)_P dx_1 dx_2 dx_3$$

$$\underbrace{\phantom{\left(\frac{\partial \underline{x}^*}{\partial x_1}\right)_P \cdot \left(\frac{\partial \underline{x}^*}{\partial x_2} \times \frac{\partial \underline{x}^*}{\partial x_3}\right)_P}}_{J|_P = det\left[\frac{\partial x_i^*}{\partial x_j}\right]} = det\begin{bmatrix} \frac{\partial x_1^*}{\partial x_1} & \frac{\partial x_1^*}{\partial x_2} & \frac{\partial x_1^*}{\partial x_3} \\ x_{2,1}^* & x_{2,2}^* & x_{2,3}^* \\ x_{3,1}^* & x_{3,2}^* & x_{3,3}^* \end{bmatrix}$$

$$dV^* = J\,dV \longrightarrow \underline{25} \quad \text{(Algebraic form of Lagrangian continuity eqn)}.$$

Incompressible $\Rightarrow e (or \tilde{D}) = 0$ and $J = 1$. (ie $dV^* = dV$).

Thus, consv of mass $\Rightarrow dm^* = dm$

$$\rho^* dV^* = \rho\, dV$$

$$\rho^* J|_P = \rho$$

If $\rho = const = \rho^*$ (ie incompr) $\Rightarrow J = 1$.

Note that cubical dilatation (linearized) is zero for strain deviatoric tensor (ie $\hat{E}_{ii} = 0$) by its definition. So strain deviatoric part causes no volumetric strain but only distortion shear. Similarly the spherical part causes only volumetric strain.

③ Constitutive Equations. (Linear)

For 3-D anisotropic solid we have

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl} \longrightarrow \text{Linear 3-D Hookes law.}$$

From symmetry of stress and strain tensors you get C_{ijkl} comprises 36 indep coefficients (elastic constants).

(i.e, interchange $i \leftrightarrow j$, note $\sigma_{ij} = \sigma_{ji}$, get $C_{ijkl} = C_{jikl}$; then interchange $k \leftrightarrow l$, note $\varepsilon_{kl} = \varepsilon_{lk}$, get $C_{ijkl} = C_{ijlk}$).

Alternatively, noting symmetry of σ_{ij} and ε_{kl} you can use extended notation for stress and strain tensor,

i.e,

$$(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) \equiv (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$$

$$(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}) \equiv (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)$$

|| Note: that these are not to be confused with earlier notation used for stress vector components or extensional strains, respectively. They are reduced notation for stress & strain tensor components.

Hence the 3-D anisotropic const. law becomes,

$$\sigma_i = C_{ij}\, \varepsilon_j \quad , \quad i,j = 1, \cdots 6$$

$$\Rightarrow C_{ij} \text{ are 36 constants. (independent consts)}$$

Later on, it can be shown from strain energy considerations that the 36 constants reduce to 21 independent constants.

Isotropic Solid. (Linear).

Constitutive law in terms of Engg consts (E, ν)

E = Youngs modulus, ν = Poissons ratio.

$$\mathcal{E}_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}), \text{ similarly for } \mathcal{E}_{22}, \mathcal{E}_{33}$$

$$\mathcal{E}_{12} = \frac{1+\nu}{E}\sigma_{12}, \text{ similarly for } \mathcal{E}_{13}, \mathcal{E}_{23}.$$

These can be combined into

$$\boxed{\mathcal{E}_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}} \rightarrow \text{(I)}, \quad \begin{array}{l} i = 1,2,3 \\ j = 1,2,3 \end{array} \quad \begin{array}{l} \delta_{ij} = 0, i \neq j \\ \delta_{ij} = 1, i = j \end{array}$$

In terms of Lame's constants,

$$\boxed{\sigma_{ij} = 2\mu\,\mathcal{E}_{ij} + \lambda\,\mathcal{E}_{kk}\,\delta_{ij}} \rightarrow \text{(II)}, \quad \text{where } \mathcal{E}_{kk} = D = \text{linear cubical dilatation.}$$

$(\mu = \text{shear modulus.})$.

Inverting (I) (either do by writing all 6 eqns or use concept of contraction operation)

we get,

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\mathcal{E}_{ij} + \frac{\nu}{E}\delta_{ij}\sigma_{kk}\right]$$

To get σ_{kk} in terms of strains, do contraction operation in (I) (i.e., identify any two indices that are distinct, i, j in this case). You get,

$$\mathcal{E}_{ii} = \frac{1+\nu}{E}\sigma_{ii} - \frac{\nu}{E}\sigma_{kk}\overset{=3}{\delta_{ii}} = \frac{1-2\nu}{E}\sigma_{ii} \quad \left(\begin{array}{l} \because \sigma_{ii} \text{ same} \\ \text{as } \sigma_{kk} \end{array}\right).$$

ie both represent summation $\sigma_{11} + \sigma_{22} + \sigma_{33}$

Substitute as shown and get, (ie, $\sigma_{kk} = \frac{E}{1-2\nu}\mathcal{E}_{kk}$)

$$\boxed{\sigma_{ij} = \frac{E}{1+\nu}\left[\mathcal{E}_{ij} + \frac{\nu}{E}\frac{E}{1-2\nu}\delta_{ij}\mathcal{E}_{kk}\right]} \rightarrow \text{(I a)}$$

Similar inversion of Ⅱ gives,

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu}\delta_{ij}\,\varepsilon_{RR}$$

Now contraction (re $i=j$) in Ⅱ gives,

$$\sigma_{ii} = 2\mu\,\varepsilon_{ii} + \lambda\,\varepsilon_{RR}\,\delta_{ii}^{\;3} = 2\mu\,\varepsilon_{ii} + 3\lambda\,\varepsilon_{RR}$$

$$= (2\mu + 3\lambda)\,\varepsilon_{ii}$$

$$\qquad\qquad\qquad\qquad\text{(or }\varepsilon_{RR})$$

$$\Rightarrow \varepsilon_{RR} = \frac{\sigma_{RR}}{(2\mu + 3\lambda)}$$

Substitute ε_{RR} in ε_{ij} and get,

$$\boxed{\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu}\frac{1}{(2\mu+3\lambda)}\sigma_{RR}\,\delta_{ij}} \rightarrow \boxed{\text{Ⅱ}a}$$

Comparing Ⅰ & Ⅱa or Ⅰa & Ⅱ, get,

$$\boxed{\begin{array}{ll} E = \dfrac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}, & \nu = \dfrac{\lambda}{2(\lambda + \mu)} \\[4mm] \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}, & \mu = \dfrac{E}{2(1+\nu)} \end{array}}$$

$\rightarrow$ Relation between Engg consts (E, ν) & Lame's consts (λ, μ)

Bulk modulus.

Define P_m = mean pressure = $-\dfrac{\sigma_{RR}}{3} = -\dfrac{(2\mu+3\lambda)}{3}\varepsilon_{RR} = -\left(\tfrac{2}{3}\mu + \lambda\right)D$.

$$= -KD = -\frac{E}{3(1-2\nu)}D.$$

where, $K = \left(\tfrac{2}{3}\mu + \lambda\right)$ = bulk modulus. $= \dfrac{E}{3(1-2\nu)}$

Incompressibility $\Rightarrow \boxed{D = 0} \Rightarrow \boxed{K \to \infty}$ to give finite P_m.

and $K \to \infty \Rightarrow \boxed{\nu = \tfrac{1}{2}}$.

So incompressible solid has $\nu = \tfrac{1}{2}$, re; no cubical dilatation $(D=0)$.
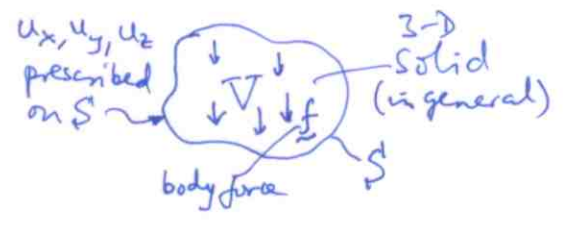
TYPES OF BOUNDARY VALUE PROBLEMS IN ELASTICITY.

(I) **Type I — Displacement formulation.**

Given : displacements, u_i, prescribed (ie, u_1, u_2, u_3) on boundary surface, S, and body forces f_i at all points. (x, y, z).

Find : u_i at all pts, ie $u_i [x, y, z]$ and $\sigma_{ij} [x, y, z]$ & $\varepsilon_{ij} (x, y, z)$.

Consider equilibrium equation,

$$\sigma_{ij,j} + f_i = 0$$

u_x, u_y, u_z prescribed on S. body force. 3-D solid (in general). S

Substitute constitutive law (Ia) or (II) (see pgs (7 (a)) in above. We get, (say using (Ia))

$$\frac{E}{1+\nu} \left[\varepsilon_{ij,j} + \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon_{kk,j} \right] + f_i = 0 .$$

Substitute strain-displ relations, $\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$ in above. We get, (also note that $\delta_{ij} \varepsilon_{kk,j} = \varepsilon_{kk,i}$)

$$\frac{E}{1+\nu} \left[\frac{1}{2} (u_{i,jj} + u_{j,ij}) + \frac{\nu}{1-2\nu} u_{k,ki} \right] + f_i = 0$$

Same as $u_{k,ik} = u_{k,ki}$

in E, ν form

$$\frac{E}{1+\nu} \left[\frac{1}{2} u_{i,jj} + \frac{1}{2(1-2\nu)} u_{j,ij} \right] + f_i = 0 .$$

$\longrightarrow$ Navier equation

(or) $$\frac{E}{2(1+\nu)} \left[\nabla^2 u_i + \frac{1}{1-2\nu} D_{,i} \right] + f_i = 0$$

($i = 1, 2, 3$, so three equations).

where $\nabla^2 =$ Laplacian operator, $D = $ linear cubical dilatation

These represent three coupled PDE's (partial differential equations) in the unknowns u_i (i.e. (u_x, u_y, u_z). They are to be solved along with the prescribe BC's on u_x, u_y, u_z (ie, PDE's & BC's should be satisfied by the solution obtained for u_x, u_y, u_z).

Alternatively you can use the constitutive equation Ⓘ on p.(17a), i.e., Lame's form, and proceed in the same manner, to get Navier's equation as

$$(\lambda + \mu)\, u_{j,ji} + \mu\, u_{i,jj} + f_i = 0$$ → Navier's eqn in λ, μ form.

i.e.,
$$(\lambda + \mu)\, D_{,i} + \mu \nabla^2 u_i + f_i = 0$$

$i = 1, 2, 3$. So three p.d.e's

Can also get this by substituting for (E, ν) in terms of (λ, μ) in the previous version of Navier's eqn given on p.(17d).

Note that to get Navier's equation we used equilibrium, constitutive, & strain displacement equations. So a solution to Navier's equation would satisfy these three sets of equations.

Q: What about satisfying compatibility.

A: Automatically satisfied if strain-displacement relations are explicitly used, as is the case in the Navier's equation (ie displacement formulation). You can check it by putting strain displ relations into the strain compatibility equations on p. 20, and seeing that the latter are identically satisfied.
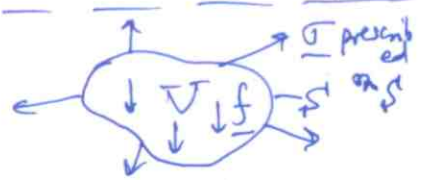
Ⓘ Type II — Stress Formulation

Given: Components of stress vector on boundary S' (ie, stress boundary conditions involving components of stress tensor on S), and body forces f_i (ie f_x, f_y, f_z) throughout V (volume — x, y, z).

<u>Find</u>: stress components σ_{ij} at all pts (x,y,z) in $\bar{V}$ and hence find e_{ij} and u_i (ie u_x, u_y, u_z) throughout $\bar{V}$.

In general, a set of stress components σ_{xx}, σ_{yy}, etc. satisfying equilibrium may not be correct unless compatibility is satisfied, since these stresses could yield strains that violate compatibility. So we use the six Beltrami − Mitchell compatibility equations, which represent 6 pde's (coupled) to be solved so as to satisfy the stress BC's prescribed. After obtaining stresses, obtain strains through constitutive law, and then displacements by integrating (numerically or analytically) the strain displacement equations. Single-valued displacements u_x, u_y, u_z are guaranteed to result from the integration of the six strain-displ relations, since compatibility was already satisfied (ie we used BM compat equations to get stress components).

(III) <u>Mixed Formulation</u>

Here $\underline{\sigma}$ is prescribed on boundary S_1, and $\underline{u}$ is prescribed on boundary S_2, where total boundary $S = S_1 \cup S_2$. We need to use all 4-sets of governing equations (ie, equilibrium, strain-displ, compatibility, constitutive) & do a mixed formulation.
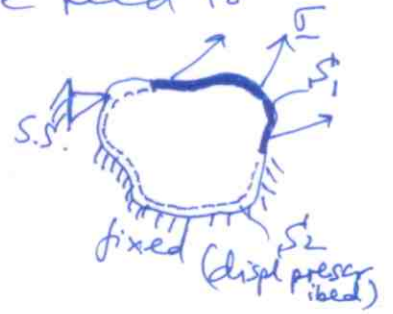
Strain Displacement Relations in orthogonal Curvilinear coords.

Results (see Boresi)

$$\varepsilon_{xx} = \frac{1}{\alpha}\left[u_{,x} + \frac{v}{\beta}\alpha_{,y} + \frac{w}{\gamma}\alpha_{,z}\right]$$

$$\varepsilon_{yy} = \frac{1}{\beta}\left[v_{,y} + \frac{w}{\gamma}\beta_{,z} + \frac{u}{\alpha}\beta_{,x}\right]$$

$$\varepsilon_{zz} = \frac{1}{\gamma}\left[w_{,z} + \frac{u}{\alpha}\gamma_{,x} + \frac{v}{\beta}\gamma_{,y}\right]$$

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{1}{\beta}u_{,y} + \frac{1}{\alpha}v_{,x} - \frac{v}{\alpha\beta}\beta_{,x} - \frac{u}{\alpha\beta}\alpha_{,y}\right]$$

$$\varepsilon_{xz} = \frac{1}{2}\left[\frac{1}{\gamma}u_{,z} + \frac{1}{\alpha}w_{,x} - \frac{w}{\alpha\gamma}\gamma_{,x} - \frac{u}{\alpha\gamma}\alpha_{,z}\right]$$

$$\varepsilon_{yz} = \frac{1}{2}\left[\frac{1}{\gamma}v_{,z} + \frac{1}{\beta}w_{,y} - \frac{w}{\beta\gamma}\gamma_{,y} - \frac{v}{\beta\gamma}\beta_{,z}\right]$$

where u, v, w are projections of displ vector onto the x, y, z coordinates at a point

Compatibility Equations. (Linear strain theory).

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \longrightarrow ①$$

If strains are given (or determined first) then to get displacements we need to solve 6 strain-displ equations to obtain the 3 displ components. Thus to get a unique solution, strains must (chosen or determined) satisfy compatibility equations. Another way of looking at it is that u_i should be single-valued, continuous, and unique (ie a particle before def cannot occupy two pts after def — no voids or tearing, and two particles before def cannot coalesce after def). So E_{ij} are not all indep and cannot be chosen arbitrarily.

Geometric explanation: medium composed of small cubes. Assign arbitrary strains. After def there should be no gaps or overlaps, ie deformed cubes must fit together.

Derivation: Consider two pts denoted (1), (2) in the medium. Then

$$u_i(2) = u_i(1) + \int_C du_i$$

where C is any path from 1 to 2 within a simply connected domain. Thus

$$u_i(2) = u_i(1) + \int_C \left(E_{ij} + \tilde{\omega}_{ij}\right) dx_j = u_i(1) + \int_C E_{ij} dx_j + \int_C \tilde{\omega}_{ij} \, d\left[x_j - x_j(2)\right]$$

Int by parts,

$$u_i(2) = u_i(1) + \int_C E_{ij} dx_j - \tilde{\omega}_{ij}\big[x_j(1) - x_j(2)\big] - \int_C \left(x_j - x_j(2)\right) \underbrace{\tilde{\omega}_{ij,R} \, dx_R}_{= d\tilde{\omega}_{ij}}$$

Now $\tilde{\omega}_{ij,R} = \frac{1}{2}\left(u_{i,jR} - u_{j,iR} + u_{R,ij} - u_{R,ij}\right) = \left(E_{iR,j} - E_{jR,i}\right)$

$$\Rightarrow u_i(2) = u_i(1) - \tilde{\omega}_{ij}(1)\big[x_j(1) - x_j(2)\big] + \int_C F_{iR} \, dx_R$$

where $F_{iR} \triangleq E_{iR} - \left(x_j - x_j(2)\right)\left(E_{iR,j} - E_{jR,i}\right)$

So for $u_i(z)$ to be single valued, $F_{ik} dx_k$ must be a perfect differential, ie,

$$F_{ik} dx_k = dG_i = G_{i,k} dx_k$$

$$\Rightarrow F_{ik} = G_{i,k} \Rightarrow \boxed{F_{ik,l} = F_{il,k}.}$$

$$\Rightarrow \cancel{\varepsilon_{ik,l}} - (x_j - x_j(z))(\varepsilon_{ik,jl} - \varepsilon_{jk,il}) - \cancel{(\varepsilon_{ik,l} - \varepsilon_{lk,i})}$$

$$= \cancel{\varepsilon_{il,k}} - (x_j - x_j(z))(\varepsilon_{il,jk} - \varepsilon_{jl,ik}) - \cancel{(\varepsilon_{il,k} - \varepsilon_{kl,i})}$$

$$\Rightarrow \varepsilon_{ik,jl} - \varepsilon_{jk,il} + \varepsilon_{jl,ik} - \varepsilon_{il,jk} = 0$$

$$\left(\because x_j - x_j(z) \text{ is arbitrary}\right).$$

$$k \rightleftharpoons j \Rightarrow \varepsilon_{ij,kl} - \varepsilon_{jk,il} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} = 0$$

$$i \rightleftharpoons j \text{ or } k \rightleftharpoons l \Rightarrow \boxed{\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{jl,ik} + \varepsilon_{ik,jl}} \longrightarrow ②$$

Only 6 of these 3^4 eqns are independent. The list is

	1st eq	2nd	3rd	4th	5th	6th
i	1	3	2	1	2	3
j	1	3	2	1	2	3
k	2	1	3	2	3	1
l	2	1	3	3	1	2

Writing in full form in classical notation, you get,

$$\boxed{\begin{aligned} &\varepsilon_{yy,xx} + \varepsilon_{xx,yy} = 2\varepsilon_{xy,xy} \qquad \text{and other two } (x \to y, y \to z, z \to x) \\ &\varepsilon_{zz,xy} + \varepsilon_{xy,zz} = \varepsilon_{yz,zx} + \varepsilon_{zx,yz} \qquad \text{and other two } (x \to y, y \to z, z \to x) \end{aligned}}$$

<u>Compat eqns in terms of stresses.</u> (Isotropic case).

We had Const law: $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{mm} \quad \left(\Rightarrow \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk}\right).$

$$\longrightarrow ③$$

From ② & ③,

$$(1+\nu)\left[\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik}\right] = \nu\left[\delta_{ij}\sigma_{mm,kl} + \delta_{kl}\sigma_{mm,ij} - \delta_{jl}\sigma_{mm,ik} - \delta_{ik}\sigma_{mm,jl}\right]$$

So form above you can use same table for indep eqns and obtain them (ie 6 indep eqns). However an easier and compact way is 'as follows:

Perform contraction $k = l$, to get,

$$(1+\nu)\left[\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik}\right] = \nu\left[\delta_{ij}\sigma_{mm,kk} + 3\sigma_{mm,ij} - \delta_{jk}\sigma_{mm,ik} - \delta_{ik}\sigma_{mm,jk}\right]$$

Subst equil eqn $\sigma_{ij,j} + B_i = 0$. in above, to get,

$$(1+\nu)\left[\sigma_{ij,kk} + \sigma_{kk,ij} + \tilde{B}_{i,j} + \tilde{B}_{j,i}\right] = \nu\left[\delta_{ij}\sigma_{mm,kk} + \sigma_{mm,ij}\right]$$

From here itself, due to symmetry of σ_{ij}, we see that there are only 6 indep eqns. However, going a step further, contracting $i = j$, solving for $\sigma_{mm,kk}$, you get,

$$\sigma_{mm,kk} = -\frac{1+\nu}{1-\nu}\tilde{B}_{i,i}$$

Subst this in above, yields,

$$\boxed{\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \tilde{B}_{i,j} + \tilde{B}_{j,i} + \frac{\nu}{1-\nu}\delta_{ij}\tilde{B}_{k,k} = 0}$$

$\hookrightarrow$ B.M. compat eqns.

PLANE STRESS.

(eg) Thin plate subject to, uniformly distributed in-plane loading
along edges. Further $\bar{B}_3 = 0$ assumed. Thus $\sigma_{i3} = 0$ on top &
bottom face, & due to thinness we assume $\sigma_{i3} = 0$ throrout and
$\frac{\partial}{\partial x_3} = 0$.

Equil: $\sigma_{11,1} + \sigma_{12,2} + \bar{B}_1 = 0 \rightarrow$ ①

$\sigma_{12,1} + \sigma_{22,2} + \bar{B}_2 = 0 \rightarrow$ ②   (3rd equil is 0=0).

BM compat:

① ← $i=1, j=1$: $\nabla^2 \sigma_{11} + \frac{1}{1+\nu}(\sigma_{11,11} + \sigma_{22,11}) + \bar{B}_{1,1} + \tilde{B}_{1,1} + \frac{\nu}{1-\nu}(\bar{B}_{1,1} + \bar{B}_{2,2}) = 0$

② ← $i=2, j=2$: $\nabla^2 \sigma_{22} + \frac{1}{1+\nu}(\sigma_{11,22} + \sigma_{22,22}) + \bar{B}_{2,2} + \tilde{B}_{2,2} + \frac{\nu}{1-\nu}(\bar{B}_{1,1} + \bar{B}_{2,2}) = 0$

③ ← $i=1, j=2$: $\nabla^2 \sigma_{12} + \frac{1}{1+\nu}(\sigma_{11,12} + \sigma_{22,12}) + \bar{B}_{1,2} + \tilde{B}_{2,1} = 0$

④ ← $i=3, j=3$: $\frac{\nu}{1-\nu}(\bar{B}_{1,1} + \bar{B}_{2,2}) = 0$

⑤,⑥ ← $i=1, j=3$ & $i=2, j=3$: $0=0$

⚹ $\begin{cases} \text{From } \frac{\partial ①}{\partial x_2} + \frac{\partial ②}{\partial x_1} = 0 \text{ we infer that ③ is violated unless } \nu = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(even if } \bar{B}_1 = \bar{B}_2 = \text{const)} \\ \text{Also ④ is violated unless } \bar{B}_1 = \bar{B}_2 = \text{const} \\ \rightarrow \text{Due to } \partial/\partial z = 0 \qquad \text{assumption.} \end{cases}$

① + ② $\Rightarrow$ $\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{2(1+\nu)}{(2+\nu)(1-\nu)}\overbrace{(\bar{B}_{1,1} + \bar{B}_{2,2})}^{\nabla \cdot \bar{B}} \longrightarrow$ ⑦

Strain compat: (eqs numbered according to table on p.19).

$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \rightarrow$ ①'

$\varepsilon_{33,11} = 0 \rightarrow$ ②' $0=0 \rightarrow$ ④'

$\varepsilon_{33,22} = 0 \rightarrow$ ③' $0=0 \rightarrow$ ⑤'

$\varepsilon_{33,12} = 0 \rightarrow$ ⑥'

where $\frac{\partial}{\partial x_3}$ and $\varepsilon_{13} = \varepsilon_{23} = 0$ have been used.

Const law is,

$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\sigma_{22}$, $\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}\sigma_{22}$, $\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$,

$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$

Insert const law in ①' :

$$①' \rightarrow \frac{1}{E}\left\{ (\sigma_{11} - \nu\sigma_{22})_{,22} + (\sigma_{22} - \nu\sigma_{11})_{,11} \right\} = \frac{2(1+\nu)}{E}\sigma_{12,12} \rightarrow (A)$$

Also $\dfrac{\partial ①}{\partial x_1} + \dfrac{\partial ②}{\partial x_2} \longrightarrow$ $\sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12} + \tilde{B}_{1,1} + \tilde{B}_{2,2} = 0 \longrightarrow (B)$

Eliminate $\sigma_{12,12}$ from (A), (B) $\longrightarrow$ $\boxed{\nabla^2(\sigma_{11} + \sigma_{22}) = -(1+\nu)\underline{\nabla}\cdot\underline{\tilde{B}}} \longrightarrow ⑦'$

III^{ly} Insert CL in ②', ③', ⑥' :

$$②' \rightarrow -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})_{,11} = 0 \rightarrow (C)$$

$$③' \rightarrow -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})_{,22} = 0 \rightarrow (D)$$

$$⑥' \rightarrow -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})_{,12} = 0 \rightarrow (E)$$

$-\overset{.}{X}.$ For (C), (D), (E) to be satisfied $\rightarrow (\sigma_{11} + \sigma_{22}) = ax_1 + bx_2 + c$

$$\qquad\qquad\qquad = -\frac{\nu}{E}\varepsilon_{33}$$

This places a severe restriction on σ_{11}, σ_{22} (hence ε_{33}), and hence the remaining strain compat eqns (ie, ②', ③', ⑥') are not satisfiable in general. Again this arises due to $\dfrac{\partial}{\partial x_3} = 0$ assumption.

Note that ⑦ & ⑦' defer by a factor. This is due to fact that BM compat & strain compat do not have one-to-one correspondence, and since some equations in each case are non satisfiable, we cannot expect the remaining eqn (which we intend to satisfy) to be identical by the two approaches. The restrictions placed on the solution (or problem data) when attempting to satisfy ③, ④ (ie $\nu = 0$ or $\tilde{B}_1$ & $\tilde{B}_2$ being constant) are very severe as compared to those placed when attempting to satisfy (C), (D), (E) (ie $\sigma_{11} + \sigma_{22} = $ linear fn of x_1, x_2). So we choose strain compat approach for plane stress & attempt to satisfy ⑦' only (as in Timoshenko etc.).

PLANE STRAIN.

(eg) Long prismatic body loaded $\perp^{ar}$ to longitudinal axis by forces that don't vary along longitudinal coordinate. Assume ends restrained from longitudinal motion. Thus $u_3 = 0$ at ends, and by symmetry $u_3 = 0$ throut (first infer $u_3 = 0$ at center and extend symmetry argument). Further $\frac{\partial}{\partial x_3} = 0$ due to loading. (Example is a dam or retaining wall.) Thus $\varepsilon_{i3} = 0$.

Thus $\sigma_{13} = \sigma_{23} = 0$ here also and 3rd equil eqn is $0 = 0$ and other two are ①, ⑪ p. 21 (as in plane stress)

Strain compat

②² − ②⁶ p. 20 identically satisfied.

CL: $\varepsilon_{33} = 0 \Rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \Rightarrow \varepsilon_{11} = \frac{1-\nu^2}{E}\left(\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right)$

$\varepsilon_{22} = \frac{1-\nu^2}{E}\left(\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right)$

Insert CL in ②′ p.21 ⇒.

$\frac{1-\nu^2}{E}\left[\left(\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right)_{,22} + \left(\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right)_{,11}\right] = 2\frac{1+\nu}{E}\sigma_{12,12}$

Eliminate $\sigma_{12,12}$ using equil, ie, (B) p.22 (ie $\frac{\partial ①}{\partial x_1} + \frac{\partial ⑪}{\partial x_2} = 0$) ⇒

$$\boxed{\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{1}{1-\nu}\nabla \cdot \underline{\tilde{B}}} \longrightarrow ①$$

Note that all other compat eqns are satisfied identically. So we expect the BM compat approach to yield same result as ① above. (see following page).

Let $\phi(x_1, x_2)$, $\psi(x_1, x_2)$ be such that

$\sigma_{11} = \phi_{,22} - \psi$, $\sigma_{22} = \phi_{,11}$, $\sigma_{12} = \phi_{,12} - \psi \longrightarrow$ ①*

$\bar{B}_1 = \psi_{,1}$, $\bar{B}_2 = \psi_{,2}$ $\longrightarrow$ ⑪*

①* ⇒ Equil eqns ① & ⑪ p. 21 identically satisfied.

①* in ① ⇒ $\boxed{\nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2\psi}$

Discussion on validity of plane stress solution

$\frac{\partial}{\partial x_3} = 0$ is the cause of non-satisfaction of compat eqn which causes a severe restriction to be posed on ε_{33} (hence $\sigma_{11} + \sigma_{22}$) as discussed in p. 23. So we relax this assumption and seek exact 3-D elasticity solutions for which $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ given $\bar{B}_3 = 0$, $\bar{B}_1$ & $\bar{B}_2$ are constant w.r.t (x_1, x_2).

$\bar{B}_1, \bar{B}_2$ const in (x, y) plane $\Rightarrow \psi [x_1, x_2] = p x_1 + q x_2 + \boxed{\alpha [x_3]}$

Equil $\rightarrow$

$\sigma_{11,1} + \sigma_{12,2} + \psi_{,1} = 0$ — (1a)

$\sigma_{12,1} + \sigma_{22,2} + \psi_{,2} = 0$ — (1b)

$0 = 0$ — (1c)

inconsequential. Can drop w/o loss of generality, as you can see in the following & the final soln for stresses.

BM compat $\rightarrow$

$(1+\nu)\nabla^2 \sigma_{11} + \theta_{,11} = 0$ — (2a)

$(1+\nu)\nabla^2 \sigma_{22} + \theta_{,22} = 0$ — (2b)

$(1+\nu)\nabla^2 \sigma_{33}^{\,0} + \theta_{,33} = 0$ — (2c)

$(1+\nu)\nabla^2 \sigma_{12} + \theta_{,12} = 0$ — (2f)

$(1+\nu)\nabla^2 \sigma_{13}^{\,0} + \theta_{,13} = 0$ — (2e)

$(1+\nu)\nabla^2 \sigma_{23}^{\,0} + \theta_{,23} = 0$ — (2d)

where $\theta = \sigma_{11} + \sigma_{22} + \sigma_{33}$

$2(c, e, d) \Rightarrow \theta_{,3} = k$ (const) $\Rightarrow \theta = k x_3 + \theta_0 [x_1, x_2] \longrightarrow$ (3)

Satisfaction of (1) $\Rightarrow$

$\sigma_{11} = \phi_{,22} - \psi \rightarrow$ (4a)

$\sigma_{22} = \phi_{,11} - \psi \rightarrow$ (4b)   where $\psi = \psi [x_1, x_2, x_3]$

$\sigma_{12} = -\phi_{,12} \rightarrow$ (4c)

(2a)+(2b)+(2c)=0 $\Rightarrow \nabla^2 \theta = 0 \rightarrow$ (5) $\left(\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$

$\Rightarrow \nabla_1^2 \theta_0 = 0 \rightarrow$ (6) $\left(\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$

(4a)+(4b) $\Rightarrow \nabla_1^2 \phi = \theta + 2\psi = k x_3 + \theta_0 + 2\psi \rightarrow$ (7).

(3), (4a) in (2a) $\Rightarrow (1+\nu)\nabla^2 \phi_{,22} + \theta_{0,11} = 0 \rightarrow$ (8)

Now $\nabla^2 \phi_{,22} = (\nabla^2 \phi)_{,22} = (\nabla_1^2 \phi + \phi_{,33})_{,22} \underset{\underset{\text{use (7)}}{\uparrow}}{=} (k x_3 + \theta_0 + 2\psi + \phi_{,33})$

$\Rightarrow \nabla^2 \phi_{,22} = (\theta_0 + \phi_{,33})_{,22} \rightarrow$ (9)

⑨ in ⑧ ⟹ $\nabla^2 \phi_{,22} + \frac{\theta_{0,11}}{1+\nu} = \left(\phi_{,33} + \frac{\nu}{1+\nu}\theta_0\right)_{,22} = 0 \longrightarrow$ ⑩a
using ⑥

III^{dy} ③, ④b in ②b & using ⑦ ⟹ $\left(\phi_{,33} + \frac{\nu}{1+\nu}\theta_0\right)_{,11} = 0 \longrightarrow$ ⑩b

③, ④c in ②f & using ⑦ ⟹ $\left(\phi_{,33} + \frac{\nu}{1+\nu}\theta_0\right)_{,12} = 0 \longrightarrow$ ⑩c

So ⑩ is reduced BM compat eqns (since they represent 2a, 2b, 2f with 2c, 2d, 2e satisfied vide ③).

⑩ ⟹ $\phi_{,33} + \frac{\nu}{1+\nu}\theta_0 = a[x_3]x_1 + b[x_3]x_2 + C[x_3]$

⟶ (integrate) (twice wrt x_3) $\phi = A[x_3]x_1 + B[x_3]x_2 + C[x_3] - \frac{\nu}{1+\nu}\theta_0\frac{x_3^2}{2} + \phi_1[x_1,x_2]x_3$
$+ \phi_0[x_1,x_2]$

↳ ⑪

Now we can set $Ax_1 + Bx_2 + C = 0$ since it wont affect stress distribution obtained via ④ ∴ Airy stress fn non-unique upto linear terms in x_1, x_2.
Further, considering loading as symmetric about midplane implies stress is symm about midplane ⟹ $\phi_1 = 0$ in ⑪ and $k = 0$ in ③.

Thus, ⑪ ⟹ $\phi = -\frac{1}{2}\frac{\nu}{1+\nu}\theta_0 x_3^2 + \phi_0 \longrightarrow$ ⑪a

⑪ in ⑦ using ⑤, ⑥ ⟹ $\nabla_1^2 \phi = \nabla_1^2 \phi_0 = \theta_0 + 2\psi \longrightarrow$ ⑫

⟹ $\nabla_1^4 \phi_0 = 0 \longrightarrow$ ⑬

Algorithm: Solve ⑬ for ϕ_0 ⟶ get θ_0 from ⑫ ⟶ ϕ from ⑪a
⟶ $\sigma_{11}, \sigma_{22}, \sigma_{12}$ from ④

Thus stresses consists of two parts, first one due to ϕ_0 (obt from $\nabla_1^4 \phi_0 = 0$) and second one varying quadratically thru thickness. So if thickness is small, $\phi \approx \phi_0$ and $\nabla^4\phi = 0$ gives good enough approximation (ie all BM compat not satisfied but good enough approx) for thin plates with $\bar{B}_3 = 0$, $\bar{B}_1, \bar{B}_2$ linear wrt (x_1, x_2), and loading symm about $x_3 = 0$. If not thin plate then use above algorithm and expect quadratic variation thru thickness for the stresses. Note that if edge loads are symm wrt midplane, but not necessarily quadratic, the solution predicts quadratic variation in interior & at edges. This is acceptable by virtue of Saint Venant's Principle.

TWO DIMENSIONAL PROBLEMS — CARTESIAN COORDINATES.

POLYNOMIAL SOLUTIONS.

Consider thin plate, i.e., plane stress w/o body forces (for convenience — w/o loss of generality).

Elementary solutions:-

(i) $\phi_1 = a_1 x + b_1 y \longrightarrow \sigma_{xx} = \sigma_{yy} = 0 = \sigma_{xy}. \Rightarrow \nabla^4 \phi \equiv 0.$

If const b.f., e.g. gravity, $\psi = px + qy = -\rho g y$, $\sigma_{xx} = \sigma_{yy} = -\psi$.

(ii) $\phi_2 = a_2 \dfrac{x^2}{2} + b_2 xy + \dfrac{c_2}{2} y^2 , \Rightarrow \nabla^4 \phi \equiv 0.$

$\boxed{\sigma_{xx} = c_2, \quad \sigma_{yy} = a_2, \quad \sigma_{xy} = -b_2}$ $\longrightarrow$ Constant stress.

(iii) $\phi_3 = \dfrac{a_3}{6} x^3 + \dfrac{b_3}{2} x^2 y + \dfrac{c_3}{2} x y^2 + \dfrac{d_3}{6} y^3 \Rightarrow \nabla^4 \phi \equiv 0$

$\boxed{\sigma_{xx} = c_3 x + d_3 y, \quad \sigma_{yy} = a_3 x + b_3 y, \quad \sigma_{xy} = -b_3 x - c_3 y}$ $\longrightarrow$ linearly varying stresses

(iv) $\phi_4 = \dfrac{a_4}{12} x^4 + \dfrac{b_4}{6} x^3 y + \dfrac{c_4}{2} x^2 y^2 + \dfrac{d_4}{6} x y^3 + \dfrac{e_4}{12} y^4$

For $\nabla^4 \phi = 0 \Rightarrow e_4 = -(2 c_4 + a_4) \leftarrow$ i.e., coeff of $x^0 y^0$ is zero

$\boxed{\begin{aligned} \sigma_{xx} &= c_4 x^2 + d_4 xy - (2 c_4 + a_4) y^2 \\ \sigma_{yy} &= a_4 x^2 + b_4 xy + c_4 y^2 \\ \sigma_{xy} &= -\dfrac{b_4}{2} x^2 - 2 c_4 xy - \dfrac{d_4}{2} y^2 \end{aligned}}$ $\leftarrow$ quadratic variation.

(eg) only $d_4 \neq 0 \rightarrow \sigma_{xx} = d_4 xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -\dfrac{d_4}{2} y^2$

parabolic
const.

(v) $\phi_5 = \dfrac{a_5}{20} x^5 + \dfrac{b_5}{12} x^4 y + \dfrac{c_5}{6} x^3 y^2 + \dfrac{d_5}{6} x^2 y^3 + \dfrac{e_5}{12} x y^4 + \dfrac{f_5}{20} y^5$

$\nabla^4 \phi = 0 \Rightarrow e_5 = -(2 c_5 + 3 a_5) \leftarrow$ setting x^1 coeff in $\nabla^4 \phi$ to zero

$\qquad\qquad\quad f_5 = -\dfrac{1}{3}(b_5 + 2 d_5) \qquad \leftarrow$ " y^1 " " $\nabla^4 \phi$ to zero.

$\Rightarrow \boxed{\begin{aligned} \sigma_{xx} &= \dfrac{e_5}{3} x^3 + d_5 x^2 y - (2 c_5 + 3 a_5) x y^2 - \dfrac{1}{3}(b_5 + 2 d_5) y^3 \\ \sigma_{yy} &= a_5 x^3 + b_5 x^2 y + c_5 x y^2 + \dfrac{d_5}{3} y^3 \\ \sigma_{xy} &= -\dfrac{1}{3} b_5 x^3 - c_5 x^2 y - d_5 x y^2 + \dfrac{1}{3}(2 c_5 + 3 a_5) y^3 \end{aligned}}$

Cubic variation.

Similarly for ϕ_6 and higher order polynomials.

Although $\nabla^4 \phi = 0$ for plane stress as well a plane strain problems, in the absence of body forces, the displacements wont be same since constitutive law is not same, ie strains are not same (due to presence of σ_{zz} in plane strain).

(Eg) <u>Tip Loaded Cantilever.</u>
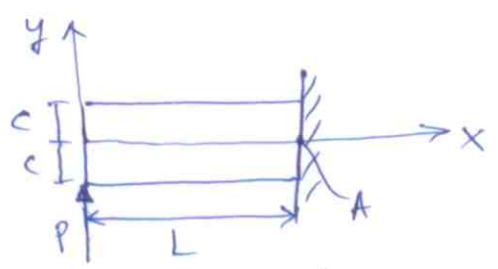
$$t=1, \quad t \ll L, \quad t \ll c, \quad I = \frac{2c^3}{3}$$

$$\phi = \phi_2 + \phi_3 + \phi_4$$

<u>BC's</u>

$\sigma_{xx}\Big|_{x=0} = 0 \Rightarrow c_2 = 0 \longrightarrow ①$

(ie set y^0, y, y^2 coeff to zero)

$\qquad d_3 = 0 \longrightarrow ②$

$\qquad 2c_4 + a_4 = 0 \longrightarrow ③ \xrightarrow{w/⑥} (c_4 = 0)$

$\sigma_{yy}\Big|_{y=\pm c} = 0 \Rightarrow a_2 \pm b_3 c + c_4 c^2 = 0 \rightarrow ④ \xrightarrow{w/⑧} (a_2 = 0) \Longleftarrow$

(set x^0, x, x^2 coeff to zero) $\qquad a_3 \pm b_4 c = 0 \longrightarrow ⑤ \rightarrow (a_3 = 0, b_4 = 0)$

$\qquad a_4 = 0 \longrightarrow ⑥$

$\sigma_{xy}\Big|_{y=\pm c} = 0 \Rightarrow -b_2 \mp c_3 c - \frac{d_4}{2} c^2 = 0 \rightarrow ⑦ \rightarrow c_3 = 0$

(set x^0, x, x^2 coeff to zero) $\quad -b_3 \mp 2c_4 c = 0 \longrightarrow ⑧ \rightarrow (b_3 = 0, c_4 = 0)$

$\qquad b_4 = 0 \longrightarrow ⑨$

$\sigma_{xy}\Big|_{x=0} = 0 \Rightarrow \begin{array}{l} b_2 = 0 \longrightarrow Ⓐ \\ c_3 = 0 \longrightarrow Ⓑ \\ d_4 = 0 \longrightarrow Ⓒ \end{array} \Big\}$ cant satisfy (see below).

(set coeff y^0, y, y^2 to zero)

Ⓐ-Ⓒ, ①-⑨ $\Rightarrow a_2 = c_2 = a_3 = b_3 = c_3 = d_3 = a_4 = b_4 = c_4 = 0$

So if you satisfy Ⓐ, Ⓒ then all coeffs. vanish & $\phi = 0$ ie, cant satisfy Ⓐ, Ⓑ.

So satisfy $\displaystyle\int_{-c}^{c} \sigma_{xy}\Big|_{x=0} dy = -P \Rightarrow 2\left[-b_2 c - \frac{d_4}{6} c^3\right] = -P \longrightarrow ⑩$

From ⑦ (w/$c_3=0$) and ⑩ $\rightarrow b_2 = \frac{3P}{4c}, \quad d_4 = -\frac{3P}{2c^3}$

$\Rightarrow \sigma_{xx} = -\frac{3}{2}\frac{P}{c^3}xy = \frac{P}{I}xy \; ; \; \sigma_{yy} = 0 \; ; \; \sigma_{xy} = -\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right) = -\frac{P}{I}\frac{1}{2}(c^2-y^2)$

So solution is exact if P applied via parabolic σ_{xy} distribution — | otherwise St. Venant's effect.

$\Rightarrow \varepsilon_{xx} = u_{,x} = \frac{\sigma_{xx}}{E} = -\frac{P}{EI}xy \longrightarrow$ (A)

$\varepsilon_{yy} = V_{,y} = -\frac{\nu\sigma_{xx}}{E} = \frac{\nu P}{EI}xy \longrightarrow$ (B)

$\varepsilon_{xy} = \frac{1}{2}u_{,x} + V_{,y} = \frac{1+\nu}{E}\sigma_{xy} = -\frac{P}{4IG}(c^2-y^2) \longrightarrow$ (C).

(A) $\longrightarrow u = -\frac{P}{2EI}x^2y + f(y) \longrightarrow$ (D)

(B) $\longrightarrow V = \frac{\nu P}{2EI}xy^2 + g(x) \longrightarrow$ (E)

(D), (E) in (C) $\longrightarrow g' - \frac{P}{2EI}x^2 = k_1 \text{ (const)} \longrightarrow$ (F)

$f' + \frac{\nu P}{2EI}y^2 - \frac{P}{2IG}y^2 = k_2 \text{ (const)} \longrightarrow$ (G)

where, $k_1 + k_2 = -\frac{Pc^2}{2IG} \longrightarrow$ (H)

(D, E, F, G) $\longrightarrow u = -\frac{P}{2EI}x^2y - \frac{\nu P}{6EI}y^3 + \frac{P}{6IG}y^3 + k_2 y + k_3 \longrightarrow$ (I)

$V = \frac{\nu P}{2EI}xy^2 + \frac{Px^3}{6EI} + k_1 x + k_4 \longrightarrow$ (J).

For k_1, k_2, k_3, k_4 set RB motion to zero. Thus, for no RB translation,

$u\Big|_{\substack{y=0\\x=L}} = V\Big|_{\substack{y=0\\x=L}} = 0 \longrightarrow k_3 = 0, \quad k_4 = -\frac{PL^3}{6EI} - k_1 L$
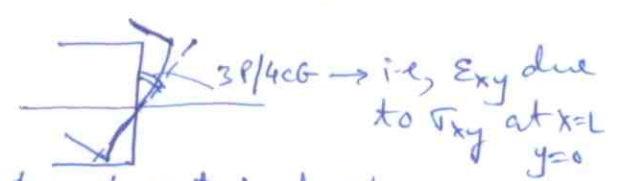
For no RB rotation, we have several possibilities, as follows:

(i) Longitudinal-axis element fixed at A.

$\Rightarrow V_{,x}\Big|_{\substack{x=L\\y=0}} = 0 \; \overset{(H)}{\Longrightarrow} \; k_1 = -\frac{PL^2}{2EI} \longrightarrow k_2 = \frac{PL^2}{2EI} - \frac{Pc^2}{2IG}$

$\Rightarrow V\Big|_{y=0} = \frac{Px^3}{6EI} - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} \quad \longleftarrow$ matches elementary beam theory at $x=0$.

$\Rightarrow u_{,y}\Big|_{\substack{y=0\\x=L}} = -\frac{3}{4}\frac{P}{cG} \longrightarrow$

$3P/4cG \rightarrow$ i.e, ε_{xy} due to σ_{xy} at $x=L$ $y=0$

Distortion of section at $x=L$

(ii) Vertical-axis element fixed at A

$$\Rightarrow \ u_{,y}\Big|_{\substack{x=L \\ y=0}} = 0 \ \overset{\text{H}}{\longrightarrow} \ k_2 = \frac{PL^2}{2EI} \ \longrightarrow \ k_1 = -\frac{PL^2}{2EI} - \frac{Pc^2}{2IG}$$

$$v\Big|_{\substack{y=0 \\ x=L}} = \frac{P}{6EI}x^3 - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} + \frac{Pc^2}{2IG}(L-x)$$

$$\underbrace{\phantom{\frac{Pc^2}{2IG}(L-x)}}_{\substack{\text{due to rotation of } \mathcal{L} \\ \text{ie effect of } \sigma_{xy}.}}$$

$$v_{,x}\Big|_{\substack{y=0 \\ x=L}} = -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{CG}$$

$\uparrow \frac{3P}{4cG}$ Distortion at x=L
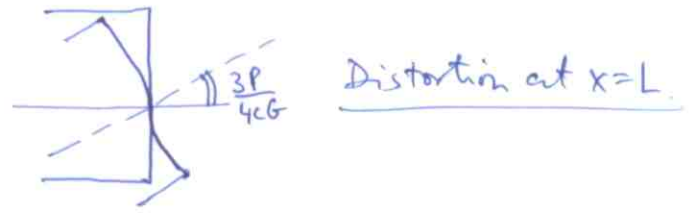
(iii) $\displaystyle v_{,x} - u_{,y}\Big|_{\substack{x=L \\ y=0}} = 0 \ \Rightarrow \ \frac{PL^2}{2EI} + k_1 + \frac{PL^2}{2EI} - k_2 = 0$

$$\Rightarrow(\text{using H}) \ \begin{array}{c} k_1 \\ k_2 \end{array}\!\bigg\} = \mp\frac{PL^2}{2EI} - \frac{Pc^2}{4IG}$$

$$v\Big|_{y=0} = \frac{Px^3}{6EI} - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI} + \frac{Pc^2}{4IG}(L-x).$$

$$u\Big|_{x=L} = -\frac{Pc^2}{4IG}y + \frac{P}{6IG}\left(\frac{2+\nu}{2(1+\nu)}\right)y^3.$$

(Eg) Bending of beam by uniform load.
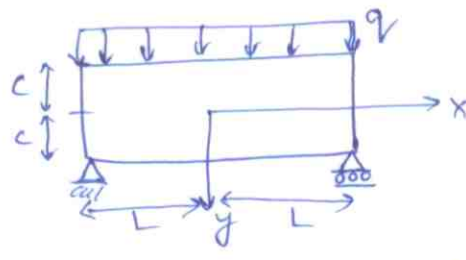
$$\phi = \phi_2 + \phi_3 + \phi_4 + \phi_5$$

$$\sigma_{xx} = c_2 + c_3 x + d_3 y + c_4 x^2 + d_4 xy - (2c_4 + a_4)y^2 + \frac{c_5}{3}x^3$$
$$+ d_5 x^2 y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$

$$\sigma_{yy} = a_2 + a_3 x + b_3 y + a_4 x^2 + b_4 xy + c_4 y^2 + a_5 x^3 + b_5 x^2 y$$
$$+ c_5 xy^2 + \frac{d_5}{3} y^3$$

$$\sigma_{xy} = -b_2 - b_3 x - c_3 y - \frac{b_4}{2}x^2 - 2c_4 xy - \frac{d_4}{2}y^2 - \frac{1}{3}b_5 x^3$$
$$- c_5 x^2 y - d_5 xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

BC's

$$\sigma_{yy}\big|_{y=-c} = -q$$

(set x^0, x^1, x^2, x^3 coeffs to zero) $\Rightarrow$

$$a_2 - b_3 c + c_4 c^2 - \frac{d_5}{3}c^3 = -q \longrightarrow ①$$
$$a_3 - b_4 c + c_5 c^2 = 0 \longrightarrow ②$$
$$a_4 - b_5 c = 0 \longrightarrow ③$$
$$a_5 = 0 \longrightarrow ④$$

$$\sigma_{yy}\big|_{y=c} = 0$$

(set x^0, x^1, x^2, x^3 coeffs to zero) $\Rightarrow$

$$a_2 + b_3 c + c_4 c^2 + \frac{d_5}{3}c^3 = 0 \longrightarrow ⑤$$
$$a_3 + b_4 c + c_5 c^2 = 0 \longrightarrow ⑥$$
$$a_4 + b_5 c = 0 \longrightarrow ⑦$$

$$\int_{-c}^{c} \sigma_{xy}\big|_{x=\pm L} \, dy = \mp qL$$

(set $x^0 y^0$ coeff to zero, odd & even power of x terms separate out)

$$\Rightarrow 2\left(-b_2 c - \frac{b_4}{2}L^2 c - \frac{d_4}{6}c^3\right) = 0 \longrightarrow ⑧$$
$$2\left(-b_3 Lc - \frac{1}{3}b_5 L^3 c - \frac{d_5}{3}Lc^3\right) = -qL \longrightarrow ⑨$$

$$\sigma_{xx}\big|_{x=\pm L} = 0$$

(set y^0, y^1, y^2, y^3 coeffs to zero)

$$\Rightarrow c_2 \pm c_3 L + c_4 L^2 \pm \frac{c_5}{3}L^3 = 0 \longrightarrow ⑩$$

$$d_3 \pm d_4 L + d_5 L^2 = 0 \longrightarrow \text{⑪}$$

$$-(2c_4 + a_4) \mp (2c_5 + 3a_5)L = 0 \longrightarrow \text{⑫}$$

$$b_5 + 2d_5 = 0 \longrightarrow \text{⑬}$$

$$\sigma_{xy}\big|_{y=\pm c} = 0$$

(set x^0, x^1, x^2, x^3 ⟹
coeff to zero)

$$-b_2 - \frac{d_4}{2}c^2 = 0 \qquad \longrightarrow \text{⑭}$$

$$-b_3 - d_5 c^2 = 0 \qquad \longrightarrow \text{⑮}$$

$$-\frac{b_4}{2} = 0 \qquad \longrightarrow \text{⑯}$$

$$-\frac{b_5}{3} = 0 \qquad \longrightarrow \text{⑰}$$

$$-c_3 c + \frac{1}{3}(2c_5 + 3a_5)c^3 = 0 \longrightarrow \text{⑱}$$

$$-2c_4 c = 0 \qquad \longrightarrow \text{⑲}$$

$$-c_5 c = 0. \qquad \longrightarrow \text{⑳}$$

solution of ① - ⑳:

④ → $d_5 = 0$.

②, ⑥ → $b_4 = 0$

③, ⑦ → $a_4 = 0, b_5 = 0$

⑪ → $d_4 = 0$

④, ⑫ → $c_5 = 0$

②, ⑥, $c_5 = 0 \rightarrow a_3 = 0$

⑧, $b_4 = 0, d_4 = 0 \rightarrow b_2 = 0$

⑩, $c_5 = 0 \rightarrow c_3 = 0$

⑬, $b_5 = 0 \rightarrow d_5 = 0$

⑫, $a_4 = 0 \rightarrow c_4 = 0$

⑪, $d_5 = 0 \rightarrow d_3 = 0$

⑮, $d_5 = 0 \rightarrow b_3 = 0$.

⑤, $b_3 = 0, c_4 = 0, d_5 = 0 \rightarrow a_2 = 0$

⑩, $c_4 = 0 \rightarrow c_2 = 0$.

$$\Rightarrow \phi = 0.$$
Trivial.

So satisfying $\sigma_{xx}|_{x=\pm L} = 0$ yields $\phi = 0$ and also that ①, ⑨ cannot be satisfied. Hence we discard $\sigma_{xx}|_{x=\pm L} = 0$ in favor of

$$\int_{-c}^{c} \sigma_{xx} \, dy \Big|_{x=\pm L} = 0$$

(coeff $x^0 y^0$ to zero, odd & even powers of x separate out)

$$\Rightarrow \quad C_2 c + C_4 L^2 c - \frac{2}{3}(C_4 + a_4) c^3 = 0 \quad \longrightarrow ㉑$$

$$C_2 L c + \frac{C_5}{3} L^3 c - (2C_5 + 3a_5) \frac{L c^3}{3} = 0 \quad \longrightarrow ㉒$$

and $$\int_{-c}^{c} \sigma_{xx} |_{x=\pm L} y \, dy = 0$$

(same here)

$$\Rightarrow \quad d_3 \frac{c^3}{3} + d_5 L^2 \frac{c^3}{3} - \frac{1}{3}(b_5 + 2d_5)\frac{c^5}{5} = 0 \longrightarrow ㉓$$

$$d_4 L \frac{c^3}{3} = 0 \quad \longrightarrow ㉔$$

So discard ⑩ — ⑬, solve ①-⑨ & ⑭ — ㉔, get,

$$b_2 = C_2 = C_3 = a_3 = b_4 = C_4 = d_4 = a_4 = b_5 = C_5 = a_5 = 0$$

and,

⑤ — ① $\rightarrow 2\left[b_3 c + \frac{d_5}{3}c^3\right] = q \quad \longrightarrow$ (a)

⑤ + ①, $C_4 = 0 \rightarrow 2a_2 = -q \quad \longrightarrow$ (b)

⑮ $\rightarrow \quad b_3 + d_5 c^2 = 0 \quad \longrightarrow$ (c)

㉓, $b_5 = 0 \rightarrow \quad d_3 + d_5 L^2 - 2d_5 \frac{c^2}{5} = 0 \rightarrow$ (d)

Solution of (a) — (d) gives,

$$a_2 = -\frac{q}{2}, \quad d_5 = -\frac{3q}{4c^3}, \quad b_3 = \frac{3q}{4c}, \quad d_3 = \frac{3q}{4c}\left[\frac{L^2}{c^2} - \frac{2}{5}\right]$$

∴ Stresses are,

$$\sigma_{xx} = \phi_{,yy} = \frac{q}{2I}(L^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2 y\right) \rightarrow$$ → first term same as Euler Bernoulli
→ second term is correction due to $\sigma_{yy} \neq 0$, it is small when $c \ll L$

$$\sigma_{yy} = \phi_{,xx} = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2 y + \frac{2}{3}c^3\right) \longrightarrow$$ $\sigma_{yy} \ll \sigma_{xx}$ when $c \ll L$
Note $\sigma_{yy} = 0$ for Euler Bernoulli beam

$$\sigma_{xy} = -\phi_{,xy} = -\frac{q}{2I}(c^2 - y^2)x \longrightarrow$$ Same as in Euler Bernoulli beam

Deflections are obtained in the usual manner
by integrating strain-displ eqn (see solution of HW#2, prob 13).
We get for $v|_{y=0}$,

$$v|_{y=0} = \frac{5}{24}\frac{qL^4}{EI}\left[\underbrace{1 - \frac{6}{5}\left(\frac{x}{L}\right)^2 + \frac{1}{5}\left(\frac{x}{L}\right)^4}_{\text{Term I}} + \underbrace{\frac{12}{5}\left(\frac{c}{L}\right)^2\left\{\left(\frac{4}{5}+\frac{\nu}{2}\right)\left(1 - \left(\frac{x}{L}\right)^2\right)\right.}_{\text{Term II}} + \underbrace{\left.\left(\frac{1}{64}-\frac{\nu}{480}\right)\left(\frac{h}{L}\right)^2\right\}}_{\text{Term II(a)}}\right]$$

Term I → Classical Euler Bernoulli beam, ie for $\frac{c}{L} \simeq 0$

Term II → correction due to shear deformation — non-negligible if
$$\frac{c}{L} \not\ll 1$$

<u>Note</u>: Timoshenko & Shames & Dyn use b.c $v|_{\substack{y=0\\x=\pm L}}=0$ in which case
Term II(a) Vanishes. In HW#2, P.13 we have used $v|_{\substack{y=-c\\x=\pm L}}=0$.
which yields the additional Term II(a)

Thus Timoshenso/Shames assume beam supported at two
ends at $\oint_{\wedge}^{(y=0)}$ whereas we have assumed supports at $y=-c$

<u>Note</u>: $v|_{\substack{y=-c\\x=\pm L}}=0$ (or alternatively $v|_{\substack{y=0\\x=\pm L}}=0$) prevents RB translation
in y-dir and RB rot in x-y plane. Additionally,
$u|_{\substack{y=-c\\x=-L}}=0$ (or alternatively $u|_{\substack{y=0\\x=0}}=0$) is also required to
be imposed to prevent RB translation in x-dir (see HW2, P.13)

<u>Note</u>: can also impose, instead, $u, v, (u_{,y}-v_{,x})$ as zero at
$y=-c, x=-L$ for $\underset{\wedge}{zero}$ RB transl in x,y dir &$\underset{\wedge}{zero}$ RB rot of infinitesimal
element at left bottom.

<u>Note</u>: For all 3 alternatives of bc's noted above the constants
of integration will be different, yielding different
solutions.
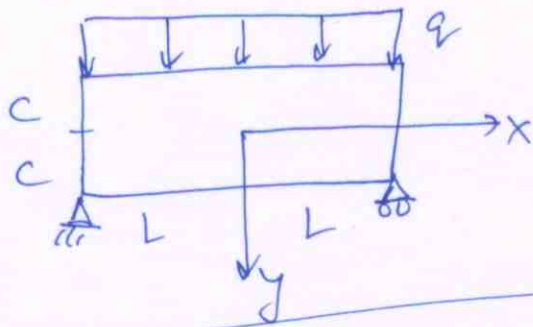
First the key steps of the semi-invese method:
Assume solution for either some or all of the stresses,
or some or all of the displacements. This assumption
is based on loading or BC's.
Then show that the governing differential equations
are satisfied. In the case when we solve by Airy's
stress function (ϕ) approach, you assume some
stress component, then obtain ϕ by integrating
suitably this stress component, and then impose
that the ϕ thus obtained must satisfy

$$\nabla^4 \phi = 0 \quad (ie \; compatibility).$$

In case the resulting solution fails to satisfy
the BC's (up to the St. Venant effect) then
Revise your assumption for the stress component(s).

—————X—————

For the problem at hand, assume that σ_{yy}
is mainly produced by the direct loading q.
Since q is constant, assume σ_{yy} is constant
w.r.t. 'x' ie,

$$\sigma_{yy} = f(y).$$

$$\Rightarrow \phi_{,xx} = f(y)$$

$$\Rightarrow \phi_{,x} = x f(y) + g(y)$$

$$\Rightarrow \phi = \frac{x^2}{2} f(y) + x\, g(y) + h(y)$$

Now ϕ must satisfy $\nabla^4 \phi = 0$. (∵ no body forces considered).

$$\Rightarrow \frac{x^2}{2} f^{IV} + x\, g^{IV} + h^{IV} + 2f'' = 0$$

Seperate powers of 'x', get,

$$f^{IV} = 0, \qquad g^{IV} = 0, \qquad h^{IV} + 2f'' = 0$$

Solution is,

$$f(y) = Ay^3 + By^2 + Cy + D \; ; \quad g(y) = Ey^3 + Fy^2 + Gy$$

where constant term in 'g' is dropped since it wont affect the stresses. Thereafter we get,

$$h^{IV} = -2f'' = --12Ay - 4B$$

$$\Rightarrow h(y) = -\cancel{12}A\frac{y^5}{\underset{10}{\cancel{120}}} - \cancel{4}B\frac{y^4}{\underset{6}{\cancel{24}}} + Hy^3 + Ky^2$$

Thus, $\phi = \dfrac{x^2}{2}\left(Ay^3 + By^2 + Cy + D\right) + x\left(Ey^3 + Fy^2 + Gy\right)$

$$- \frac{A}{10}y^5 - \frac{B}{6}y^4 + Hy^3 + Ky^2$$

Stresses are,

$$\phi_{,yy} = \sigma_{xx} = \frac{x^2}{2}(6Ay + 2B) + x(6Ey + 2F) - 2Ay^3 - 2By^2$$
$$+ 6Hy + 2K.$$

$$\phi_{,xx} = \sigma_{yy} = Ay^3 + By^2 + Cy + D$$

$$-\phi_{,xy} = \sigma_{xy} = -x(3Ay^2 + 2By + C) - (3Ey^2 + 2Fy + G)$$

Apply BC's :

(I) $\left.\sigma_{xy}\right|_{y=\pm c} = 0 \longrightarrow$
$\dot{x}$ coeff : $3Ac^2 \pm 2Bc + C = 0$
x^0 coeff : $3Ec^2 \pm 2Fc + G = 0$

$$\Rightarrow \boxed{B=F=0}, \quad 3Ac^2 + C = 0 \xrightarrow{(i)}, \quad 3Ec^2 + G = 0 \searrow_{(ii)}$$

$(\text{II}) \ \tau_{yy}\big|_{y=c} = 0 \longrightarrow Ac^3 + \underset{\text{0 used}}{\cancel{B}c^2} + Cc + D = 0 \quad \Big\}$

$(\text{III}) \ \tau_{yy}\big|_{y=-c} = -q \longrightarrow -Ac^3 + \underset{\text{0 used}}{\cancel{B}c^2} - Cc + D = q.$

$$\Rightarrow \boxed{D = -\frac{q}{2}}, \quad Ac^3 + Cc = \frac{q}{2} \longrightarrow (iii)$$

$(i),(ii) \longrightarrow \boxed{A = -\frac{q}{4c^3}, \quad C = \frac{3q}{4c}}$

$(\text{IV}) \ \displaystyle\int_{-c}^{c} \tau_{xx} dy\Big|_{x=\pm L} = 0 \longrightarrow$ after inserting $B=F=0$ you get,

$$2K * 2c = 0 \Rightarrow \boxed{K=0}$$

$\displaystyle\int_{-c}^{c} \tau_{xx} y\, dy\Big|_{x=\pm L} = 0 \longrightarrow \frac{L^2}{2}\left(6*2\frac{Ac^3}{3}\right) \pm L\left(6*2 E\frac{c^3}{3}\right)$

$$-2*2\frac{Ac^5}{5} + 6*2*\frac{Hc^3}{3} = 0$$

$$\Rightarrow \boxed{E=0}, \quad \boxed{H = \frac{qL^2}{8c^3} - \frac{q}{20c}}$$

Then $(ii) \longrightarrow \boxed{G=0}$

So we have got all the constants, ie,

$$B=F=K=E=G=0, \quad A = -\frac{q}{4c^3}, \quad C = \frac{3q}{4c}, \quad D = -\frac{q}{2}$$

$$H = \frac{qL^2}{8c^3} - \frac{q}{20c}$$

NOTE: After inserting constants in the equations for the stresses, you will see that result

matches exactly with the one of HW #2 problem
#13 (after replacing $c \to \frac{h}{2}$ and $L \to \frac{L}{2}$). It also
matches the result obtained by the direct
polynomial approach (see bottom of page 34).

NOTE: The integral BC, $\int_{-c}^{c} \sigma_{xy} \, dy \Big|_{x=\pm L} = 0$ was
not used. So we expect that our solution
will identically satisfy this BC. Check
that it does satisfy this BC. If it doesn't,
there is an error. Of course it will satisfy
it !!

NOTE: From loading & geometric symmetry about y-axis,
we expect σ_{xx} to be an even function of
'x' and σ_{xy} to be an odd function of 'x'

from symmetry.
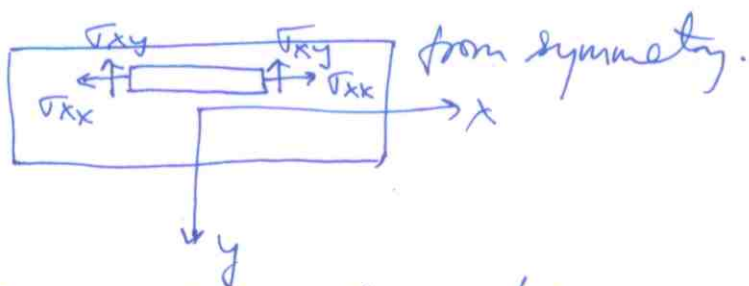
Thus from the stresses on bottom of pg 35b
we could have directly obtained $E = F = G = 0$.
This is an after thought, and is not required.
It would only have simplified the working on
pg 35c, but we did not use it.

Fourier series solutions.

Polynomial solutions restricted to simple loading, and they cannot handle point loads. Thus we use Fourier series which can handle more general loading, including discontinuities also.

Choose $\phi = \sin \alpha x\, f(y)$ and $\phi = \cos \alpha x\, f(y) \longrightarrow$ 1(a,b)
(where $\alpha =$ arbitrary as yet)
which satisfy $\nabla^4 \phi = 0. \longrightarrow$ (2).

(Note:- This choice is appropriate when loading applied only on faces parallel to x-axis)

Subst 1(a) in (2) or 1(b) in (2), get

$$f^{IV} - 2\alpha^2 f'' + \alpha^4 f = 0$$

which has solution

$$f = A \sinh \alpha y + B \cosh \alpha y + C \alpha y \sinh \alpha y + D \alpha y \cosh \alpha y.$$

$$\Rightarrow \phi(x,y) = \sum_{m=1}^{\infty}\Big[\sin \alpha_m x \left(A_m \sinh \alpha_m y + B_m \cosh \alpha_m y + C_m \alpha_m y \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y \right)$$
$$+ \cos \alpha_m' x \left(A_m' \sinh \alpha_m' y + B_m' \cosh \alpha_m' y + C_m' \alpha_m' y \sinh \alpha_m' y + D_m' \alpha_m' y \cosh \alpha_m' y \right)\Big]$$

∴ Stresses are,

$$\sigma_{xx} = \sum_{m=1}^{\infty}\Big\{ \alpha_m^2 \sin \alpha_m x \Big[(A_m + 2D_m) \sinh \alpha_m y + (B_m + 2C_m) \cosh \alpha_m y + C_m \alpha_m y \sinh \alpha_m y$$
$$+ D_m \alpha_m y \cosh \alpha_m y \Big]$$
$$+ \alpha_m'^2 \cos \alpha_m' x \begin{bmatrix} A_m \to A_m',\ B_m \to B_m' \\ C_m \to C_m',\ D_m \to D_m' \\ \alpha_m \to \alpha_m' \end{bmatrix} \Big\}$$

$$\sigma_{yy} = -\sum_{m=1}^{\infty} \alpha_m^2 \sin \alpha_m x \Big[A_m \sinh \alpha_m y + B_m \cosh \alpha_m y + C_m \alpha_m y \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y \Big]$$
$$+ \alpha_m'^2 \cos \alpha_m' x \Big[A_m \to A_m',\ etc \Big]$$

$$\sigma_{xy} = -\sum_{m=1}^{\infty} \alpha_m^2 \cos \alpha_m x \Big[(B_m + C_m) \sinh \alpha_m y + (A_m + D_m) \cosh \alpha_m y + D_m \alpha_m y \sinh \alpha_m y + C_m \alpha_m y \cosh \alpha_m y \Big]$$
$$+ \alpha_m'^2 \sin \alpha_m' x \Big[A_m \to A_m'\ etc \Big].$$

$\left. \right\} \longrightarrow$ (X)

Above stresses satisfy equilibrium & compatibility. BC's satisfied by choosing A_m, B_m etc.

Consider beam with arbitrary transverse loads.
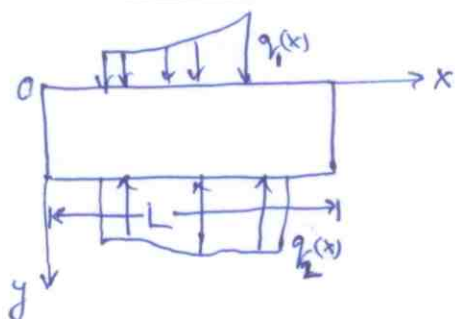
BC's $\left. \sigma_{xx} \right|_{x=0} = \left. \sigma_{xx} \right|_{x=L} = 0 \longrightarrow$ (3)

$\left. \sigma_{xy} \right|_{y=0} = \left. \sigma_{xy} \right|_{y=h} = 0 \longrightarrow$ (4)

$\left. \sigma_{yy} \right|_{y=0} = -q_1(x), \quad \left. \sigma_{yy} \right|_{y=h} = -q_2(x) \longrightarrow$ (5)

$(3) \rightarrow A_m' = B_m' = C_m' = D_m' = 0, \quad \alpha_m = \frac{m\pi}{L} \quad \longrightarrow (6)$

$(4) \rightarrow \begin{cases} \displaystyle\sum_{i=1}^{m} m^2 \cos \frac{m\pi x}{L} (A_m + D_m) = 0 \quad \longrightarrow (7) \\[4mm] \displaystyle\sum_{i=1}^{m} m^2 \cos \frac{m\pi x}{L} \left[(B_m + C_m) \sinh \frac{m\pi h}{L} + (A_m + D_m) \cosh \frac{m\pi h}{L} + D_m \frac{m\pi h}{L} \sinh \frac{m\pi h}{L} \right. \\[4mm] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + C_m \frac{m\pi h}{L} \cosh \frac{m\pi h}{L} \right] = 0 \rightarrow (8) \end{cases}$

$(5) \rightarrow \begin{cases} \displaystyle\frac{\pi^2}{L^2} \sum_{i=1}^{m} m^2 \sin \frac{m\pi x}{L} (B_m) = q_1(x) \quad \longrightarrow (9) \\[4mm] \displaystyle\frac{\pi^2}{L^2} \sum_{i=1}^{m} m^2 \sin \frac{m\pi x}{L} \left[A_m \sinh \frac{m\pi h}{L} + B_m \cosh \frac{m\pi h}{L} + C_m \frac{m\pi h}{L} \sinh \frac{m\pi h}{L} + D_m \frac{m\pi h}{L} \cosh \frac{m\pi h}{L} \right] \\[4mm] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = q_2(x). \rightarrow (10) \end{cases}$

In (7), (8), equating coeffs of $\cos \frac{m\pi x}{L}$ to zero (or alternatively multiply series $_{(7)(8)}$ by $\cos \frac{k\pi x}{L}$ and do $\int_0^L (\ ) dx = 0$) we get,

$$A_m + D_m = 0 \quad \longrightarrow (11)$$

$A_m \cosh \frac{m\pi h}{L} + B_m \sinh \frac{m\pi h}{L} + C_m \left(\sinh \frac{m\pi h}{L} + \frac{m\pi h}{L} \cosh \frac{m\pi h}{L} \right)$

$\qquad\qquad + D_m \left(\cosh \frac{m\pi h}{L} + \frac{m\pi h}{L} \sinh \frac{m\pi h}{L} \right) = 0 \rightarrow (12)$

Expand q_1 and q_2 in Fourier sine series (ie, consider q_1 (and q_2) to be a piece of a function that is odd & periodic), we get
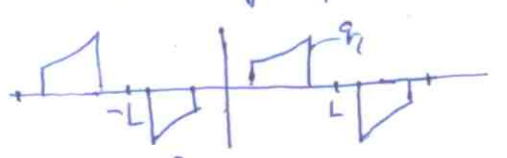
$q_1(x) = \left\{ \displaystyle\sum_{m=1}^{\infty} \left[\frac{2}{L} \int_0^L q_1(x) \sin \frac{m\pi x}{L} dx \right] \sin \frac{m\pi x}{L} \right\}, \quad q_2(x) = \left\{ \quad \right\} w / q_1 \rightarrow q_2$

Comparing coeffs of $\sin \frac{m\pi x}{L}$ (or eqvtly multiplying$_{(9)(10)}$ by $\sin \frac{k\pi x}{L}$ and $\int_0^L (\ ) = 0$) we get from series (9), (10),

$$B_m = \frac{2L}{m^2 \pi^2} \int_0^L q_1(x) \sin \frac{m\pi x}{L} dx \quad \longrightarrow (13)$$

$A_m \sinh \frac{m\pi h}{L} + B_m \cosh \frac{m\pi h}{L} + C_m \frac{m\pi h}{L} \sinh \frac{m\pi h}{L} + D_m \frac{m\pi h}{L} \cosh \frac{m\pi h}{L}$

$\qquad\qquad\qquad = \frac{2L}{m^2 \pi^2} \int_0^L q_2(x) \sin \frac{m\pi x}{L} dx . \rightarrow (14)$

Evaluate definite integrals in (13), (14) & then solve (11)-(14) for $A_m - D_m$, and get stress solution, then displ solution.

Note that we don't <u>need</u> to satisfy explicitly the conditions,

$$R_1 = \int_0^L \tau_{xy}\big|_{x=0} \, dy \quad , \quad R_2 = -\int_0^L \tau_{xy}\big|_{x=L} \, dy \quad \longrightarrow (15)$$

where R_1 & R_2 are given from external (static) equilibrium as

$$R_1 = \frac{1}{L}\int_0^L (q_1 - q_2)(L-x) \, dx \quad , \quad R_2 = \frac{1}{L}\int_0^L (q_2 - q_1) x \, dx \quad \longrightarrow (16)$$

since the τ_{xy} in the series form ⊛ satisfy $\nabla^4 \phi = 0$ (ie equil & compat).

However, you can use (15) & (16) as a check to see that coeffs $A_m - D_m$ were correctly obtained. (ie, evaluate R_1, R_2 from (15), (16) and check they are the same).

For loading on all four side, the above solution is ← ie I(a,b) not general. In that case we consider $\phi = \phi_x + \phi_y$

where $\phi_x = \sum_m (\cos \alpha_m' x \, f_m(y) + \sin \alpha_m x \, f_m(y))$

$$\phi_y = \sum_m (\cos \beta_m' y \, g_m(x) + \sin \beta_m x \, g_m(x))$$

where $^{(')}$ <u>does not</u> denote derivative but primed coeffs A_m', etc and $f_m(y)$, $f_m(y)$ $g_m(x)$, $g_m(x)$ have same functional form. ⋯ This is the double Fourier series method.

2-D Problems in Polar Coordinates.

Transformation of equilibrium equations

$$\boxed{\begin{array}{l} \sigma_{r,r} + \dfrac{\sigma_r - \sigma_\theta}{r} + \dfrac{\sigma_{r\theta,\theta}}{r} + B_r = 0 \\[3mm] \sigma_{r\theta,r} + \dfrac{2\sigma_{r\theta}}{r} + \dfrac{\sigma_{\theta,\theta}}{r} + B_\theta = 0 \end{array}} \qquad \begin{array}{l} (\Sigma F_r = 0) \longrightarrow \text{①} \\[3mm] (\Sigma F_\theta = 0) \longrightarrow \text{②} \end{array}$$

Transformation of derivatives & Compatibility Eqn.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\left(\frac{\partial r}{\partial x} = \frac{x}{r} = c\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = s\theta, \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{s\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{c\theta}{r}\right)$$

$$\frac{\partial}{\partial x} = c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}\right)\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial \theta}\right) \qquad \text{③}$$

$$= c^2\theta\frac{\partial^2}{\partial r^2} + s^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) - 2s\theta c\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

For $\dfrac{\partial^2}{\partial y^2}$ & $\dfrac{\partial}{\partial y}$ do $\left\{\begin{array}{l} s\theta \to -c\theta \\ c\theta \to s\theta \end{array}\right\}$ in $\dfrac{\partial^2}{\partial x^2}$ & $\dfrac{\partial}{\partial x}$, respectively. $\longrightarrow$ ④
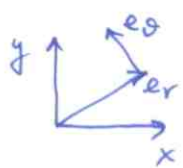
$$\frac{\partial^2}{\partial x \partial y} = -s\theta c\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial r^2}\right) + (c^2\theta - s^2\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$\text{⑤}$$

Thus $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = \nabla^2_{xy} = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2} = \nabla^2_{r\theta}$

$$\nabla^4_{xy} = \nabla^2_{xy}\nabla^2_{xy} \longrightarrow \boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0} \longrightarrow \text{⑥}$$

$$\text{COMPATIBILITY EQN.}$$

Transformation of Stress & stress function relations.

$$\Rightarrow \boxed{\begin{array}{l} \sigma_r = \sigma_{xx}c^2\theta + \sigma_{yy}s^2\theta + 2\sigma_{xy}s\theta c\theta \\[2mm] \sigma_\theta = \sigma_{xx}s^2\theta + \sigma_{yy}c^2\theta - 2\sigma_{xy}s\theta c\theta \\[2mm] \sigma_{r\theta} = -(\sigma_{xx} - \sigma_{yy})s\theta c\theta + \sigma_{xy}(c^2\theta - s^2\theta) \end{array}} \longrightarrow \text{⑦}$$

Substitute $\sigma_{xx} = \phi_{,xx}$, $\sigma_{yy} = \phi_{,yy}$, $\sigma_{xy} = -\phi_{,xy}$ in ⑦ and use ③–⑤ to transform second-partials from x.y.z to r-θ system. We get,

$$\boxed{\begin{array}{l} \sigma_r = \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} \\[2mm] \sigma_\theta = \phi_{,rr} \\[2mm] \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\phi_{,\theta}\right) \end{array}} \longrightarrow \text{(8)} \cdot (\text{in absence of body forces}).$$

Transformation of Strain-displacement relations.

Let (u,v) & (u_r, u_θ) be Cartesian & polar comp's of displacement, respectively.

$$\Rightarrow \quad \begin{cases} u = c\theta\, u_r - s\theta\, u_\theta \\ v = s\theta\, u_r + c\theta\, u_\theta \end{cases} \longrightarrow \text{(9)}$$

Now, using transf of strain comp's, transf. of displ. comp's, & transf. of derivatives,

$$\varepsilon_{xx} = u_{,x} \underset{\uparrow}{=} c^2\theta\,\varepsilon_r + s^2\theta\,\varepsilon_\theta - 2\varepsilon_{r\theta}\,s\theta c\theta = c^2\theta\, u_{r,r} - c\theta s\theta\, u_{\theta,r}$$

(used transf. of strains)   (used transf of displs, and derivatives).

$$-\frac{s\theta}{r}\left(-s\theta\, u_r - c\theta\, u_\theta + c\theta\, u_{r,\theta} - s\theta\, u_{\theta,\theta}\right) \quad \text{(i)}$$

$$\begin{array}{l} s\theta \to -c\theta \\ c\theta \to s\theta \end{array} \bigg\} \to \varepsilon_{yy} = v_{,y} = s^2\theta\,\varepsilon_r + c^2\theta\,\varepsilon_\theta + 2\varepsilon_{r\theta}\,s\theta c\theta = s^2\theta\, u_{r,r} + s\theta c\theta\, u_{\theta,r} + \frac{c\theta}{r}\Big(c\theta\, u_r - s\theta\, u_\theta$$

$$+ s\theta\, u_{r,\theta} + c\theta\, u_{\theta,\theta}\Big) \to \text{(ii)}$$

$$\varepsilon_{xy} = \tfrac{1}{2}(u_{,y} + v_{,x}) = (\varepsilon_r - \varepsilon_\theta)s\theta c\theta + \varepsilon_{r\theta}(c^2\theta - s^2\theta) = \tfrac{1}{2}\Big(2s\theta c\theta\{u_{r,r} - \tfrac{1}{r}u_{\theta,\theta} - \tfrac{u_r}{r}\}$$

$$+ [c^2\theta - s^2\theta]\{u_{\theta,r} + \tfrac{1}{r}u_{r,\theta} - \tfrac{u_\theta}{r}\}\Big) \quad \text{(iii)}$$

Solve (i)–(iii) [details on reverse] and get,

$$\boxed{\begin{array}{l} \varepsilon_r = u_{r,r} \\[2mm] \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}u_{\theta,\theta} \\[2mm] \varepsilon_{r\theta} = \tfrac{1}{2}\left(u_{\theta,r} + \tfrac{1}{r}u_{r,\theta} - \tfrac{u_\theta}{r}\right) \end{array}} \longrightarrow \text{(10)}$$

Transformation of Constitutive Relations.

They remain same with $\varepsilon_{xx} \to \varepsilon_r$, $\sigma_{xx} \to \sigma_r$, $\varepsilon_{yy} \to \varepsilon_\theta$, $\sigma_{yy} \to \sigma_\theta$, $\varepsilon_{xy} \to \varepsilon_{r\theta}$, $\sigma_{xy} \to \sigma_{r\theta}$.

$$\boxed{\begin{array}{l} \varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \\[2mm] \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \\[2mm] \varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta} \\[2mm] \varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) \end{array}} \longrightarrow \text{(11)}$$

(Plane Stress)

$$\boxed{\begin{array}{l} \varepsilon_r = \frac{1-\nu^2}{E}\left[\sigma_r - \frac{\nu}{1-\nu}\sigma_\theta\right] \\[2mm] \varepsilon_\theta = \frac{1-\nu^2}{E}\left[\sigma_\theta - \frac{\nu}{1-\nu}\sigma_r\right] \\[2mm] \varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta} \end{array}} \longrightarrow \text{(12)}$$

(Plane Strain)
i.e $E \to \frac{E}{1-\nu^2}$
$\nu \to \frac{\nu}{1-\nu}$

Details of solution of (i) – (iii).

Let $P = u_{r,r}$, $Q = \dfrac{u_{\theta,\theta} - u_r}{r}$, $R = u_{\theta,r} + \dfrac{u_{r,\theta}}{r} - \dfrac{u_\theta}{r}$

(i)+(ii) $\rightarrow$ $\varepsilon_r + \varepsilon_\theta = P + Q \longrightarrow$ (iv)

$c^2\theta*$(ii) $- s^2\theta*$(i) $\rightarrow 2\cos\theta\, \varepsilon_{r\theta} + (c^2\theta - s^2\theta)\varepsilon_\theta = \cos\theta R + (c^2\theta - s^2\theta)Q \longrightarrow$ (v)

(iii) remains same $\rightarrow$ (vi).

Now solve (iv) – (vi), ie,

$$\frac{s2\theta}{2} * (vi) - c2\theta * (v) \xrightarrow{2\theta \equiv \alpha} s^2\alpha\, \varepsilon_r - (1 + c^2\alpha)\varepsilon_\theta = s^2\alpha\,(P - Q) - 2c^2\alpha\, Q \Big\}$$

(vii)

Then put (iv) in (vii) and get (10a), then from (iv) get (10 b), and from (v) get (10 c).

Axisymmetric Problems.

Loading, hence stresses, are invariant about an axis.

$\Rightarrow \phi = \phi(r)$. $\Rightarrow$ stresses, strains indep of θ, but not so for displacements.

$$\nabla^4 \phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 = \phi^{IV} + \frac{2}{r}\phi^{III} - \frac{1}{r^2}\phi'' + \frac{1}{r^3}\phi' = 0.$$

put $r = e^t$ to convert to ODE with const coeffs, ie,

details only
$$\begin{cases} \frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = e^{-t}\frac{d\phi}{dt}, \quad \frac{d^2\phi}{dr^2} = e^{-2t}\left[-\frac{d\phi}{dt} + \frac{d^2\phi}{dt^2}\right] \\[2mm] \frac{d^3\phi}{dr^3} = e^{-3t}\left[2\frac{d\phi}{dt} - 3\frac{d^2\phi}{dt^2} + \frac{d^3\phi}{dt^3}\right], \quad \frac{d^4\phi}{dr^4} = e^{-4t}\left[-6\frac{d\phi}{dt} + 11\frac{d^2\phi}{dt^2} - 6\frac{d^3\phi}{dt^3} + \frac{d^4\phi}{dt^4}\right] \end{cases}$$

$$\Rightarrow \nabla^4\phi = \frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

$$\phi = e^{\lambda t} \rightarrow \lambda^4 - 4\lambda^3 + 4\lambda^2 = 0 \rightarrow \lambda = 0, 0, 2, 2$$

$$\Rightarrow \phi = A^* + B^* t + C^* e^{2t} + D^* t e^{2t} = \boxed{A \ln r + B r^2 \ln r + C r^2 + D} = \phi.$$

$\rightarrow$ (13)

Stresses are (use (8))

$$(\frac{1}{r}\phi_{,r}) \quad \boxed{\begin{aligned} \sigma_r &= \frac{A}{r^2} + B(1+2\ln r) + 2C \\[2mm] \sigma_\theta &= \phi_{,rr} = -\frac{A}{r^2} + B(3+2\ln r) + 2C \\[2mm] \sigma_{r\theta} &= 0 \end{aligned}} \rightarrow (14)$$

Note: If $r=0$ is part of domain, then finite stress $\Rightarrow A=B=0 \Rightarrow$ stresses are constant, i.e, for plate w/o hole we have only one possible axisymm stress distribution wherein $\sigma_r = \sigma_\theta = 2C = $ const, ie representing plate under uniform tension or compression in all directions in its plane.

Displacements:

$$(14, 11^a, 10^c) \rightarrow u_{\theta,r} = \frac{1}{E}\left[\frac{(1+\nu)A}{r^2} + 2(1-\nu)B\ln r + (1-3\nu)B + 2(1-\nu)C\right]$$

$$\int \rightarrow u_r = \frac{1}{E}\left[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\ln r - B(1+\nu)r + 2C(1-\nu)r\right] + f(\theta)$$

$$(14, 11^b, 10^b, u_r) \rightarrow u_{\theta,\theta} = \frac{4Br}{E} - f(\theta)$$

$$u_{r,r} = \varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{1}{E}\left(\frac{1}{r}\phi_{,r} - \nu\phi_{,rr}\right)$$

$$u_r = \frac{1}{E}\left(A\left[-\frac{1}{r} - \frac{\nu}{r}\right] + B\left[r + 2r(\ln r - 1) - \nu 2r\ln r - \nu r\right] + 2C[r - \nu r]\right)$$

$$= \frac{1}{E}\left(-\frac{A}{r}(1+\nu) + B(1-\nu)2r\ln r - B(1+\nu)r + 2C(1-\nu)r\right) + f(\theta)$$

$$u_{\theta,\theta} = r\varepsilon_\theta - u_r = r\frac{1}{E}(\sigma_\theta - \nu\sigma_r) - u_r$$

$$= \frac{1}{E}\left(-\frac{A}{r}(1+\nu) + B(1-\nu)2r\ln r + B(3-\nu)r + 2C r(1-\nu)\right) - u_r$$

$$= \frac{1}{E}(4Br) - f(\theta)$$
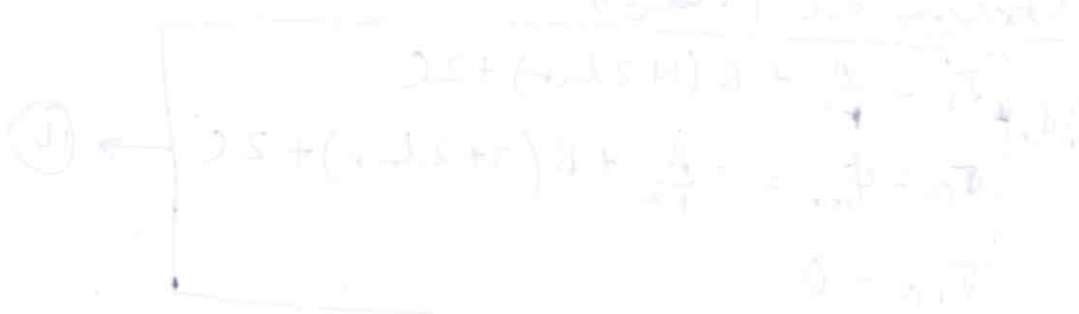
$$\int \rightarrow \quad u_\theta = \frac{4Br\theta}{E} - \int f(\theta)d\theta + g(r)$$

$$(u_r, u_\theta, 10^c, 14^c, 11^c) \rightarrow \quad \frac{1}{r}f_{,\theta} + g_{,r} + \frac{1}{r}\int f d\theta - \frac{1}{r}g = 0.$$

$$\Rightarrow \quad g - rg_{,r} = Z \ (const).$$

$$f_{,\theta} + \int f d\theta = Z$$

$$\Rightarrow f'' + f = 0 \rightarrow \boxed{f = H\sin\theta + K\cos\theta} \rightarrow Z = H\!\!\!\!/\theta - K\!\!\!/s\theta - H\!\!\!\!/\theta + K\!\!\!/s\theta$$
$$= 0$$

$$\rightarrow r\frac{dg}{dr} = g \rightarrow \boxed{g = Gr} \ (G = const)$$

$$\Rightarrow \boxed{\begin{aligned} u_r &= \frac{1}{E}\Big[-\frac{(1+\nu)}{r}A + 2(1-\nu)Br\ln r - B(1+\nu)r + 2C(1-\nu)r\Big] \\ &\qquad\qquad\qquad + H\sin\theta + K\cos\theta \\ u_\theta &= \frac{4Br\theta}{E} + Gr + H\cos\theta - K\sin\theta \end{aligned}} \quad \downarrow \text{(15)}$$

(I) Thick Walled Cylinder.

Case A Ends open, unconstrained $\rightarrow \sigma_{zz} = 0$ $\Big\}$ $\Rightarrow$ Plane Stress.
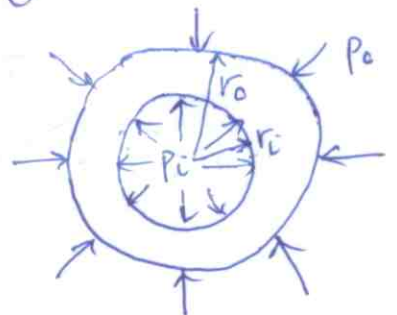
Symmetry along z-axis $\rightarrow \sigma_{\theta z} = \sigma_{rz} = 0$ $\Big/$

Axisymmetry $\rightarrow \sigma_{r\theta} = 0$ (from 14^c, or physical symmetry).

(also from compatibility – single valuedness).

Symmetry $\rightarrow u_\theta = 0 \Rightarrow B = 0 \Rightarrow Gr = K\sin\theta - H\cos\theta = const.$

$u_r \neq u_r[\theta] \Rightarrow K = H = 0 \rightarrow G = 0$

Stress BC's $\rightarrow \sigma_r\big|_{r=r_0} = -P_0, \quad \sigma_r\big|_{r=r_i} = -P_i$

$$(14) \rightarrow A = \left(\frac{r_i^2 r_0^2}{r_0^2 - r_i^2}\right)(P_0 - P_i)$$

$$2C = \frac{P_i r_i^2 - P_0 r_0^2}{r_0^2 - r_i^2}$$

Thus, $\boxed{\begin{aligned} \sigma_{r/\theta} &= \pm\frac{r_i^2 r_0^2 (P_0 - P_i)}{r_0^2 - r_i^2}\frac{1}{r^2} + \frac{P_i r_i^2 - P_0 r_0^2}{r_0^2 - r_i^2} \\ u_\theta &= 0, \quad u_r = \left(\frac{1+\nu}{E}\right)\frac{r_i^2 r_0^2 (P_i - P_0)}{(r_0^2 - r_i^2)r} + \left(\frac{1-\nu}{E}\right)\left(\frac{P_i r_i^2 - P_0 r_0^2}{r_0^2 - r_i^2}\right)r \end{aligned}} \rightarrow \text{(16)}$

indep of E, ν

- $\sigma_{r/\theta}$ indep of E, ν.

- $\varepsilon_\theta = \dfrac{\sigma_\theta - \nu\sigma_r}{E} = \dfrac{u_r}{r} + \dfrac{u_{\theta,\theta}}{r}^{\,0}$ checks out from (15^c).

- $u_\theta = 0 \not\Rightarrow \varepsilon_\theta = 0$

- $\sigma_r + \sigma_\theta = \text{const} \rightarrow \varepsilon_z = -\dfrac{\nu}{E}(\sigma_r + \sigma_\theta) = \text{const} \rightarrow$ sections $\perp^a$ to z-axis remain plane.

- If $p_o = 0 \rightarrow \sigma_r < 0$ (compressive), $\sigma_\theta > 0$ (tensile).

$$(\sigma_\theta)_{max} = (\sigma_\theta)_{r=r_i} = p_i \frac{r_i^2 + r_o^2}{r_o^2 - r_i^2} > p_i$$

$$(\sigma_r)_{min} = (\sigma_r)_{r=r_i} = -p_i$$

$\left.\begin{array}{c}\end{array}\right\}$ max shear stress at $r = r_i = \dfrac{\sigma_\theta - \sigma_r}{2}$

- Thin walled $\rightarrow r_o - r_i = t$, $\dfrac{t}{r_i} \ll 1 \xrightarrow{\text{use } r_o \approx r_i \approx r \text{ appropriately}} \sigma_\theta \approx \dfrac{p_i r_i}{t}$, $\sigma_r \approx 0$
  ($p_o = 0$)

Same result from SoM for closed tw cyl wherein $\sigma_z = \dfrac{pr}{2t}$ additionally.

Case B.
- Ends restrained $\Rightarrow \varepsilon_{zz} = 0$
  also $\sigma_{\theta z} = \sigma_{rz} = 0$ as before from symm. argument $\xrightarrow{} \varepsilon_{rz} = \varepsilon_{\theta z} = 0$

So Plane strain case.
So results are same as (15) w/ $\nu \rightarrow \dfrac{\nu}{1-\nu}$, $E \rightarrow \dfrac{E}{(1+\nu)(1-\nu)}$

$$\sigma_{zz} = \nu(\sigma_r + \sigma_\theta) = 2\nu \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} = \text{const thru } r$$

Case C. Ends capped : (see over)
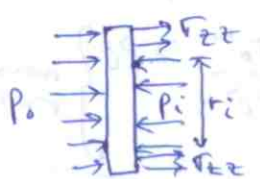
(II) Pure Bending of Curved Beams.
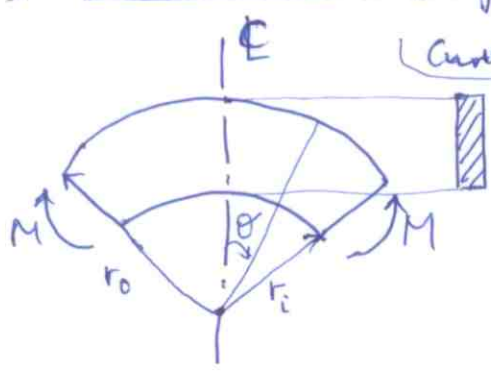
Curved Beam with narrow rectangular section, and axis being a circular arc. Loaded by end couples M acting in plane of beam axis. Hence bending moment is same at each section. Since $M \neq M(\theta)$, problem is axisymmetric & stresses dont vary across θ.

BC's: $\sigma_r = 0$ for $r = r_i$ and $r = r_o$ $\longrightarrow$ (1)

$$\int_{r_i}^{r_o} \sigma_\theta \, dr = 0 \qquad\qquad \longrightarrow (2)$$

$$\int_{r_i}^{r_o} \sigma_\theta \, r \, dr = -M \qquad \longrightarrow (3)$$

$\sigma_{r\theta} = 0$ on entire boundary. $\longrightarrow$ (4)

<u>Case C</u>: Ends capped. (Thick walled cylinder):

For restrained ends we got $\sigma_{zz} = $ const. Thus if capped rigidly, but ends unrestrained, we still expect $\sigma_{zz} = $ const. This is true for sections away from end-caps.

Thus external equilibrium for end caps yields,

$$\sigma_{zz} \pi (r_0^2 - r_i^2) + p_0 \pi r_0^2 = p_i \pi r_i^2$$

$$\sigma_{zz} = \frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2}$$

σ_r, σ_θ same as for previous two cases.

① ⟹ $\dfrac{A}{r_i^2} + B(1 + 2\ln r_i) + 2C = 0$ ⎫

$\dfrac{A}{r_0^2} + B(1 + 2\ln r_0) + 2C = 0$ ⎬ ⟶ ⑤

② ⟶ $\displaystyle\int_{r_i}^{r_0} \sigma_\theta \, dr = \int_{r_i}^{r_0} \dfrac{\partial^2 \phi}{\partial r^2} \, dr = \dfrac{d\phi}{dr}\Big|_{r_i}^{r_0} = r\sigma_r\Big|_{r_i}^{r_0} = 0 \; \text{if} \; ① \; \text{satisfied}.$

(w/⑧ p.40)

ie ② satisfied identically if ① satisfied. (ie, ① ⟹ ②).

(This was expected ∵ usage of stress fn. means equil. satisfied).

③ ⟶ $\displaystyle\int_{r_i}^{r_0} \sigma_\theta \, r \, dr = -M = \int \phi_{,rr} \, r \, dr = \phi_{,r} \, r \Big|_{r_i}^{r_0} - \phi\Big|_{r_i}^{r_0} = \phi[r_i] - \phi[r_0]$

$= 0$ from ① being satisfied

$\overset{13, p.41}{\Longrightarrow} M = A \ln \dfrac{r_0}{r_i} + B(r_0^2 \ln r_0 - r_i^2 \ln r_i) + C(r_0^2 - r_i^2) \longrightarrow ⑥$

$5^a, 5^b, 6 \longrightarrow A = -\dfrac{4M}{N} r_0^2 r_i^2 \ln \dfrac{r_0}{r_i}$, $B = -\dfrac{2M}{N}(r_0^2 - r_i^2)$, $C = \dfrac{M}{N}\left(r_0^2 - r_i^2 + 2\{r_0^2 \ln r_0 - r_i^2 \ln r_i\}\right)$

$N \triangleq (r_0^2 - r_i^2)^2 - 4 r_0^2 r_i^2 \left(\ln \dfrac{r_0}{r_i}\right)^2$

(14) p.41 ⟶ $\sigma_r = -\dfrac{4M}{N}\left(\dfrac{r_0^2 r_i^2}{r^2} \ln \dfrac{r_0}{r_i} + r_0^2 \ln \dfrac{r}{r_0} + r_i^2 \ln \dfrac{r_i}{r}\right)$

$\sigma_\theta = -\dfrac{4M}{N}\left(-\dfrac{r_0^2 r_i^2}{r^2} \ln \dfrac{r_0}{r_i} + r_0^2 \ln \dfrac{r}{r_0} + r_i^2 \ln \dfrac{r_i}{r} + b^2 - a^2\right)$

$\sigma_{r\theta} = 0$

(see over for ⑦)

Soln valid if M applied via σ_θ as above, else St. Venant effects.

Comparing stresses with S.o.M. solution, which assumes plane sections plane & $\sigma_r = 0$, we see that stresses compare well (max, min values comparison done in Timoshenko & Goodier p.74 table, see Popov p.362-363 for SoM solution). The discrepancy is due to $\sigma_r = 0$ assumption in the SoM solution.
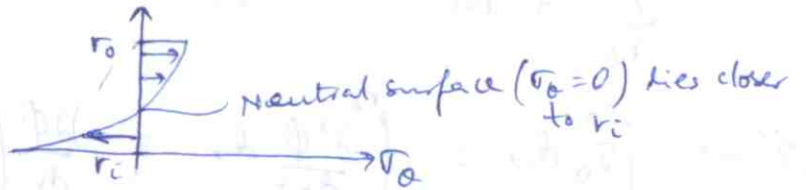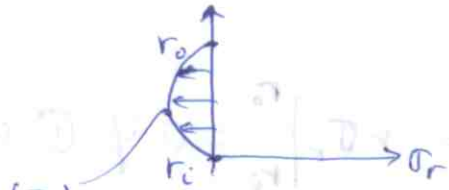
Displacements:
$\theta = 0$ is ℄ which is fixed due to symmetry considerations.

⟹ $u_r = 0, u_\theta = 0, \dfrac{du_\theta}{dr} = 0$ for $r = \dfrac{r_0 + r_i}{2} = \bar{r}$ & $\theta = 0$

(15) p.42 ⟶ $u_r\big|_{\substack{\theta=0 \\ r=\bar{r}}} = 0 \longrightarrow K = -\dfrac{1}{E}\left(\dfrac{-(1+\nu)A}{\bar{r}} + 2(1-\nu)B\bar{r}\ln\bar{r} - b(1+\nu)\bar{r} + 2C(1-\nu)\bar{r}\right)$

Stress distr in curved beam bending.

r_o

r_i

σ_r

$(\sigma_r)_{max}$ lies closer to r_i

r_o

r_i

σ_θ

Neutral surface $(\sigma_\theta = 0)$ lies closer to r_i

$u_\theta \big|_{\substack{r=\bar{r} \\ \theta=0}} = 0 \rightarrow G\bar{F} + H = 0$

$\dfrac{\partial u_\theta}{\partial r}\bigg|_{\substack{r=\bar{r} \\ \theta=0}} \longrightarrow G = 0 \rightarrow H = 0$

$\implies u_r = \dfrac{1}{E}\left[\dfrac{-(1+\nu)}{r}A + 2(1-\nu)Br\ln r - B(1+\nu)r + 2C(1-\nu)r\right] + K\cos\theta$

$\underbrace{\qquad\qquad\qquad}_{\text{function of 'r'}}$

$u_\theta = \dfrac{4Br\theta}{E} - K\sin\theta \longrightarrow$ ie transl $(-K\sin\theta)$ + rotation thru

angle $\dfrac{4B\theta}{E} \rightarrow$ ie plane sections

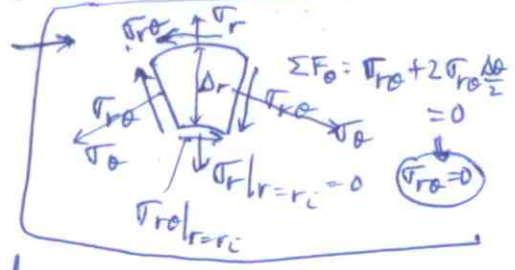remain plane (so Popov assumption OK

except $\sigma_r \neq 0$).

(III) Rotating disks.
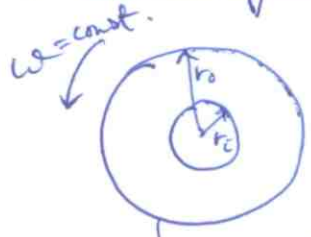
$\omega = $ const.

constant thickness, t.

(closed)

Assume thin disk, $t \ll r_i \implies \sigma_z = \sigma_{rz} = \sigma_{\theta z} = 0$ (∴ closed disk)

$\implies$ plane stress. Problem is obviously axisymmetric

$\implies \sigma_{r\theta} = 0.$ (Explanation   FBD of slice

also shows $\sigma_{r\theta} = 0$ if you start with slice at

$r = r_i^{\text{ or }r_0}$ for which $\sigma_r\big|_{\substack{r=r_i \\ \text{or } r=r_0}} = 0 \rightarrow$

and $\sigma_{r\theta}\big|_{\substack{r=r_i \\ \text{or } r=r_0}} = 0$)

$\sum F_\theta = \sigma_{r\theta} + 2\sigma_{r\theta}\dfrac{\Delta\theta}{2}$

$= 0$

$\sigma_r\big|_{r=r_i} = 0 \quad \boxed{\sigma_{r\theta}=0}$

$\sigma_{r\theta}\big|_{r=r_i}$

Put $B_r = \delta\omega^2 r$, $B_\theta = 0$

θ equil eqn ((2) p. 39) $\longrightarrow$ identically satisfied.

r equil eqn ((1) p. 39) $\longrightarrow \dfrac{d}{dr}(r\sigma_r) - \sigma_\theta + \delta\omega^2 r^2 = 0.$

strain displ (10) p 40) $\longrightarrow \varepsilon_r = \dfrac{du_r}{dr}$

$\varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\underbrace{\dfrac{du_\theta}{d\theta}}_{=0 \text{ due to axisymm}}$

stresses $\longrightarrow \sigma_r = \dfrac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) = \dfrac{E}{1-\nu^2}\left(\dfrac{du_r}{dr} + \dfrac{\nu u_r}{r}\right)$

(plane stress
constitutive
law).

$\sigma_\theta = \dfrac{E}{1-\nu^2}\left(\dfrac{u_r}{r} + \nu\dfrac{du_r}{dr}\right)$

Put stress-displ relations in r-equil eqn (Navier formulation),

$\longrightarrow r^2\dfrac{d^2 u_r}{dr^2} + r\dfrac{du_r}{dr} - u = -\dfrac{1-\nu^2}{E}\delta\omega^2 r^3 \longrightarrow \text{①}$

Solve ① $\longrightarrow u_r = \underbrace{(u_r)_h}_{\text{put } r=e^t} + \underbrace{(u_r)_p}_{\text{proportional to } r^3 \text{ (easily seen)}}$

① $r = e^t \Longrightarrow$ $\dfrac{d^2 u_r}{dt^2} - u_r = -\left(\dfrac{1-\nu^2}{E}\right)\rho\omega^2 e^{3t}$

$$u_r = \underbrace{C_1 e^t + C_2 e^{-t}}_{(u_r)_h} - \underbrace{\left(\dfrac{1-\nu^2}{E}\right)\dfrac{\rho\omega^2}{8}e^{3t}}_{(u_r)_p}$$

$$u_r = C_1 r + \dfrac{C_2}{r} - \dfrac{1-\nu^2}{E}\dfrac{\rho\omega^2}{8}r^3 \qquad .$$

(C.L, ie stress–displ relations) $\Longrightarrow$

$$\sigma_r = \dfrac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\dfrac{C_2}{r^2} - (3+\nu)\left(\dfrac{1-\nu^2}{8E}\right)\rho\omega^2 r^2\right]$$

$$\sigma_\theta = \dfrac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\dfrac{C_2}{r^2} - (1+3\nu)\left(\dfrac{1-\nu^2}{8}\right)\rho\omega^2 r^2\right]$$

<u>Solid Disk</u> $(r_i = 0)$
<u>Stress BC's</u> : Finite σ_r, σ_θ at $r=0 \Rightarrow C_2 = 0$ (also get it from finite u_r at $r=0$).

$$(\sigma_r)_{r=r_0} = 0 \rightarrow C_1 = \dfrac{(3+\nu)(1-\nu)}{8E}\rho\omega^2 r_0^2$$

$$\rightarrow \sigma_r = \dfrac{(3+\nu)}{8}\rho\omega^2(r_0^2 - r^2)$$

$$\sigma_\theta = \dfrac{(3+\nu)}{8}\rho\omega^2 r_0^2 - \dfrac{1+3\nu}{8}\rho\omega^2 r^2$$

$$\rightarrow \sigma_r, \sigma_\theta \text{ max at } r = r_i = 0, \quad (\sigma_r)_{max} = (\sigma_\theta)_{max} = \dfrac{3+\nu}{8}\rho\omega^2 r_0^2$$

<u>Disk with hole</u> $(r_i \neq 0)$
<u>Stress BC's</u> : $(\sigma_r)_{r=r_i} = (\sigma_r)_{r=r_0} = 0 \rightarrow$ get C_1, C_2

Stresses are $\rightarrow$

$$\sigma_r = \dfrac{3+\nu}{8}\rho\omega^2\left(r_0^2 + r_i^2 - \dfrac{r_0^2 r_i^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \dfrac{3+\nu}{8}\rho\omega^2\left(r_0^2 + r_i^2 + \dfrac{r_0^2 r_i^2}{r^2} - \dfrac{1+3\nu}{3+\nu}r^2\right) \qquad .$$

Results $\rightarrow$ $\dfrac{d(\sigma_r)}{dr} = 0 \rightarrow (\sigma_r)_{max}$ at $r = \sqrt{r_0 r_i}$, $(\sigma_r)_{max} = \dfrac{3+\nu}{8}\rho\omega^2(r_0 - r_i)^2$

$(\sigma_\theta)_{max}$ at $r = r_i$, $(\sigma_\theta)_{max} = \dfrac{3+\nu}{4}\rho\omega^2\left(r_0^2 + \dfrac{1-\nu}{3+\nu}r_i^2\right)$

$(\sigma_\theta)_{max} > (\sigma_r)_{max}$ (by mere observation $\because r_0^2 > (r_0 - r_i)^2$)

$r_i \rightarrow 0 \Rightarrow (\sigma_\theta)_{max} = \dfrac{3+\nu}{4}\rho\omega^2 r_0^2 = 2 * (\sigma_\theta)_{max}$ of solid disk.
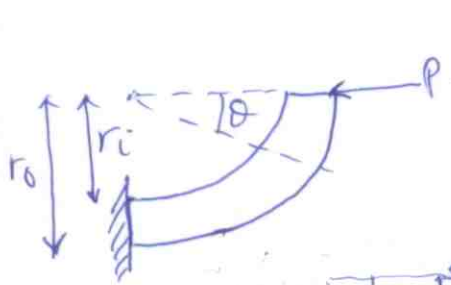
Displacements

u_r : put C_1, C_2 in u_r eqn or use $u_r = r \varepsilon_\theta$.

$u_r = u_r[r]$.

u_θ : $\sigma_{r\theta} = 0 \to \varepsilon_{r\theta} = 0 = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial \cancel{u_r}^{0}}{\partial \theta} - \frac{u_\theta}{r}\right)$

$\Rightarrow u_\theta = kr$, $k = \omega dt$,

ie RB displ.

NON-AXISYMMETRIC PROBLEMS.

(I.) Bending of curved bar with force at the end.

Circular arc with narrow rectangular cross section. Force applied in radial direction. Narrow section $\Rightarrow$ plane stress assumption.
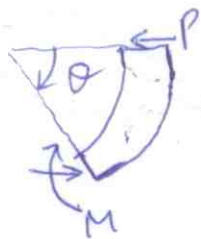
External equilibrium $\Rightarrow M \propto \sin\theta$

$\Rightarrow \sigma_\theta \propto \sin\theta$

$\Rightarrow \phi_{,rr} \propto \sin\theta$

So try $\phi = f(r)\sin\theta$

$\nabla^4 \phi = 0 \to \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2}\right) = 0$

$f^{IV} + \frac{2}{r}f^{III} - \frac{3}{r^2}f^{II} + \frac{3}{r^3}f^{I} - \frac{3}{r^4}f = 0$

$r = e^t \to e^{-4t}\left[\frac{d^4 f}{dt^2} - 4\frac{d^3 f}{dt^3} + 2\frac{d^2 f}{dt^2} + 4\frac{df}{dt} - 3f\right] = 0$

$f = e^{st} \to s^4 - 4s^3 + 2s^2 + 4s - 3 = 0 \to s = 3, -1, +1, +1.$

$f(r) = Ar^3 + B\frac{1}{r} + Cr + Dr\ln r$

Stresses $\to \sigma_r = \left(\frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta}\right) = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$

$\sigma_\theta = \phi_{,rr} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta$

$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\phi_{,\theta}\right) = -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta$

Stress BC's : $\quad \sigma_r = \sigma_{r\theta} = 0 \quad$ at $r = r_o$, $r = r_i \longrightarrow$ gives only two

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longleftarrow$ indep equs, i.e,

$$2Ar_i - \frac{2B}{r_i^3} + \frac{D}{r_i} = 0 \quad \longrightarrow ⓐ \qquad \text{(see form of } \sigma_r, \sigma_{r\theta}).$$

$$2Ar_o - \frac{2B}{r_o^3} + \frac{D}{r_o} = 0 \quad \longrightarrow ⓑ$$

Also, $\displaystyle\int_{r_i}^{r_o} \sigma_{r\theta}\Big|_{\theta=0} dr = -\int_{r_i}^{r_o} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\Big|_{\theta=0} dr = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{r_o}^{r_i}\Big|_{\theta=0} = P$

$$\Longrightarrow \quad -A(r_o^2 - r_i^2) + \frac{B(r_o^2 - r_i^2)}{r_i^2 r_o^2} - D \ln \frac{r_o}{r_i} = P \quad \longrightarrow ⓒ$$

$(a, b, c) \longrightarrow \quad A = \dfrac{P}{2N} \; , \quad B = \dfrac{-P r_i^2 r_o^2}{2N} \; , \quad D = \dfrac{-P}{N}(r_i^2 + r_o^2), \quad \boxed{N \overset{\triangle}{=} r_i^2 - r_o^2 + (r_i^2 + r_o^2)\ln\left(\frac{r_o}{r_i}\right)}$

\# Note: $\sigma_\theta\Big|_{\theta=0} = 0$ identically satisfied.

Soln is exact only if P applied via $\sigma_{r\theta}$ as above, else
St. Venant's effect.

Displacements.

$$\varepsilon_r = u_{r,r} = \frac{\sin\theta}{E}\left[2Ar(1-3\nu) - \frac{2B}{r^3}(1+\nu) + \frac{D}{r}(1-\nu)\right] \longrightarrow ⓓ$$

$$u_{\theta,\theta} = r\varepsilon_\theta - u_r = \frac{r}{E}(\sigma_\theta - \nu\sigma_r) - u_r = \frac{r}{E}\sin\theta\left[Ar(6-2\nu) + \frac{2B}{r^3}(1+\nu) + \frac{D}{r}(1-\nu)\right] - u \longrightarrow ⓔ$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right] \longrightarrow ⓕ$$

$\displaystyle\int ⓓ \longrightarrow \quad u_r = \frac{\sin\theta}{E}\left[Ar^2(1-3\nu) + \frac{B}{r^2}(1+\nu) + D(1-\nu)\ln r\right] + f(\theta) \longrightarrow ⓖ$

$\displaystyle\int ⓔ \longrightarrow \quad u_\theta = -\frac{\cos\theta}{E}\left[Ar^2(5+\nu) + \frac{B}{r^2}(1+\nu) - D\log r(1-\nu) + D(1-\nu)\right] - \int f\,d\theta + F(r) \longrightarrow ⓗ$

$(g),(h)$ in $ⓕ \longrightarrow \displaystyle\int f\,d\theta + f' + rF' - F = \frac{-4D\cos\theta}{E}$

$\Longrightarrow \displaystyle\int f\,d\theta + f' + \frac{4D\cos\theta}{E} = F - rF' = \text{const}. \overset{\alpha}{\nearrow}$

$$f + f'' = \frac{4D\sin\theta}{E} \Rightarrow f = \underbrace{K\sin\theta + L\cos\theta}_{f_h} - \underbrace{\frac{2D}{E}\theta\cos\theta}_{f_p} \quad \text{(49)}$$

$$\Rightarrow \alpha = 0. \Rightarrow F = rF' \Rightarrow F = Hr.$$

Thus u_r, u_θ are obtained (they contain A, B, D, H, K, L).

Displ. BC's: $\quad u_\theta = 0, \quad \frac{\partial u_\theta}{\partial r} = 0 \quad$ at $\theta = \frac{\pi}{2}, \forall r.$

$$\text{gives } H = 0, \quad L = \frac{D\pi}{E}$$

$$u_r = 0 \text{ at } \theta = \frac{\pi}{2}, \quad r = \bar{r} = \frac{r_0 + r_i}{2} \longrightarrow \text{gives } K.$$

$$u_r\Big|_{\theta = 0} = L = \frac{D\pi}{E} = -\frac{P\pi(r_i^2 + r_0^2)}{E\left[(r_i^2 - r_0^2) + (r_i^2 + r_0^2)\ln(r_0/r_i)\right]}$$

When $r_0 \simeq r_i$, $\ln(r_0/r_i) = \ln\left(1 + \frac{h}{r_i}\right) \simeq \frac{h}{a} + O\left(\frac{h}{a}\right)^2$, where $h \triangleq r_0 - r_i$

$$\Rightarrow u_r\Big|_{\theta=0} = -\frac{3\pi a^3 P}{E h^3}$$

- Take $\phi = f(r)\cos\theta$ to get case where vertical force P and couple are applied (ie $BM = M + P\bar{r} - P\bar{r}\cos\theta$). Then subtract solution due to pure couple ($\because$ p.43 pure bending of curved bars) to get soln due to pure vertical load. Then do combination of soln due to vertical & horizontal load to get solution for arbitrary inclined load.

- Can also use this approach when $\sigma_r, \sigma_{r\theta}$ on curved faces are non-zero. However, it would apply only when loading on curved face is $(\sigma_{rr})_{applied} \propto \sin\theta$ and $(\sigma_{r\theta})_{applied} \propto \cos\theta$ ($\because$ b.m. at section, M is $\propto \sin\theta$ only for such loading and we started (see p.47) with this fact.
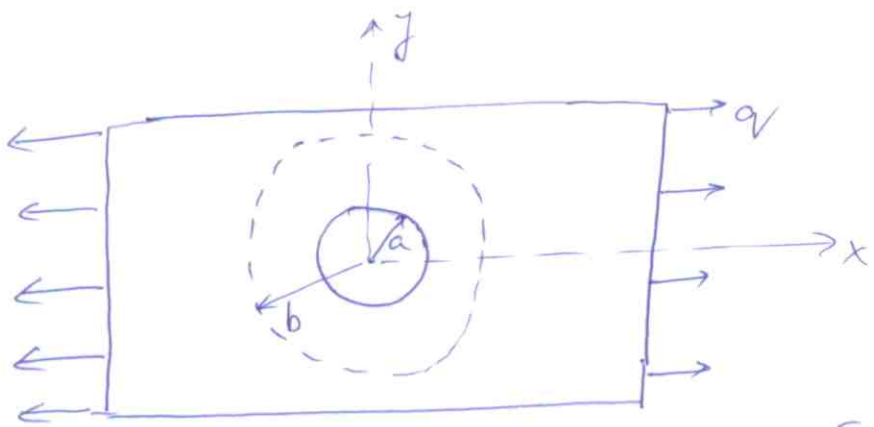
II Plate with circular hole.

Plate uniformly loaded in x-direction. Fichtitious boundary at b where $b \gg a$, $a =$ hole radius

$\therefore b \gg a$, $\left.\sigma_x\right|_{r=b} = q$, $\left.\sigma_y\right|_{r=b} = 0$, $\left.\sigma_{xy}\right|_{r=b} = 0$

$$\Rightarrow \left.\sigma_r\right|_{r=b} = \frac{q}{2} + \frac{q}{2}\cos 2\theta, \quad \left.\sigma_{r\theta}\right|_{r=b} = -\frac{q}{2}\sin 2\theta.$$

$$\left.\sigma_\theta\right|_{r=b} = \frac{q}{2} - \frac{q}{2}\cos 2\theta$$

Equivalent problem is a annular disk loaded with $\left.\sigma_r\right|_{r=b}$ and $\left.\sigma_{r\theta}\right|_{r=b}$ on its outer boundary.

Part(A) : Axisymm loading, $\left.\sigma_r\right|_{r=b} = \frac{q}{2}$ applied

$\Rightarrow$ put $p_o = -\frac{q}{2}$, $p_i = 0$ in Thick-walled cylinder soln, and use $b \gg a$, get

$$\sigma_r = \frac{q}{2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = \frac{q}{2}\left(1 + \frac{a^2}{r^2}\right) \quad \left. \right\} \text{ Part(A) solution.}$$

$$\sigma_{r\theta} = 0.$$

Part(B) $\left.\sigma_r\right|_{r=b} = \frac{q}{2}\cos 2\theta$, $\left.\sigma_{r\theta}\right|_{r=b} = -\frac{q}{2}\sin 2\theta$ applied.

Semi inverse method — try $\phi = f(r)\cos 2\theta$.

$$\nabla^4\phi = \cos 2\theta\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0.$$

$$\nabla^4\phi = 0 \longrightarrow \cos 2\theta \left[f^{IV} + \frac{2}{r} f^{III} - \frac{9}{r^2} f^{II} + \frac{9}{r^3} f^{I} \right] = 0.$$

$$r = e^t \longrightarrow \frac{d^4 f}{dt^4} - 4 \frac{d^3 f}{dt^3} - 4 \frac{d^2 f}{dt^2} + 16 \frac{df}{dt} = 0.$$

$$s^4 - 4s^3 - 4s^2 + 16 = 0, \quad s = 0, 2, -2, 4.$$

$$f(r) = A r^4 + B r^2 + C + \frac{D}{r^2}$$

$$\sigma_r = -\cos 2\theta \left[2B + \frac{4C}{r^2} + \frac{6D}{r^4} \right]$$

$$\sigma_\theta = \cos 2\theta \left[12 A r^2 + 2B + \frac{6D}{r^4} \right]$$

$$\sigma_{r\theta} = \sin 2\theta \left[6 A r^2 + 2B - \frac{2C}{r^2} - \frac{6D}{r^4} \right]$$

BC's : $\sigma_r \big|_{r=a} = \sigma_{r\theta} \big|_{r=a} = 0$

and $\sigma_r \big|_{r=b} = \frac{q}{2} \cos 2\theta$, $\sigma_{r\theta} \big|_{r=b} = -\frac{q}{2} \sin 2\theta$

$$\Rightarrow \quad 2B + \frac{4C}{b^2} + \frac{6D}{b^4} = -\frac{q}{2}$$

$$6 A b^2 + 2B - \frac{2C}{b^2} - \frac{6D}{b^4} = -\frac{q}{2}$$

$$2B + \frac{4C}{a^2} + \frac{6D}{a^4} = 0$$

$$6 A a^2 + 2B - \frac{2C}{a^2} - \frac{6D}{a^4} = 0$$

Solution $\left(\text{use } \frac{a}{b} \cong 0 \right) \longrightarrow A = 0, \ B = -\frac{q}{4}, \ C = \frac{q a^2}{2},$

$$D = -q \frac{a^4}{4}$$

Part (A) + (B)) solution : $\sigma_r = \frac{q}{2}\left(1 - \frac{a^2}{r^2} \right) + \frac{q}{2} \cos 2\theta \left(1 - \frac{a^2}{r^2} \right)\left(1 - 3 \frac{a^2}{r^2} \right)$
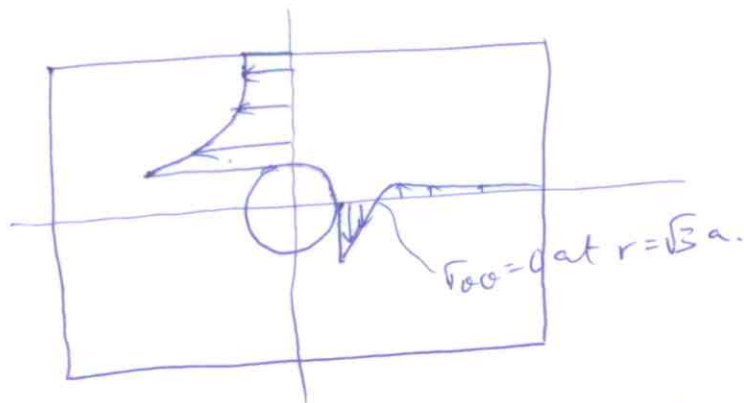
$$\sigma_\theta = \frac{q}{2}\left(1 + \frac{a^2}{r^2} \right) - \frac{q}{2} \cos 2\theta \left(1 + \frac{3 a^4}{r^4} \right)$$

$$\sigma_{r\theta} = -\frac{q}{2} \sin 2\theta \left(1 - \frac{a^2}{r^2} \right)\left(1 + 3 \frac{a^2}{r^2} \right)$$

at $r = a$, $\sigma_r = 0$, $\sigma_\theta = q(1 - 2\cos 2\theta)$

$$= 3q, \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$= -q, \quad \theta = 0, \pi.$$

at $\theta = \frac{\pi}{2}$, $\sigma_\theta = q\left(1 + \frac{1}{2}\frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)$

$\theta = 0$, $\sigma_\theta = -\frac{q}{2}\frac{a^2}{r^2}\left(\frac{3a^2}{r^2} - 1\right)$

$\sigma_{\theta\theta} = 0$ at $r = \sqrt{3}\,a$.

If q_2 (uniform) acts in $-y$-direction, superpose

$(q_1 \text{ soln}) + (q_1 \text{ soln with } q_1 \to q_2, \theta \to \theta - \pi/2)$.

In general for arbitrary shaped plate with arbitrary in-plane loads, approximate algorithm is:

(i) Find stresses for plate w/o hole.

(ii) Find stress comp's for plate w/o hole at coordinates of hole.

(iii) Find principal stresses corresponding to solution in (ii) above. Let them be q_1, q_2.

(iv) Put origin at hole center, and superpose solutions due to q_1 & q_2.

2D PROBLEMS IN CURVILINEAR COORDINATES USING COMPLEX POTENTIALS & CONFORMAL MAPPING

(I) Stress function in terms of complex potentials.

$\phi(x,y)$ = real potential function satisfying $\nabla^4\phi = 0$

(i.e., body forces neglected).

$z = x + iy \rightarrow$ complex variable., $\bar{z} = x - iy$.

Thus $(x,y) \leftrightarrow (z,\bar{z})$ is transformation from real to complex independent variable

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial \phi}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\phi$$

$$\frac{\partial \phi}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\phi \qquad \right\} \rightarrow ①$$

$$\Rightarrow \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = 2\frac{\partial \phi}{\partial \bar{z}} \quad , \quad \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = 2\frac{\partial \phi}{\partial z} \longrightarrow ②$$

$$\Rightarrow \nabla^2_{xy}\phi = 4\frac{\partial^2 \phi}{\partial z \partial \bar{z}} \quad , \quad \nabla^4_{xy}\phi = 16\frac{\partial^4 \phi}{\partial z^2 \partial \bar{z}^2} = 0. \longrightarrow ③$$

Solution to biharmonic is,

$$\phi = f_1(z) + \bar{z}f_2(z) + f_3(\bar{z}) + z f_4(\bar{z})$$

$\because \phi$ is real, $f_1 = \bar{f_3}$ and $f_2 = \bar{f_4}$.

Re-naming $f_1 \equiv \dfrac{\chi(z)}{2}$, $f_2 \equiv \dfrac{\psi(z)}{2}$ we obtain,

$$\boxed{\phi = \tfrac{1}{2}\left(\bar{z}\psi + z\bar{\psi} + \chi + \bar{\chi}\right) = \operatorname{Re}\left[\bar{z}\psi(z) + \chi(z)\right]}$$ Goursats formula. $\rightarrow ④$

Here ψ and χ are analytic functions, i.e., they satisfy CR eqns.

Aside on CR eqns.

Let $f(z) = \alpha(x,y) + i\beta(x,y)$ be analytic, ie possessing partial derivatives wrt x & y.

$$\Rightarrow \frac{df(z)}{dx} = f'(z) \quad , \quad \frac{\partial f(z)}{\partial y} = if'(z) \longrightarrow ⑤$$

Also, $\dfrac{\partial f(z)}{\partial x} = \left(\dfrac{\partial \alpha}{\partial x} + i\dfrac{\partial \beta}{\partial x}\right)$, $\dfrac{\partial f}{\partial y} = \left(\dfrac{\partial \alpha}{\partial y} + i\dfrac{\partial \beta}{\partial y}\right) \longrightarrow ⑥$

⑤ & ⑥ $\longrightarrow \dfrac{\partial \alpha}{\partial x} = +\dfrac{\partial \beta}{\partial y}$, $\dfrac{\partial \alpha}{\partial y} = -\dfrac{\partial \beta}{\partial x} \longrightarrow ⑦$ CR eqns

⑦ → $\nabla^2 \alpha = 0$, $\nabla^2 \beta = 0$ → ie α, β are Harmonic.

(II) Stress & displacement in terms of ψ & χ.

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

(3a, 4) →

$$\sigma_y + \sigma_x = \nabla^2 \phi = 4 \frac{\partial^2 \phi}{\partial z \partial \bar{z}} = 4 \, \text{Re} \, \psi'(z) \longrightarrow (7a)$$

(2b),(4) →

$$\sigma_y - \sigma_x + 2i \sigma_{xy} = \phi_{,xx} - \phi_{,yy} - 2i \phi_{,xy} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^2 \phi$$

$$= 4 \frac{\partial^2 \phi}{\partial z^2} = 2[\bar{z}\psi'' + \chi''] \longrightarrow (7b)$$

Gives Cartesian components of stress in terms of $\psi(z), \chi(z)$

Using Strain-displ & CL, for plane stress we have,

$$E \frac{\partial u}{\partial x} = (\sigma_x + \sigma_y) - (1+\nu)\sigma_y = 2(\psi' + \bar{\psi}') - (1+\nu)\phi_{,xx} \quad \Big\}$$
$$E \frac{\partial v}{\partial y} = (\sigma_x + \sigma_y) - (1+\nu)\sigma_x = 2(\psi' + \bar{\psi}') - (1+\nu)\phi_{,yy} \quad \Big\} (8)$$

$$\frac{E}{2(1+\nu)}(v_{,x} + u_{,y}) = \sigma_{xy}$$

Now $\frac{\partial}{\partial x}(\psi \pm \bar{\psi}) = \psi'(z) + \bar{\psi}'(z)$, $\frac{\partial}{\partial y}(\psi \pm \bar{\psi}) = i\left(\psi'(z) \mp \bar{\psi}'(z)\right)$ (9)

(9, 8a,b) →

$$E u_{,x} = 2(\psi + \bar{\psi})_{,x} - (1+\nu)\phi_{,xx} \quad \Big\} \longrightarrow (10)$$
$$E v_{,y} = -2i(\psi - \bar{\psi})_{,y} - (1+\nu)\phi_{,yy} \quad \Big\}$$

∫(10) →

$$E u = 2(\psi + \bar{\psi}) - (1+\nu)\phi_{,x} + U(y) \quad \Big\} \longrightarrow (11)$$
$$E v = -2i(\psi - \bar{\psi}) - (1+\nu)\phi_{,y} + V(x) \quad \Big\}$$

(11) in (8c) w/(9) → $-U_{,y} = V_{,x} \Rightarrow U = u_0 - \omega y$, $V = v_0 + \omega x \longrightarrow (12)$

(details on reverse) ie U, V are RB displ, so discard them sina we consider only elastic displ.

(11, 2a, 4a) → $\boxed{E(u+iv) = 4\psi - (1+\nu)(\phi_{,x} + i\phi_{,y}) = (3-\nu)\psi - (1+\nu)(z\bar{\psi}' + \bar{\chi}')}$

Gives Cartesian comp of displ. in terms of $\psi(z), \chi(z)$. →(details on reverse) (13)

Details of (11) in §(C) w/(9):

$$\frac{E}{2(1+\nu)}(u_{,y}+v_{,x}) = \frac{1}{1+\nu}\left[(\psi+\overline{\psi})_{,y} - i(\psi-\overline{\psi})_{,x}\right] - \frac{1}{2}\,2\phi_{,xy} + U_{,y} + V_{,x} = \sigma_{xy}$$

$$\Rightarrow U_{,y} = -V_{,x}$$

Details of (13) (after discarding U, V)

$$\overline{E(u+iv)} = 4\psi - (1+\nu)[\phi_{,x}+i\,\phi_{,y}] = 4\psi - \frac{z}{z}(1+\nu)(\psi + z\overline{\psi'} + \overline{\chi'})$$

$$2\phi_{,\overline{z}} \qquad = (3-\nu)\psi - (1+\nu)(z\overline{\psi'} + \overline{\chi'})$$

- $(7a, 7b) \rightarrow$ If $(iCz + \gamma_1)$ & $(\gamma_2 z + \gamma_3)$ are added to $\psi(z)$ & $\chi(z)$, **55**
  respectively, where C is _{any} real const and γ_1,
  γ_2, γ_3 are _{any} complex constants, then stresses are
  not affected.

- $(13) \rightarrow$ If γ_4 and $\left(\frac{3-\nu}{1+\nu} \bar{\gamma}_4 z + \gamma_5\right)$ are added to ψ & χ, resply,
  where γ_4, γ_5 are _{any} complex const then displ's. are
  not affected. Further, _{(7a,7b) show that} even the stresses are not affected.

- For plane strain do $E \rightarrow E/(1-\nu^2)$, $\nu \rightarrow \nu/(1-\nu)$

(III) <u>Boundary conditions in terms of ψ, χ.</u>
<u>STRESS BC'S</u>

$(\mathbf{X}, \mathbf{Y})$ are tractions applied on boundary S.

$\ell = \cos\theta = \frac{dy}{ds}$, $m = \sin\theta = -\frac{dx}{ds}$.
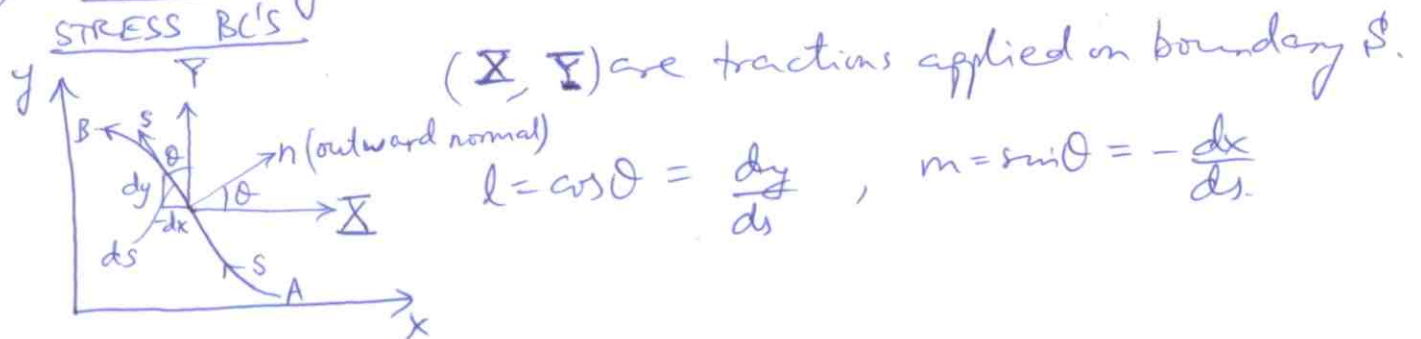
$$\mathbf{X} = \ell \sigma_x \Big|_S + m \tau_{xy} \Big|_S = \ell \phi_{,yy}\Big|_S - m \phi_{,xy}\Big|_S$$

$$= \frac{dy}{ds} \phi_{,yy}\Big|_S + \frac{dx}{ds} \phi_{,xy}\Big|_S = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)\Big|_S$$

$||^{arly},\ \mathbf{Y} = \ell \sigma_{xy}\Big|_S + m \sigma_y\Big|_S = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)\Big|_S$

(2) & above $\rightarrow \mathbf{X} + i\mathbf{Y} = \frac{d}{ds}\left(\phi_{,y} - i\phi_{,x}\right)\Big|_S = -2i\frac{d}{ds}\left(\frac{d\phi}{d\bar{z}}\right)\Big|_S$

$$= -i\frac{d}{ds}\left(\overline{\chi'} + z\overline{\psi'} + \psi\right)$$

$$\Rightarrow \left[\overline{\chi'} + z\overline{\psi'} + \psi\right]\Big|_A^B = i\int_A^B (\mathbf{X} + i\mathbf{Y})\, ds$$

Let B be generic point z. Let $\left(\overline{\chi'} + z\overline{\psi'} + \psi\right)\Big|_A = k$, a _{complx} constant.
Choose additive terms for ψ & χ as γ_4 and $\frac{3-\nu}{1+\nu}\bar{\gamma}_4 z$, resply,
with γ_4 chosen such that $\left(\frac{3-\nu}{1+\nu}\gamma_4 + \bar{\gamma}_4\right) = k$. Then the extra contribution
in $\left[\overline{\chi'} + z\overline{\psi'} + \psi\right]\Big|_B$ cancels out $(-k)$ arising from $\left[\overline{\chi'} + z\overline{\psi'} + \psi\right]_A$, ie
k drops out.

Thus

$$\overline{\left[\overline{X'(z)} + z\,\overline{\psi'(z)} + \psi(z)\right)_S} = i\int(\overline{X} + i\overline{Y})\,ds \Big] \longrightarrow (14)$$

- Eqn (14) represents stress b.c's in terms of ψ, X.
- It relates the resultant of surface forces between generic pt & datum (i-e, surface integral term) to the bracketed term containing complex potentials evaluated at a generic pt on S.

DISPLACEMENT BC'S

When $\hat{u}$ and $\hat{v}$ are applied displ's on S, they must satisfy (13), i.e.,

$$E(\hat{u}+i\hat{v}) = \left[(3-\nu)\,\psi(z) - (1+\nu)\left(z\,\overline{\psi'(z)} + \overline{X'}\right)\right]_S \longrightarrow (15)$$

(IV) Conformal Mapping & Curvilinear Coordinates.

- Used for problems with irregular (e.g. non-circular) boundaries.

Consider mapping from ζ plane (non-physical) to z-plane (physical):

(16) $\longleftarrow$ $\boxed{z = f(\zeta)}$, $z = x+iy$, $\zeta = \xi(x,y) + i\eta(x,y)$

- $f(\zeta)$ is analytic $\Rightarrow$ ξ & η satisfy C.R. eqns.
- $f(\zeta)$ suitably chosen to map regular boundary (eg. circle) in ζ-plane to irregular (i-e, less simple) one in z-plane (eg. ellipse). This makes the application of b.c's more manageable.

Use polar coords in ζ-plane,
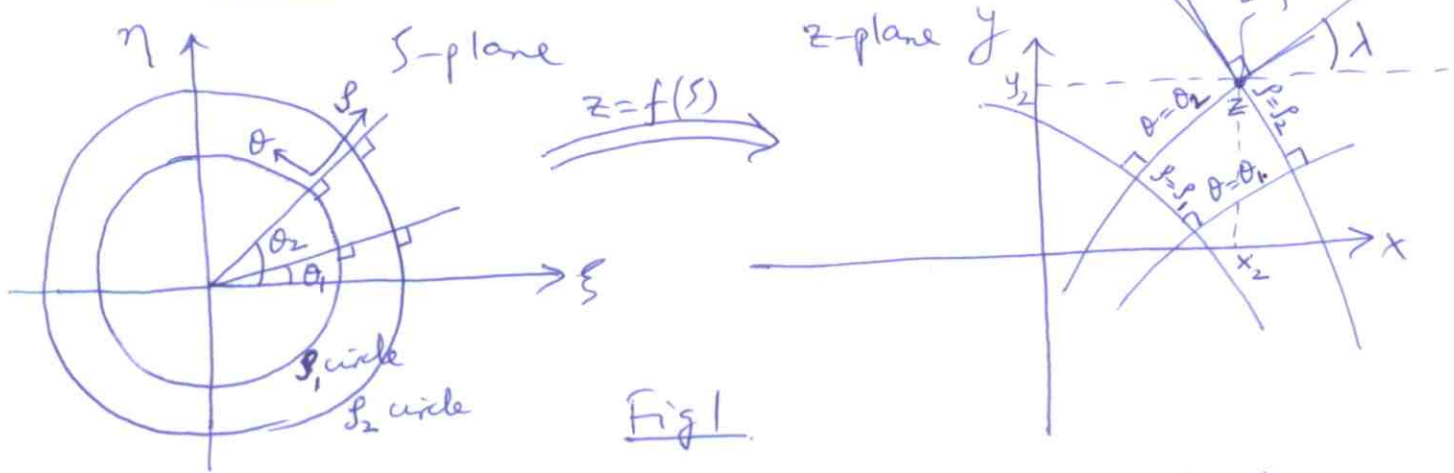
$$\boxed{\zeta = \xi + i\eta = \rho e^{i\theta}} \longrightarrow (17)$$

ζ-plane

$z = f(\zeta)$

z-plane

Fig 1.

- $\rho = \text{const}$ are θ coordinate curves, while $\theta = \text{const}$ are ρ coordinate curves. Hence (ρ, θ) are curvilinear coordinates.

- $\because f$ is analytic the transformation across ζ and z planes is conformal (ie, angle preserving — see eg. Kreyzig).

- (ρ, θ) are curvilinear coordinates and (x_2, y_2) are cartesian coordinates of same point in physical plane.
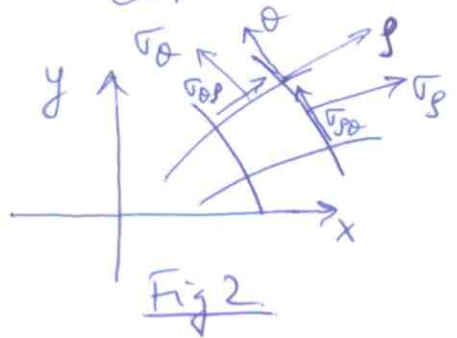
Fig 2.

$$\sigma_{\rho/\theta} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sigma_x - \sigma_y}{2}\cos 2\lambda \pm \sigma_{xy}\sin 2\lambda \Bigg\}$$

$$\sigma_{\rho\theta} = \frac{\sigma_x - \sigma_y}{2}\sin 2\lambda + \sigma_{xy}\cos 2\lambda. \Bigg\} \quad (18)$$

Fig 1 $\Rightarrow$
$$dz = |dz|\, e^{i\lambda} \quad \bigg\}\text{ for } \theta = \text{const curve.}$$
$$d\zeta = |d\zeta|\, e^{i\theta} \quad \bigg\} \longrightarrow (19)$$

$(16, 17, 19) \longrightarrow e^{i\lambda} = \dfrac{\zeta}{\rho}\dfrac{f'(\zeta)}{|f'(\zeta)|} \Rightarrow e^{2i\lambda} = \dfrac{\zeta^2}{\rho^2}\dfrac{f'(\zeta)}{\overline{f'(\zeta)}} \longrightarrow (20)$

$(18, 7, 20) \longrightarrow$

Gives curvilinear components of stress as function of Cartesian coords + curvilinear coords

$$\boxed{\begin{aligned} \sigma_\theta + \sigma_\rho &= 4\,\mathrm{Re}\,\psi'(z) \\ \sigma_\theta - \sigma_\rho + 2i\sigma_{\rho\theta} &= 2\left[\bar{z}\psi''(z) + \chi''(z)\right]\dfrac{\zeta^2}{\rho^2}\dfrac{f'(\zeta)}{\overline{f'(\zeta)}} \end{aligned}} \longrightarrow (21a)$$

Defining, $\psi(z) = \psi(f(S)) \triangleq \tilde{\psi}(S)$

$\underbrace{\chi'(z) \triangleq \chi_1(z)}$, $\chi_1(z) = \chi_1(f(S)) \triangleq \tilde{\chi}(S)$

Can introduce this from Eqn (7) onwards ∴ only $\chi'(z)$ & higher derivatives appear.

$\psi'(z) = \tilde{\psi}'(S)/f'(S) \triangleq \hat{\psi}(S)$

$\chi''(z) = \chi_1'(z) = \tilde{\chi}'(S)/f'(S) \triangleq \hat{\chi}(S)$

$\psi''(z) = \hat{\psi}'(S)/f'(S)$

$\Big\}$ (21b)

$(21a,b) \rightarrow$

$$\sigma_\theta + \sigma_\rho = 2\left[\hat{\psi}(S) + \overline{\hat{\psi}(S)}\right] = 4\,\mathrm{Re}\left[\hat{\psi}(S)\right]$$

$$\sigma_\theta - \sigma_\rho + 2i\,\sigma_{\rho\theta} = \frac{2S^2}{\rho^2\,\overline{f'(S)}}\left[\overline{f(S)}\,\hat{\psi}'(S) + f'(S)\,\hat{\chi}(S)\right]$$

$\longrightarrow$ (21c)

Gives curvilinear components of stresses of a point in physical plane as a function of its curvilinear coordinates (ie its coords in non-physical plane).

Curvilinear components of displacements are,

$$u_\rho = u\cos\lambda + v\sin\lambda \quad, \quad u_\theta = -u\sin\lambda + v\cos\lambda \longrightarrow (22)$$

$(22, 13, 20, 21b) \rightarrow$

$$E(u_\rho + i u_\theta) = \left[(3-\nu)\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)}\right]\frac{S}{\rho}\frac{f'(S)}{|f'(S)|}$$

$$E(u_\rho + i u_\theta) = \left[(3-\nu)\tilde{\psi}(S) - f(S)\overline{\hat{\psi}(S)} - \overline{\tilde{\chi}(S)}\right]\frac{S}{\rho}\frac{f'(S)}{|f'(S)|}$$

$\Big\}$ (23) (a,b)

(V) <u>Solution method for Infinite body with non-circular holes.</u>

The method of Mushkelishvili is outlined below. Here we map a unit circle in the ζ-plane, (centered at $\zeta = 0$) to the desired shaped hole in z-plane using,

$$\boxed{z = f(\zeta) = R\left(\frac{1}{\zeta} + c_0 + c_1 \zeta + \cdots + c_n \zeta^n\right).} \longrightarrow (24)$$

where n is tve integer, R = real constant, $c_0 \cdots c_n$ are cmplx consts and $|c_1| + \cdots |c_n| < 1$.

From conditions of finite & single-valued stress & displ. solutions, [*] it <u>can be shown</u> (ref. Sokolnikoff) using (24) that $\widetilde{\psi}(\zeta)$ and $\widetilde{X}(\zeta)$ should be of the form,

$$\boxed{\begin{aligned} \widetilde{\psi}(\zeta) &= \frac{1+\nu}{8\pi}(X+iY)\ln\zeta + B f(\zeta) + \widetilde{\psi}_0(\zeta) \\ \widetilde{X}(\zeta) &= -\frac{3-\nu}{8\pi}(X-iY)\ln\zeta + (B^*+iC^*)f(\zeta) + \widetilde{X}_0(\zeta) \\ \text{where } \widetilde{\psi}_0(\zeta) &= \sum_{k=1}^{\infty}\alpha_k \zeta^k, \quad \widetilde{X}_0(\zeta) = \sum_{k=1}^{\infty}\beta_k \zeta^k \end{aligned}} \longrightarrow (25)$$

(left margin) (* lengthy & non-trivial)

(left margin) see (29) for B, B^*, C^* definitions.

$(25) \Rightarrow \widetilde{\psi}, \widetilde{X}$ analytic inside unit circle $|\zeta| = 1$.

Here $\boxed{X = \sum_{k=1}^{m} X_k, \quad Y = \sum_{k=1}^{m} Y_k, \quad X_k = \oint X ds, \quad Y_k = \oint Y ds}$

i.e, X_k, Y_k represent total applied surface force (26) components on k^{th} hole and X, Y are total applied surface force components (ie, summed over all holes).

The boundary conditions are assumed as stress bc's, ie (14). This transforms to (use 16, 21(b), 25)

$$\boxed{\widetilde{\psi}_0(\sigma) + \frac{f(\sigma)}{\overline{f'(\sigma)}}\,\overline{\widetilde{\psi}_0'(\sigma)} + \overline{\widetilde{X}_0(\sigma)} = f_0} \longrightarrow (27)$$

where $\sigma = \left. S \right|_{\rho = 1} = 1 \cdot e^{i\theta}$ and,

(28)
$$f_0 = i \int (\overline{X} + i\overline{Y}) ds - \frac{X + iY}{2\pi} \ln \sigma - \frac{1+\nu}{8\pi} (X - iY) \frac{f(\sigma)}{\overline{f'(\sigma)}} \sigma$$
$$- 2B f(\sigma) - (B^* - iC^*) \overline{f(\sigma)}$$

(29)
$$B = \frac{1}{4} (\sigma(1) + \sigma(2)), \quad B^* + iC^* = -\frac{1}{2} (\sigma(1) - \sigma(2)) e^{-2i\alpha}$$

- $\sigma(1), \sigma(2)$ are principal stresses in far field and α is angle of $\sigma(1)$ with x-axis.

- Thus f_0 is known f^n of σ, i.e. known from the applied load (bc's) at hole and the applied far-field load and the mapping function.

- $\overline{X} = \overline{Y} = 0$, $X = Y = 0$ if hole is not loaded.

- $B = B^* = C^* = 0$ when far-field is not loaded.
  $\therefore$ in that case loading only the hole wont much affect the far-field which will remain almost stress free.

- (28) and onwards assumes a single hole.

Now recall Cauchy integral formulas (as applied to our unit circle):
(i) If $F(S)$ is analytic inside the unit circle then

$$\frac{1}{2\pi i} \oint_\sigma \frac{F(\sigma)}{\sigma - S} d\sigma = F(S) \longrightarrow \text{(30a)}$$

(ii) If $F(S)$ is analytic outside the unit circle,

$$\frac{1}{2\pi i} \oint \frac{F(\sigma)}{\sigma - S} d\sigma = F(\infty) \longrightarrow \text{(30b)}$$

Analytic $\Longleftrightarrow$ $F(S)$ should be defined and have a derivative at every pt. inside unit circle.

(eg) $\widetilde{\psi}_0$ (see (27)) analytic inside unit circle, while $\overline{\widetilde{\psi}}_0$ analytic outside unit circle.

Thus (27) and its conjugate yield (using 25(c,d) & 30(a,b))

$$\widetilde{\psi}_0(\varsigma) = \frac{1}{2\pi i} \oint_\sigma \left(f_0 - \frac{f(\sigma)}{\overline{f'(\sigma)}} \overline{\widetilde{\psi}'_0(\sigma)} \right) \frac{d\sigma}{\sigma - \varsigma}$$

$$\widetilde{X}_0(\varsigma) = \frac{1}{2\pi i} \oint_\sigma \left(\overline{f}_0 - \frac{\overline{f(\sigma)}}{f'(\sigma)} \widetilde{\psi}'_0(\sigma) \right) \frac{d\sigma}{\sigma - \varsigma}$$

$\longrightarrow$ (31)

Algorithm:

① Choose mapping $f(\varsigma)$ to map $|\varsigma| = 1$ to physical hole.

② Find f_0 using (28) (also (29)).

③ Find $\widetilde{\psi}_0(\varsigma)$ using (31a). Here for $\widetilde{\psi}'_0(\sigma)$ required in integral, use (25c).

④ Find $\widetilde{X}_0(\varsigma)$ using (31b)

⑤ Find $\widetilde{\psi}(\varsigma)$, $\widetilde{X}(\varsigma)$ using (25 a,b)

⑥ Find stresses and displacements using (21b) (21c), (23).

Application to plate with elliptic hole.

Step① Choose $z = f(\mathfrak{J}) = R\left(\frac{1}{\mathfrak{J}} + m\mathfrak{J}\right)$

$\left(\; R = \dfrac{a+b}{2}, \quad m = \dfrac{a-b}{a+b}, \quad a,b \text{ are semi-axes} \atop \qquad\qquad\qquad\qquad\qquad\text{and } m < 1 \text{ (obvious)} \right.$

$z = x + iy, \quad \mathfrak{J} = \mathfrak{f} e^{i\theta}$ yields,

$x = R\left(\frac{1}{\mathfrak{f}} + m\mathfrak{f}\right)\cos\theta, \quad y = -R\left(\frac{1}{\mathfrak{f}} - m\mathfrak{f}\right)\sin\theta$

$\Rightarrow \dfrac{x^2}{R^2\left(\frac{1}{\mathfrak{f}} + m\mathfrak{f}\right)^2} + \dfrac{y^2}{R^2\left(\frac{1}{\mathfrak{f}} - m\mathfrak{f}\right)^2} = 1 \longrightarrow$ family of ellipses for different $\mathfrak{f}$'s.

$\dfrac{x^2}{4R^2 m \cos^2\theta} - \dfrac{y^2}{4R^2 m \sin^2\theta} = 1 \longrightarrow$ family of hyperbolae for different θ's.

Take $\mathfrak{f} = 1$, θ varying (ie elliptical coords).

$z = f(\mathfrak{J})$.

$D'\,(\theta = \pi/2)$

$B'\,(\theta = -\pi/2)$
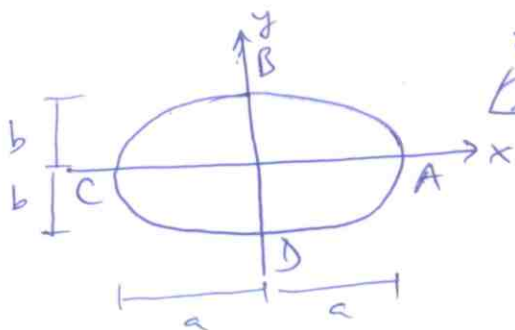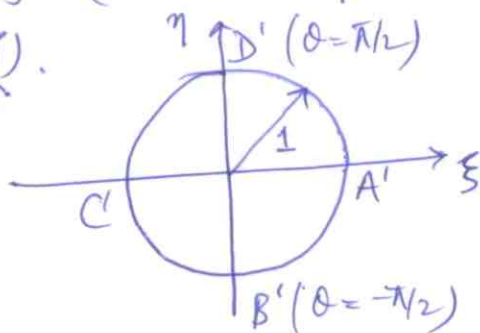
Step 2
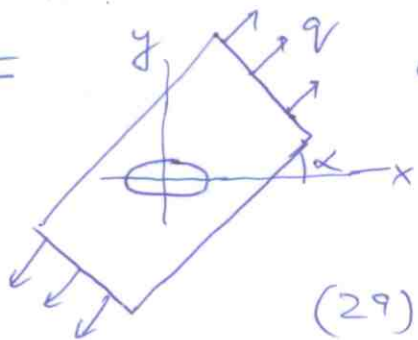
as Plate with elliptic hole loaded at angle α wrt major axis.

$\sigma(1) = q, \quad \sigma(2) = \overline{X} = \overline{Y} = X = Y = 0$

$(29) \Rightarrow B = \dfrac{q}{4}, \quad B^* - iC^* = -\dfrac{q}{2} e^{2i\alpha}. \longrightarrow \underline{(a)}$

$(30) \Rightarrow f_0 = -\dfrac{qR}{2}\left(\frac{1}{\sigma} + m\sigma\right) + \dfrac{qR}{2}\left(\sigma + \frac{m}{\sigma}\right)e^{2i\alpha} \searrow \underline{(b)}$

Step 3. (31a) & (25c) yield,   integral = 0. 63

$$\tilde{\psi}_0(s) = \frac{1}{2\pi i} \oint \frac{f_0}{\sigma - s} d\sigma - \frac{1}{2\pi i} \oint_\sigma \left(-\frac{1}{\sigma} \frac{m\sigma^2 + 1}{\sigma^2 - m}\right)\left(\bar{\alpha}_1 + \frac{2\bar{\alpha}_2}{\sigma} + \frac{3\bar{\alpha}_3}{\sigma^2} + \cdots\right) \frac{d\sigma}{\sigma - s}$$

corresponding $F(s)$ is analytic outside unit circle so evaluate $F(\infty)$ which is zero.

$$= -\frac{qR}{2}\left[\left(1 - me^{-2i\alpha}\right)s + \left(m - e^{2i\alpha}\right)\frac{m+s}{1-ms^2}s\right]$$

$\longrightarrow$ (c)

Step 4. (c) & (31b) yield $\tilde{\chi}_0(s)$

Step 5 (25a,b), (c), $\tilde{\chi}_0(s)$ yield

$$(d) \longleftarrow \begin{cases} \tilde{\psi}(s) = \frac{qR}{4}\left[\frac{1}{s} + (2e^{2i\alpha} - m)s\right] \\[2mm] \tilde{\chi}(s) = -\frac{qR}{2}\left[\frac{1}{s}e^{-2i\alpha} + \frac{s^3 e^{2i\alpha} + (me^{2i\alpha} - m^2 - 1)s}{ms^2 - 1}\right] \end{cases}$$

Step 6. (d), (21 b,c), yield. $\sigma_\theta, \sigma_\rho, \sigma_{\rho\theta}$ as a function of s (ie real function of ρ, θ after separating real & imaginary parts in (21c)).

At the hole boundary, ie putting $s = \sigma$, get,

$$\sigma_\theta = q\left(\frac{1 - m^2 + 2m\cos 2\alpha - 2\cos 2(\theta + \alpha)}{1 + m^2 - 2m\cos 2\theta}\right)$$

$$\sigma_{\rho\theta} = \sigma_\rho = 0 \quad (\text{ie bc satisfied})$$

For $\underline{\alpha = 0}$, $(\sigma_\theta)_{max} = q\left(1 + \frac{2b}{a}\right)$, occurs at $\theta = \pm\frac{\pi}{2}$

$(\sigma_\theta)_{min} = -q$, occurs at $\theta = 0, \pi$.

$\alpha = \frac{\pi}{2}$, $(\sigma_\theta)_{max} = q\left(1 + \frac{2a}{b}\right)$, occurs at $\theta = 0, \pi$

$(\sigma_\theta)_{min} = -q$, occurs at $\theta = 0, \pi/2$

If only hole loaded with q,
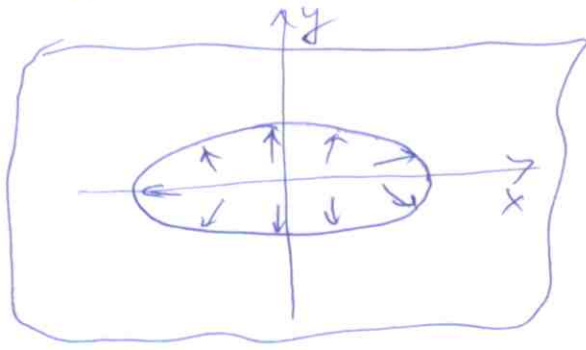
$$X + iY = -lq - imq$$

$$(X + iY)ds = -q(l\,ds + im\,ds)$$
$$= -q(dy - i\,dx)$$
$$= i\,q(dx + i\,dy) = iq\,dz$$

∴ resultant surface forces on hole are zero (due to symmetry), $X = Y = 0$. Also no applied far field load $\Rightarrow B = B^* = C^* = 0$.

$$\Rightarrow f_0(s) = i\int(X + iY)ds = -q\int dz = -qz = -qR\left(\frac{1}{s} + ms\right)$$

ie $f(\sigma) = -qR\left(\frac{1}{\sigma} + m\sigma\right)$

Results: $\widetilde{\psi}_0(s) = \widetilde{\psi} = -qRms$

$$\widetilde{X}_0(s) = \widetilde{X} = -qR(1 + m^2)\frac{s}{1 - ms^2}$$

$$\sigma_\theta = q\left(\frac{1 - 3m^2 + 2m\cos 2\theta}{1 + m^2 - 2m\cos 2\theta}\right)$$

$$\sigma_\rho = -q$$

$(\sigma_\theta)_{max} = q\left(\frac{2a}{b} - 1\right)$ at $\theta = 0, \pi$

$(\sigma_\theta)_{min} = q\left(-\frac{2b}{a} - 1\right)$ at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

$(|\sigma|)_{max} = (\sigma_\theta)_{max}$

$(|\sigma|)_{min} = |\sigma_\rho| = q$

Note that unlike circular sections, here you <u>cant</u>
assume that $\tau_{\theta z}$ is the only shear stress in the z-plane.
If you did so, you would get a non-zero $(\tau_{\theta z})_n$ (ie normal-to-
boundary component of $\tau_{\theta z}$) which remains unbalanced &
hence violates bc on longitudinal surface which are stress free.
You can also see it in the following manner. If only
$\tau_{\theta z}$ exists as the shear stress, then

$$\tau_{xz} = -\frac{y}{r}\tau_{\theta z} \quad, \quad \tau_{yz} = \frac{x}{r}\tau_{\theta z}.$$

lateral face bc $\Rightarrow$ $\tau_{xz}\underbrace{\ell}_{dy/ds} + \tau_{yz}\underbrace{m}_{-dx/ds} = 0 \Rightarrow \frac{\tau_{\theta z}}{r}\left(-ydy - xdx\right) = 0$

can be zero for circle only.

I) <u>Solution by Prandtl stress function (ϕ) formulation</u>
Guided by Solid Mechanics, assume τ_{xz} and τ_{yz} are
only non-zero stresses.

Equilibrium $\longrightarrow$ $\tau_{zx,z} = 0$
(zero body forces)
$\left.\begin{array}{c} \\ \tau_{zy,z} = 0 \end{array}\right\} \longrightarrow \tau_{xz}, \tau_{yz}$ f^n of (x,y).

$$\tau_{zx,x} + \tau_{zy,y} = 0$$

The stress function ϕ defined by

$$\boxed{\tau_{xz} = \phi_{,y} \quad, \quad \tau_{yz} = -\phi_{,x}}, \quad \phi = \phi(x,y) \longrightarrow ①$$

satisfies 3rd equil eqn.

BM compatibility eqns (see p.21, $\sigma_{ij,kk} = 0$) reduce to $\qquad$ ⑥⑥

$$\nabla^2 \sigma_{xz} = 0, \quad \nabla^2 \sigma_{yz} = 0, \quad \nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \longrightarrow ②$$

(can also get them from strain compatibility as on reverse).

$$①, ② \longrightarrow \frac{\partial}{\partial x}(\nabla^2 \phi) = 0, \quad \frac{\partial}{\partial y}(\nabla^2 \phi) = 0$$
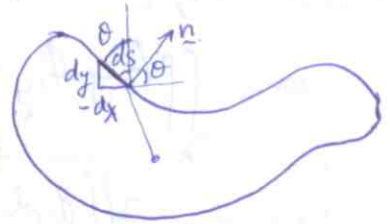
$$\Rightarrow \boxed{\nabla^2 \phi = K \text{ (const)}} \underset{\text{Poisson's eqn}}{\longrightarrow} ③$$

BC's on lateral face :

$$\underline{n} = (l, m, 0), \quad l = \frac{dy}{ds}, \quad m = -\frac{dx}{ds}$$

$$(\sigma_{ij} \cdot n_j)_s = \sigma_i^* \text{ (applied tractions)}$$

$i=1, \ i=2$ BC's are i.s. (identically satisfied)

$i=3 :: (l \sigma_{xz} + m \sigma_{yz})_s = 0$

$$\Rightarrow \frac{dy}{ds}\left(\frac{\partial \phi}{\partial y}\right)_s + \frac{dx}{ds}\left(\frac{\partial \phi}{\partial x}\right)_s = \left(\frac{d\phi}{ds}\right)_s = 0 \Rightarrow \boxed{(\phi)_s = \text{const}} \quad (4a)$$

$\therefore$ addition of constant to ϕ does not affect stresses,

so take $\boxed{(\phi)_s = 0}$ — valid for simply connected domain. $(4b)$

[For multiply connected domain, take $\phi=0$ along one
boundary and find the non-zero constant value on
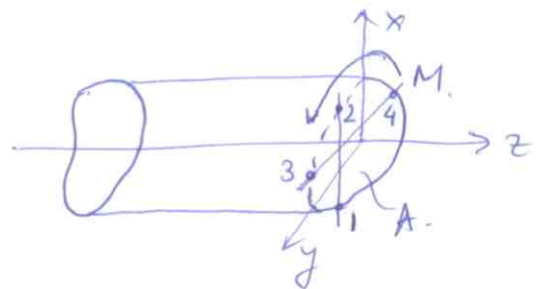other boundaries by single valuedness of displacements].

BC's on end faces.

$$\underline{n} = (0, 0, \pm 1)$$

$$\iint_A \sigma_{xz} \, dx \, dy = 0 \longrightarrow (5a)$$

$$\iint_A \sigma_{yz} \, dx \, dy = 0 \longrightarrow (5b)$$

$$\iint_A (-y \sigma_{xz} + x \sigma_{yz}) \, dx \, dy = M \longrightarrow (5c)$$

$$(5a) \longrightarrow \iint_A \frac{\partial \phi}{\partial y} \, dx \, dy = \int (\phi_{(3)} - \phi_{(4)}) \, dx = 0 \quad \left(\text{ie } (5a) \text{ is i.s.,} \atop \text{and } ||| \text{ only so is } (5b)\right)$$

Strain compat reduce to :

$$0 = 0 \quad -(i), \quad 0 = 0 \quad -(ii), \quad 0 = 0 \quad -(iii)$$

$$\varepsilon_{yz,xx} = \varepsilon_{xz,xy} \quad -(iv), \quad \varepsilon_{xz,yy} = \varepsilon_{yz,xy} - (v), \quad 0 = 0 \quad -(vi).$$

$$(iv) + 3^{rd} \text{ equil} \rightarrow \varepsilon_{yz,xx} = -\varepsilon_{zy,yy} \rightarrow \nabla^2 \varepsilon_{yz} = 0 \rightarrow \nabla^2 \tau_{yz} = 0$$

$$\text{||}^{aly} \ (v) + 3^{rd} \text{ equil} \rightarrow \nabla^2 \tau_{xz} = 0$$

————————— X X —————————

Alternative derivation for Eqn ⑥ :

$$M = \iint_A (-y \, \phi_{,y} - x \, \phi_{,x}) \, dx \, dy = -\iint \left((y\phi)_{,y} + (x\phi)_{,x} - 2\phi \right) dx \, dy$$

$$\boxed{M = 2\iint_A \phi \, dx \, dy - \oint_S (xl + ym) \, \phi \, ds} \longrightarrow ⑥^*$$

$$\underset{w/o, \ \phi = 0 \text{ on } S}{}$$

$$= 0 \text{ on } S$$

$$= 2\iint_A \phi \, dx \, dy$$

(5c)→

alternative (more elegant) for Eqs⑥ in p.6+6.

$$\iint_A \left(-y\frac{\partial \phi}{\partial y} - x\frac{\partial \phi}{\partial x}\right)dx\,dy = \int\left(-y(3)\phi(3) + y(4)\phi(4)\right)dx$$
$$+ \iint \phi\,dx\,dy + \int\left(-x(2)\phi(2) + x(1)\phi(1)\right)dy$$
$$+ \iint \phi\,dx\,dy$$

$$\Rightarrow \boxed{2\iint_A \phi\,dx\,dy = M.} \to ⑥ \to \underline{\text{for simply connected}}_{\text{only}} \overline{\left(\because \phi(3) = \phi(4) = 0 \text{ implied}\right)}.$$

Summary: solve ③ subject to ④b & ⑥ for $\phi\cdot(x,y)$.

solve ③ subject to ④b & ⑥ for $\phi\cdot(x,y)$. Then use ① to get stresses. Then get displacements.

Displacements:

$$\frac{\partial u}{\partial x} = 0 \quad, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = -\frac{1}{G}\frac{\partial \phi}{\partial x}, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{1}{G}\frac{\partial \phi}{\partial y} \Bigg\} ⑦$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

7(a,b,c) → $u = f_1(y,z), \quad v = f_2(x,z), \quad w = f_3(x,y) \to ⑧$

Subst this in 7(d,e,f) →

$$\frac{\partial f_3}{\partial y} + \frac{\partial f_2}{\partial z} = -\frac{1}{G}\frac{\partial \phi}{\partial x}$$
$$\frac{\partial f_1}{\partial z} + \frac{\partial f_3}{\partial x} = \frac{1}{G}\frac{\partial \phi}{\partial y} \Bigg\} ⑨$$
$$\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} = 0$$

$\frac{\partial (9b)}{\partial z} \Rightarrow \frac{\partial^2 f_1}{\partial z^2} = 0, \quad \frac{\partial (9c)}{\partial y} \Rightarrow \frac{\partial^2 f_1}{\partial y^2} = 0$

$$f_1 = ayz + by + cz + d \Bigg\} \to ⑪$$
$$f_2 = exz + fx + gz + h$$

11ally

(11, 9c) → $(e+a)z + f + b = 0 \Rightarrow e = -a, \ f = -b.$

$\Rightarrow u = f_1 = ayz + by + cz + d, \quad v = f_2 = -axz - bx + gz + h$

The constant & linear terms in u, v represent rigid body displacements & rotations, respectively. Assuming that the end $z=0$ is restrained to prevent RB motions, ie

$$b = c = d = g = h = 0,$$

$$\boxed{u = ayz, \qquad v = -axz.} \longrightarrow \quad \textcircled{12}$$

$$u_r = v\sin\theta + u\cos\theta = az\left(\underbrace{-x\sin\theta}_{r\cos\theta} + \underbrace{y\cos\theta}_{r\sin\theta}\right) \doteq 0 \Bigg\}$$

$$u_\theta = v\cos\theta - u\sin\theta = \alpha z r = \beta r \qquad \Bigg\} \longrightarrow \textcircled{13}$$

(where $\boxed{a = -\alpha}$ is used, $\beta = \alpha z =$ twist of section,

$\alpha = \dfrac{d\beta}{dz} =$ twist per length $=$ constant).

$7(c,d), (12) \longrightarrow$

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{G}\frac{\partial \phi}{\partial y} + \alpha y \\[2mm] \frac{\partial w}{\partial y} &= -\frac{1}{G}\frac{\partial \phi}{\partial x} - \alpha x \end{aligned}} \longrightarrow \textcircled{14} \longrightarrow \text{use to determine}$$
$w -$ warping displ.

$14(a,b) \longrightarrow \boxed{\nabla^2 \phi = -2G\alpha.} \longrightarrow \textcircled{15} \cdot \longrightarrow$ use to determine α from ϕ.

This implies that kinematics is inplane kinematics (ie u, v) resulting from rotation of points in a section thru angle β, and then out-of-plane kinematics (ie w) superposed. So plane sections dont stay plane.

MEMBRANE ANALOGY

Prandtl observed analogy with static deformation of a uniformly tensioned membrane subject to uniform (but small) transverse pressure.

transverse displacement.
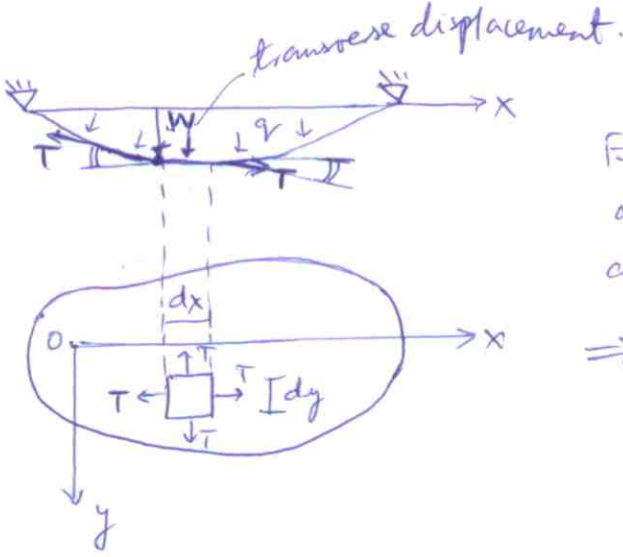
Equilibrate vertical components of T acting on edges of element $dxdy$ with the applied pressure.

$$\Rightarrow -T dy \frac{\partial w}{\partial x} + T dy \left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) dx \right)$$

$$-T dx \frac{\partial w}{\partial y} + T dx \left(\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) dy \right)$$

$$+ q\, dxdy = 0$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{q}{T} \qquad \Rightarrow \nabla^2\left(\frac{T}{q}w\right) = -1 \left.\right\} \rightarrow \boxed{16}$$

b.c. $(w)_s = 0$

Compare with Prandtl formulation ie $(8, 4b, 6)$, i.e.,

$$\nabla^2\left(\frac{\phi}{2G\alpha}\right) = -1, \quad \left(\frac{\phi}{2G\alpha}\right)_s = 0. \qquad \longrightarrow \boxed{17}$$

$(16, 17) \longrightarrow \quad \phi = \dfrac{2G\alpha T}{q} w.$

vol displaced by membrane

$(6) \rightarrow \quad M = 2\iint \phi\, dxdy = \dfrac{2G\alpha T}{q} 2 \underbrace{\overbrace{\iint_A w\, dxdy}}_{} = \dfrac{2G\alpha T}{q} 2V$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = \frac{2G\alpha T}{q} \frac{\partial w}{\partial y}$$

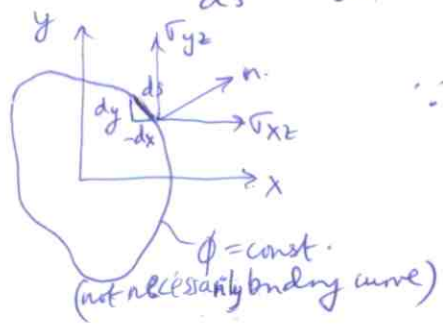If G, α, T, q, adjusted : $\dfrac{2G\alpha T}{q} = 1$, then

$$\boxed{\phi = w, \quad M = 2V, \quad \tau_{zx} = \frac{\partial w}{\partial y}, \quad \tau_{zy} = -\frac{\partial w}{\partial x}} \rightarrow \boxed{18}$$

Lines of shearing stress.

$$\nabla \phi \cdot (\sigma_{xz}, \sigma_{yz}) = (\phi_{,x}, \phi_{,y}) \cdot (\phi_{,y}, -\phi_{,x}) = 0$$

Thus on $\phi = const$ curves, total shear stress $(\sigma_{xz}, \sigma_{yz})$ lies along tangent to curves $(\because \nabla \phi$ is along normal$)$.

Thus, $l = \dfrac{dy}{ds} = \dfrac{\partial x}{\partial n} = \dfrac{\sigma_{yz}}{\sqrt{\sigma_{xz}^2 + \sigma_{yz}^2}}$, $\qquad m = -\dfrac{dx}{ds} = \dfrac{\partial y}{\partial n} = \dfrac{\sigma_{xz}}{\sqrt{\sigma_{xz}^2 + \sigma_{yz}^2}}$ ⑦⓪

y, σ_{yz}, n, σ_{xz}, dy, ds, $-dx$, x

$\phi = $ const.
(not necessarily bndry curve)

$\therefore$ resultant shear $\tau = (\sigma_{xz}, \sigma_{yz})$ lies tangential to contour we have

$$\sigma_{nz} = 0$$

$$\sigma_{sz} = \tau = \sigma_{yz}\frac{dy}{ds} - \sigma_{xz}\left(-\frac{dx}{ds}\right).$$

$$= -\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial n} - \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial n} = -\frac{\partial\phi}{\partial n}$$

$\Rightarrow$ magnitude of shearing stress is $\dfrac{\partial\phi}{\partial n}$

$\therefore$ τ is tangential to $\phi = $ const,

$$\boxed{\tau = -\frac{\partial\phi}{\partial n} = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}} \longrightarrow \text{⑲}$$

Now, $\nabla^2\tau^2 = 2\left(\phi_{,xx}^2 + \phi_{,x}\,\phi_{,xxx} + \phi_{,xy}^2 + \phi_{,y}\,\phi_{,xxy}\right.$

$\left. + \phi_{,yy}^2 + \phi_{,x}\,\phi_{,xyy} + \phi_{,y}\,\phi_{,yyy}\right)$

($\because$ $\nabla^2\phi = $ const the underlined pair of terms vanish.)

$\Rightarrow \nabla^2\tau^2 \geqslant 0 \Rightarrow \tau^2$ is subharmonic, and its maximum occurs on the boundary.

Note :- (i) $\nabla^2\tau^2 = 0$ & $\tau^2 \neq$ const inside region A, then max and min of τ^2 occur on boundary & not inside A

s, A

(ii) $\nabla^2\tau^2 \geqslant 0$ & $\tau^2 \neq$ const inside A, then τ^2 is subharmonic & max (τ^2) occurs on boundary & not inside A

(iii) $\nabla^2\tau^2 \leqslant 0$ & $\tau^2 \neq$ const inside A, then τ^2 is superharmonic and min (τ^2) occurs on bndry & not inside A.
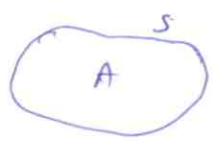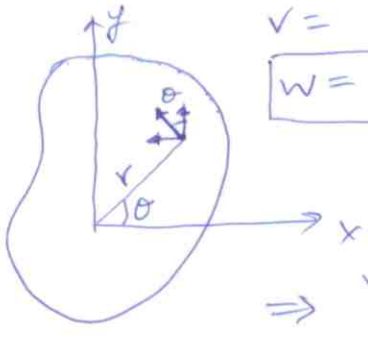
aside from Maths.

∴. Thus max shear stress occurs on boundary, so shear failure, will occur on lateral face.
if it occurs

(II) St. VENANT's semi-inverse method & formulation (ψ, ϕ)

Guided by kinematics of circular shaft, St. Venant

assumed $u = -u_\theta \sin\theta = -r\beta \sin\theta = \boxed{-y\alpha z = u}$ } → (20)

$v = u_\theta \cos\theta = r\beta \cos\theta = \boxed{x\alpha z = v}$ → (12)

$\boxed{w = \alpha \, \psi(x,y)}$

→ WARPING FUNCTION

Observe that (20 a,b) match (12).

$\Rightarrow \gamma_{xz} = \alpha\left(\dfrac{\partial\psi}{\partial x} - y\right)$, $\gamma_{yz} = \alpha\left(\dfrac{\partial\psi}{\partial y} + x\right)$ } → (21)

$\gamma_{xy} = \varepsilon_x = \varepsilon_y = \varepsilon_z = 0$

$\Rightarrow \boxed{\sigma_{xz} = G\alpha(\psi_{,x} - y) \quad , \quad \sigma_{yz} = G\alpha(\psi_{,y} + x)}$ → (22)

$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0 \Rightarrow$ no distortion in plane of $\otimes$ section.

Equilibrium → $1^{st}, 2^{nd}$ are i.s. 3rd yields

$\boxed{\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0}$ ⟶ (23)

(1), (22) → $\boxed{\phi_{,y} = G\alpha(\psi_{,x} - y) \quad, \quad \phi_{,x} = -G\alpha(\psi_{,y} + x)}$ → (24)

or from (14)

So eliminating ϕ from (24), you can get back (23).

Lateral face BC's :

$\left.\left(\ell\sigma_{xz} + m\sigma_{yz}\right)\right|_S = 0 \Rightarrow \left(\dfrac{dy}{ds}\right)\left(\dfrac{\partial\psi}{\partial x} - y\right) + \left(-\dfrac{dx}{ds}\right)\left(\dfrac{\partial\psi}{\partial y} + x\right) = 0$

$\overbrace{\partial x/\partial n}$ $\overbrace{\partial y/\partial n}$

$\Rightarrow \boxed{\dfrac{\partial\psi}{\partial n} = y\dfrac{\partial x}{\partial n} - x\dfrac{\partial y}{\partial n} = y\ell - xm}$ on S (lateral face's closed curve). → (25)

$\underbrace{}_{n.\nabla\psi}$

(23),(24) represent Neumann problem for ψ. Soln of ψ unique upto a constant (since bc involves $\nabla\psi$).

Now (23) alone $\Rightarrow \oint_S \dfrac{\partial\psi}{\partial n}\,ds = 0$. The details are:

$\oint_S \dfrac{\partial\psi}{\partial n}\,ds = \oint_S n.\nabla\psi\,ds = \iint_A \nabla.\nabla\psi\,dA = 0.$ ⟶ (A).

2D div thm.   A $=0$ from (23)

aside.

Does ㉕ satisfy condition (A). Yes it does ! (details ㉒

are :)

aside. ㉕ $\Rightarrow$ $\oint_S \frac{\partial \psi}{\partial n} ds = \oint_S (y\underset{\frac{dy}{ds}}{\ell} - x\underset{(-\frac{dx}{ds})}{m}) ds = \underset{S}{\oint} (ydy + xdx) \rightarrow d(\frac{x^2+y^2}{2}) = 0.$

End-face bc's :

The force resultant bc's will be i.s. as was the case in the stress-function approach. Details are :

$$\iint_A \sigma_{xz} dA \nearrow = \iint_A \left[\frac{\partial}{\partial x}(x\sigma_{xz}) + \frac{\partial}{\partial y}(x\sigma_{yz})\right] dA \underset{\text{2D Div Thom.}}{\overset{\nearrow}{=}} \int_S x(\ell\sigma_{xz} + m\sigma_{yz}) = 0.$$

add 3rd equil eqn $\underset{\substack{=0 \text{ from lateral face} \\ bc}}{\swarrow}$

and similarly for $\iint_A \sigma_{yz} dA = 0.$

Moment resultant **BC** becomes,

㉖ $M = \iint_A (-y\sigma_{xz} + x\sigma_{yz}) dxdy = G\alpha \iint_A \left(x^2+y^2 + x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right) dxdy = C\alpha$

ie $\boxed{M = C\alpha}$ $\underbrace{\qquad\qquad}_{\triangleq C/G}$ ㉗a

whee $\boxed{C = \text{torsional rigidity} := G\iint_A (x^2+y^2 + x\psi_{,y} - y\psi_{,x}) dxdy}$

Now consider $I = \iint_A \left(x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right)\frac{dx}{dy} = \iint_A \left(\frac{\partial(x\psi)}{\partial y} - \frac{\partial(y\psi)}{\partial x}\right) dxdy$

$\underset{\text{Div Thom}}{\overset{\nearrow}{=}} \oint_S (-\ell y + mx)\psi\, ds = -\oint_S \left(\ell\frac{\partial\psi}{\partial x} + m\frac{\partial\psi}{\partial y}\right)\psi\, ds$

$\underset{\substack{\text{Div Thom} \\ + ㉓}}{\overset{\nearrow}{=}} -\iint_A \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] dxdy \leq 0.$

$\Rightarrow \frac{C}{G} \leq \iint_A (x^2+y^2) dA$, ie $\boxed{C \leq GJ}$

For $\psi_{,x} = \psi_{,y} = 0$ in A, ie $\psi = \text{const}$ in A, $C = GJ$

Now $\psi = \underset{(\text{in A})}{\text{const}} \Rightarrow \left(\frac{\partial\psi}{\partial n}\right)_S = 0 \overset{\nearrow}{=} (y\underset{(\frac{dy}{ds})}{\ell} - x\underset{(-\frac{dx}{ds})}{m})_S = \left(\frac{d(\frac{x^2+y^2}{2})}{ds}\right)_S$

bc ㉕

$\Rightarrow S$ is a circle.

$\therefore \Rightarrow$ circle has max torsional rigidity when compared to another section having same J

$$\Rightarrow \boxed{C = GJ - G\iint [(\psi_{,x})^2 + (\psi_{,y})^2]\,dx\,dy} \longrightarrow \text{(27.b)} \qquad \text{(73)}$$
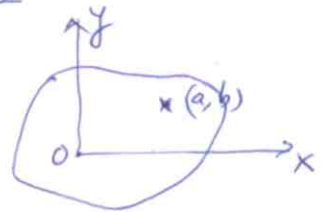
$$\text{(27a)} \times 2 - \text{(27b)}$$
$$\Rightarrow C = G\iint_A (x^2 + y^2 + (\psi_{,x})^2 + (\psi_{,y})^2 + 2x\psi_{,y} - 2y\psi_{,x})\,dA$$

$$= G\iint (\psi_{,y} + x)^2 + (\psi_{,x} - y)^2)\,dA = \frac{1}{\alpha^2 G}\iint_A (\sigma_{yz}^2 + \sigma_{xz}^2)\,dA$$

$$\text{ie} \quad \boxed{C = \frac{1}{\alpha^2 G}\iint (\sigma_{yz}^2 + \sigma_{xz}^2)\,dA} \longrightarrow \text{(27c)}$$

Change of torsion axis (ie reference origin)

New Axis passes thru (a,b), old one thru 0

$$\text{(28a)} \quad \begin{bmatrix} u = -\alpha(y-b)z, & v = \alpha(x-a)z, & w = \alpha\psi_1(xy) \\[4pt] \sigma_{xz} = G\alpha\left(\dfrac{\partial\psi_1}{\partial x} - y + b\right), & \sigma_{yz} = G\alpha\left(\dfrac{\partial\psi_1}{\partial y} + x - a\right) \end{bmatrix}$$

$$\longrightarrow \text{(28b)}$$

$$\sigma_{xz,x} + \sigma_{yz,y} = 0 \Rightarrow \nabla^2\psi_1 = 0.$$

$$\underline{\text{Lateral bc}}: \quad l\sigma_{xz} + m\sigma_{yz} = 0 \Rightarrow \frac{\partial}{\partial n}(\psi_1 + bx - ay) = ly - mx \underset{\text{on } s}{} \text{(28c)}$$

$$\text{So} \quad \boxed{\psi_1 = \psi - bx + ay + \text{const}} \underset{\longrightarrow \text{(29)}}{\text{ solves (28b,c) if } \psi \text{ solves (23,25)}}$$

$$\text{(28a)} \Rightarrow \sigma_{xz} = G\alpha(\psi_{,x} - y), \quad \sigma_{yz} = G\alpha(\psi_{,y} + x) \longrightarrow \text{i.e., } \underline{\text{stresses}}$$

remain invariant to change of torsion axis. So only effect is that displ's differ by rigid body comp.

These results are to be expected $\therefore$ origin wont affect stresses. In fact we presumed invariance by keeping $\boxed{\alpha \text{ the same}}$

Hence C is also invariant (see 27c).

$\therefore$ Now $\because C \le GJ \Rightarrow C \le GJ_{min}$ $\underset{J_c \text{ (centroidal axis)}}{}$ for

$$\nabla^2 \psi = 0 \implies \exists \, g : \boxed{\nabla^2 g = 0} \xrightarrow{\text{(30a)}} \text{where,}$$

$$\psi_{,x} = g_{,y} \, , \quad \psi_{,y} = -g_{,x} \longrightarrow CR \text{ eqns.}$$

lateral bc: $\quad \lambda \psi_{,x} + m \psi_{,y} = yl - xm \quad$ on S'

$$\implies \frac{\partial g}{\partial y} \frac{dy}{ds} + \frac{\partial g}{\partial x} \frac{dx}{ds} = y \frac{dy}{ds} + x \frac{dx}{ds}$$

$$\implies \boxed{g = \frac{x^2 + y^2}{2} + C \,(\text{const})} \longrightarrow \text{(30b)}$$

(30a,b) is the Dirichlet problem — unique solution for chosen constant. Stresses are,

$$\sigma_{xz} = G\alpha (g_{,y} - y) \, , \quad \sigma_{yz} = G\alpha (-g_{,x} + x)$$

They are unaffected by constant. Displacements are affected upto rigid body motions only.

Summary of formulations.

(i) Prandtl's stress function (ϕ)

$$\nabla^2 \phi = -2 G\alpha \, , \quad \phi = 0 \quad \text{on } S \text{ (simply-connected)}.$$

$$M = 2 \iint \phi \, dA \text{ (simply connected)}, \quad \sigma_{xz} = \phi_{,y}, \, \sigma_{yz} = -\phi_{,x}$$

(ii) St Venants Warping function (ψ)

$$\nabla^2 \psi = 0 \, , \quad \frac{d\psi}{dn} = yl - xm \quad \text{on } S \, , \quad M = C\alpha,$$

$$\sigma_{xz} = G\alpha (\psi_{,x} - y), \, \sigma_{yz} = G\alpha (\psi_{,y} + x) \quad \left| C = G \iint_A (x^2 + y^2 - (\psi_{,x})^2 - (\psi_{,y})^2) \, dA \right.$$

(iii) Conjugate Warping $f^n (g)$

$$\underline{\quad u = -\alpha y z, \, v = \alpha x z, \, w = \alpha \psi (x,y) \quad}$$

$$\nabla^2 g = 0, \quad g = \frac{x^2 + y^2}{2} + C \quad \text{on } S \, , \quad M = C\alpha ,$$

$$\sigma_{xz} = G\alpha (g_{,y} - y), \, \sigma_{yz} = G\alpha (-g_{,x} + x) \quad \left| C = G \iint_A (x^2 + y^2 - (g_{,x})^2 - (g_{,y})^2) \, dA. \right.$$

Solution Methods.

3 main methods exist.

(a) Semi-inverse method: Assume ϕ or ψ or g satisfying b.c's & check whether governing eqn is satisfied.

(b) Separation of variables based solution.

(c) Complex variable method.

Torsion of Elliptic sections

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$
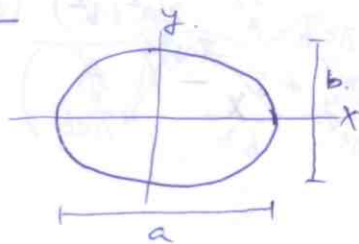
Prandtl's ϕ fn. approach.

③ & BC (4(a)) satisfied by choosing $\phi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

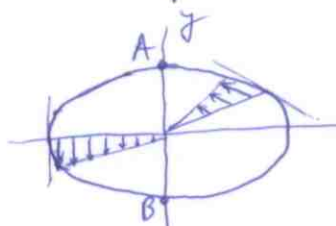or ⑮

③ / ⑮ $\Rightarrow$ $m = \frac{a^2 b^2}{2(a^2+b^2)} \underbrace{K}_{-2G\alpha}$

$\left.\begin{array}{l} \text{(over for details)} \\ \text{-- routine} \end{array}\right\}$

BC ⑥ $\Rightarrow$ $K = -2G\alpha = -2M\frac{(a^2+b^2)}{\pi a^3 b^3}$

$\Rightarrow \phi = -\frac{M}{ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

$\phi, ① \Rightarrow$ $\boxed{\sigma_{xz} = -\frac{2My}{\pi ab^3}, \quad \sigma_{yz} = \frac{2Mx}{\pi a^3 b}}$

$\Rightarrow \frac{\sigma_{xz}}{\sigma_{yz}} \propto \frac{y}{x}$ so direction of τ is constant along a radial line and coincides with the tangent at the boundary since bndry is a $\phi = $ const curve (recall lines of shearing stress).

$\sigma_{yz} = 0$ on vertical
$\sigma_{xz} = 0$ on horizontal axis.

on $\mathcal{S}$, $\tau^2 = \frac{4M^2}{\pi^2 a^2 b^2}\left(\frac{y^2}{b^4} + \frac{x^2}{a^4}\right) \overset{\text{on } \mathcal{S}, \; y^2/b^2 = 1 - x^2/a^2}{=} \frac{4M^2}{\pi^2 a^2 b^2}\left(\frac{1}{b^2} - x^2\left[\frac{1}{a^2 b^2} - \frac{1}{a^4}\right]\right)$

$> 0 \; \therefore \; a > b$

$\Rightarrow$ max τ^2 occurs for $x = 0$ on bndry, ie pts A, B

This matches membrane analogy which indicates max slope $\frac{\partial w}{\partial y}$, analogous to max σ_{xz}, at A & B.

Further, since $\tau = -\frac{\partial \phi}{\partial n}$ on the boundary (since it is a $\phi = $ const curve) since $\frac{\partial w}{\partial n}$ (see p. 70) we are certain that $\frac{\partial w}{\partial n}$ is max at A & B, then τ is also max at A & B

Details for elliptic section

$$\nabla^2 \phi = m\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = K \quad \Rightarrow \quad m = K\frac{a^2 b^2}{2(a^2+b^2)}$$

$$M = 2\iint_A \phi \, dx \, dy = 2\iint_A K\frac{a^2 b^2}{2(a^2+b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx \, dy$$

$$= K\frac{a^2 b^2}{(a^2+b^2)}\left(\underset{a^2}{\underbrace{\frac{I_y}{a^2}}_{\frac{1}{4}\pi a^3 b}} + \underset{b^2}{\underbrace{\frac{I_x}{b^2}}_{\pi a b^3/4}} - \underset{\pi a b}{\underbrace{A}}\right) = -\frac{\pi a^3 b^3 K}{2(a^2+b^2)}$$

Thus $\boxed{T_{max} = \dfrac{2M}{\pi a b^2}}$ $\left(\text{at } x=0, \; y=\pm b/2\right)$ with direction tgt to bndry.

Now $\boxed{C = \dfrac{M}{\alpha} = \dfrac{G\pi a^3 b^3}{a^2+b^2}}$ (from BC ⑥).

$\underset{\text{torsional rigidity}}{}$

Displacements, from ⑫, ⑭ are :

$$u = -\frac{(a^2+b^2)M}{\pi a^3 b^3 G}\, yz \quad , \qquad v = -\frac{(a^2+b^2)M}{\pi a^3 b^3 G}\, xz$$

$$\frac{\partial w}{\partial x} = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G}\, y \quad , \qquad \frac{\partial w}{\partial y} = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G}\, x$$

$$\Rightarrow \boxed{w = -\frac{(a^2-b^2)M}{\pi a^3 b^3 G}\, xy} + w_0$$

$\left(\text{Const, RB motion, so neglect since we will constrain appropriately}\right)$.

∴ $w = $ Const represent hyperbolas.

$w \,(\equiv \psi) = $ const.

------ concave

——— convex
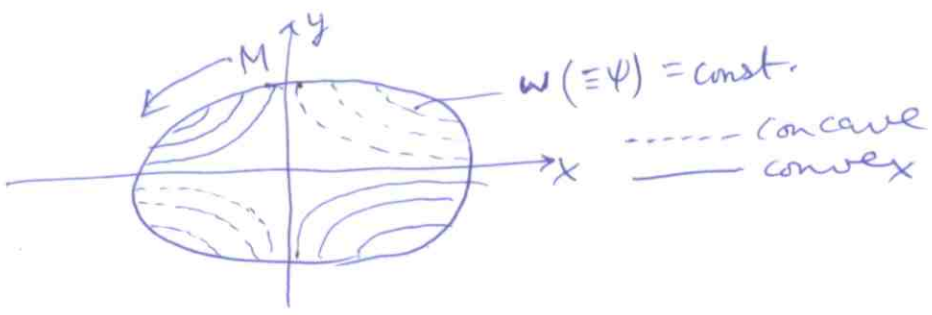
Complex variable approach :

Mapping function $\quad W(z) = i\left(c^2 z^2 + k^2\right)$,

$\quad z = x+iy, \qquad w = \psi + ig$.

$\Rightarrow \psi, g$ satisfy C.R. eqns so if $\nabla^2 \psi = 0 \Rightarrow \nabla^2 g = 0$.

Separating Re and Im parts,

$$\psi = -2xy\,c^2$$

$$g = (x^2-y^2)\,c^2 + k^2 \overset{bc}{=} \tfrac{1}{2}(x^2+y^2) \text{ on } S'$$

∵ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ on S we get,

$$c^2 = \frac{(a^2-b^2)}{2(a^2+b^2)} \quad , \qquad k^2 = \frac{a^2 b^2}{(a^2+b^2)} \;\rightarrow\; \begin{array}{l}\text{then}\\ \text{get stresses and}\\ \text{displ.}\end{array}$$

Torsion of Triangular section

$$y = -\frac{1}{\sqrt{3}}\left(x - \frac{2}{3}a\right)$$

$$y = \frac{1}{\sqrt{3}}\left(x - \frac{2}{3}a\right)$$
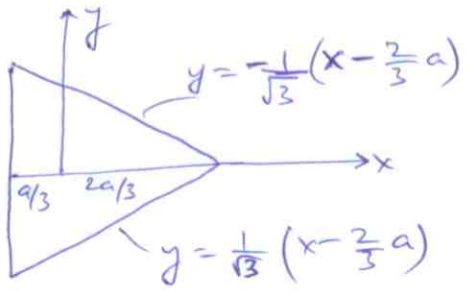
Eqn of boundary is,

$$F(x,y) = \left(x + \frac{a}{3}\right)\left(x - \frac{2}{3}a + \sqrt{3}y\right)\left(x - \frac{2}{3}a - \sqrt{3}y\right) = 0$$

Try,

$$\phi = m\,F(x,y) = m\left(x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right)$$

$$\nabla^2\phi = -2GK \implies m = \frac{G\alpha}{2a}$$

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -\frac{G\alpha y}{a}(3x + a)$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = -\frac{G\alpha}{2a}(3x^2 - 2ax - 3y^2)$$

$\tau_{xz} = 0$ on $x = -\frac{a}{3}$ (as it should be $\because x = -\frac{a}{3}$ is a stress curve so $\tau_{xz} = 0$ along it)

$\tau_{xz} = \tau_{yz} = 0$ at corners (as expected $\because \tau$ which lies along boundary curve cant have two directions)

$\tau_{xz} = \tau_{yz} = 0$ at $(x,y) = (0,0)$, ie centroid.

$$\boxed{M = \frac{3}{5}G\alpha J_0}, \quad J_0 = \frac{\sqrt{3}}{27}a^4 = \text{polar MI about centroid.}$$

on $y = 0$, $\tau_{xz} = 0$, $\tau_{yz} = -\frac{G\alpha x}{a}\left(\frac{3}{2}x - a\right)$
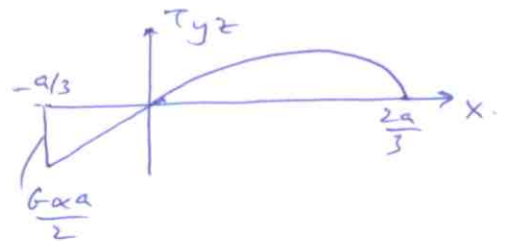
$$\boxed{\tau_{max} = \frac{G\alpha a}{2} \text{ at } x = -\frac{a}{3}}$$

Complex variable method

Try $w(z) = i(cz^3 + k)$

$\implies g = c(x^3 - 3xy^2) + k = \frac{1}{2}(x^2 + y^2)$ on S'

Choosing $c = \frac{1}{2a}$, $k = \frac{2}{27}a^2$, get

$$g - \frac{(x^2+y^2)}{2} = x^3 - 3xy^2 - a(x^2+y^2) + \frac{4}{27}a^3 = 0 = \phi$$

$\quad\quad\quad\quad \hookrightarrow$ ie eqn of triangle.

$$\implies g = \frac{1}{2a}(x^3 - 3xy^2) + \frac{2}{27}a^2$$

$$\boxed{\psi = -\frac{1}{2a}(3x^2y - y^3)}$$
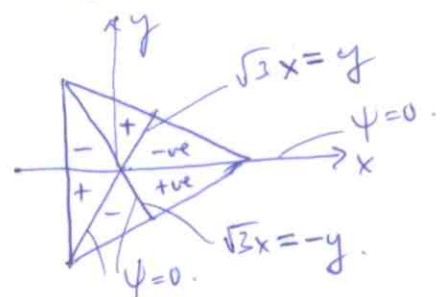
Circular shaft with cutout

Effect of grooves or slots or key ways, on the max shearing stress can be gauged from this example.

Try $\quad g = a\left(x - b^2 \dfrac{x}{x^2+y^2}\right) + \dfrac{1}{2}b^2 = a\left(r\cos\theta - \dfrac{b^2\cos\theta}{r}\right) + \dfrac{1}{2}b^2$

On $\hat{s}$, $\quad g = \dfrac{1}{2}(x^2+y^2) = \dfrac{1}{2}r^2$

$$\left(r^2 - b^2\right)\left(1 - \dfrac{2a\cos\theta}{r}\right) = 0$$

Thus boundary comprises two circles, $r = b$, and $r = 2a\cos\theta$

$\left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$
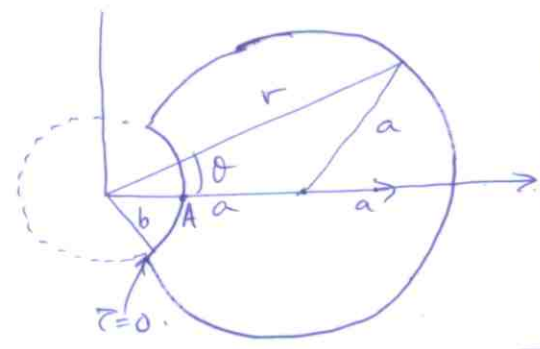
$$\begin{array}{l} r = 2a\cos\theta \\ \therefore r^2 = 2ax \\ \therefore x^2+y^2 = 2ax \\ \therefore (x-a)^2 + y^2 = a^2 \end{array}$$

$\tau = 0$

τ_{max} at A $\longrightarrow$ $\boxed{\tau_{max} = G\alpha\,(2a-b)}$ $\approx 2b\alpha a$ if $b \ll a$

$= 2 \times$ stress on periphery of circular shaft of radius a

Thus stress concentration due to cutout is evident.

→ Equivalently can choose $\phi = (x^2+y^2-b^2)\left([x-a]^2+y^2-a^2\right)$

Narrow Rectangular Bar

$a \gg b$.
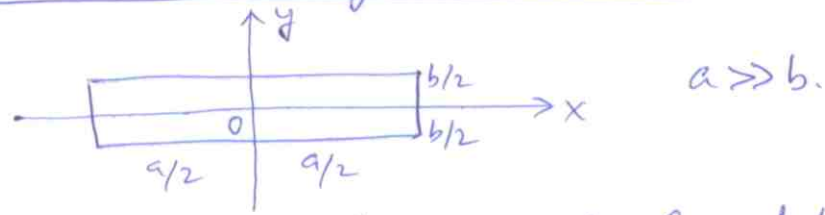
membrane analogy $\Rightarrow$ cylindrical bending $\Rightarrow w = w(y)$

$\Rightarrow \tau_{zy} = 0$, $\quad \phi = \phi(y)$

$\Rightarrow \nabla^2\phi = \dfrac{d^2\phi}{dy^2} = K$

integrating & using b.c's $\phi|_{y = \pm b/2} = 0$, get

$$\phi = \dfrac{K}{2}\left(y^2 - \dfrac{b^2}{4}\right)$$

$$M = 2 \iint_A \phi \, dxdy = 2 \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \frac{K}{2} \left(y^2 - \frac{b^2}{4} \right) dxdy = -\frac{ab^3 K}{6}$$

$$\phi = \frac{3M}{ab^3} \left(\frac{b^2}{4} - y^2 \right)$$

$$\boxed{\tau_{zx} = \phi_{,y} = -\frac{6M}{ab^3} y = 2G\alpha y}$$

Membrane analogy $\Rightarrow \tau_{max}$ at $y = \pm \frac{b}{2}$

$$\boxed{\tau_{max} = \frac{3M}{ab^2} = b\,G\alpha} = 2 \times \left(\text{max } \tau \text{ for circular shaft of diameter } b \right)$$

$$\boxed{\alpha = -\frac{\nabla^2 \phi}{2G} = \frac{3M}{ab^3 G}} \qquad \Rightarrow \qquad \boxed{C = G\frac{ab^3}{3}}$$

Calculating M from approximate shear stresses, ie $\tau_{zy} = 0$,

$$M = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left(x \underset{=0}{\tau_{yz}} - y\, \tau_{xz} \right) dxdy \cong \frac{1}{6} G\alpha\, ab^3 \longrightarrow \text{ie half of actual } M \text{ (see above).}$$

Thus we conclude that although $\tau_{yz} \cong 0$ (ie small), it contributes to half of M since the lever arm $(a/2)$ is large.

Rolled Sections — Open Thin Walled members.

L T C

Membrane analogy: If narrow rectangular membrane is loaded and then bent, the volume bounded and the slopes $\nearrow$ displaced will remain unchanged (physical argument). Hence the narrow rectangular bar, when bent into a curved cross-section $\wedge$ one will have the same torsional moment and shear stresses. So treat all curved narrow rectangles as straight ones when solving such problems.

$a_i, b_i \rightarrow$ length & width of ith narrow rectangle in the cross section

$M_i \rightarrow$ twisting moment on i^{th} narrow rectangle, $M = \sum M_i$.
total moment.

$T_i \rightarrow$ max stress on i^{th} rectangle

$\alpha \rightarrow$ twist per unit length of bar.

$$\Rightarrow \quad T_i = \frac{3 M_i}{a_i b_i^2} \quad , \quad \alpha = \frac{3 M_i}{a_i b_i^3 G} \quad , \quad M = \sum M_i = \frac{G\alpha}{3} \sum a_i b_i^3$$

boxed eqns.

(32) $\leftarrow$

$$M_i = \frac{a_i b_i^3}{\sum a_i b_i^3} M \quad , \quad \boxed{T_i = \frac{3 M b_i}{\sum a_i b_i^3}} \quad , \quad \boxed{\alpha = \frac{3M}{G \sum a_i b_i^3}}$$

The formula for $T_{max}^{(i\,e\,T_i)}$ does not hold at a corner where high stress concentrations exist.
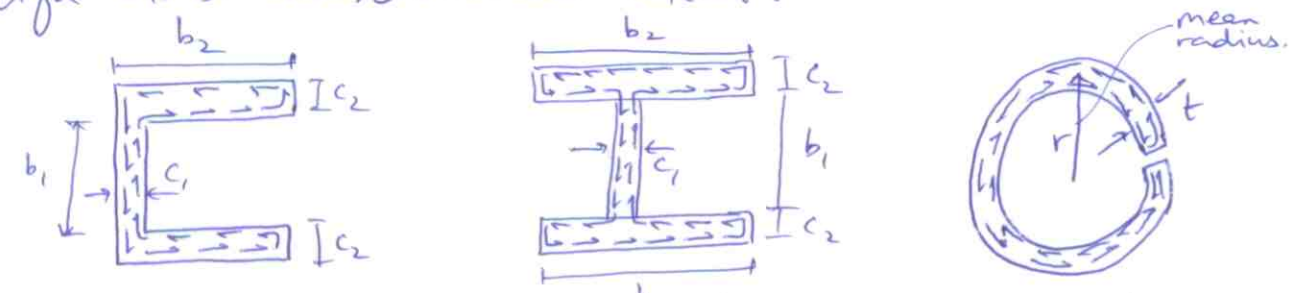
$$\alpha = \frac{3M}{(b_1 c_1^3 + 2 b_2 c_2^3) G}$$

$$\alpha = \frac{3M}{2\bar{A} r t^3 G}$$

$$\longrightarrow \quad \alpha = \frac{3M}{(2a-c) c^3 G}$$
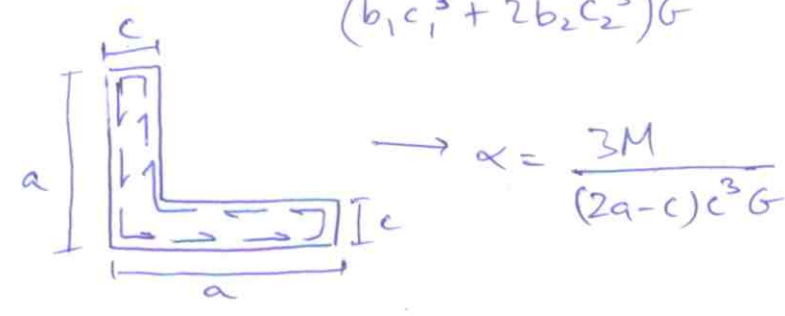
Thin Walled Tubes.

(A) Single celled. TW Tube
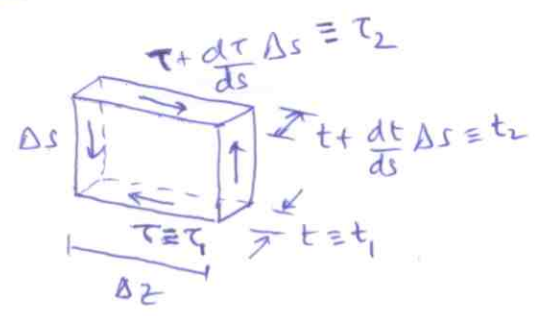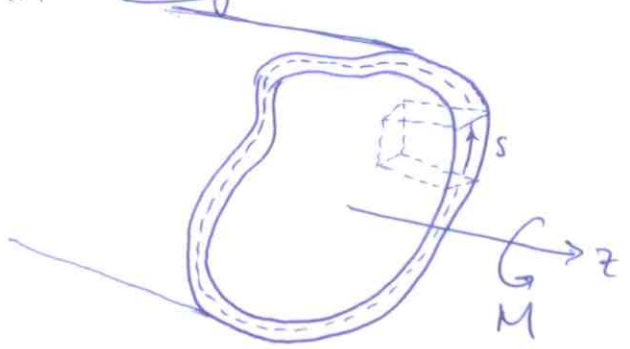
Assume : $\sigma_{zs} = \tau = $ const thru thickness and directed along t,
even when $t = t(s)$.

$$\sum F_z: \quad -\tau t \Delta z + \left(\tau + \frac{d\tau}{ds}\Delta s\right)\left(t + \frac{dt}{ds}\Delta s\right)\Delta z = 0$$

$$\Rightarrow \frac{d}{ds}(\tau t) = 0 \quad \Rightarrow \quad q = \tau t = \text{constant.} = \tau_1 t_1 = \tau_2 t_2$$
$$(\text{shear flow is const}).$$

$$M\underline{k} = \oint \underline{r} \times \tau t \, d\underline{s} = q \oint \underline{r} \times d\underline{s} = q \int_A 2 d\underline{A} = 2qA\underline{k}$$

$$\boxed{M = 2qA.} \longrightarrow \text{Bredt's formula.}$$

(33)

$$\boxed{\tau = \tau_{zs} = \frac{M}{2tA}}$$

(34)
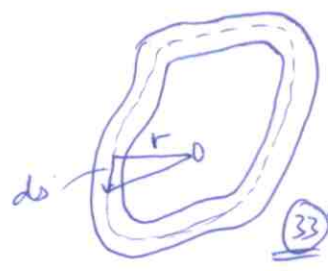
$$\boxed{\begin{array}{l} A = \text{area enclosed by } \oint \text{ curve} \\ \propto \text{area encl by inner perimeter} \end{array}}$$

$$U_T = \text{Strain energy in torsion} = \frac{1}{2}\int_V \frac{\tau^2}{G} dV = \frac{1}{2}\int_0^z \oint \frac{M^2}{4t^2 A^2 G} t \, ds \, dz$$

$$= \frac{1}{2}\frac{M^2 z}{4GA^2}\oint \frac{1}{t} ds$$

Using Castigliano's theorem,

$$\frac{1}{z}\frac{\partial U_T}{\partial M} = \boxed{\alpha = \frac{M}{4GA^2}\oint \frac{1}{t}ds \quad = \frac{q}{2GA}\oint \frac{1}{t}ds} \longrightarrow (35)$$

If q changes over $\oint$ of cell then $\boxed{\alpha_i = \frac{1}{2GA}\oint \frac{q}{t}ds} \longrightarrow (35)^*$ for use in multicelled case. Arises from compatibility of α for all cells.

(B) <u>Multi celled thin walled tube</u>
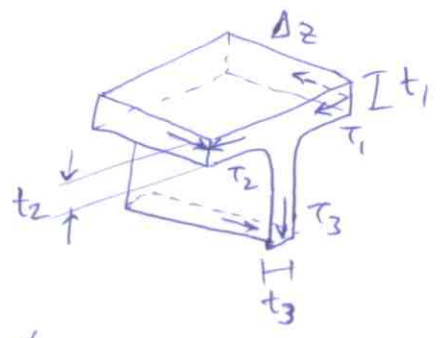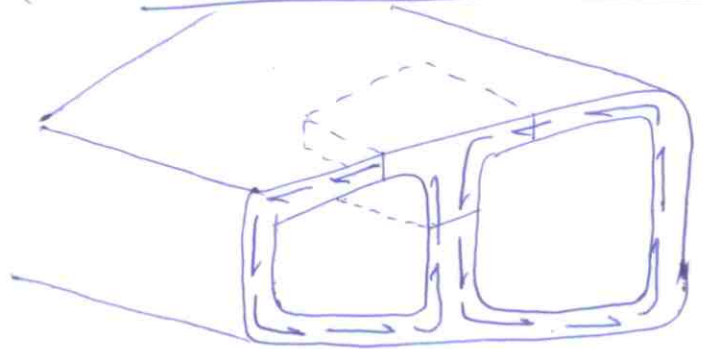
$$\sum F_z: \quad (\tau_2 t_2 + \tau_3 t_3 - \tau_1 t_1)\Delta z = 0$$

$$q_2 + q_3 = q_1$$

$$M_1 = 2q_1 A_1 = \text{moment abt 0 due to } q_1$$

$$M_2 = \underbrace{2q_2(A_2 + A_1^*)}_{} - \underbrace{2q_2 A_1^*}_{} = \text{moment abt 0 due to } q_2$$
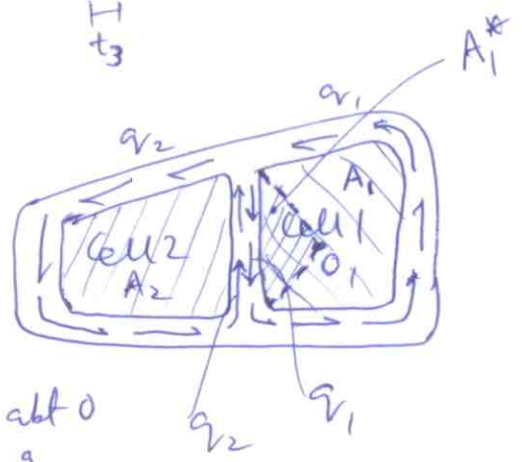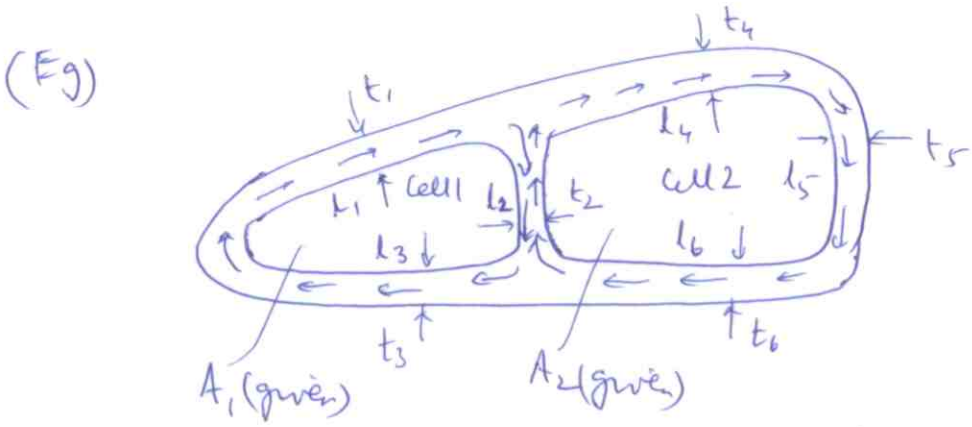

$$\boxed{M = M_1 + M_2 = 2q_1 A_1 + 2q_2 A_2} \longrightarrow (36)$$

Now rate of twist (α) is same for all cells

(ie compatibility condition). So using ③⑤*,

For cell 1,

$$2G\alpha = \frac{1}{A_1}\left(a_1 q_1 - a_{12} q_2\right)$$

For cell 2,

$$2G\alpha = \frac{1}{A_2}\left(a_2 q_2 - a_{12} q_1\right)$$ ⟶ ③⑦

where, $a_1 = \oint \frac{ds}{t}$ for cell 1 (including web). ⎫ contour
integrals
over $\mathcal{L}$.

$a_2 = \oint \frac{ds}{t}$ for cell 2 (including web). ⎬

$a_{12} = \oint \frac{ds}{t}$ for web. ⎭

(Eg)

M applied CW. (given)

$a_1 = \frac{l_1}{t_1} + \frac{l_2}{t_2} + \frac{l_3}{t_3}$

$a_2 = \frac{l_4}{t_4} + \frac{l_5}{t_5} + \frac{l_6}{t_6} + \frac{l_2}{t_2}$

$a_{12} = \frac{l_2}{t_2}$

A_1 (given) A_2 (given)

Solve ③⑥, ③⑦ for α, q_1, q_2. (ie rate of twist & shear flows).

———————— ✗ ————————

SOLID RECTANGULAR BAR   (using separation of variables approach).
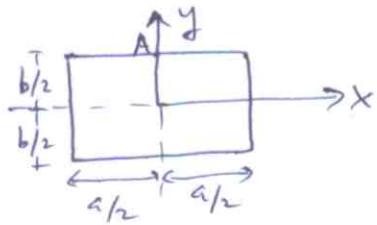
$a \gg b$.

$\nabla^2 \phi = -2G\alpha$.

BC's: $\phi_{(x = \pm a/2)} = \phi_{(y = \pm b/2)} = 0$

Consider $\phi(x,y) = \{\Phi(y)\}_{\text{narrow rect bar}} + F(x,y) = G\alpha\left(\frac{b^2}{4} - y^2\right) + F$

This transformation from $\phi(x,y)$ to $F(x,y)$ affords homogenization of the DE, ie we get Laplace eqn for $F(x,y)$, as follows.

$\nabla^2 \phi = \nabla^2 \left\{ G\alpha\left(\frac{b^2}{4} - y^2\right) + F \right\} = -2G\alpha$

$\Rightarrow \nabla^2 F = 0$, bc's: $F_{(x = \pm a/2)} = G\alpha\left(y^2 - b^2/4\right)$ ⟶ (a)

$F_{(y = \pm b/2)} = 0$ ⟶ (b)

Assume variable-separable solution

$$F(x,y) = X_{(x)} Y_{(y)}$$

$$\nabla^2 F = 0 = X''Y + XY'' \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = const \overset{a\lambda}{}$$

From b.c.(b), $Y_{(y)}$ cannot be exponential (hyperbolic) so it must be circular (trigonometric), i.e. $\lambda > 0$; in which case $X_{(x)}$ is hyperbolic.

From b.c.(a), $F(x,y)$ is even in x, y, i.e, $X_{(x)} = X_{(-x)}, Y_{(y)} = Y_{(-y)}$, so $X_{(x)} = \cosh \lambda x$ and $Y_{(y)} = \cos \lambda y$. To satisfy b.c.(a), $\lambda = \frac{(2m+1)\pi}{b} = \frac{n\pi}{b}$, $m = 1, 2, 3, \cdots$,

$n = 1, 3, 5, \cdots$. Thus,

$$F(x,y) = \sum_{n=1,3,5\cdots}^{\infty} A_n \cosh\left(\frac{n\pi}{b}x\right)\cos\left(\frac{n\pi}{b}y\right) \longrightarrow (c)$$

Thus b.c (b) becomes,

$$\sum_{n=1,3,5,\cdots}^{\infty} A_n \cosh\frac{n\pi a}{2b} \cos\frac{n\pi y}{b} = G\alpha\left(y^2 - \frac{b^2}{4}\right) \longrightarrow \circledast$$

Expanding the rhs above as a $\cos\frac{n\pi y}{b}$ series,

$$G\alpha\left(y^2 - \frac{b^2}{4}\right) = \sum_{n=1,3,5\cdots}^{\infty} B_n \cos\frac{n\pi y}{b}, \quad B_n = \frac{2}{b}\int_{-b/2}^{b/2} G\alpha\left(y^2 - \frac{b^2}{4}\right)\cos\frac{n\pi y}{b}\,dy \to \circledast\circledast$$

Equating coeffs of $\cos\frac{n\pi y}{b}$ from $\circledast$ & $\circledast\circledast$,

$$A_n \cosh\frac{n\pi a}{2b} = \frac{2}{b}\int_{-b/2}^{b/2} G\alpha\left(y^2 - \frac{b^2}{4}\right)\cos\frac{n\pi y}{b}\,dy$$

$$= -\frac{8G\alpha b^2}{\pi^3 n^3}(-1)^{\frac{n-1}{2}} \longrightarrow (d)$$

$$\Rightarrow \phi = G\alpha\left[\frac{b^2}{4} - y^2 - \frac{8b^2}{\pi^3}\sum_{n=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^3}\frac{\cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)}{\cosh\left(\frac{n\pi a}{2b}\right)}\right]$$

From membrane analogy, max torsional shear stress occurs at midpoint of longer side, ie, pt A if $a > b$. Thus,

$$\tau_{max} = (\tau_{zx})_{\substack{x=0 \\ y=-b/2}} = \left(\frac{\partial\phi}{\partial y}\right)_{\substack{x=0 \\ y=-b/2}} = G\alpha b\left[1 - \frac{8}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2\cosh\frac{n\pi a}{2b}}\right]$$

α in terms of M.

$$M = 2 \iint \phi \, dx \, dy = 2 \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \phi \, dx \, dy =$$

$$= G\alpha \, ab^3 \left[\frac{1}{3} - \frac{64}{\pi^5} \frac{b}{a} \sum_{n=1,3,5\cdots}^{\infty} \frac{\tanh \frac{n\pi a}{2b}}{n^5} \right]$$

i.e, $\alpha = \dfrac{M}{ab^3 G \left[\frac{1}{3} - \frac{64}{\pi^5} \frac{b}{a} \displaystyle\sum_{n=1,3,5\cdots\infty}^{\infty} \frac{\tanh \frac{n\pi a}{2b}}{n^5} \right]} = \dfrac{M}{ab^3 G \beta}$

Thus,

$$\tau_{max} = \frac{M \left[1 - \frac{8}{\pi^2} \displaystyle\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2 \cosh \frac{n\pi a}{2b}} \right]}{ab^2 \left[\frac{1}{3} - \frac{64}{\pi^5} \frac{b}{a} \displaystyle\sum_{n=1,3,5\cdots}^{\infty} \frac{\tanh \frac{n\pi a}{2b}}{n^5} \right]} = \frac{M}{ab^2 \beta_1}$$

The coefficients β, β_1 are dimensionless and depend on ratio a/b only. For $\frac{a}{b} \to \infty$, $(\beta, \beta_1) \to \frac{1}{3}$, and we recover result for narrow rectangular section. For other a/b, β, β_1 are tabulated below.

a/b	β	β_1	a/b	β	β_1
1.0	0.141	0.208	3.0	0.263	0.267
1.2	0.166	0.219	4.0	0.281	0.282
1.5	0.196	0.230	5.0	0.291	0.291
2.0	0.229	0.246	10.0	0.312	0.312
2.5	0.249	0.258	∞	0.333	0.333

Torsion of Hollow Shafts (ie multi-celled, not thin walled)

Let exterior contour be denoted c_0.
Contours $c_1, c_2, \ldots c_N$ correspond to
cavities of shaft.
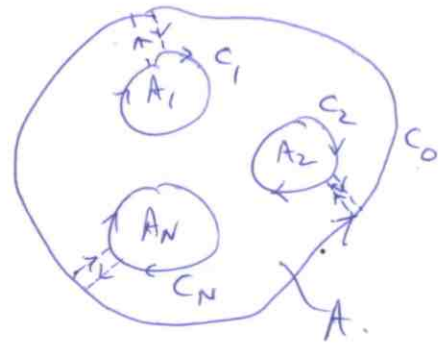
Thus,

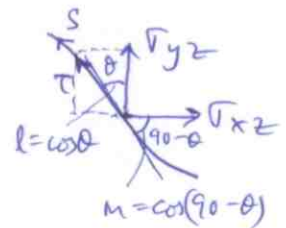$$\nabla^2 \phi = -2G\alpha \quad \text{in } A$$

$$\phi = 0 \quad \text{on } c_0$$

$$\phi = K_m \quad \text{on } c_m, \quad m = 1, \ldots N.$$

The K_m are determined from single-valuedness of ψ (hence w, since u, v are single valued by their definition). Thus,

$$G\alpha \left(\oint_{c_m} d\psi = 0 \right)$$

$$\Rightarrow G\alpha \oint_{c_m} (\psi_{,x} dx + \psi_{,y} dy) = \oint_{c_m} [(\phi_{,y} + G\alpha y) dx - (\phi_{,x} + G\alpha x) dy] = 0$$

$$\oint_{c_m} \left(\phi_{,y} \frac{dx}{ds} - \phi_{,x} \frac{dy}{ds} \right) ds = G\alpha \oint_{c_m} \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right) ds$$

$\sigma_{xz} \, (-m) \quad -\sigma_{yz} \quad \ell$

Now $\ell \sigma_{yz} - m \sigma_{xz} = \tau$ (since $\phi = $ const on c_m, hence τ is tangential to c_m. Note +ve τ is along $+s$ $\therefore$ CCW +ve for (4)).

$$\left\{ \text{or alternatively, } \phi_{,y} \frac{dx}{ds} - \phi_{,x} \frac{dy}{ds} = -\phi_{,y} \frac{dy}{dn} - \phi_{,x} \frac{dx}{dn} = -\frac{d\phi}{dn} = \tau \atop \text{from eq. (19), p. 70}) \right\}$$

So, $\dfrac{1}{G\alpha} \oint_{c_m} \tau \, ds = \oint (\ell x + m y) \, ds \Rightarrow \dfrac{1}{G\alpha} \oint_{c_m} \tau \, ds = \iint_{A_m} (1+1) dA = 2A_m$

ie $\boxed{\oint_{c_m} \tau \, ds = 2G\alpha A_m} \longrightarrow \text{③⑧} \rightarrow \text{(gives } N \text{ equations)}$

Now $M = 2\iint_A \phi \, dx \, dy - \oint_{s} \phi (x\ell + ym) \, ds$.

$s \not> \text{ or } \equiv C$

where $s \not\equiv c_0 \cup c_1 \cup \cdots \cup c_N$ traversed in direction shown (ie c_0 traversed CCW, c_m traversed CW).

A = area of hollow section.

Choose $\phi = 0$ on C_0 , $\phi = K_m$ on C_m

$$\therefore \quad M = 2 \iint_A \phi \, dA + \sum_{i=1}^{N} K_i \underbrace{\oint_{C_i} (\ell x + m y) \, ds}_{2 A_i}$$

$$\boxed{ M = 2 \iint_A \phi \, dA + \sum_{i=1}^{N} 2 K_i A_i } \quad \longrightarrow \widehat{39}$$

where we use $\phi = 0$ on C_0 and $\oint_{C_m} = -\oint_{C_m}$ in obtaining $\widehat{39}$.

$\boxed{\text{Now solve } (38,39) \text{ for } \alpha, K_i \text{'s: } \text{ie } N+1 \text{ unknowns}}$

By membrane analogy, $M \equiv$ vol displaced by membrane with flat rigid plates over A_m , ie,

$$2 \iint_A \phi \, dA \equiv 2 \, (\text{vol displaced by membrane part})$$

$$2 \sum_{i=1}^{N} K_i A_i \equiv 2 \, (\text{vol displaced by flat plates}).$$

(i.e, K_i is analogous to the vertical displ of the ith flat rigid plate).

$(\phi = 0) \equiv w = 0$

$(\phi = K_1) \equiv w = K_1$

$K_1 A_1 \qquad K_2 A_2$

$K_2 = w \, (\equiv \phi = K_2)$

cross-hatched $= \iint w \, dA$, $\equiv \phi$.
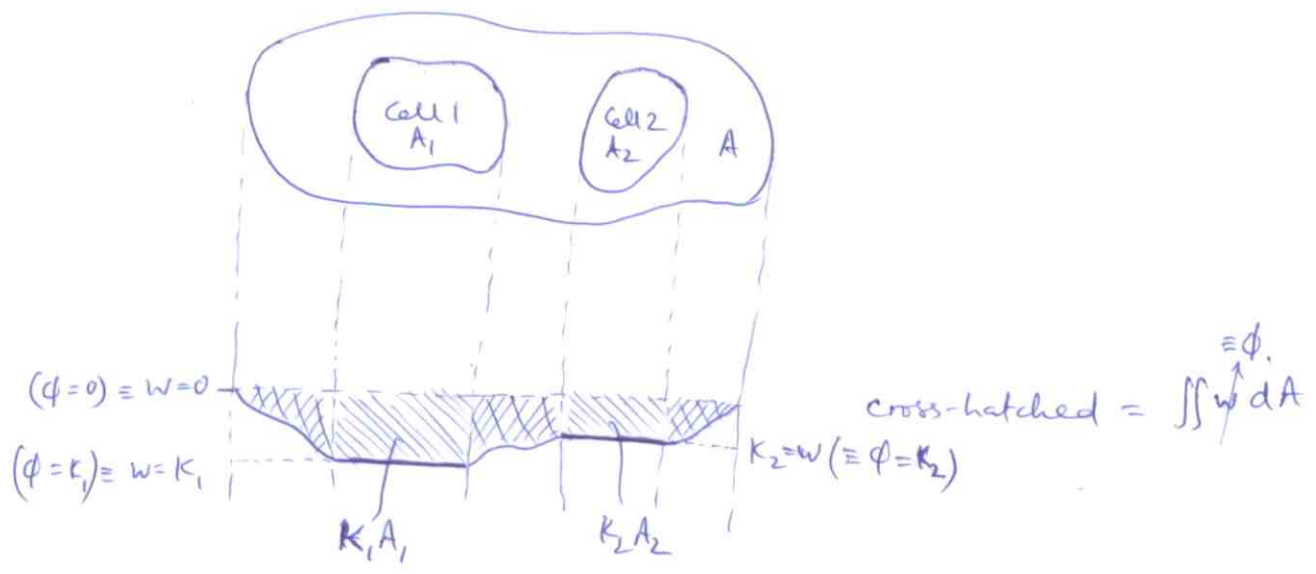
(Eg) Hollow cylinder.

$$\phi = -\frac{G\alpha}{2} (x^2 + y^2 - b^2)$$

satisfies $\nabla^2 \phi = -2 G \alpha$

$\phi = 0$ on C_0

$\phi = K_1 = \frac{G\alpha}{2} (b^2 - a^2)$ on C_1
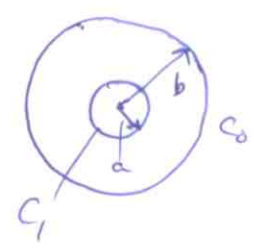
$$M = 2\int_0^{2\pi}\int_a^b -\frac{G\alpha}{2}(r^2-b^2)\,r\,dr\,d\theta \;+\; 2\frac{G\alpha}{2}(b^2-a^2)\,\pi a^2 \qquad \text{⑧⑥}$$

$$= \frac{G\alpha}{2}\,\pi(b^4-a^4) \quad\longrightarrow\quad \text{matches with result from basic solid}$$
$$\text{mechanics course.}$$

(Eg) Using ㉟, ㊴ obtain (by specializing) the same results for multi-celled TW tube, as were obt by using Eq (36), (37) in the example on p. 82.

$$(\phi_0 \equiv)\; w_0 = 0$$
$$(\phi_1 \equiv)\; w_1 = K_1$$
$$(\phi_2 \equiv)\; w_1 = K_2$$
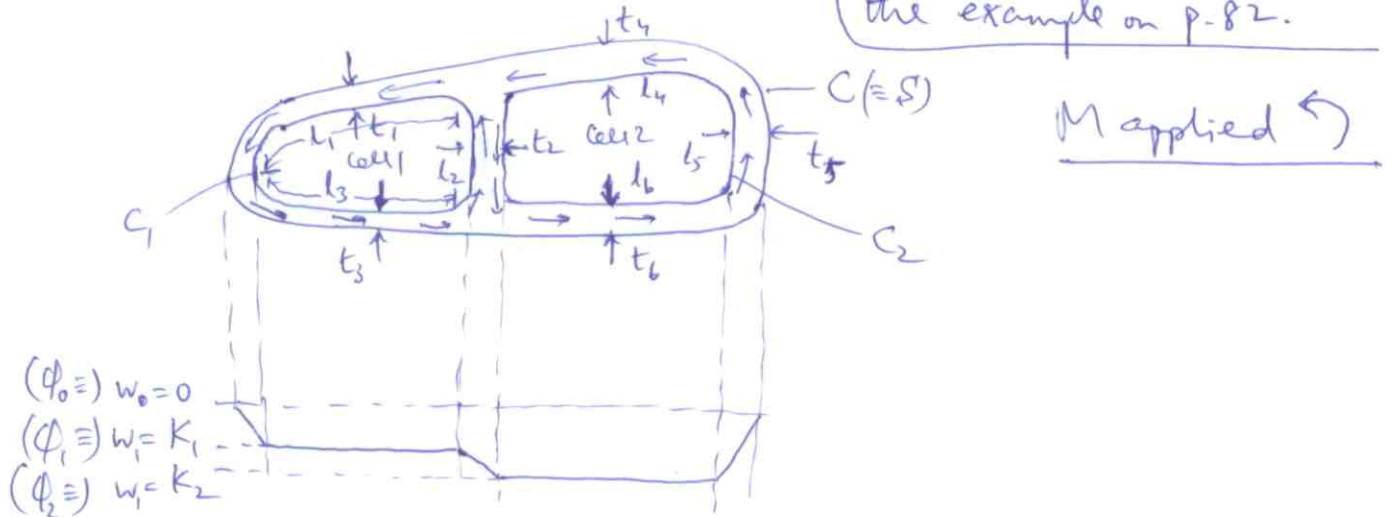
Since thin walled, assume (from membrane analogy) that $\phi\;(\equiv w)$ varies linearly over the thickness.

$$\tau = -\frac{\partial\phi}{\partial n} \quad\longrightarrow\quad \text{leg } l_1:\; \tau_1 = -\left(\frac{\phi_0-\phi_1}{t_1}\right) = \frac{K_1}{t_1}$$

$$l_2:\; \tau_2 = -\left(\frac{\phi_2-\phi_1}{t_2}\right) = \frac{K_1-K_2}{t_2}$$

$$l_3:\; \tau_3 = -\frac{(\phi_0-\phi_1)}{t_3} = \frac{K_1}{t_3}$$

$$l_4:\; \tau_4 = -\frac{(\phi_0-\phi_2)}{t_4} = \frac{K_2}{t_4}$$

$$l_5:\; \tau_5 = -\frac{(\phi_0-\phi_2)}{t_5} = K_2/t_5$$

$$l_6:\; \tau_6 = -\frac{(\phi_0-\phi_2)}{t_6} = K_2/t_6$$

$$C_1:\; \oint_{C_1}\tau\,ds = 2G\alpha A_1 \;\Rightarrow\; \frac{K_1}{t_1}l_1 + \frac{(K_1-K_2)}{t_2}l_2 + \frac{K_1}{t_3}l_3 = 2G\alpha A_1 \;\longrightarrow (A)$$

$$C_2:\; \oint_{C_2}\tau\,ds = 2G\alpha A_2 \;\Rightarrow\; \frac{K_2}{t_4}l_4 + \frac{K_2}{t_6}l_6 + \frac{K_2}{t_5}l_5 + \frac{K_2-K_1}{t_2}l_2 = 2G\alpha A_2 \;\longrightarrow (B)$$
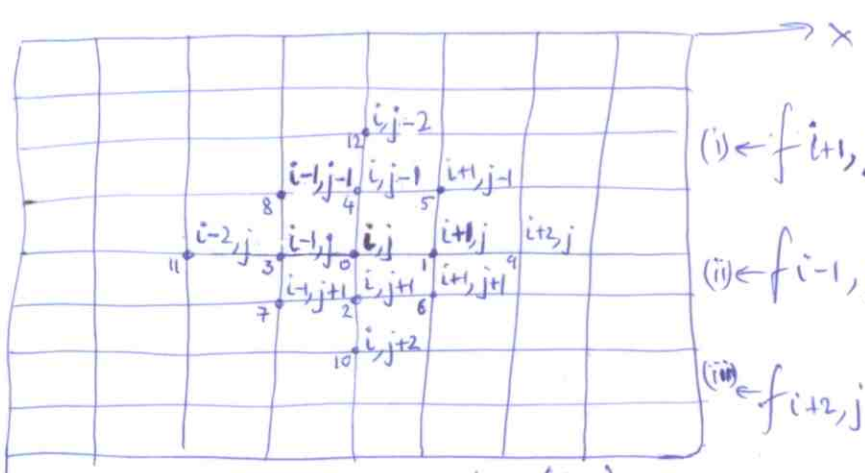
$$M = 2 \iint_A \phi \, dA + 2K_1 A_1 + 2K_2 A_2 \quad \rightarrow (C)$$

$\underbrace{}_{\simeq 0 \; \therefore \; A_i \text{'s small due to thin wall.}}$

From (A), (B), (C), get K_1, K_2, α. Comparing with (36, 37)

see that $K_1 = q_1$, $K_2 = q_2$ are the shear flows.

FINITE DIFFERENCE METHOD FOR TORSION.

Consider simply connected domain only (formulation is more complex for multiply connected domains — ie need to do displacement formulation).

In FDM, the gov eqn and bc's are converted to their finite difference approximations, which are algebraic eqns.

$(i) \leftarrow f_{i+1,j} = f_{i,j} + h \frac{\partial f}{\partial x}\Big|_{i,j} + \frac{h^2}{2} \frac{\partial^2 f}{\partial x^2}\Big|_{i,j}$

$(ii) \leftarrow f_{i-1,j} = f_{i,j} - h \frac{\partial f}{\partial x}\Big|_{i,j} + \frac{h^2}{2} \frac{\partial^2 f}{\partial x^2}\Big|_{i,j}$

$(iii) \leftarrow f_{i+2,j} = f_{i,j} + 2h \frac{\partial f}{\partial x}\Big|_{i,j} + 2h^2 \frac{\partial^2 f}{\partial x^2}\Big|_{i,j}$

Central Difference Formulae (CD)

$(i), (ii) \longrightarrow \frac{\partial f}{\partial x}\Big|_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$

$\frac{\partial^2 f}{\partial x^2}\Big|_{i,j} = \frac{f_{i+1,j} + f_{i-1,j} - 2f_{i,j}}{h^2}$

(CD-5)

similarly, $\frac{\partial f}{\partial y}\Big|_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2h}$
$i \rightleftharpoons j$

$\frac{\partial^2 f}{\partial y^2}\Big|_{i,j} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{h^2}$

not reqd for Torsion $\leftarrow \frac{\partial^2 f}{\partial x \partial y}\Big|_{i,j} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\Big|_{i,j}\right) = \frac{1}{4h^2}\left[\left(f_{i+1,j+1} + f_{i-1,j-1}\right) - \left(f_{i+1,j-1} + f_{i-1,j+1}\right)\right]$

$$\frac{\partial^4 f}{\partial x^4}_{i,j} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}_{i,j}\right) = \frac{1}{h^4}\left[f_{i+2,j} + f_{i,j} - 2f_{i+1,j} + f_{i,j} + f_{i-2,j}\right.$$
$$\left. - 2f_{i-1,j} - 2f_{i+1,j} - 2f_{i-1,j} + 4f_{i,j}\right]$$
$$= \frac{1}{h^4}\left[6f_{i,j} - 4\left(f_{i+1,j} + f_{i-1,j}\right) + f_{i+2,j} + f_{i-2,j}\right]$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}_{i,j} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}_{i,j}\right) = \frac{1}{h^4}\left[f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j} + f_{i-1,j+1} + f_{i-1,j-1}\right.$$
$$\left. - 2f_{i-1,j} - 2f_{i,j+1} - 2f_{i,j-1} + 4f_{i,j}\right]$$
$$= \frac{1}{h^4}\left[f_{i+1,j+1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i-1,j-1} - 2\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}\right.\right.$$
$$\left.\left. - 2f_{i,j}\right)\right]$$

similarly $(i \rightleftarrows j)$

$$\frac{\partial^4 f}{\partial y^4} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial y^2}_{i,j}\right) = \frac{1}{h^4}\left[6f_{i,j} - 4\left(f_{i,j+1} + f_{i,j-1}\right) + f_{i,j+2} + f_{i,j-2}\right]$$

Formulae CD_{5-8} are required for $\underset{\times}{\overset{eg}{\text{plane}}}$ stress probls, ie not for torsion problems.

Forward/Backward difference Formulae
_____

$(i),(iii) \rightarrow$
$$\frac{\partial f}{\partial x}_{i,j} = \frac{4f_{i+1,j} - 3f_{i,j} - f_{i+2,j}}{2h} \qquad \longrightarrow (FD_1)$$

Similarly (replace h by $(-h)$ and $(+)$ by $(-)$ in $(F.D_1)$)
$$\frac{\partial f}{\partial x}_{i,j} = \frac{-4f_{i-1,j} + 3f_{i,j} + f_{i-2,j}}{2h} \qquad \longrightarrow (BD_1)$$

Similarly $(i \rightleftarrows j)$
$$\frac{\partial f}{\partial y}_{i,j} = \frac{4f_{i,j+1} - 3f_{i,j} - f_{i,j+2}}{2h} \qquad \longrightarrow (FD_2)$$

$$\frac{\partial f}{\partial y}_{i,j} = \frac{-4f_{i,j-1} + 3f_{i,j} + f_{i,j-2}}{2h} \qquad \longrightarrow (BD_2)$$

$(i,iii) \rightarrow$
$$\frac{\partial^2 f}{\partial x^2}_{i,j} = \frac{-2f_{i+1,j} + f_{i,j} + f_{i+2,j}}{h^2} \qquad \longrightarrow (FD_3)$$

similarly (replace h by $(-h)$ and $(+)$ by $(-)$ in (FD_3))
$$\frac{\partial^2 f}{\partial x^2}_{i,j} = \frac{-2f_{i-1,j} + f_{i,j} + f_{i-2,j}}{h^2} \qquad \longrightarrow (BD_3)$$

similarly
$(i \rightleftharpoons j)$
$$\frac{\partial^2 f}{\partial y^2}_{i,j} = \frac{-2f_{i,j+1} + f_{i,j} + f_{i,j+2}}{h^2} \longrightarrow (FD_v)$$

$$\frac{\partial^2 f}{\partial y^2}_{i,j} = \frac{-2f_{i,j-1} + f_{i,j} + f_{i,j-2}}{h^2} \longrightarrow (BD)_v$$

In general we use CD formulae (∵ more accurate as step size is smaller ie h, and derivative is obtained based on function evaluation on either side of node in question). Use FD/BD formulae only when function evaluations on either side of node in question are not easily done — eg. at boundaries.

∴ $\nabla^2 \phi (i,j) = -2G\alpha$

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -2G\alpha h^2 \longrightarrow \boxed{40}$$

Take nodal value of ϕ on boundary as zero (∵ $\phi = 0$ on S)

So (40) can be solved for internal nodal values of ϕ, ie $\phi_{m,n}$.
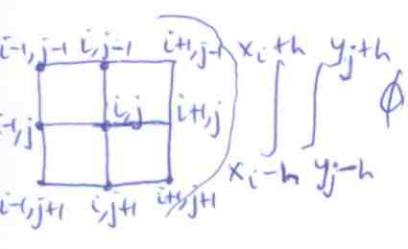
$$\tau_{zx} = \frac{\partial \phi}{\partial y}_{ij} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h} \quad \text{for interior node } (i,j)$$

$$= \frac{\pm 4\phi_{i,j\pm1} \mp 3\phi_{i,j} \pm \phi_{i,j\pm2}}{2h}$$

for boundary node (use + or − sign depending on whether boundary node is to left or right).

$$\tau_{zy} = -\frac{\partial \phi}{\partial x}_{ij} = -\text{ve of above formulae with } i \rightleftharpoons j.$$

Thus $\phi_{m,n}$ and $\tau_{xz(m,n)}$, $\tau_{yz(m,n)}$ can be expressed (solved) in terms of $G\alpha h^2$ and $G\alpha h$, respectively. To express α in terms of applied M consider $M = 2\int_A \phi \, dx\, dy$. Using Simpson's formula in 2-D,

$$\int\int_{x_i-h, y_j-h}^{x_i+h, y_j+h} \phi \, dx\, dy = \frac{h^2}{9}\Big[16\phi_{i,j} + 4(\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i-1,j} + \phi_{i,j-1}) + \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\Big]$$

Adding the integral over all squares of sides $2h \times 2h$ we get
$M = \text{number} \times G\alpha h^4$ from which you solve for α in terms of M (ie numerical value).

(Fig).
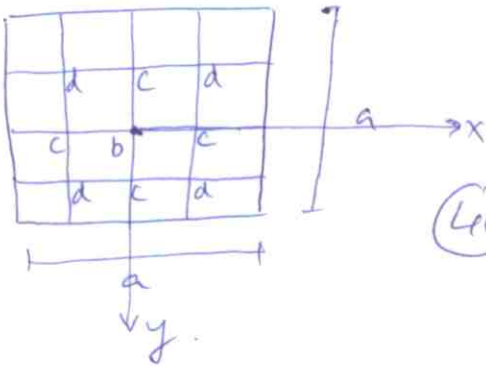
Choose $h = a/4$

Symmetry $\Rightarrow$ for interior nodes, only distinct values are ϕ_b, ϕ_c, ϕ_d

$(4.0) \Rightarrow$

$$4\phi_b - 4\phi_c = 2G\alpha h^2$$
$$4\phi_c - \phi_b - 2\phi_d = 2G\alpha h^2$$
$$4\phi_d - 2\phi_c = 2G\alpha h^2$$

$$\Rightarrow \phi_b = \frac{9}{4}G\alpha h^2, \quad \phi_c = \frac{7}{4}G\alpha h^2, \quad \phi_d = \frac{11}{8}G\alpha h^2$$

$$\frac{M}{2} = 4 * \int_{x_d-h}^{x_d+h} \int_{y_d-h}^{y_d+h} \phi \, dx \, dy = \left(16\phi_d + 4(2\phi_c) + \phi_b\right)\frac{h^2}{9} * 4 = 17 G\alpha h^4$$

$$\Rightarrow \alpha = \frac{M}{34 G h^4} = \frac{M}{0.133 G a^4} \quad \left(\text{exact value of coeff in denom is } 0.141\right).$$

$$T_{max} = \sigma_{zy}\Big|_{\substack{x=a/2 \\ y=0}} = -\frac{\partial\phi}{\partial x}\Big|_{\substack{x=a/2 \\ y=0}} = -\frac{-4\phi_c + \phi_b}{2h} = \frac{19}{32}G\alpha a$$

$$\Rightarrow T_{max} = \frac{19}{32}\frac{M}{0.133 a^3} = \frac{M}{0.224 a^3} \quad \left(\text{exact value of coeff is } 0.208\right).$$

BENDING OF BEAMS.

We orient the coordinate system as below (for convenience later on)
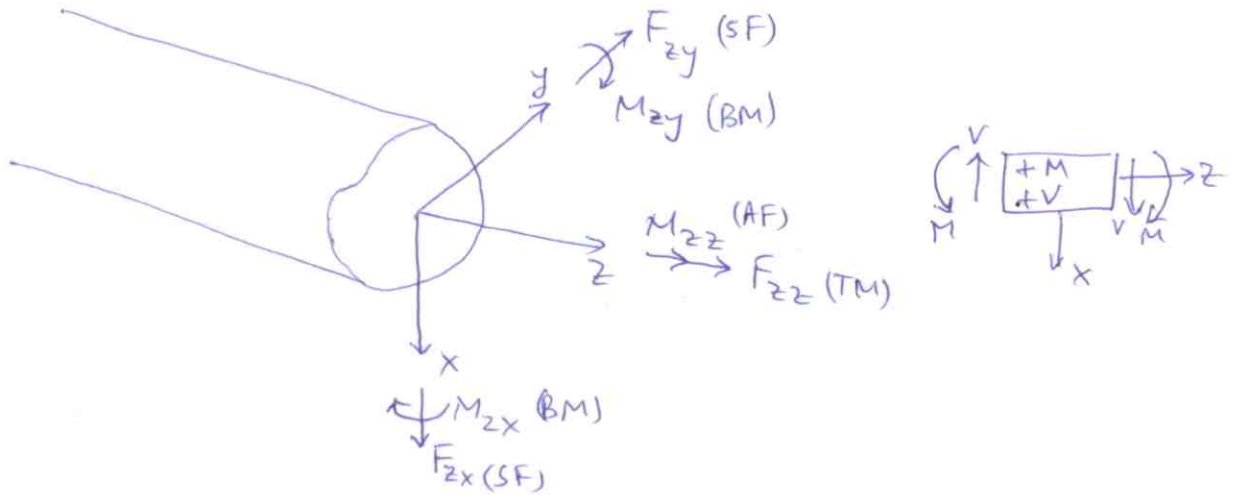
Relation between q, V, M.
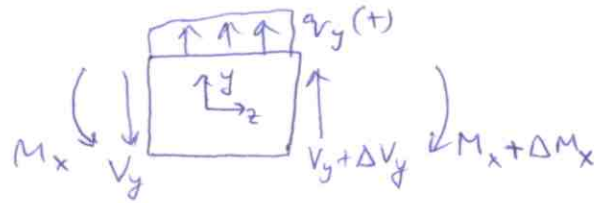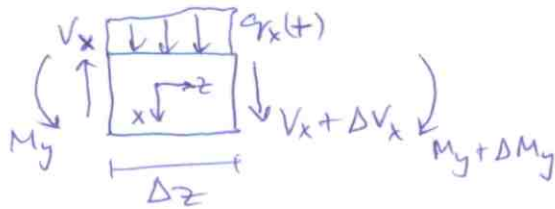
Equilibrium $\Rightarrow \quad \dfrac{dV_x}{dz} = -q_x \qquad\qquad \dfrac{dV_y}{dz} = -q_y$

$$\dfrac{dM_y}{dz} = -V_x \qquad\qquad \dfrac{dM_x}{dz} = V_y \qquad \Biggr\} \rightarrow \text{①}$$

$$\dfrac{d^2 M_y}{dz^2} = q_x \qquad\qquad \dfrac{d^2 M_x}{dz^2} = -q_y$$

M_x and M_y are max/min when V_y and V_x are zero, resply, in case of distributed loading.

When only point loading is applied along the beam length, such that the loads lie in xy plane, i.e.,
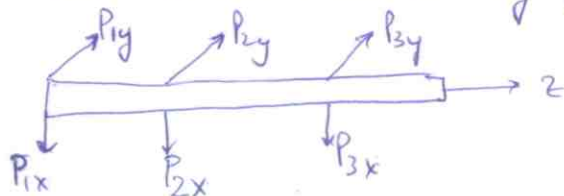
the SF's (V_x, V_y) are cnst between loading points and BM's (M_x, M_y) are linearly varying between load points. Hence max M_x and max M_y occur at load pts. We have,

$$M_b^2 = M_x^2 + M_y^2 = (Az + B)^2 + (Cz + D)^2$$

$$\Rightarrow \quad \frac{d^2(M_b^2)}{dz^2} = 2(A^2 + C^2) \geq 0$$

ie M_b^2 is sub-harmonic $\Rightarrow (M_b^2)_{max}$ occurs at load pt.
(ref. p. 70).

Area Moments of Inertia.

$$I_{x'x'} = \int_A y'^2 \, dA$$
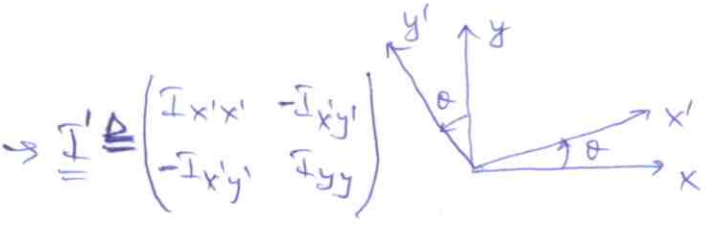
$$I_{y'y'} = \int_A x'^2 \, dA \quad \Biggr\} \rightarrow \underline{\underline{I}}' \triangleq \begin{pmatrix} I_{x'x'} & -I_{x'y'} \\ -I_{x'y'} & I_{yy} \end{pmatrix}$$

$$I_{x'y'} = \int_A x'y' \, dA$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{a}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{\underline{I}}' = \underline{\underline{a}} \, \underline{\underline{I}} \, \underline{\underline{a}}^T$$

ie, $\left. \begin{array}{l} I_{x'x'} \\ I_{y'y'} \end{array} \right/ = \dfrac{I_{xx} + I_{yy}}{2} \pm \dfrac{I_{xx} - I_{yy}}{2}\cos 2\theta \mp I_{xy}\sin 2\theta \quad \Biggr\} \rightarrow (2)$

$$I_{x'y'} = \dfrac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$
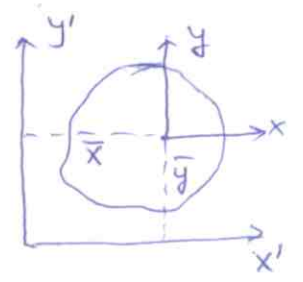
Mohr's circle : (i) Take I_{xy} axis as upward +ve ($\because$ off-diagonal comp's are negative).

(ii) Center at $(I_{xx} + I_{yy})/2$
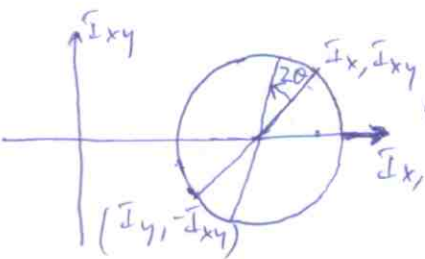
(iii) Radius $= \left[(I_{xx} - I_{yy}/2)^2 + (I_{xy})^2 \right]^{1/2}$

(iv) Rot in physical plane $= +\frac{1}{2}($Rot in Mohr's plane$)$

Parallel axis theorem.

$$I_{y'y'} = I_{yy} + \bar{x}^2 A$$

$$I_{x'x'} = I_{xx} + \bar{y}^2 A$$

$$I_{x'y'} = I_{xy} + \bar{x}\bar{y} A$$
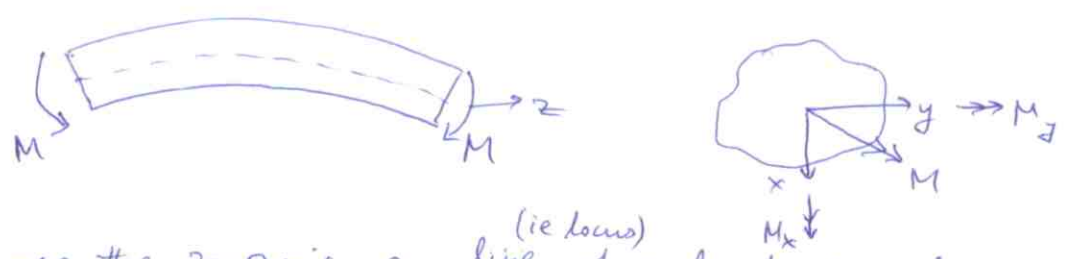

(eg) S.T. $I_{x'x'} I_{y'y'} - I_{x'y'}^2 > 0$

Let x, y be p-axes $\Rightarrow \det(I) = \det \begin{pmatrix} I_{xx} & 0 \\ 0 & I_{yy} \end{pmatrix} = I_{xx} I_{yy} > 0$

Now $\det(I') = \underset{=1}{\det(a)} \det(I) \underset{=1}{\det(a^T)} = \det(I)$

$\Rightarrow I_{x'x'} I_{y'y'} - I_{x'y'}^2 > 0.$

(I) PURE BENDING OF PRISMATIC BEAM.

Consider pure bending of prismatic bar by terminal couples M.

(ie locus)

Choose the z-axis as line of centroids of sections.

Use semi-inverse method. Hence assume, (guided by introductory S.M).

$$\sigma_z = -\frac{E}{R_x}x + \frac{E}{R_y}y \quad , \quad \text{other stresses} = 0$$

We will show that for appropriate R_x, R_y, the above is an exact solution. Observe that,

i) Above stress distribution satisfies equilibrium ($\because \sigma_z \neq f^n$ of z).

ii) Resulting strains are linear in x, y, hence compatibility satisfied.

iii) Traction free bc's on lateral surface satisfied ($\because n_z^{n_3} = 0$).
   (Stress distribution $\Rightarrow \sigma_1, \sigma_2 = 0, n_z = 0 \Rightarrow \sigma_3 = 0$)

iv) On end faces, $F_x = \int_A \sigma_{xz}\, dA = 0;$ $\qquad F_y = \int_A \sigma_{yz}\, dA = 0$

$$F_z = \int_A \sigma_z\, dA = -\frac{E}{R_x}\underbrace{\int_A x\, dA}_{=0} + \frac{E}{R_y}\underbrace{\int_A y\, dA}_{=0} = 0$$

$$(\because z \text{ is line of centroids})$$

v) Resultant moments:

$$M_x = \int_A y\,\sigma_z\, dA = -\frac{E}{R_x}I_{xy} + \frac{E}{R_y}I_x$$

$$M_y = -\int_A x\,\sigma_z\, dA = \frac{E}{R_x}I_{yy} - \frac{E}{R_y}I_{xy}$$

$\longrightarrow$ ③

Solve ③ $\rightarrow$

$$\frac{1}{R_x} = \frac{M_x I_{xy} + M_y I_x}{E(I_x I_y - I_{xy}^2)} \quad ; \quad \frac{1}{R_y} = \frac{M_x I_y + M_y I_{xy}}{E(I_x I_y - I_{xy}^2)}$$

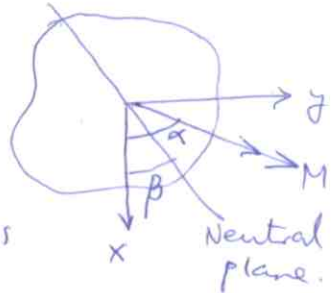④ (Euler-Bernoulli Law).

Thus, the stress distribution

$$\sigma_z = -\frac{(M_x I_{xy} + M_y I_x)}{(I_x I_y - I_{xy}^2)} x + \frac{(M_x I_y + M_y I_{xy})}{(I_x I_y - I_{xy}^2)} y \longrightarrow (5)$$

(other stresses zero)

solves the problem of pure bending of a prismatic bar, with applied terminal couples M_x, M_y about x, y axes, resply, and z-axis being line of centroids.

Along the neutral plane we have $\sigma_z = 0$

$$\Rightarrow \tan \beta = \frac{y}{x} = \frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}} \longrightarrow (6)$$

where β is $\angle$ of neutral plane with x-axis

$$\tan \alpha = \frac{M_y}{M_x} \longrightarrow (7)$$

So $\alpha = \beta$ iff,

$$M_x M_y (I_x - I_y) + (M_x^2 - M_y^2) I_{xy} = 0$$

If x, y are p-axes,

$$\sigma_z = -\frac{M_y}{I_y} x + \frac{M_x}{I_x} y; \qquad \frac{1}{R_x} = \frac{M_y}{E I_y}; \quad \frac{1}{R_y} = \frac{M_x}{E I_x} \longrightarrow (8)$$

Note: $I_{xy} = 0$ for areas possessing one (or two) axes of symmetry.

Displacements:

$$\varepsilon_x = \frac{-\nu \sigma_z}{E} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = -\nu \frac{\sigma_z}{E} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\sigma_z}{E} = -\frac{x}{R_x} + \frac{y}{R_y} = \frac{\partial w}{\partial z}$$

$$\gamma_{xz} = 0 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yx} = 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{zy} = 0 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\longrightarrow (10)$$

$(10c) \rightarrow \quad W = -\dfrac{xz}{R_x} + \dfrac{yz}{R_y} + g(x,y) \longrightarrow \text{(i)}$

$(10d, i) \rightarrow \quad \dfrac{\partial u}{\partial z} = \dfrac{z}{R_x} - \dfrac{\partial g}{\partial x} \longrightarrow u = \dfrac{z^2}{2R_x} - z\dfrac{\partial g}{\partial x} + f(x,y) \longrightarrow \text{(ii)}$

$(10f, i) \rightarrow \quad \dfrac{\partial v}{\partial z} = -\dfrac{z}{R_y} - \dfrac{\partial g}{\partial y} \longrightarrow v = -\dfrac{z^2}{2R_y} - z\dfrac{\partial g}{\partial y} + h(x,y) \longrightarrow \text{(iii)}$

$(10e, ii, iii) \rightarrow \quad -2z\dfrac{\partial^2 g}{\partial x \partial y} + \underbrace{\dfrac{\partial f}{\partial y} + \dfrac{\partial h}{\partial x}}_{\to f \, g(x,y) \, only} = 0$

$\Rightarrow \dfrac{\partial^2 g}{\partial x \partial y} = 0, \quad \dfrac{\partial f}{\partial y} = -\dfrac{\partial h}{\partial x} \quad\longrightarrow \text{(iv, v)}$

$(10a, ii, iii) \rightarrow \quad -z\dfrac{\partial^2 g}{\partial x^2} + \underbrace{\dfrac{\partial f}{\partial x} = \nu\dfrac{x}{R_x} - \nu\dfrac{y}{R_y}}_{\to f \, g(x,y) \, only}$

$\Rightarrow \dfrac{\partial^2 g}{\partial x^2} = 0 \longrightarrow \text{(vi)}$

$\dfrac{\partial f}{\partial x} = \nu\dfrac{x}{R_x} - \nu\dfrac{y}{R_y} \Rightarrow f = \nu\dfrac{x^2}{2R_x} - \nu\dfrac{xy}{R_y} + m(y) \longrightarrow \text{(vii)}$

$(10b, ii, iii) \longrightarrow \quad -z\dfrac{\partial^2 g}{\partial y^2} + \underbrace{\dfrac{\partial h}{\partial y} = \nu\dfrac{x}{R_x} - \nu\dfrac{y}{R_y}}_{\to g(x,y) \, only}$

$\Rightarrow \dfrac{\partial^2 g}{\partial y^2} = 0 \longrightarrow \text{(viii)}$

$\dfrac{\partial h}{\partial y} = \nu\dfrac{x}{R_x} - \nu\dfrac{y}{R_y} \Rightarrow h = \nu\dfrac{xy}{R_x} - \nu\dfrac{y^2}{2R_y} + n(x) \longrightarrow \text{(ix)}$

$(v, vii, ix) \longrightarrow \quad \underbrace{-\nu\dfrac{x}{R_y} + \dfrac{dm}{dy}}_{fn \, g \, y} = \underbrace{-\nu\dfrac{y}{R_x} - \dfrac{dn}{dx}}_{fn \, g \, x}$

$\Rightarrow m = -\dfrac{\nu y^2}{2R_x} + c_1 y + c_2$

$n = +\dfrac{\nu x^2}{2R_y} - c_1 x + c_3$

Apply bc's $u = v = w = 0$ at $(x,y,z) = (0,0,0)$, and rotations $\left(\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\right)$, $\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right)$, $\left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right)$ are also zero at $(x,y,z) = (0,0,0)$ (to get rid of RB motions).

Now $(iv, vi, viii) \rightarrow$ $g = c_4 x + c_5 y$

$\Rightarrow u = \dfrac{z^2}{2R_x} - c_4 z + \dfrac{\nu}{2R_x} x^2 - \dfrac{\nu}{R_y} xy - \dfrac{\nu}{2R_x} y^2 + c_1 y + c_2$

$v = -\dfrac{z^2}{2R_y} - z c_5 + \dfrac{\nu}{R_x} xy - \dfrac{\nu}{2R_y} y^2 + \dfrac{\nu x^2}{2R_y} - c_1 x + c_3$

$w = -\dfrac{xz}{R_x} + \dfrac{yz}{R_y} + c_4 x + c_5 y$.

BC'S $\Rightarrow$ $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ (could have got it by noting that const & linear terms in displ represent RB translation & rotation, so drop them).

$\Rightarrow$ $\left.\begin{array}{l} u = \dfrac{1}{2R_x}\left[z^2 + \nu(x^2 - y^2)\right] - \dfrac{\nu}{R_y} xy \\[4mm] v = \dfrac{1}{2R_y}\left[-z^2 + \nu(x^2 - y^2)\right] + \dfrac{\nu}{R_x} xy \\[4mm] w = -\dfrac{xz}{R_x} + \dfrac{yz}{R_y} \end{array}\right\} \longrightarrow \boxed{11}$

$\longrightarrow$ • add plane sections from w.
plane interp
• add anticlastic surface interp

$\Rightarrow \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{R_x}$, $\dfrac{\partial^2 v}{\partial z^2} = -\dfrac{1}{R_y}$ $\longrightarrow \boxed{12}$

Interpretation of R_x, R_y:
Deflection of centroidal line $(x = y = 0)$ is,

$$u = \dfrac{z^2}{2R_x} , \qquad v = -\dfrac{z^2}{2R_y} , \qquad w = 0 .$$
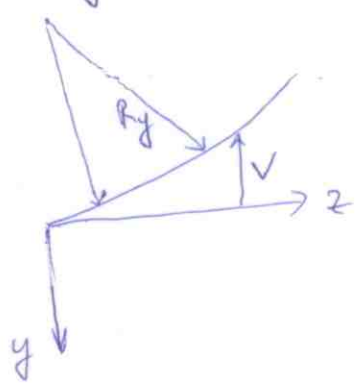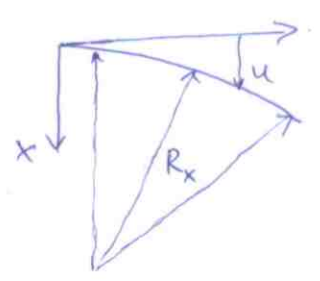
Rad of curvature in xz plane $= \dfrac{1}{K_x} = \dfrac{\left[1 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{3/2}}{\partial^2 u / \partial z^2}$

This justifies
Terminology
R_x, R_y :

$\Rightarrow R_x \approx$ rad of curvature of centroidal deformed line in xz plane $\quad \simeq \dfrac{1}{\left(\frac{\partial^2 u}{\partial z^2}\right)}$, if $\left(\frac{\partial u}{\partial z}\right) \ll 1$
$\simeq R_x$

Similarly $R_y \simeq$ −ve of rad of curv of centroidal line in yz plane.

Filaments lying in neutral plane undergo no extension
(w=0)

$$w=0 \implies y = \frac{R_y}{R_x} x \longrightarrow \text{Same condition obtained by } \sigma_{\Sigma}=0$$
(see eqn (6)).

If x, y are p-axes, ie $I_{xy}=0$,

(12), (4) $\implies \frac{1}{R_x} = \frac{\partial^2 u}{\partial z^2} = \frac{M_y}{EI_y}$, $\frac{1}{R_y} = -\frac{\partial^2 v}{\partial z^2} = \frac{M_x}{EI_x} \longrightarrow$ (13)

Eqn (13) is used in "engineering theory" to find deflection of centerline (which is assumed as deflection of whole beam as well) even when bending moment is not constant. Similarly (5) is used to find stresses in 'engineering theory', even when shear forces are present, resulting in a varying BM along the beam length. Experimental evidence, for thin metallic beams, shows this to be a reasonably good approximation.

Eqn (13) shows that more the 'EI' less are the deflections. Thus 'EI' are moduli of flexural rigidity. To increase rigidity make moment of inertia large. Assuming $M_x=0$, some crossections that have large I_y are shown below.
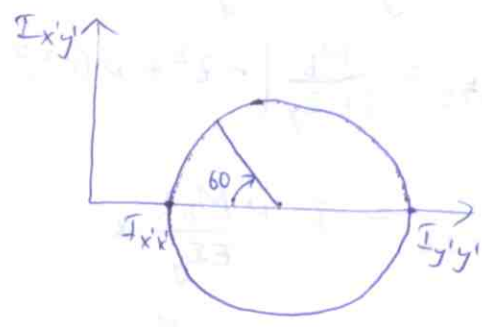
$M_x = 0$
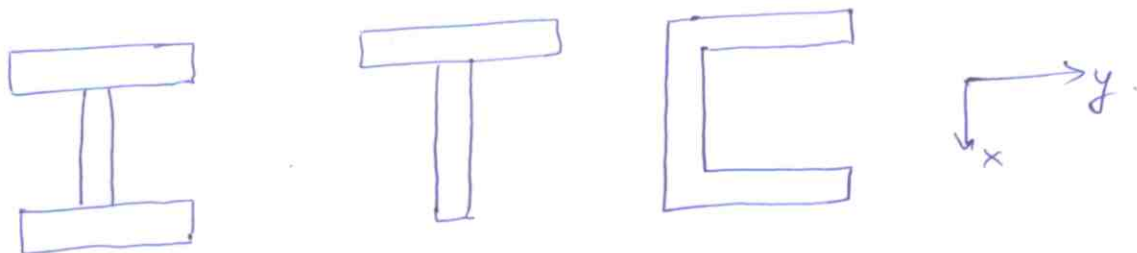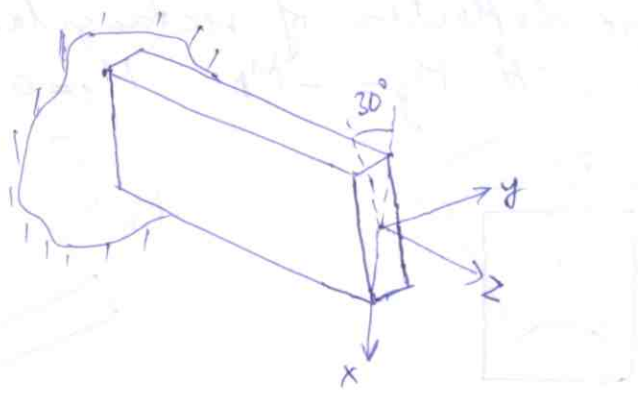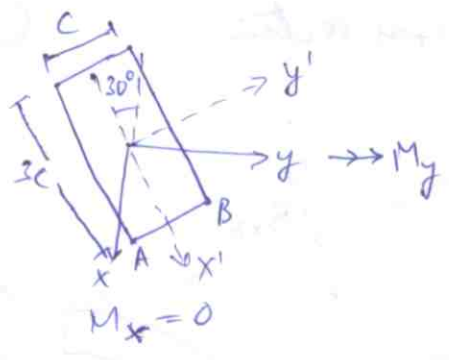
$$I_{y'y'} = \frac{c(3c)^3}{12} = \frac{9c^4}{4}, \qquad \bar{I}_{x'x'} = \frac{3c(c^3)}{12} = \frac{c^4}{4}, \qquad \bar{I}_{x'y'} = 0$$

$$R = \frac{I_{y'y'} - I_{x'x'}}{2} = c^4$$

$$I_{xy} = R\sin 60 = 0.87\,c^4$$

$$I_{xx} = I_{av} - R\cos 60 = 0.75\,c^4$$

$$I_{yy} = I_{av} + R\cos 60 = 1.75\,c^4$$

Alternatively, $\begin{bmatrix} I_{xx} & -I_{xy} \\ -I_{xy} & I_{yy} \end{bmatrix} = \begin{bmatrix} \cos(30) & \sin(-30) \\ -\sin(30) & \cos(-30) \end{bmatrix} \begin{bmatrix} c^4/4 & 0 \\ 0 & 9c^4/4 \end{bmatrix} \begin{bmatrix} \cos(30) & -\sin(-30) \\ \sin(30) & \cos(-30) \end{bmatrix}$

gives same result.

use $M_x = 0$ in ⑤, get,

$$\sigma_z = \frac{-M_y(0.75)c^4}{(0.75)(1.75)c^8 - (0.87)^2 c^8}\, x \;+\; \frac{M_y(0.87c^4)}{(0.75)(1.75)c^8 - (0.87)^2 c^8}\, y$$

$$= \frac{-1.35\,M_y}{c^4}\, x \;+\; \frac{1.566\,M_y}{c^4}\, y$$
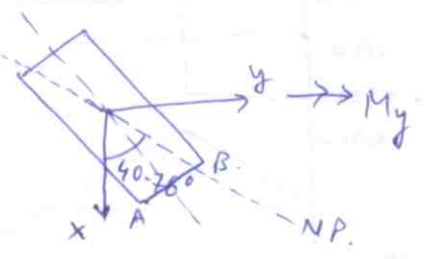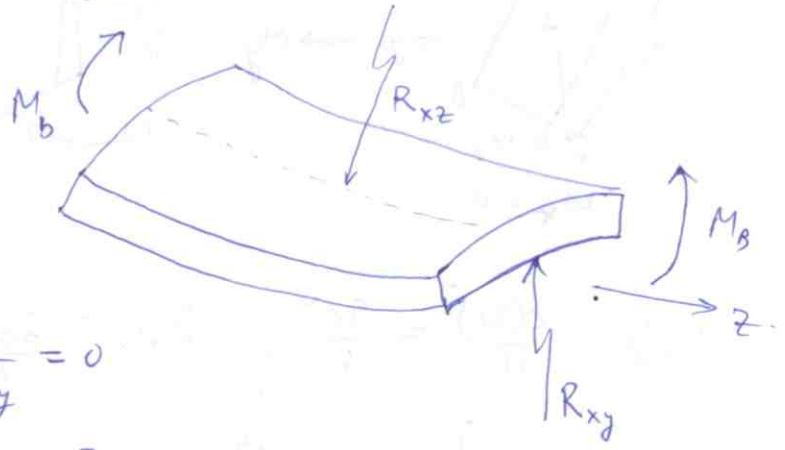
$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 3c/2 \\ -c/2 \end{pmatrix} \quad \Rightarrow x_A = 1.55c, \quad y_A = 0.317c$$

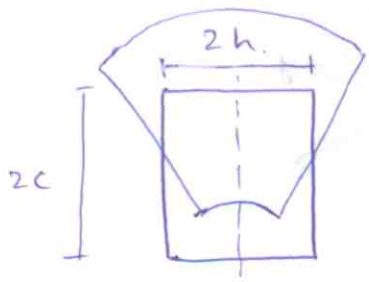$$(\sigma_z)_A = \frac{-1.6}{c^3}\, M_y \quad (\text{compressive})$$

$$\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 3c/2 \\ c/2 \end{pmatrix} = \begin{pmatrix} 1.05c \\ 1.18c \end{pmatrix}$$

$$(\sigma_z)_B = \frac{0.43}{c^3}\, M_y \quad (\text{Tensile}).$$

$$\tan\beta = \frac{0.75}{0.87} \longrightarrow \beta = 40.763°$$

(Ex) Analyze deflection of rectangular cross section beam with $M_y = -M_b$, $M_x = 0$

$$\frac{1}{R_x} = \frac{M_y}{EI_y} = \frac{-M_b}{EI_y} \quad , \quad \frac{1}{R_y} = 0$$
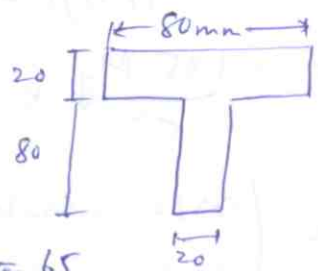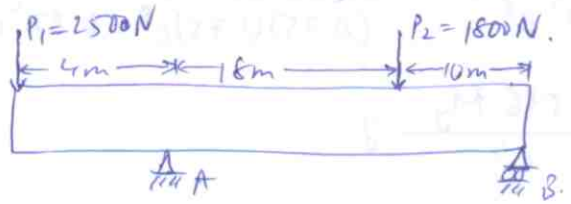
$$u\big|_{x=\pm c} = \frac{M_b}{2EI_y}\left[-z^2 + \nu(-c^2 + y^2)\right]$$

$$v\big|_{y=\pm h} = \mp \nu \frac{M_b h}{EI_y} x$$

Surfaces originally parallel to yz plane are deformed into saddle-shaped (anticlastic) surfaces. Radius of curvature in xy plane is R/ν where R is rad. of curv in xz plane.

$$\left(ie, \quad \frac{\partial^2 u}{\partial y^2} = \frac{M_b}{EI_y}\nu = \frac{1}{R_{xy}} \quad ; \quad \frac{\partial^2 u}{\partial z^2} = -\frac{M_b}{EI_y} = \frac{-1}{R_{xz}}\right.$$

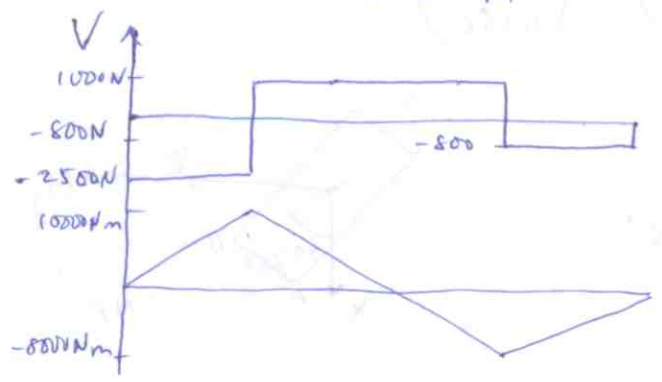$$\left. \Rightarrow R_{xy} = R_{xz}/\nu \right).$$

(Ex)

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{(20)(80)(40) + (20)(80)(90)}{(20)(80) + (20)(80)} = 65$$

$$I_y = \left(\frac{80(20)^3}{12} + (80)(20)(25)^2 + \frac{(20)(80)^2}{12} + (80)(20)(25)^2\right) \times 10^{-12}$$

$$= 2.907 \times 10^{-6} \, m^4$$

$R_A = 3500 \, N$

$R_B = 800 \, N$

From SFD, BMD, you see that critical sections are $z = 4m$, and $22m$. Critical planes, at each z, are $x = 65mm, 0mm, -35mm$

Assume $\sigma_z = -\dfrac{Mx}{I_y}$ holds despite shear force being present. (102b)

$\underline{z=4}$ $M = 10000 \; N \cdot m$.

$$\sigma_z \Big|_{x=0.065} = -\frac{10000\,(0.065)}{2.9017 \times 10^{-6}} = -223.6 \; MPa$$

$$\sigma_z \Big|_{x=-0.035} = 120.4 \; MPa = -\frac{10000\,(-0.035)}{2.9067 \times 10^{-6}}$$

$$\tau_{xz} \Big|_{x=0} = \frac{VQ}{Ib} \quad , \qquad Q = \bar{x}A = \frac{(65)(65)(20)}{2} \times 10^{-9} = 4.225 \times 10^{-5}$$

$$= \frac{-2500\,(4.225 \times 10^{-5})}{(2.907 \times 10^{-6})(0.02)} = -1.816 \; MPa$$

$\underline{z=22 \, m}$ $M = -8000 \; N \cdot m$.

$$\sigma_z \Big|_{x=0.065} = \frac{(8000)\,(0.065)}{2.9067 \times 10^{-6}} = 178.9 \; MPa$$

$$\sigma_z \Big|_{x=-0.035} = \frac{(8000)\,(-0.035)}{2.9067 \times 10^{-6}} = -96.3 \; MPa$$

$$\tau_{xz} \Big|_{\substack{x=0 \\ z=22}} < \tau_{xz} \Big|_{\substack{z=4m \\ x=0}} \quad (\text{in magnitude})$$

$\therefore \; \sigma_{max} = 178.9 \; MPa$, at $z = 22$, $x = 0.065 \, m$.

$\sigma_{min} = -223.6 \; MPa$, at $z=4$, $x = 0.065 \, m$

$\tau_{max} = \dfrac{223.6}{2} = 111.8 \; MPa$ at $z = 4 \, m$, $x = 0.065 \, m$.

Pure Bending of Composite Beams.
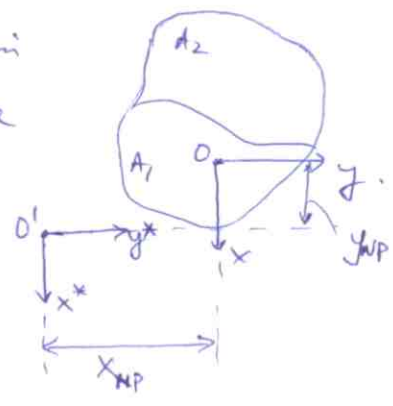

E_1, A_1

E_2, A_2

$\nu_1 = \nu_2$

Beam is prismatic and cross sections are uniform (geometry and material-distribution wise) thrount.

We follow semi-inverse procedure as before. So assume stress field of the form,

$$\boxed{\begin{aligned} \sigma_z &= -E_1 K_x \, x + E_1 K_y \, y \quad \text{in } A_1 \\ &= -E_2 K_x \, x + E_2 K_y \, y \quad \text{in } A_2 \end{aligned}} \longrightarrow \text{(14a)}$$

remaining stress components $= 0$.

Note that z-axis is not the line of centroids, in general, now. Locate z-axis by zero axial force condition. ($F_z = 0$ on end face).

$$\int_{A_1+A_2} \sigma_z \, dA = -E_1 K_x \int_{A_1} x \, dA + E_1 K_y \int_{A_1} y \, dA - E_2 K_x \int_{A_2} x \, dA$$

$$+ E_2 K_y \int_{A_2} y \, dA = 0 \longrightarrow \circledast$$

Put $x = x^* - x_{NP}, \quad y = y^* - y_{NP}$,

$$-E_1 K_x \bar{x}_1^* A_1 - E_2 K_x \bar{x}_2^* A_2 + (E_1 A_1 + E_2 A_2) K_x x_{NP}$$

$$+ E_1 K_y \bar{y}_1^* A_1 + E_2 K_y \bar{y}_2^* A_2 + (E_1 A_1 + E_2 A_2) K_y y_{NP} = 0.$$

The above should be valid for $K_x \neq 0$, $K_y = 0$ and vice-versa (ie one-plane bending also). Hence,

$$\text{(14b)} \leftarrow \quad x_{NP} = \frac{E_1 A_1 \bar{x}_1^* + E_2 A_2 \bar{x}_2^*}{E_1 A_1 + E_2 A_2}, \qquad y_{NP} = \frac{E_1 A_1 \bar{y}_1^* + E_2 A_2 \bar{y}_2^*}{E_1 A_1 + E_2 A_2}$$

So fix z-axis at (x_{NP}, y_{NP}) w.r.t. O' (the arbitrary r/f frame). Here $\bar{x}_1^*, \bar{x}_2^*, \bar{y}_1^*, \bar{y}_2^*$ are coordinates of centroid of areas A_1, A_2 measured from O'.

K_x and K_y determined from,

$$M_x = \int_{A_1} y \sigma_z \, dA + \int_{A_2} y \sigma_z \, dA = -E_1 K_x (I_{xy})_1 + E_1 K_y (I_x)_1$$

$$- E_2 K_x (I_{xy})_2 + E_2 K_y (I_x)_2 \longrightarrow \circledast\circledast$$

$$M_y = -\int_{A_1} x\sigma_z\, dA - \int_{A_2} x\sigma_z\, dA = E_1 K_x (I_y)_1 - E_1 K_y (I_{xy})_1$$
$$+ E_2 K_x (I_y)_2 - E_2 K_y (I_{xy})_2 \longrightarrow (\boxed{***})$$

Solve $\boxed{**}$, $\boxed{***}$,

$$K_x = \frac{F_3 M_x + F_1 M_y}{F_1 F_2 - F_3^2} \quad , \quad K_y = \frac{F_2 M_x + F_3 M_y}{F_1 F_2 - F_3^2} \longrightarrow \boxed{1.4c}$$

where,
$$F_1 = E_1 (I_x)_1 + E_2 (I_x)_2$$
$$F_2 = E_1 (I_y)_1 + E_2 (I_y)_2 \qquad I_x, I_y, I_{xy} \text{ are about}$$
$$F_3 = E_1 (I_{xy})_1 + E_2 (I_{xy})_2 \qquad \text{axes thru } O.$$

It is readily seen that equilibrium, compatibility, and bc's (lateral, $F_x = F_y = F_z = 0$ on end faces) are satisfied.

Now $\varepsilon_x = \varepsilon_y = -\nu\sigma_z = -\nu(-x K_x + y K_y)$ if $\nu_1 = \nu_2$. If $\nu_1 \neq \nu_2$ strains will be discontinuous across interface (ie delamination occurs).

(II) BENDING OF BEAMS BY TERMINAL LOADS.

(plane stress solution done on p.29-31).
Consider the problem of a cantilever beam having uniform cross section, and one end $(z=0)$ fixed and the other end $(z=l)$ loaded by some stress distribution that is statically equivalent to a single force $(W_x, W_y, 0)$ lying in the plane $z=l$ and acting at the load point (x_0, y_0, l). The z-axis is the line of centroids, and x,y are arbitrary axes. Lateral surface is traction free and body forces are negligible.
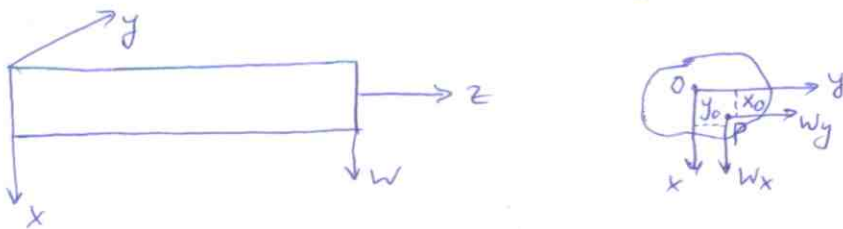
Using semi-inverse method, assume,
$$\sigma_x = \sigma_y = \tau_{xy} = 0$$
Guided by the fact that moment M_y produced by W_x acting alone is

$$M_y = W_x (l-z)$$

and that for bending of beams by terminal couple M_y we have,

$$\sigma_z = -\frac{M_y x}{I_y}$$

we assume for the present problem,

$$\boxed{\sigma_z = -E(l-z)(k_x x + k_y y)} \longrightarrow \text{(15)}$$

where k_x, k_y are constants to be determined from bc's

$$\int_A \tau_{xz} dA = W_x \quad , \quad \int_A \tau_{yz} dA = W_y .$$

Equilibrium equations reduce to

(16) $\longleftarrow$
$$\begin{cases} \dfrac{\partial \tau_{xz}}{\partial z} = 0 \quad , \quad \dfrac{\partial \tau_{yz}}{\partial z} = 0 \\[2mm] \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = 0 \Rightarrow \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + E(k_x x + k_y y) = 0 \end{cases}$$

BC's on lateral face are

$$\tau_{xz} n_x + \tau_{yz} n_y = 0 \text{ on } S \text{ (other two i.s)} \longrightarrow \text{(17)}$$

Determination of k_x, k_y:

$$W_x = \int_A \tau_{xz} dA = \int_A \left[\tau_{xz} + x \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \overset{0}{\cancel{\frac{\partial \sigma_z}{\partial z}}} \right) \right] dA$$

$$= \int_A \left[\frac{\partial}{\partial x}(x \tau_{xz}) + \frac{\partial}{\partial y}(x \tau_{yz}) + \frac{\partial}{\partial z}(x \sigma_z) \right] dA$$

$$= \int_S x (\tau_{xz} n_x + \tau_{yz} n_y) ds + \int_A x \frac{\partial \sigma_z}{\partial z} dA = \int_A x E(k_x x + k_y y) dA$$

$$\underset{=0 \text{ on } S \text{ (from (17))}}{}$$

$$= E k_x I_y + E k_y I_{xy} \longrightarrow \circledast$$

Similarly, $W_y = \int_A \tau_{yz} dA = E k_x I_{xy} + E k_y I_x \quad \circledast\circledast$

$$\circledast, \circledast\circledast \longrightarrow \boxed{E k_x = \frac{I_x W_x - I_{xy} W_y}{I_x I_y - I_{xy}^2} \quad , \quad E k_y = \frac{I_y W_y - I_{xy} W_x}{I_x I_y - I_{xy}^2}}$$

put $M_y = W_x(l-z)$
$M_x = -W_y(l-z)$
into (15)
and (18) into (5). So,
and get (5). So,
σ_z has same form
for pure bending
or for
(18) bend
by
terminal
loads.

Origin at centroid $\Rightarrow \int_A x\,dA = \int_A y\,dA = 0 \Rightarrow \int_A \sigma_z\,dA = 0$

$\int_A \sigma_z y\,dA = -E(\ell-z)(K_x I_{xy} + K_y I_x) = -W_y(\ell-z) = M_x$ (checks out)

$-\int_A \sigma_z x\,dA = +E(\ell-z)(K_x I_y + K_y I_{xy}) = W_x(\ell-z) = M_y$ (checks out)

Now ⑯ $\overset{\text{(Equilibrium)}}{\underline{\text{satisfied}}}$ if stresses expressed as,

$$\left.\begin{array}{l} \tau_{xz} = \dfrac{\partial \phi}{\partial y} + f(y) - \frac{1}{2}E K_x x^2 \\[2mm] \tau_{yz} = -\dfrac{\partial \phi}{\partial x} - g(x) - \frac{1}{2}E K_y y^2 \end{array}\right\} \longrightarrow ⑲$$

where $\phi(x,y)$ is stress function.

Compatibility :

B M compat equs reduce to,

$$\nabla^2 \sigma_{ij} + \frac{1}{(1+\nu)}\Delta_{,ij} = 0 \quad , \qquad \Delta = \sigma_x + \sigma_y + \sigma_z = \sigma_z$$

$$\therefore \ \nabla^2 \tau_{yz} + \frac{E}{(1+\nu)}K_y = 0 \ \overset{⑲}{\Longrightarrow}\ \frac{\partial}{\partial x}(\nabla^2 \phi) = -\frac{E\nu}{(1+\nu)}K_y - \frac{\partial^2 g}{\partial x^2} \longrightarrow ⊛$$

$$\nabla^2 \tau_{xz} + \frac{E}{(1+\nu)}K_x = 0 \ \overset{⑲}{\Longrightarrow}\ \frac{\partial}{\partial y}(\nabla^2 \phi) = \frac{E\nu}{(1+\nu)}K_x - \frac{\partial^2 f}{\partial y^2} \longrightarrow ⊛⊛$$

$⊛⊛, ⊛ \Rightarrow \boxed{\nabla^2 \phi = -2G\nu K_y x - \dfrac{\partial g}{\partial x} + 2G\nu K_x y - \dfrac{\partial f}{\partial y} - 2G\alpha} \longrightarrow ⑳$

where constant $(-2G\alpha)$ determined from twisting moment condition, i.e,.

$$\int_A (x\tau_{yz} - y\tau_{xz})\,dA = x_0 W_y - y_0 W_x \longrightarrow ㉑$$

For lateral bc ⑰ to be satisfied,

$$\tau_{xz}n_x + \tau_{yz}n_y = 0 \Rightarrow \underbrace{\frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial x}\frac{dx}{ds}}_{\frac{d\phi}{ds}} = \left[\frac{1}{2}E K_x x^2 - f(y)\right]\frac{dy}{ds}$$
$$- \left[\frac{1}{2}E K_y y^2 + g(x)\right]\frac{dx}{ds}$$

㉒ⓐ $\leftarrow \Rightarrow \boxed{\dfrac{d\phi}{ds} = 0}$ on S if we choose

㉓ $\leftarrow \boxed{g(x) = -\frac{1}{2}E K_y y^2}$ and $\boxed{f(y) = \frac{1}{2}E K_x x^2}$ on S

Note that $g(x)$ and $f(y)$ can be chosen as any value on that part of S for which $dx/ds = 0$ and $dy/ds = 0$, respectively.

$\Rightarrow \boxed{\phi = \text{const} = 0}$ (∵ value of const doesn't affect stresses) $\longrightarrow$ ②②⑦ ⑩⑦

on ϕ if g, f chosen as in ②①.

Summarizing, we have verified $F_z = 0$, $F_x = W_x$, $F_y = W_y$,

$M_x = -W_y(l-z)$, $M_y = W_x(l-z)$.

We <u>choose</u> α such that proper twisting
moment $M_z = x_0 W_y - y_0 W_x$ is obtained.

We <u>chose</u> ϕ to automatically satisfy
equilibrium.
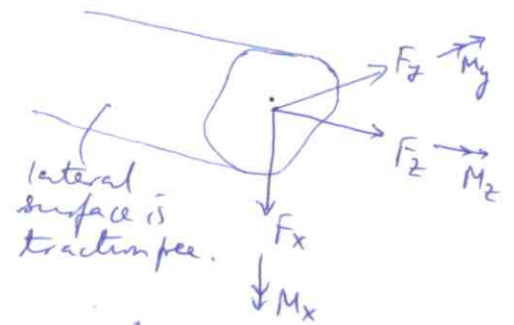
Compatibility & traction-free bc's will be satisfied if we
solve ②⓪ & ②③ with g, f chosen as per ②②.

Thus, a solution of ②⓪ & ②③ with ②② gives exact solution
to the problem.

lateral
surface is
traction free.

Neutral plane: $\varepsilon_z = \sigma_z = 0 \longrightarrow \tan\beta = \dfrac{y}{x} = -\dfrac{K_x}{K_y}$

(eg): $I_{xy} = W_y = 0 \Rightarrow K_y = 0 \Rightarrow \beta = \dfrac{\pi}{2}$, ie yz is the **N.P.**

<u>Interpretation of α.</u>

Rotation (infinitesimal) of line/area element, in xy plane is

$$\phi_z = \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$\omega_{21}$$

Define <u>local twist</u> at point (x,y) of a crosssection as

②④ $\longleftarrow \boxed{\dfrac{\partial \omega}{\partial z}} = \frac{1}{2}\left(\dfrac{\partial^2 v}{\partial x \partial z} - \dfrac{\partial^2 u}{\partial y \partial z}\right) = \left(\dfrac{\partial \varepsilon_{yz}}{\partial x} - \dfrac{\partial \varepsilon_{xz}}{\partial y}\right) \boxed{= \dfrac{1}{2G}\left(\dfrac{\partial \tau_{zy}}{\partial x} - \dfrac{\partial \tau_{xz}}{\partial y}\right)}$

For pure torsion,

$$\dfrac{\partial \omega}{\partial z} = \frac{1}{2G}\left(\overbrace{-\dfrac{\partial^2 \phi}{\partial x^2} - \dfrac{\partial^2 \phi}{\partial y^2}}^{2G\alpha}\right) = \alpha \qquad \left\{\text{alternatively } \dfrac{\partial \omega}{\partial z} = \frac{1}{2}\dfrac{\partial}{\partial z}\left(\overbrace{\dfrac{\partial v}{\partial x}}^{\alpha xz} - \overbrace{\dfrac{\partial u}{\partial y}}^{-\alpha yz}\right) = \alpha\right\}$$

So local twist for pure torsion is constant over section (and also
over z, if warping restraint is ignored).

For problem at hand, from ⑲, ②④, ②⓪

$$\dfrac{\partial \omega}{\partial z} = -\frac{1}{2G}\left[\nabla^2 \phi + \dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial y}\right] = \alpha + \nu(K_y x - K_x y)$$

$\Rightarrow$ Mean twist $= \dfrac{\int_A \frac{\partial w}{\partial z}\, dA}{A} = \alpha = $ local twist at origin (centroidal line).

Thus twisting occurs in addition to bending. The general flexure problem can hence be resolved into:

(a) A flexure problem with $\alpha = 0$ (no mean local twist). The position of load point that yields this stress distribution (in which τ_{xz}, τ_{yz} determined for $\alpha = 0$) is obtained by twisting moment condition
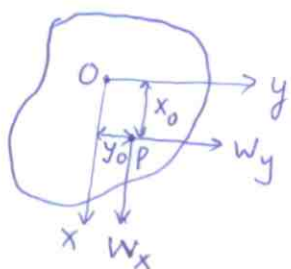
$$\int_A (x\,\tau_{yz} - y\,\tau_{xz})\, dA = X_{CF}\, W_y - Y_{CF}\, W_x \longrightarrow \boxed{25}$$

(here τ_{yz}, τ_{xz} determined for $\alpha=0$)

Eqn $\boxed{25}$ holds for arbitrary loading. The load point $(X_{CF}, Y_{CF}, 1)$ corresponding to zero mean twist $(\alpha = 0)$ is called 'center of Flexure' or 'Shear Center'.

(b) A torsion problem with mean twist α due to applied couple $M = W_y (X_0 - X_{CF}) - W_x (Y_0 - Y_{CF})$, and with shear stresses determined (from $\boxed{19}$) for $k_x = k_y = 0$, and hence $f(y) = g(x) = 0$. __Note__ that in this case eqns $\boxed{19} - \boxed{23}$ specialize to the ones for pure torsion (Prandtl's stress function formulation).

So we can think of $\underline{W} = (W_x, W_y)$ at $P = (x_0, y_0, 1)$ as being replaced by $\underline{W}$ at $(X_{CF}, Y_{CF}, 1)$ and a couple M_2.
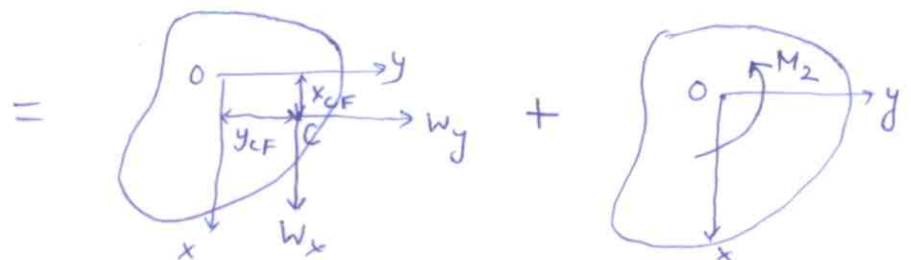
General bending $\quad=\quad$ Bending w/o Twist $\quad+\quad$ Torsion w/o Flexure.

<u>Determination of Shear Center. (CF)</u>

Use (25) to determine (x_{CF}, y_{CF}) as follows. So (25), (19), (22b), (23)
yield,

$$x_{CF} W_y - y_{CF} W_x = \int_A (x \bar{\tau}_{yz} - y \bar{\tau}_{xz}) \, dA$$

$$= \int_A \left[x \left(-\frac{\partial \phi}{\partial x} - g(x) - \frac{1}{2} E K_y y^2 \right) - y \left(\frac{\partial \phi}{\partial y} + f(y) - \frac{1}{2} E K_x x^2 \right) \right] dA$$

$$= \int_A \left[2\phi + \frac{\partial}{\partial x} \left[-x\phi - xy f(y) + \frac{1}{6} E K_x x^3 y \right] - \frac{\partial}{\partial y} \left[y\phi + xy g(x) + \frac{1}{6} E K_y xy^3 \right] \right] dA$$

$$= \int_A 2\phi \, dA + \oint_C \left[y\phi + xy g(x) + \frac{1}{6} E K_y xy^3 \right] dx + \oint_C \left[-x\phi - xy f(y) + \frac{1}{6} E K_x x^3 y \right] dy$$

$$= \int_A 2\phi \, dA - \frac{1}{3} E \left[K_y \oint_C xy^3 \, dx + K_x \oint_C x^3 y \, dy \right] \qquad \text{(used (22b),(23))}$$

$$= \int_A (2\phi - E K_x x^2 y + E K_y xy^2) \, dA$$

where ϕ is determined from (20), (22b), (23), setting $\alpha = 0$. Inserting
(18) in the above & equating coefficients of W_x, W_y ($\because$ they are
independent), we obtain

$$x_{CF} = \frac{1}{W_y} \int_A \left[2 W_y \phi_2 + \frac{I_{xy}}{F} W_y x^2 y + \frac{I_y}{F} W_y xy^2 \right] dA$$

where $F = I_x I_y - I_{xy}^2$

and $$y_{CF} = -\frac{1}{W_x} \int_A \left[2 W_x \phi_1 - \frac{I_x}{F} W_x x^2 y - \frac{I_{xy}}{F} W_x xy^2 \right] dA$$

$\longrightarrow$ (26).

In the above we have used the fact that since (20) is
linear in ϕ, its solution is $\phi = W_y \phi_2 + W_x \phi_1$, where ϕ_2 and
ϕ_1 are solutions of (20), (22b), (23) with $(W_y, W_x) = (1, 0)$
and $(0, 1)$, respectively. Thus ϕ_1 and ϕ_2 depend on the geometry
of the cross-section only, and consequently so does (x_{CF}, y_{CF}).

<u>Section having one axis of symmetry</u>:
For e.g. consider x-axis as the axis of symm. (ie $I_{xy} = 0$)
Let $W_x, W_y = 0$ be applied thro' CF, so that
$(\alpha = 0)$ no twisting occurs.
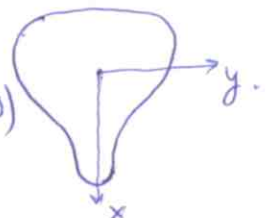
Thus, $K_y = 0$, ie bending about y-axis ⑩
(ie in xz plane) only w/o twist. Due to symmetry of
cross-section about x-axis, τ_{xz} is even in y and $\bar{\tau}_{yz}$
$\underset{\wedge}{}$ we conclude that
is odd in y (the latter correctly yields $W_y = \int_A \tau_{yz} dA = 0$ as a
check of this conclusion). From ㉕,

$$-y_{CF} W_x = \int_A \underbrace{(x\,\tau_{yz} - y\,\tau_{xz})}_{= \text{odd in } y} dA \;\underset{\nearrow}{=}\; 0 \quad\Rightarrow\quad \underset{\text{x is symm axis and vice versa}}{y_{CF} = 0 \text{ when}}$$

Thus a doubly symmetric section has CF as its centroid.

Physical interpretation of K_x, K_y

$$\frac{\partial u}{\partial z} = \frac{\tau_{xz}}{G} - \frac{\partial w}{\partial x} \quad\longleftarrow \overset{\text{used}}{(\text{strain displ} + \text{const. law})}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2} = \frac{1}{G} \underbrace{\frac{\partial \tau_{xz}}{\partial z}}_{=0 \,(\text{equil})} - \frac{\partial^2 w}{\partial x \partial z} \;=\; -\frac{1}{E}\frac{\partial \sigma_z}{\partial x} = (l-z) K_x$$

Now, for $\dfrac{\partial u}{\partial z} \ll 1$, $\quad \dfrac{\partial^2 u}{\partial z^2} \simeq \dfrac{1}{R_x} = (l-z) K_x$

Thus K_x is inversely proportional to radius of curvature in xz
plane.

For special case when $W_y = 0$, $I_{xy} = 0$, ie loading parallel
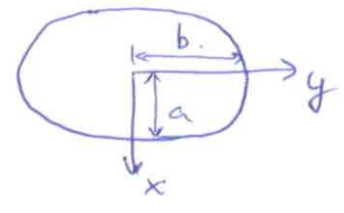to a principal axis, $K_x = \dfrac{W_x}{E I_y}$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2} = (l-z)\frac{W_x}{E I_y} = \frac{M_y}{E I_y} \quad\Rightarrow\quad \boxed{M_y = E I_y \frac{\partial^2 u}{\partial z^2}}$$

Thus the Bernoulli Euler law applies even when a
transverse load is applied if it is parallel to a
p-axis.

(Ex 1) Elliptic Ω-section, $(W_x, 0)$ applied at centroid. $(0,0)$

$\therefore I_{xy} = 0$, $K_y = 0$, $\sigma_z = -E(l-z) K_x x$.

$\because CF = (0,0) \Rightarrow x = 0$ as load applied at shear center.

Thus no twisting occurs.

Boundary eqn : $S \to \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$

$\therefore K_y = 0$, $g(x) = 0$ on S (and hence choose $g(x) = 0$ on A).

$K_x = \dfrac{W_x}{E I_y} = \dfrac{4 W_x}{E \pi a^3 b}$

$f(y) = \tfrac{1}{2} E K_x x^2 = \tfrac{1}{2} \dfrac{W_x}{I_y} x^2$ on S $\nearrow^{a^2(1 - y^2/b^2)}$

so chose $f(y) = \dfrac{W_x}{2 I_y} \dfrac{a^2}{b^2} (b^2 - y^2)$ on A

$\therefore \nabla^2 \phi = \dfrac{W_x}{I_y} \left(\dfrac{\nu}{1+\nu} + \dfrac{a^2}{b^2} \right) y$ on A —(a)

$\phi = 0$ on S —(b)

$\therefore$ solution of the form $\phi = my \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 \right)$ will work. (c)

Subst (c) in (a) to get m. We get,

$\phi = \underbrace{\dfrac{a^2 \left[(1+\nu) a^2 + \nu b^2 \right]}{2(1+\nu)(3a^2 + b^2)}}_{= m} \dfrac{W_x}{I_y} \, y \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 \right)$

$\therefore \tau_{xz} = \phi_{,y} + f(y) - \tfrac{1}{2} E K_x x^2 = \dfrac{2 W_x}{\pi a^3 b} \dfrac{\left[2(1+\nu) a^2 + b^2 \right]}{(1+\nu)(3a^2 + b^2)} \left[a^2 - x^2 - \dfrac{(1-2\nu) a^2 y^2}{2(1+\nu) a^2 + b^2} \right]$

$\tau_{yz} = -\phi_{,x} - g(x)^0 - \tfrac{1}{2} E K_y^0 y^2 = -\dfrac{4 W_x}{\pi a^3 b} \dfrac{(1+\nu) a^2 + \nu b^2}{(1+\nu)(3a^2 + b^2)} xy$

$\sigma_z = -\dfrac{4 W_x}{\pi a^3 b} (l-z) x.$

$\therefore \tau_{yz} = 0$ on $x = 0, y = 0$ and on

For $(\tau_{yz})_{max}$ we need $(xy)_{max}$. Obviously $(xy)_{max}$ occurs on boundary S.

$xy = \underbrace{(a \cos\theta)}_{x} \underbrace{(b \sin\theta)}_{y} = \dfrac{ab}{2} \sin 2\theta \Rightarrow (xy)_{max} = \dfrac{ab}{2}$ for $\theta = \dfrac{\pi}{4}$
   (parametric eqn of ellipse)

$\therefore \left| (\tau_{yz})_{max} = \dfrac{2 W_x}{A} \dfrac{b}{a} \dfrac{(1+\nu) a^2 + \nu b^2}{(1+\nu)(3a^2 + b^2)} \right|$, where $A = \pi ab$.

consider last terms (marked ⊢——⟶) in τ_{xz}.

$$\left(\frac{(1-2\nu)\,a^2 y^2}{2(1+\nu)\,a^2 + b^2}\right)_{max} \overset{?}{\gtrless} a^2 \;,\; ie\; b^2 \overset{?}{\gtrless} \frac{2(1+\nu)a^2 + b^2}{(1-2\nu)} \longrightarrow$$ obviously '<' is correct choice.

$$\therefore \quad \left.\begin{matrix} a^2 - x^2 - \dfrac{(1-2\nu)\,a^2 y^2}{2(1+\nu)a^2 + b^2} \end{matrix}\right\} \begin{matrix} > -a^2 \\ \leqslant a^2 \end{matrix} \;\Rightarrow\; (\tau_{xz})_{max}$$ occurs at $(x,y) = (0,0)$

$$\boxed{(\tau_{xz})_{max} = \frac{2W_x}{A}\left[\frac{2(1+\nu)a^2 + b^2}{(1+\nu)(3a^2 + b^2)}\right]}$$

For circular section $(a=b)$, elementary beam theory, which assumes $\tau_{xz} \quad \tau_{xz}[y]$, gives

$$(\tau_{xz})_{max} = \frac{4}{3}\frac{W_x}{A} \qquad \left\{ \text{from } \tau_{xz} = \frac{VQ}{I_y t} = \frac{4W_x(a^2-x^2)}{3\pi a^4} \right\}$$

with $W_x \;\; \overset{\int_x^a x' dA}{\nwarrow} \;$, $t\; \underset{2\sqrt{a^2-x^2}}{\downarrow}$

$$\therefore \quad \frac{\left[(\tau_{xz})_{max}\right]_{exact}}{\left[(\tau_{xz})_{max}\right]_{approx}} = \frac{3}{8}\left(\frac{3+2\nu}{1+\nu}\right) \longrightarrow \text{for } \nu = 0.3 \;,\; error \simeq 4\%.$$

(you get same result from elementary beam theory using VQ/It, $t=2b$, $Q = \int_0^a x' dA$)

For $b \ll a$, $(\tau_{xz})_{max} \simeq \frac{4}{3}\frac{W_x}{A}$, $(\tau_{yz})_{max} \simeq \frac{4}{3}\frac{W_x}{A}\frac{b}{2a}$
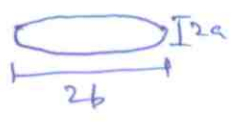
Also, putting $y = \varepsilon$ you get,

$$\tau_{xz} \simeq \frac{4W_x}{3Aa^2}\left[a^2 - x^2 - O(\varepsilon^2)\right] \;,\; \tau_{yz} = -\frac{4W_x}{3Aa^2} x\varepsilon = O(\varepsilon)$$

$$\Rightarrow \tau_{xz} > \tau_{yz} \quad \text{(even when } x-a = \varepsilon \text{ you get } \tau_{xz} \simeq 2\tau_{yz}).$$

For $b \gg a$, $(\tau_{xz})_{max} \simeq \frac{2}{1+\nu}\frac{W_x}{A}$, $(\tau_{yz})_{max} \simeq \frac{2}{(1+\nu)}\frac{W_x}{a}\frac{\nu b}{a} \gg (\tau_{xz})_{max}$

for $b \gg a$

Thus τ_{yz} is <u>not</u> negligible. Also, for $b \gg a$, $(\tau_{xz})_{max}$ has a large error, compared to elementary beam theory.

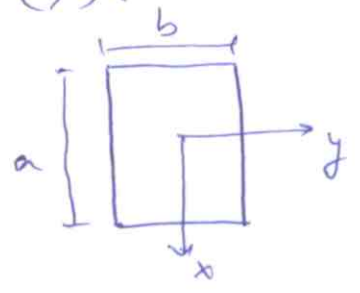~~(assuming flat ellipse as narrow rectangle) which gives $(\tau_{xz})_{max} \simeq \frac{3}{2}\frac{W_x}{A}$ (based on $\frac{VQ}{It}$ formula~~

(Ex 2) <u>Rectangular ⊗ section</u>, $(W_x, 0)$ applied at $(0,0)$.

As in (Ex 1), $g(x) = 0$, $k_y = 0$, $\alpha = 0$.

Choose $f(y) = \frac{1}{2}Ek_x\left(\frac{a}{2}\right)^2$ on A.

($\because$ for vertical boundaries, $\frac{dy}{ds} = 0$ so on that part $f(y)$ can be anything — ref p. 106).

$$\Rightarrow \nabla^2 \phi = 2 G \tau K_x y = \frac{\nu}{(1+\nu)} \frac{W_x y}{I_y} \longrightarrow \text{(a)}$$

$$= \text{even in } x, \text{ odd in } y.$$

$$\phi = 0 \quad \text{on } S' \longrightarrow \text{(b)}$$

$$\Rightarrow \text{Take } \phi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos(2m+1)\frac{\pi x}{a} \sin 2n\frac{\pi y}{b} \longrightarrow \text{(c)} \quad \begin{array}{l}\text{(satisfies} \\ \text{bc \& even-odd} \\ \text{condt).}\end{array}$$

Representing y as double Fourier series,

$$y = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos(2m+1)\frac{\pi x}{a} \sin 2n \frac{\pi y}{b} \left.\rule{0pt}{30pt}\right\} \longrightarrow \text{(d)}$$

$$B_{mn} = \frac{4}{ab} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y \cos(2m+1)\frac{\pi x}{a} \sin 2n \frac{\pi y}{b} \, dx \, dy.$$

Subst (c), (d) in (a), and equating coeffs of $\cos(2m+1)\frac{\pi x}{a} \sin 2n \frac{\pi y}{b}$,

$$- A_{mn} \pi^2 \left[\left(\frac{2m+1}{a}\right)^2 + \left(\frac{2n}{b}\right)^2 \right] = \frac{\nu}{1+\nu} \frac{W_x}{I_y} \frac{4}{ab} \underbrace{\int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y \cos(2m+1)\frac{\pi x}{a} \sin 2n \frac{\pi y}{b} \, dx \, dy}_{= \frac{4(-1)^{m+n+1} b}{(2m+1)n \left[\frac{(2m+1)^2}{a^2} + \frac{4n^2}{b^2} \right]}}$$

Thus proceed with soln using this A_{mn}.

$$\tau_{xz} = \phi_{,y} + f(y) - \frac{1}{2}\frac{W_x}{I_y} x^2$$

$$= \phi_{,y} + \frac{1}{2}\frac{W_x}{I_y} \left[\left(\frac{a}{2}\right)^2 - x^2 \right]$$

$$\tau_{yz} = -\phi_{,x}$$

Elementary beam theory gives,

$$\tau_{xz} = \frac{W_x}{2I_y}\left[\left(\frac{a}{2}\right)^2 - x^2 \right] \quad \text{based on } \frac{VQ}{It} \text{ (ref eg. Popov)}, \ \& \ \tau_{yz} = 0.$$

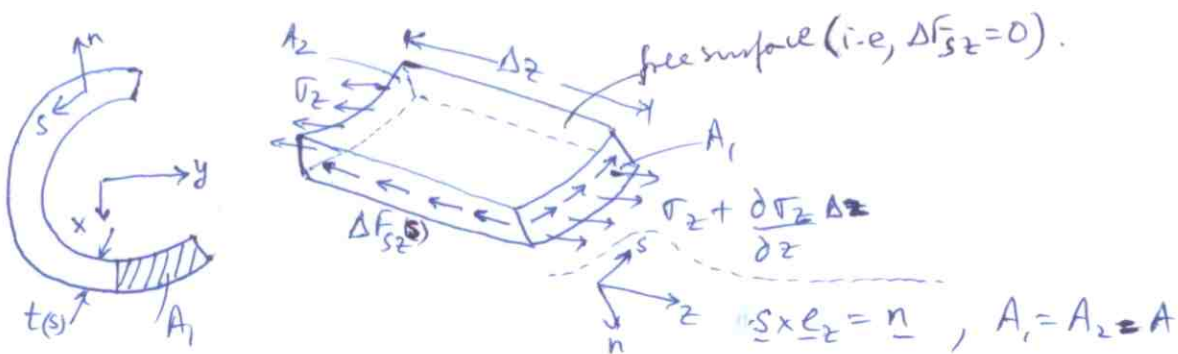So $\phi_{,y}$ and $-\phi_{,x}$ appear to be corrections to elementary beam theory solution

TRANSVERSE SHEAR : 1-DIMENSIONAL SHEAR FLOWS. FOR THIN-WALLED BEAMS

Although the expression for σ_z in ⑤ is for pure bending, we assume that it holds for general loading with shear force present (in fact in ⑱ we saw that it exactly holds true for case of terminal loads). So ⑤ is <u>assumed</u> valid when SF present & BM varies along span. We consider only beams whose thickness is small compared to length. In such beams, the shear flow can be <u>assumed</u> to be in the lengthwise direction. The shear stresses are assumed to be uniformly distributed across the thickness. All stresses apart from transverse shear and axial bending stresses are <u>assumed</u> zero. We <u>assume</u> that the point loads W_x, W_y are applied thru the shear center (x_{cF}, y_{cF}) so that no twisting occurs.

$(s : \text{anticlockwise} +)$

$(\underline{s} \times \underline{e_z} = \underline{n})$.
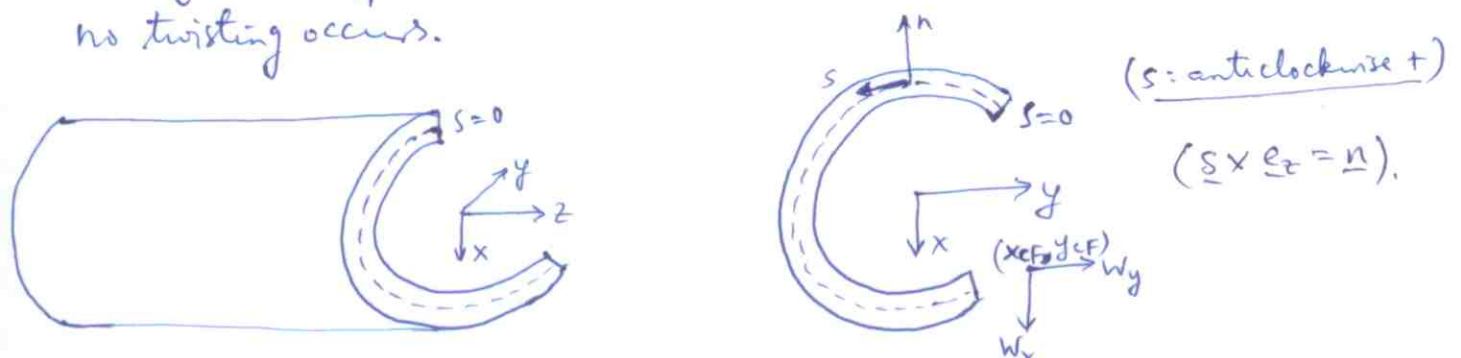
Only σ_z and τ_{sz} are non-zero (assumed) and as per <u>assumption</u>,

$$\sigma_z = -\frac{M_x I_{xy} + M_y I_x}{I_x I_y - I_{xy}^2} x + \frac{M_x I_y + M_y I_{xy}}{I_x I_y - I_{xy}^2} y \longrightarrow \underset{\text{(repeated)}}{⑤}$$

free surface (i.e, $\Delta F_{sz} = 0$).

$\underline{s} \times \underline{e_z} = \underline{n}$, $A_1 = A_2 = A$

$\sum F_z : \quad \Delta F_{sz}(s) - 0 = \int_A \left(\sigma_z + \frac{\partial \sigma_z}{\partial z} \Delta z \right) dA - \int_A \sigma_z \, dA = \int_A \frac{\partial \sigma_z}{\partial z} \Delta z \, dA$

$\Rightarrow \quad q_{sz}(s) \triangleq \frac{\Delta F_{sz}(s)}{\Delta z} \bigg|_{\lim \Delta z \to 0} = \int_A \frac{\partial \sigma_z}{\partial z} dA$

shear flow !

$$\frac{\partial \sigma_z}{\partial z} = -\frac{\overset{V_y}{\overbrace{\frac{\partial M_x}{\partial z}}} I_{xy} + \overset{(-V_x)}{\overbrace{\frac{\partial M_y}{\partial z}}} I_x}{I_x I_y - I_{xy}^2} x + \frac{\frac{\partial M_x}{\partial z} I_y + \frac{\partial M_y}{\partial z} I_{xy}}{I_x I_y - I_{xy}^2} y$$
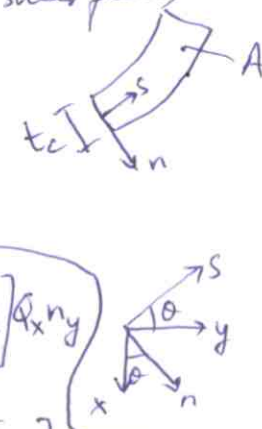
$$\Rightarrow q_{sz} = \left[-\frac{V_y I_{xy}}{F} + \frac{V_x I_x}{F}\right]\int_A x\,dA + \left[\frac{V_y I_y}{F} - \frac{V_x I_{xy}}{F}\right]\int_A y\,dA$$

$$\underbrace{\qquad}_{Q_y} \qquad \underbrace{\qquad}_{Q_x}$$

$$F = I_x I_y - I_{xy}^2$$

$\tau_{sz} = q_{sz}/t_c$ if we assume shear stress is thru thickness (i.e, uniformly distributed shear flow)

$$\Rightarrow \boxed{\tau_{sz} = \left[\frac{-V_y I_{xy} + V_x I_x}{F t_c}\right]Q_y + \left[\frac{V_y I_y - V_x I_{xy}}{F t_c}\right]Q_x}$$

$$\tau_{xz} = -\tau_{sz}\,n_y = \left[\frac{V_y I_{xy} - V_x I_x}{F t_c}\right]Q_x n_y + \left(\frac{V_x I_{xy} - V_y I_y}{F t_c}\right)Q_x n_y$$

$$\tau_{yz} = \tau_{sz}\,n_x = \left[\frac{-V_y I_{xy} + V_x I_x}{F t_c}\right]Q_y n_x + \left[\frac{V_y I_y - V_x I_{xy}}{F t_c}\right]Q_x n_x \qquad \boxed{27\,a,b,c}$$

Shear Center

Equate moment generated by loads to moment generated by shear stresse distribution arising from bending only (w/o twist), i.e.,

$$x_{cF}\,W_y - y_{cF}\,W_x = \int_A (x\,\tau_{yz} - y\,\tau_{xz})\,dA \qquad \longrightarrow \boxed{25}\ \text{(repeated)}$$

Subst 27(b, c) in (25), put $V_x = W_x$, $V_y = W_y$, equate coefficients of W_x (and of W_y) on either side — equivalent to putting $W_y = 0$, $W_x = 0$ independently, since cf does not depend on loads. We get, on using $dA = t_c\,ds$,

$$x_{cF} = -\frac{I_{xy}}{F}\left[\int_c Q_y\,(x n_x + y n_y)\,ds\right] + \frac{I_y}{F}\left[\int_c Q_x\,(x n_x + y n_y)\,ds\right]$$

$$\left(\text{use } n_x = \frac{dy}{ds},\ n_y = -\frac{dx}{ds}\right)$$

$$= -\frac{I_{xy}}{F}\left[\int_c Q_y\,(x\,dy - y\,dx)\right] + \frac{I_y}{F}\left[\int_c Q_x\,(x\,dy - y\,dx)\right]$$

$$y_{cF} = -\frac{I_x}{F}\left[\int_c Q_y\,(x n_x + y n_y)\,ds\right] + \frac{I_{xy}}{F}\left[\int_c Q_x\,(x n_x + y n_y)\,ds\right] \qquad \boxed{28}$$

$$= -\frac{I_x}{F}\left[\int_c Q_y\,(x\,dy - y\,dx)\right] + \frac{I_{xy}}{F}\left[\int_c Q_x\,(x\,dy - y\,dx)\right]$$

The contour integrals are evaluated from $s=0$ to $s=s_{max}$.
It is important to note that when load W_x, W_y is applied at (x_{CF}, y_{CF}), there is a twisting moment $(x_{CF} W_y - y_{CF} W_x)$ at the centroid, but the mean twist, ie, local twist 'α' at centroid (see p. 107-108) is zero $(\alpha=0)$. Similarly, if load applied at centroid and CF $\neq$ centroid, then there is no twisting moment about centroid but local twist $\alpha \neq 0$.

Guidelines for calculation of Q_x, Q_y.

$$Q_x = \int_A y\, dA, \quad Q_y = \int_A x\, dA$$

Q_x, Q_y can be either +ve or -ve.
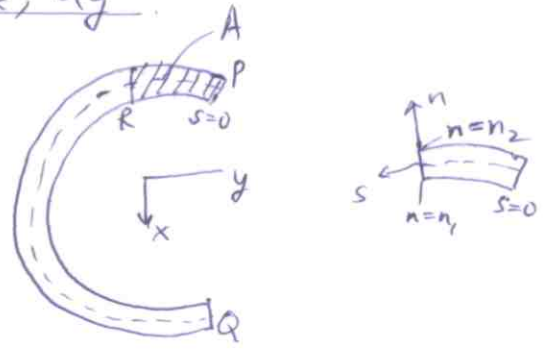
Now, $dA = ds\,dn$ (for thin walled section).
The limits of integration for s are from the value of s at cut section to value of s at free surface. Limits for n are from least value of n (ie n_1) to greatest value of n (ie, n_2). It does not matter if for calculation of shear stresses at section R, above, we consider, area A to be section RP or section RQ, as long as the end other than R is free.
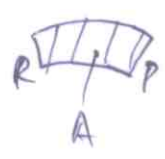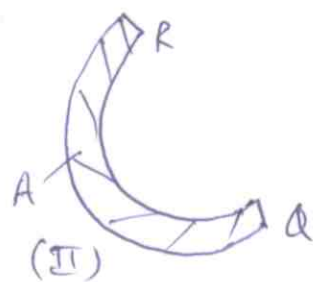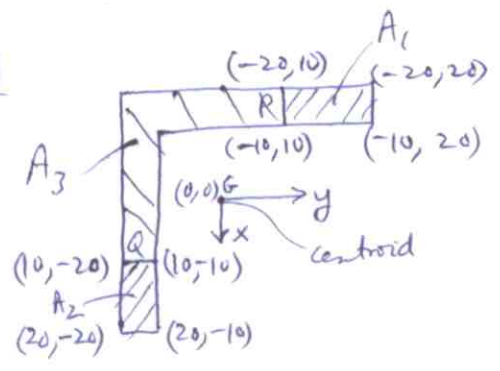

(I)   (II)

In case (I) limits on s are from R to P. In (II) it is from R to Q. In many cases we require to convert the integral originally in terms of s,n to one in terms of x,y. For that we use,

$$dA = ds\,dn = \begin{vmatrix} \dfrac{ds}{dx} & \dfrac{ds}{dy} \\[2mm] \dfrac{dn}{dx} & \dfrac{dn}{dy} \end{vmatrix} dx\,dy = J\,dx\,dy.$$
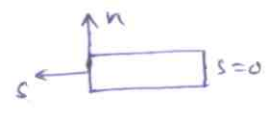
The limits of integration for x, y are decided based on those for s, n.

(Ex)

$(-20,10)$ A_1
   $(-20,20)$

R

A_3

$(-10,10)$ $(-10,20)$

$(0,0)G \longrightarrow y$

centroid

$(10,-20)$ $(10,-10)$

A_2

$(20,-20)$ $(20,-10)$

Find: Q_x, Q_y at sections $P \& Q$.

All coordinates in cm.

At P.: $s = -y, \quad n = -x$

$ds\,dn = dx\,dy.$

$$Q_x = \int_{10}^{20}\int_{-10}^{-20} y\,dx\,dy \;=\; \frac{-10\,(20^2 - 10^2)}{2} \;=\; -1500 \text{ cm}^3.$$

$$Q_y = \int_{10}^{20}\int_{-10}^{-20} x\,dx\,dy \;=\; \frac{10\left((-20)^2 - (-10)^2\right)}{2} \;=\; 1500 \text{ cm}^3$$

$\left. \begin{array}{l} \\ \\ \end{array} \right\}$ for direct use in (27)

$n_x = -1, \quad n_y = 0 \;\Rightarrow\; T_{xz} = 0, \quad T_{yz} = -T_{sz}$
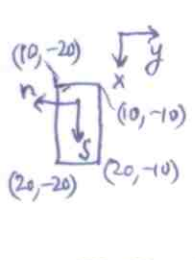

Note: If we take statical first moment of area A_1, you get

$\boxed{Q_x = \bar{y}A_1 = 15*(10*10) = 1500, \quad Q_y = \bar{x}A_1 = -15*(10*10) = -1500}$

(for use in negative of (27))

Both these are negative of Q_x, Q_y obtained above. If we wish to use these then T_{sz}, T_{xz}, T_{yz} results in (27) will have to be negative of what appears in (27). This arises due to our derivation of $q_{sz}(s) = \int_A \frac{\partial \sigma_z}{\partial z} dA$ which implies tacitly that limits are $(s,0)$ and (n_1, n_2). However, if we instead chose limits as $(0,s)$ and (n_1, n_2) for integrating over A, then we would have gotten $q_{sz}(s) = -\int_A \frac{\partial \sigma_z}{\partial z} dA$, as in Srinath, in which Q_x, Q_y are statical moments. In our convention, Q_x, Q_y are negative of statical moments.

At Q: $s = x, \quad y = -n, \quad ds\,dn = -dx\,dy$

$(10,-20)$ $(10,-10)$

$(20,-20)$ $(20,-10)$

$$Q_x = \int_{10}^{-20}\int_{10}^{20} y\,dx\,dy \;=\; (-10)*\frac{(20^2 - 10^2)}{2} = -1500 \text{ cm}^3$$

$$Q_y = \int_{-10}^{-20}\int_{10}^{20} x\,dx\,dy \;=\; (10)*\frac{\left((-20)^2 - (-10)^2\right)}{2} = 1500 \text{ cm}^3.$$

Note: If we take statical first moment of A_2, you get same Q_x and Q_y as above. This is because statical moment of A_2 is

negative of statical moment of compliment of A_2 $(= A_1 + A_3)$, due to fact that $\int_{\text{Total area}} x \, dA = \int_{\text{Total area}} y \, dA = 0$. Now if we use Srinath's convention then we need to use statical moment of $(A_1 + A_3)$ (ie: limits are $(0, s)$, (n_1, n_2)) along with negative of (27).

→ So if we use present convention then no need to bother whether our section is A_2 or $A_1 + A_3$, since Q_x, Q_y will come out the same provided we follow the guidelines for limits, ie, s from cut-section to free-surface, n from min to max. However, if we use Srinath convention with negative of (27), the Q_x, Q_y are based on $\underset{\text{first statical moment}}{\underset{\text{area corresponding}}{}}$ to $\int^s$, ie, $(A_1 + A_3)$, so care should be taken to reverse sign on Q_x, Q_y in case we compute it based on first statical moment of A_2 ie, corresponding to $\int_s^{S_{\text{end}}}$.

- -

For section at Q, $n_x = 0$, $n_y = -1$, ie, $\tau_{xz} = \tau_{sz}$, $\tau_{yz} = 0$.

Note: If centerline is straight, then s need not be counterclockwise +ve. Instead, for convenience, we could align s along either the x or the y axis.

(Ex)

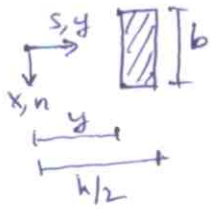$(s \equiv x)$, $(n \equiv -y)$

Case I

$(s \equiv y)$, $(n \equiv x)$
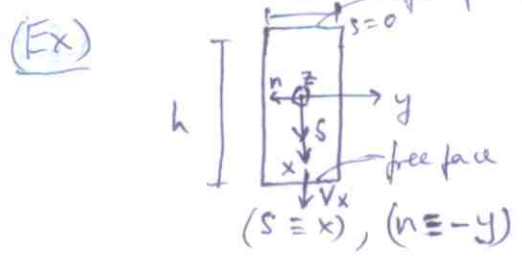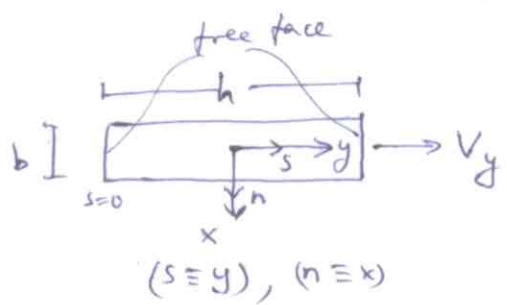
Case II

Case I: $\tau_{sz} = \tau_{xz}$, $\tau_{yz} = 0$, $V_y = 0$, $I_{xy} = 0$, $ds \, dn = -dx \, dy$

$\Rightarrow \tau_{xz} = \tau_{sz} = \dfrac{V_x Q_y}{I_y b}$

$Q_y = \int \int x \, ds \, dn = -\int_{+b/2}^{-b/2} \int^{h/2} x \, dx \, dy = \dfrac{b \left(\frac{h^2}{4} - x^2 \right)}{2}$
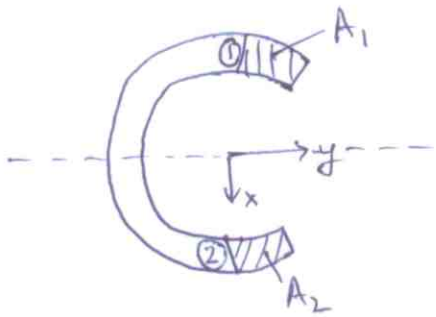
$y = -\frac{b}{2}$ $y = b/2$

Case II : $\tau_{sz} = \tau_{yz}$, $\tau_{xz} = 0$, $V_x = 0$, $I_{xy} = 0$, $dsdn = dxdy$.

$$\Rightarrow \tau_{yz} = \tau_{sz} = \frac{V_y Q_x}{I_x b}$$

$$Q_x = \iint y\, ds\, dn = \int_{-b/2}^{b/2} \int_y^{h/2} y\, dx\, dy = b\left(\frac{h^2}{4} - y^2\right) \cdot \frac{1}{2}$$

(Ex) CF for symmetric sections: Already discussed on p.110 for general thick walled sections. We reconfirm the conclusions for the special case of thin walled sections, as follows.

$I_{xy} = 0$, y-axis is symm axis.

From (28)

$$x_{CF} = \frac{I_y}{F} \int_C (x n_x + y n_y) Q_x\, ds = \frac{1}{I_x} \int_C (x\, dy - y\, dx) Q_x$$

Considering points ① and ② with coordinates $(-x, y)$ & (x, y) resply. Now $(dsdn)_{A_1} = -(dsdn)_{A_2}$.

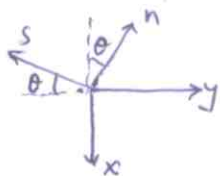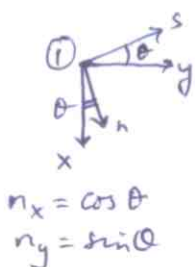$$\Rightarrow (Q_x)_① = \iint_{A_1} y\, ds\, dn = -\iint_{A_2} y\, ds\, dn = (-Q_x)_②$$

$$(Q_y)_① = \iint_{A_1} x\, ds\, dn = \iint_{A_2} (-x)(-ds\, dn) = (Q_y)_②$$

} can also be seen by statical moment in a more direct manner, ie

$$(Q_x)_② = -(Q_x)_①$$
$$(Q_y)_② = -(-(Q_y)_①)$$

Also n_x at ① & ② have opposite signs, but n_y has same sign as seen below

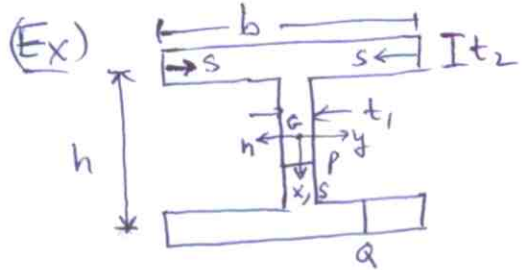$n_x = \cos\theta$
$n_y = \sin\theta$

$n_x = \cos(2\pi - (\pi + \theta))$
$\quad = -\cos\theta$
$n_y = \sin(2\pi - (\pi + \theta))$
$\quad = \sin\theta$

Thus $x Q_x n_x$ and $y Q_x n_y$ have opposite signs at symmetrical points above and below the y-axis. Considering all such pairs of symmetrical points, we see that $x_{CF} = 0$, in agreement with result on p.110. Similarly $y_{CF} = 0$ if x-axis is symmetry axis, and for doubly symmetric section $(x_{CF}, y_{CF}) = $ centroid $= (0, 0)$
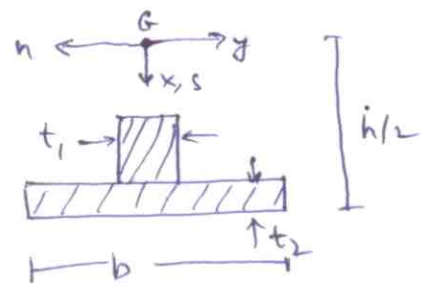
(Ex)

Given: Symm I section, $V_y = 0$.

Find: shear stresses at P & Q.

Doubly symm $\Rightarrow x_{GF} = y_{GF} = 0$

At section P:

$$\tau_{xz} = -\tau_{sz} \overset{\sin 270°}{n_y} = \tau_{sz} = \frac{V_x Q_y}{I_y t_1}$$

Q_y computed based on shaded $\otimes$ section as follows:

$$Q_y = \iint x \, dA = \int\limits_{\frac{t_1}{2}}^{\frac{t_1}{2}} \int\limits_{X}^{h/2} (-dx \, dy) x + 2 \int\limits_{0}^{b/2} \overset{\overset{x}{\frac{h}{2}} \overset{dx}{t_2}}{} dy = t_1 \left(\frac{h^2}{4} - x^2\right) + \underbrace{\frac{2b}{2} \frac{h}{2} t_2}_{2}$$

Alternatively, first statical moment gives,

$$Q_y = \underbrace{-}_{\text{for complimentary area}} \left(\underbrace{-\text{first statical moment}}_{\text{for present convention}}\right) = \text{first statical moment} \qquad \text{(They match !!)}$$

v/s Srinath's convention

$$= b t_2 \frac{h}{2} + \left(\frac{h}{2} - x\right) t_1 \left(\underbrace{\frac{\frac{h}{2} - x}{2}}_{} + x\right) = b t_2 \frac{h}{2} + \frac{t_1}{2}\left(\frac{h^2}{4} - x^2\right)$$
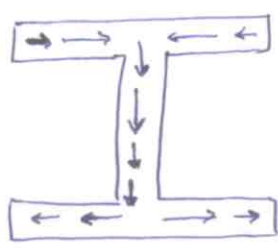
At section Q:

$$\tau_{yz} = \tau_{sz} \overset{\cos 0°}{n_x} = \tau_{sz} = \frac{V_x Q_y}{I_y t_2}$$
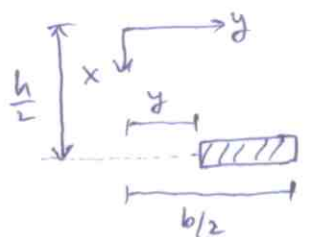
Q_y computed based on shaded area, as follows:

$$Q_y = \iint\limits_A x \, dx \, dy = \int\limits_y^{b/2} \frac{h}{2} t_2 \, dy = \frac{h}{2} t_2 \left(\frac{b}{2} - y\right)$$

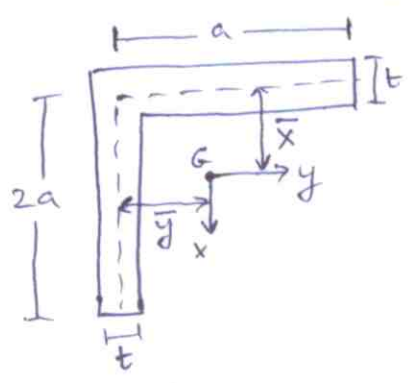The shear flow is represented as,

If either web or flange are made too thin, we could have buckling of web or flange, respectively, seen as follows:

eg $\tau = \tau_{max}$ at neutral axis
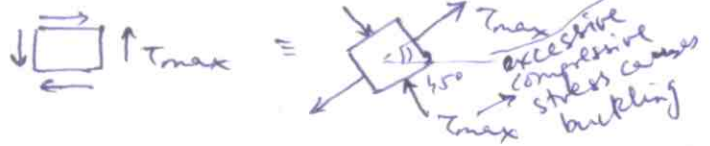
(Ex) Find distribution of shear flows, and the shear center location. $V_y = 0$ given.

Assume $t \ll a$.

$\bar{x} = \dfrac{a(2at)}{3at} = \dfrac{2}{3}a$; $\bar{y} = \dfrac{\dfrac{a}{2}(at)}{3at} = \dfrac{a}{6}$

$I_{yy} = \dfrac{at^3}{12} + at\left(\dfrac{4}{9}a^2\right) + \dfrac{(2a)^3 t}{12} + 2at\left(a - \dfrac{2}{3}a\right)^2$

$\underbrace{}_{\text{neglect}}$

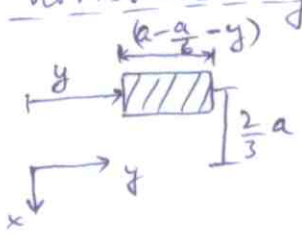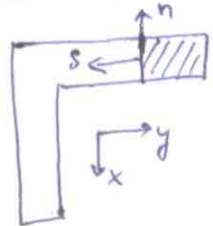$I_{yy} \simeq a^3 t\left(\dfrac{4}{9} + \dfrac{8}{12} + \dfrac{2}{9}\right) = \dfrac{4}{3}a^3 t$

$I_{xx} = \dfrac{2a t^3}{12} + 2at\left(\dfrac{a^2}{36}\right) + \dfrac{a^3 t}{12} + at\left(\dfrac{a}{2} - \dfrac{a}{6}\right)^2 \simeq a^3 t\left(\dfrac{1}{18} + \dfrac{1}{12} + \dfrac{1}{9}\right) = \dfrac{a^3 t}{4}$

$\underbrace{}_{\text{neglect}}$

$I_{xy} = at\left(-\dfrac{2}{3}a\right)\left(\dfrac{a}{2} - \dfrac{a}{6}\right) + 2at\left(-\dfrac{a}{6}\right)\left(a - \dfrac{2}{3}a\right) = \left(-\dfrac{2}{9} - \dfrac{1}{9}\right)a^3 t = -\dfrac{a^3 t}{3}$

$F = I_x I_y - I_{xy}^2 = a^6 t^2\left(\dfrac{1}{3} - \dfrac{1}{9}\right) = \dfrac{2}{9}a^6 t^2$

Shear flow in __horizontal leg__ :

$Q_x = \iint y\, ds\, dn = \displaystyle\int_{(-\frac{2}{3}a+\frac{t}{2})}^{(-\frac{2}{3}a-\frac{t}{2})} \int_{y}^{5a/6} y\, dx\, dy$

$= -t \dfrac{\left(\dfrac{25a^2}{36} - y^2\right)}{2}$

$Q_y = \displaystyle\int_{(-\frac{2}{3}a+\frac{t}{2})}^{(-\frac{2}{3}a-\frac{t}{2})} \int_{y}^{5a/6} x\, dx\, dy = \left(\dfrac{5a}{6} - y\right)\dfrac{2}{3}at \cdots$
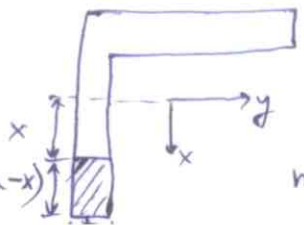
$\tau_{yz} = -\tau_{sz} = \dfrac{V_x I_{xy} Q_x}{Ft} - \dfrac{V_x I_x Q_y}{Ft} = \dfrac{V_x}{Ft}\left(I_{xy} Q_x - I_x Q_y\right)$

$\tau_{yz} = \dfrac{V_x\, a^3 t^2\left(\dfrac{25a^2}{6\times36} - \dfrac{y^2}{6} + \dfrac{1}{6}ay - \dfrac{5}{36}a^2\right)}{\dfrac{2}{9}a^6 t^2\, t} = \dfrac{V_x\left(-5a^2 - 36y^2 + 36ay\right)}{48a^3 t}$ ▶

$\tau_{xz} = 0$

You can easily verify that $\displaystyle\int_{-a/6}^{5a/6} \tau_{yz}\, t\, dy$ = nett resultant shear force in y-direction

$= 0$.

Shear flow in vertical leg:

$ds\, dn = -dx\, dy$, $Q_x = \iint y\, ds\, dn = -\displaystyle\int_{(-\frac{a}{6}+\frac{t}{2})}^{(-\frac{a}{6}-\frac{t}{2})} \int_{x}^{\frac{4a}{3}} y\, dx\, dy = \dfrac{at}{6}\left(x - \dfrac{4a}{3}\right)$

$Q_y = -\displaystyle\int_{(-\frac{a}{6}+\frac{t}{2})}^{(-\frac{a}{6}-\frac{t}{2})} \int_{x}^{\frac{4a}{3}} x\, dx\, dy = \dfrac{t}{2}\left(\dfrac{16a^2}{9} - x^2\right)$

$$\tau_{xz} = \tau_{sz} = \frac{V_x}{Ft}\left(I_x Q_y - I_{xy} Q_x\right) = \frac{V_x a^3 t^2 \left(\frac{2}{9}a^2 - \frac{x^2}{8} + \frac{ax}{18} - \frac{4a^2}{54}\right)}{\frac{2}{9}a^6 t^2 t} \qquad \boxed{122}$$

$$= \frac{V_x}{48 a^3 t}\left(32a^2 - 27x^2 + 12ax\right)$$

$$\tau_{yz} = 0$$

Readily verified that $\displaystyle\int_{-\frac{2}{3}a}^{\frac{4}{3}a} \tau_{xz}\, t\, dx = V_x = $ SF in x-direction.

<u>Shear Center :</u>

$$x_{CF} = -\frac{I_{xy}}{F}\int_C Q_y\,(x\,dy - y\,dx) + \frac{I_y}{F}\int_C Q_x\,(x\,dy - y\,dx)$$

$$C = C_h + C_v = \text{midcontour of (horz + vertical) legs}$$

On C_h, $dx = 0$;   on C_v, $dy = 0$.

$$x_{CF} = -\frac{I_{xy}}{F}\left[\int_{C_h} Q_y\left(-\tfrac{2}{3}a\right)dy + \int_{C_v} Q_y\left(\tfrac{a}{6}\right)dx\right]^{\circledast}\!\!\!\!^{\circledast\circledast} + \frac{I_y}{F}\left[\int_{C_h} Q_x\left(-\tfrac{2}{3}a\right)dy + \int_{C_v} Q_x\left(\tfrac{a}{6}\right)dx\right]$$

Note: Contour is from $s = 0$ to $s = s_{max}$, so limits on x, y are for increasing s.

$$a^2 x_{CF} = \frac{3}{2}\left[\int_{5a/6}^{-a/6}\left(-\tfrac{2}{3}\right)\!\cdot\!\left(\tfrac{2}{3}a\right)\!\left(\tfrac{5a}{6} - y\right)dy + \int_{-\frac{2}{3}a}^{\frac{4a}{3}}\left(\tfrac{1}{6}\right)\!\cdot\!\left(\tfrac{1}{2}\right)\!\left(\tfrac{16a^2}{9} - x^2\right)dx\right]$$

$$+ 6\left[\int_{5a/6}^{-a/6}\left(-\tfrac{2}{3}\right)\!\left(-\tfrac{1}{2}\right)\!\left(\tfrac{25a^2}{36} - y^2\right)dy + \int_{-\frac{2}{3}a}^{\frac{4a}{3}}\left(\tfrac{1}{6}\right)\!\left(\tfrac{a}{6}\right)\!\left(x - \tfrac{4a}{3}\right)dx\right]$$

$$= a^3 \tfrac{3}{2}\left[\left(-\tfrac{20}{54}\right)(-1) - \left(-\tfrac{4}{9}\right)\!\left(\tfrac{1}{2}\right)\!\left(-\tfrac{24}{36}\right) + \left(\tfrac{16}{108}\right)(2) - \left(\tfrac{1}{12}\right)\!\left(\tfrac{1}{3}\right)\!\left(\tfrac{72}{27}\right)\right]^{\circledast}$$

$$+ a^3 6\left[\tfrac{25}{108}(-1) - \left(\tfrac{1}{3}\right)\!\left(\tfrac{1}{3}\right)\!\left(-\tfrac{126}{216}\right) + \left(\tfrac{1}{36}\right)\!\left(\tfrac{1}{2}\right)\!\left(\tfrac{12}{9}\right) - \left(\tfrac{4}{108}\right)(2)\right]^{\circledast\circledast}$$

$$= -\tfrac{2}{3}a^3$$

$$\Rightarrow \quad x_{CF} = -\tfrac{2}{3}a. \quad \blacktriangleleft$$

(see above)

$$y_{CF} = -\frac{I_x}{F}\int_C Q_y\,(x\,dy - y\,dx) + \frac{I_{xy}}{F}\int_C Q_x\,(x\,dy - y\,dx) = -\frac{I_x}{F}\left[\;\;\right]^{\circledast} + \frac{I_{xy}}{F}\left[\;\;\right]^{\circledast\circledast}$$

$$= a\left[-\tfrac{9}{8}(\circledast) - \tfrac{3}{2}(\circledast\circledast)\right] = a\left[-\tfrac{9}{8}\times\tfrac{144}{324} - \tfrac{3}{2}\left(\tfrac{-72}{324}\right)\right] = -\tfrac{a}{6} \quad \blacktriangleleft$$

Thus CF lies at the corner of the angle, ie (x_{CF}, y_{CF})
$= (-\frac{2}{3}a, -\frac{a}{6})$. This fact is obvious since the shear flows (along horz & vert legs) intersect at the corner, whereby the resultant forces also intersect at the corner, and the shear center is obviously located at such an intersection. More generally we have,

CF

α

CF
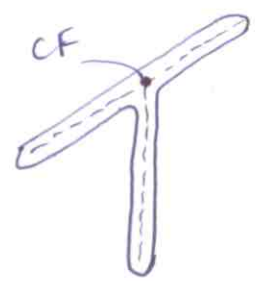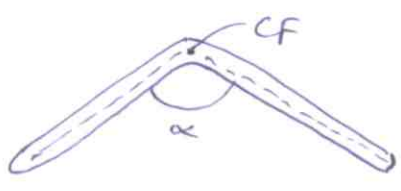
CF definition (alternative): Resultant shear flows, and hence shear forces, have zero moment about CF

(Ex)
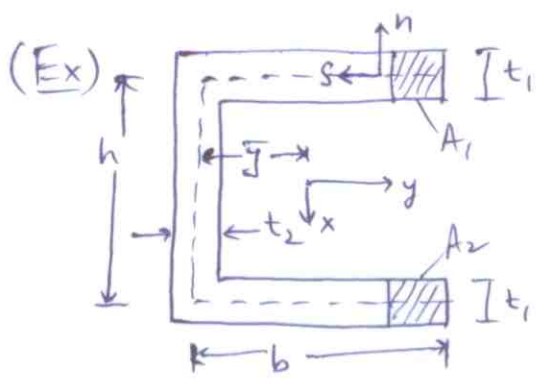
$\bar{y} = \dfrac{2(bt_1)(b/2)}{2bt_1 + ht_2} = \dfrac{b^2 t_1}{2bt_1 + ht_2}$

$V_y = 0$ given. , $(t_1, t_2) \ll (h, b)$

Upper horizontal leg. $(ds\,dn = dx\,dy)$

$Q_x = \displaystyle\int_{y}^{b-\bar{y}} \int_{-h/2+\frac{t_1}{2}}^{-h/2-t_1/2} y\,dx\,dy = -\frac{t_1}{2}\left[(b-\bar{y})^2 - y^2\right]$

$Q_y = \displaystyle\int_{y}^{b-\bar{y}} \int_{-\frac{h}{2}+\frac{t_1}{2}}^{-h/2-t_1/2} x\,dx\,dy = (b-\bar{y}-y)\,\frac{ht_1}{2}$

Lower horizontal leg. $(ds\,dn = dx\,dy)$

$Q_x = \displaystyle\int_{y}^{b-\bar{y}} \int_{\frac{h}{2}-\frac{t_1}{2}}^{h/2+t_1/2} y\,dx\,dy = \frac{t_1}{2}\left[(b-\bar{y})^2 - y^2\right]$ (expected that $[Q_x]_{lower\ leg}$ is negative of $[Q_x]_{upper\ leg}$ ∵ statical x-moments are same for A_1, A_2).

$Q_y = \displaystyle\int_{y}^{b-\bar{y}} \int_{h/2-t_1/2}^{h/2+t_1/2} x\,dx\,dy = (b-\bar{y}-y)\,\frac{ht_1}{2}$ (expected same as for upper leg ∵ y-statical moments of A_1, A_2 differ by negative sign).

Lower vertical leg. $ds\,dn = -dx\,dy$

(x>0)
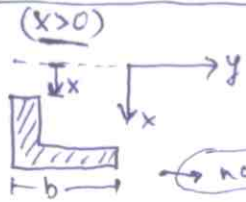
$Q_x = -\displaystyle\int_{x}^{h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} y\,dy\,dx + [Q_x]_{lower\ horz\ leg,\ y=-\bar{y}}$

→ noted as Case (a) later on

$$Q_x = \bar{y}\, t_2 \left(x - \tfrac{h}{2}\right) + \tfrac{t_1}{2}\left(b^2 - 2b\bar{y}\right) \quad, \quad \boxed{x > 0} - \text{relaxed later} \atop \text{to } -\tfrac{h}{2} \le x \le \tfrac{h}{2}$$

$$Q_y = -\int_x^{h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} x \, dy \, dx \; + \; [Q_y]_{\text{Lower Horz leg, } y = -\bar{y}}$$

(⊛)

$$= \tfrac{t_2}{2}\left(\tfrac{h^2}{4} - x^2\right) + \tfrac{bh t_1}{2} \quad, \quad x > 0 \;\; \genfrac{}{}{0pt}{}{(\text{relaxed later})}{\text{to } -\tfrac{h}{2} \le x \le \tfrac{h}{2}}$$

uh = upper horz
lh = Lower horz
uv = upper vert
lv = lower vert.

Upper vertical leg. (x ≤ 0)

By considering statical moments (shortcut) and noting that $[Q_x]_{lv} = -[Q_x]_{uv}$ (hence)

and $[Q_y]_{lv} = [Q_y]_{lv}$, we can directly write

(introduced ∵ for uv leg we have x<0, so the sign on odd power terms in x should remain same in expression for $[Q_x]_{lv}$ and $[Q_x]_{uv}$ so that their numerical values are -ve of each other)
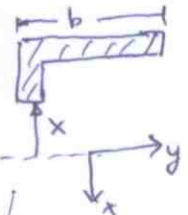
$$Q_x = -\bar{y}\, t_2 \left(\ominus x - \tfrac{h}{2}\right) - \tfrac{t_1}{2}\left(b^2 - 2b\bar{y}\right) \quad, \quad \genfrac{}{}{0pt}{}{x < 0 \text{ (relaxed later)}}{\text{to } -\tfrac{h}{2} \le x \le \tfrac{h}{2}}$$

$$Q_y = \tfrac{t_2}{2}\left(\tfrac{h^2}{4} - x^2\right) + \tfrac{bh t_1}{2}$$

or we get same result by first principles c. follows,

$$Q_x = -\int_x^{-h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} y \, dy \, dx \; + \; [Q_x]_{uh,\, y=-\bar{y}}$$

$$= \bar{y}\, t_2 \left(\tfrac{h}{2} + x\right) - \tfrac{t_1}{2}\left(b^2 - 2\bar{y} b\right)$$

$$Q_y = -\int_x^{h/2} \int_{-\bar{y}+t_2/2}^{-\bar{y}-t_2/2} x \, dy \, dx \; + \; [Q_y]_{uh,\, y=-\bar{y}}$$

$$= \tfrac{t_2}{2}\left(\tfrac{h^2}{4} - x^2\right) + \tfrac{bh t_1}{2}$$

(they match! as expected)

$\quad, \quad \genfrac{}{}{0pt}{}{x < 0 \text{ (relaxed later)}}{\text{to } -\tfrac{h}{2} \le x \le \tfrac{h}{2}}$

noted as Case (c) later on

Not required for solution. It is for explanation/understanding only

In fact we expect the expressions for Q_x, Q_y for uv and lv legs to be identical since the following integrations over paths as indicated should yield identical results

(a)   (b)   (c)
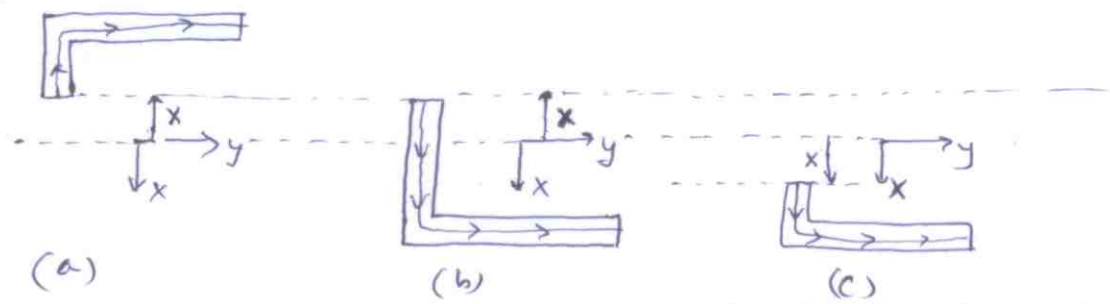
(a), (b) yield identical results (ie, numerical values for Q_x, Q_y)
(b), (c) yield identical expressions for Q_x, Q_y, respectively.
(a), (c) yield Q_x values differing by -ve sign.

Thus using any one of the expressions obtained from (a) or (b) or (c) for Q_x and using $Q_x[x] + Q_x[-x] = 0$ you get,

$$-\bar{y} t_2 \frac{h}{2} + \frac{t_1}{2} b^2 - \frac{t_1}{2} 2b\bar{y} = 0 \implies \bar{y} = \frac{b t_1^2}{2bt_1 + t_2 h} \quad \text{which we had}$$

from before. Hence, for entire vertical leg, we could directly write from ⊛ on p. 12;

$$Q_x = \bar{y} t_2 x, \qquad -\frac{h}{2} < x \le \frac{h}{2}$$

$$Q_y = \frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + \frac{bh t_1}{2}$$
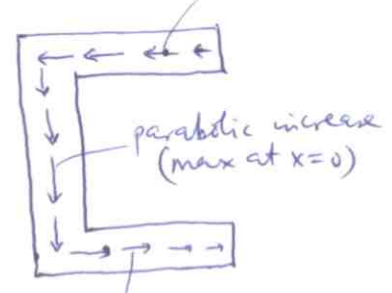
Shear flows /stresses

Upper horizontal leg.

$$\tau_{sz} = \frac{V_x Q_y}{I_y t} = \frac{V_x}{I_y t_1} \cdot \frac{h t_1}{2}(b - \bar{y} - y) = \frac{V_x h}{2 I_y}(b - \bar{y} - y) \;\blacktriangleleft$$

Linear increase

Vertical leg.

$$\tau_{sz} = \frac{V_x Q_y}{I_y t} = \frac{V_x}{I_y t_2}\left[\frac{t_2}{2}\left(\frac{h^2}{4} - x^2\right) + \frac{bh t_1}{2}\right] \;\blacktriangleleft$$

parabolic increase (max at x=0)

linear decrease.

Shear Center.

▶ $x_{CF} = 0$ from symmetry. Verify it as follows. $x_{cf} = \frac{1}{I_x}\int_C Q_x (x\,dy - y\,dx)$

$$x_{cf} = \frac{1}{I_x}\left[\underbrace{\int (Q_x)\left(-\frac{b}{2}\right)dy}_{C_{uh}} + \underbrace{\int \left(\frac{b}{2}\right)(Q_x)_{lh}\,dy}_{C_{lh}} + \bar{y}\underbrace{\int (Q_x)_v\,dx}_{C_v}\right] \qquad (C = C_{uh} + C_v + C_{lh})$$

upper horz   vert   lower horz

$\because (Q_x)_{uh} = -(Q_x)_{lh}$, and $\int_{C_{uh}} = \int_{b-\bar{y}}^{-\bar{y}}$, and $\int_{C_{lh}} = \int_{-\bar{y}}^{b-\bar{y}}$, the first two integrals cancel out.

The third integral is zero $\because$ it contains integrand that is odd power in x and limits are $\left(-\frac{h}{2}, \frac{h}{2}\right)$. Hence verified.

$$y_{cf} = -\frac{1}{I_y}\int_C Q_y (x\,dy - y\,dx) = -\frac{1}{I_y}\left[\int_{b-\bar{y}}^{-\bar{y}}(Q_y)\left(\frac{b}{2}\right)dy + \int_{-\bar{y}}^{b-\bar{y}}(Q_y)\left(\frac{b}{2}\right)dy + \int_{-h/2}^{h/2}(Q_y)_v\,\bar{y}\,dx\right]$$

Note; $(Q_y)_{uh} = (Q_y)_{lh}$

$$\implies y_{cf} = -\frac{1}{I_y}\left[\frac{2b}{2}\int_{-\bar{y}}^{b-\bar{y}}(Q_y)_{uh}\,dy + \bar{y}\int_{-h/2}^{h/2}(Q_y)_v\,dx\right] = -\frac{1}{I_y}\left[\frac{h^2 t_1}{2}\int_{-\bar{y}}^{b-\bar{y}}(b-\bar{y}-y)\,dy + \bar{y}\int_{-h/2}^{h/2}\left[\frac{t_2}{2}\left(\frac{h^2}{4}-x^2\right) + \frac{bh t_1}{2}\right]dx\right]$$

$$= -\frac{1}{I_y}\left[\frac{h^2 t_1}{2}\left\{(b-\bar{y})(b) - \frac{1}{2}(b^2 - 2b\bar{y})\right\} + \bar{y}\left\{\frac{t_2}{2}\left(\frac{h^2}{4}\cdot h - \frac{h^3}{12}\right) + \frac{bh^2 t_1}{2}\right\}\right]$$

$$y_{cf} = -\frac{1}{I_y}\left[\frac{h^2 t_1}{2}b^2 - \frac{h^2 t_1}{2}b\bar{y} - \frac{h^2 t_1}{2}\frac{b^2}{2} + \frac{h^2 t_1}{2}b\bar{y}\right.$$
$$\left. + \frac{h^3}{12}t_2\bar{y} + \frac{h^2 t_1}{2}b\bar{y}\right]$$

$$= -\frac{1}{I_y}\left[\frac{h^2 b^2 t_1}{4} + \bar{y}\left(\frac{h^2 t_1 b}{2} + \frac{h^3 t_2}{12}\right)\right]$$

$$I_y = b t_1 \frac{h^2}{2} + \frac{h^3 t_2}{12}$$

$$\Rightarrow y_{cf} = -\frac{1}{I_y}\frac{h^2 b^2 t_1}{4}$$

$$= -y_e - \bar{y}$$

$\bar{y}$
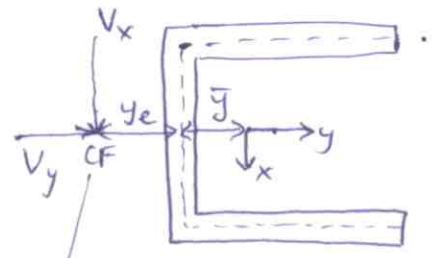
apply V_x, V_y thru cf
to have bending w/o
twisting (ie $\alpha = 0$)

Curved beams. (Symmetric Sections)
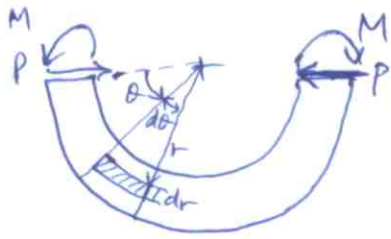
→ **Exact**
solution was done as plane stress
problem in (r, θ) coordinates to get $\sigma_r, \sigma_\theta, \sigma_{r\theta}$.
→ However, that
solution valid only for thin rectangular
sections.

Now we discuss approximate (engineering) solution
for non-rectangular but symmetric sections based on assumptions of:

(i) Plane sections remaining plane

(ii) σ_r, $\sigma_{r\theta}$ are negligible, and plane state of stress as before (in elasticity soln).

(iii) applied loads lie in plane of symmetry.

Refer Fig 1 showing the loading, cross section and forces
on beam element.

Fig 1

<u>Kinematics</u>: $F^*B^*C^*H^*$ is deformed element, FBCH is undeformed element. For simplicity we take B^*C^* <u>and FH rotated by $\Delta(d\theta)$</u> to coincide with BC (alternatively take FH and BC to rotate by $\frac{\Delta(d\theta)}{2}$ each in opposite sense so that curvature reduces, and then rotate deformed element thus obtained by $\frac{\Delta(d\theta)}{2}$ ccw to make B^*C^* coincide with BC — this way you get the same as fig 1.).
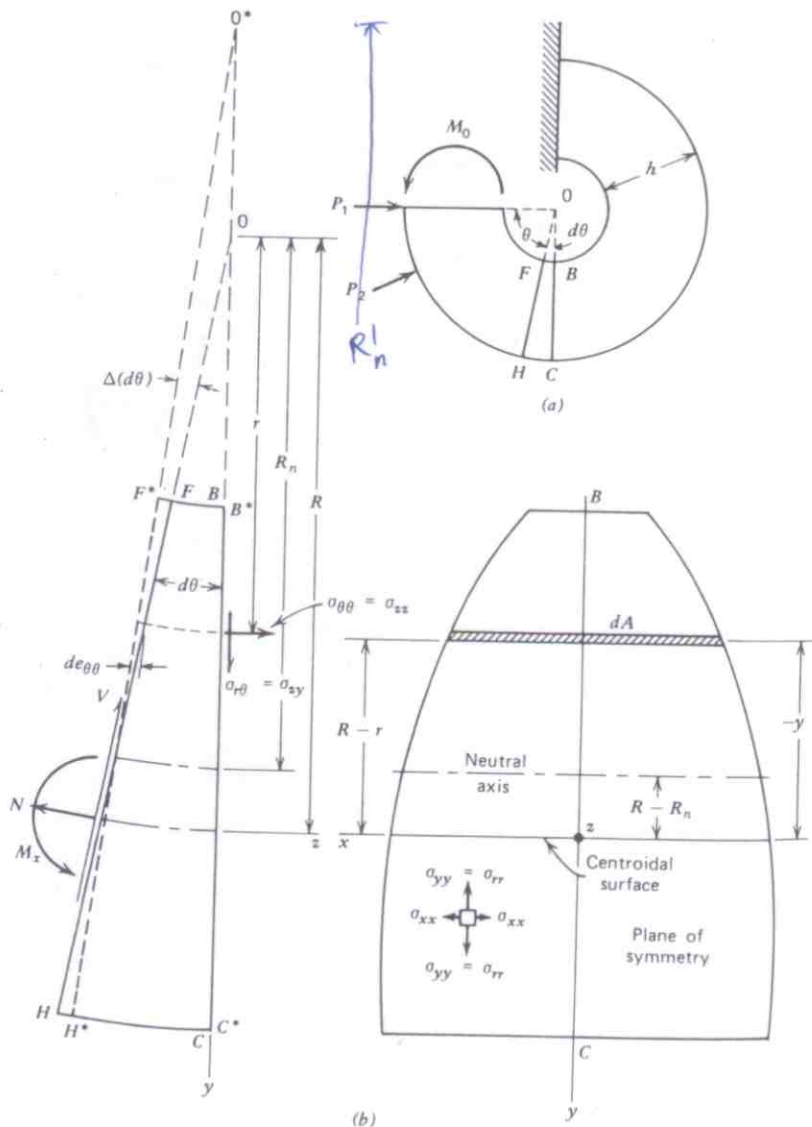
R_n, R, r, denote position of NA, centroidal, nominal fiber from original (undeformed) center of curvature. Also, $de_{\theta\theta}$ is change in lengths fiber.

$$\boxed{\varepsilon_{\theta\theta}} = \frac{de_{\theta\theta}}{r d\theta} = \frac{(R_n - r)\Delta(d\theta)}{r d\theta} \boxed{= \left(\frac{R_n}{r} - 1\right)\omega} \longrightarrow ①$$

where, $\omega \triangleq \frac{\Delta(d\theta)}{d\theta} = \left(\frac{R_n}{R_n'} - 1\right) \longrightarrow ②$ $\longrightarrow$ from $R_n'(d\theta + \Delta(d\theta)) = R_n d\theta$ ie, neutral axis length unchanged).

Here R_n' is position of NA in deformed beam (ie measured from 0^* to NA). Note that although $de_{\theta\theta}$ varies linearly with r (as in straight beams) $\varepsilon_{\theta\theta}$ is nonlinear in r, due to undeformed lengths ($r d\theta$) being function of depth (r) unlike in straight beams.

<u>Hookes law</u>: $\sigma_{rr} \simeq 0$, $\sigma_{xx} \simeq 0$ $\Rightarrow$ $\sigma_{\theta\theta} = E\varepsilon_{\theta\theta} = \frac{R_n - r}{r} E\omega \longrightarrow ③$

<u>Equilibrium</u>: $\sum F_\theta:$ $N = \int_A \sigma_{\theta\theta} dA$

$\sum M_x:$ $M_x = \int_A \sigma_{\theta\theta}(R-r) dA$

$\Rightarrow N = E\omega \int_A \frac{R_n - r}{r} dA = R_n E\omega A_m - E\omega A \longrightarrow ④$

$M_x = E\omega \int_A \left(\frac{R_n - r}{r}\right)(R-r) dA = R_n E\omega (RA_m - A) \longrightarrow ⑤$
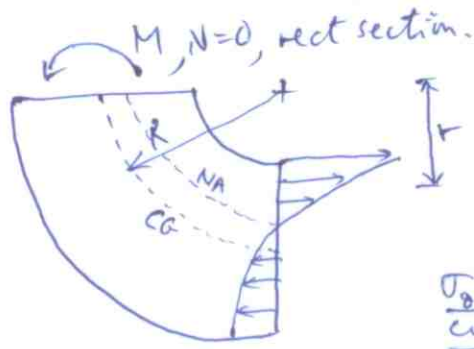
where $A_m \triangleq \int_A \frac{dA}{r} \longrightarrow ⑥$

(4,5) $\Rightarrow E\omega = \frac{A_m M_x}{A(RA_m - A)} - \frac{N}{A} \longrightarrow ⑦$

$(3),(5,7) \Rightarrow \sigma_{\theta\theta} = \dfrac{N}{A} + \dfrac{M_x(A - rA_m)}{Ar(RA_m - A)} \longrightarrow \boxed{8}$

$(\sigma_{\theta\theta})_{\substack{=0 \\ r=R_n}} \Rightarrow R_n = \dfrac{A M_x}{A_m M_x + N(A - RA_m)} \longrightarrow$ location of NA $\left. \begin{array}{c} \\ \\ \end{array} \right\} \boxed{9}$

$\qquad\qquad = \dfrac{A}{A_m}$ (for $N=0$, pure bending)

pure bending $\Rightarrow \sigma_{\theta\theta} = \dfrac{M_x(A - rA_m)}{Ar(RA_m - A)} = \dfrac{M_x(R_n - r)}{Ar(R - R_n)} \left. \begin{array}{c}\\\\ \end{array}\right]$
$(N=0)$ (use 9b)
$\qquad\qquad\qquad\qquad\qquad\qquad = \dfrac{-M_x \, y}{Ae(R_n - y)} \left. \begin{array}{c}\\\\\end{array}\right\} \longrightarrow \boxed{10}$

ie $e = R - R_n$
$-y = R_n - r$

ie, y measured
$-$ve from NA to
center of curvature

The $\sigma_{\theta\theta}$ distribution is hyperbolic ($\frac{1}{r}$) in form.

M, N=0, rect section.

$\sigma_{\theta\theta}$ distribution for Rectangular sectioned curved beam.

Comparison with elasticity for **Rectangular** section case:
• For pure bending (M) and pure shear (P) loads, comparison of $(\sigma_{\theta\theta})_{max}^{\text{obt by (8)}}$ with the plane stress elasticity soln show that the pure bending results compare better than the pure shear loading ones. Further, as R/h increases, the comparison gets better for both types of loading. In fact for R/h=5, even the straight-beam formula gives reasonably good results. (see table 8-2-1, Boresi + Sidebottom, p. 357). Thus for curved beams with R/h > 5 we use straight-beam flexure formula.

Table 8-2-1 →
(Boresi)

Ratios of the Maximum Circumferential Stress in
Rectangular Section Curved Beams as Computed by Elasticity
Theory, by the Curved Beam Formula and by the Flexure Formula

$\dfrac{R}{h}$	Pure Bending		Shear Loading	
	$\dfrac{\sigma_{\theta\theta(CB)}}{\sigma_{\theta\theta(elast)}}$	$\dfrac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(elast)}}$	$\dfrac{\sigma_{\theta\theta(CB)}}{\sigma_{\theta\theta(elast)}}$	$\dfrac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(elast)}}$
0.65	1.046	0.439	0.855	0.407
0.75	1.012	0.526	0.898	0.511
1.0	0.997	0.654	0.946	0.653
1.5	0.996	0.774	0.977	0.776
2.0	0.997	0.831	0.987	0.834
3.0	0.999	0.888	0.994	0.890

In fact we can show that second term on RHS of ⑧ reduces to $-M_x y/I_x$ when $R/h \to \infty$, ie, ⑧ approaches the straight beam formula. The details are:

$$R A_m = \int R \frac{dA}{r} = \int \frac{R}{R+y} dA = \int \left(1 - \frac{y}{R+y}\right) dA = A - \int \frac{y}{R+y} dA$$

$$(R A_m - A) = - \int \frac{y}{R+y} dA \longrightarrow \circledast$$

$$\Rightarrow Ar(R A_m - A) = -A(R+y) \int \frac{y}{R+y} dA = -A \int \frac{Ry + y^2 - y^2}{R+y} dA - Ay \int \frac{y}{R+y} dA$$

$$= \underbrace{-A \int y \, dA}_{=0} + A \int \frac{y^2}{R+y} dA - Ay \int \frac{y}{R+y} dA$$

($\because y$ measured from centroid)

For $\frac{R}{h} \to \infty$, $\quad \cong \frac{A}{R} \int \frac{y^2}{\underbrace{1 + y/R}_{\to \cong 0}} dA - \frac{Ay}{R} \int \frac{y}{\underbrace{1 + y/R}_{\cong 0}} dA \cong \frac{A}{R} I_x$

Thus, for $R/h \to \infty$, $\underset{\wedge}{\overset{\text{second term on RHS of}}{⑧}}$ approximates to

$$\frac{M_x R}{A I_x} (A - r A_m) = \frac{M_x R}{A \bar{I}_x} (A - R A_m - y A_m)$$

$$= \frac{M_x R}{A I_x} \left[\int \frac{y}{R+y} dA - y \int \frac{dA}{R+y} \right] = \frac{M_x R}{A \bar{I}_x} \left[\frac{1}{R} \int \underbrace{\frac{y \, dA}{1 + y/R}}_{\cong 0} - \frac{y}{R} \int \underbrace{\frac{dA}{1 + \frac{y}{R}}}_{\cong A} \right]$$

$$\left. \frac{M_x (A - r A_m)}{Ar (R A_m - A)} \right|_{\frac{R}{h} \to \infty} \cong - \frac{M_x y}{I_x}$$
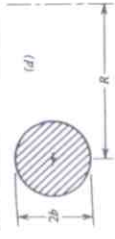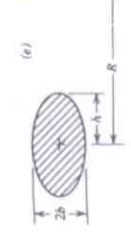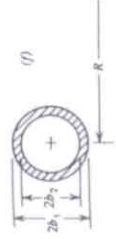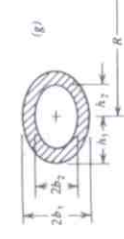
Note: $\because R A_m \to A$ as $R/h \to \infty$ (see $\circledast$ above), we must the calculation of A_m quite accurately and retain sufficient significant digits.

Composite curved beams: They are made up of a combination of fundamental areas. So we have,

$$A = \sum_{i=1}^{n} A_i \quad , \quad A_m = \sum_{i=1}^{n} A_{mi} \quad , \quad R = \frac{\sum_{i=1}^{n} R_i A_i}{\sum_{i=1}^{n} A_i} \quad ,$$

$n =$ nos of fundamental areas comprising section. See Table 8-2-2 (Boresi) for A_i, A_{mi}, R_i of regular sections.

Table 8-2.2
Continued

(a)
$$A = \pi b^2$$
$$A_m = 2\pi(R - \sqrt{R^2 - b^2})$$

(e)
$$A = \pi bh$$
$$A_m = \frac{2\pi b}{h}(R - \sqrt{R^2 - h^2})$$

(f)
$$A = \pi(b_1^2 - b_2^2)$$
$$A_m = 2\pi\left(\sqrt{R^2 - b_2^2} - \sqrt{R^2 - b_1^2}\right)$$

(g)
$$A = \pi(b_1 h_1 - b_2 h_2)$$
$$A_m = 2\pi\left(\frac{b_1 R}{h_1} - \frac{b_2 R}{h_2} - \frac{b_1}{h_1}\sqrt{R^2 - h_1^2} + \frac{b_2}{h_2}\sqrt{R^2 - h_2^2}\right)$$
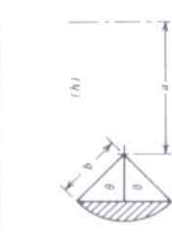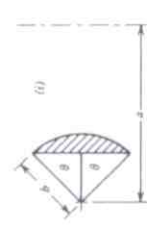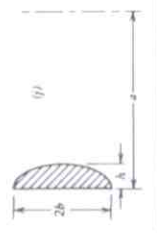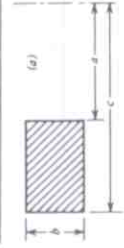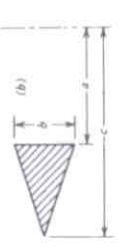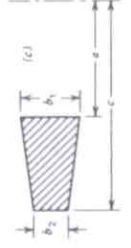
(h)
$$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; \quad R = a + \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$$
For $a > b$,
$$A_m = 2a\theta - 2b\sin\theta - \pi\sqrt{a^2 - b^2} + 2\sqrt{a^2 - b^2}\,\sin^{-1}\left[\frac{b + a\cos\theta}{a + b\cos\theta}\right]$$
For $b > a$,
$$A_m = 2a\theta - 2b\sin\theta + 2\sqrt{b^2 - a^2}\,\ln\left[\frac{b + a\cos\theta + \sqrt{b^2 - a^2}\,\sin\theta}{a + b\cos\theta}\right]$$

(i)
$$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; \quad R = a - \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$$
$$A_m = 2a\theta + 2b\sin\theta - \pi\sqrt{a^2 - b^2} - 2\sqrt{a^2 - b^2}\,\sin^{-1}\left[\frac{b - a\cos\theta}{a - b\cos\theta}\right]$$

(j)
$$A = \frac{\pi bh}{2}; \quad R = a - \frac{4h}{3\pi}$$
$$A_m = 2b + \frac{\pi b}{h}(a - \sqrt{a^2 - h^2}) - \frac{2b}{h}\sqrt{a^2 - h^2}\,\sin^{-1}\left(\frac{h}{a}\right)$$

Table 8-2.2

Analytical Expressions for A, R, and $A_m = \int \frac{dA}{r}$

(a)
$$A = b(c - a); \qquad R = \frac{a + c}{2}$$
$$A_m = b\ln\frac{c}{a}$$

(b)
$$A = \frac{b}{2}(c - a); \qquad R = \frac{2a + c}{3}$$
$$A_m = \frac{bc}{c - a}\ln\frac{c}{a} - b$$

(c)
$$A = \frac{b_1 + b_2}{2}(c - a); \qquad R = \frac{a(2b_1 + b_2) + c(b_1 + 2b_2)}{3(b_1 + b_2)}$$
$$A_m = \frac{b_1 c - b_2 a}{c - a}\ln\frac{c}{a} - b_1 + b_2$$

(eg) Crane hook. (Boresi, Ex 8.2.2).
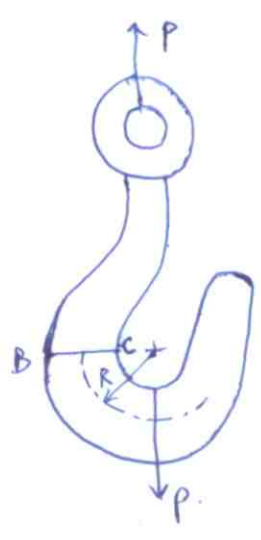
Given: Yield stress, $Y = 500 MPa$, $SF = 2$

Find = Max P.

sector of circle.

half ellipse
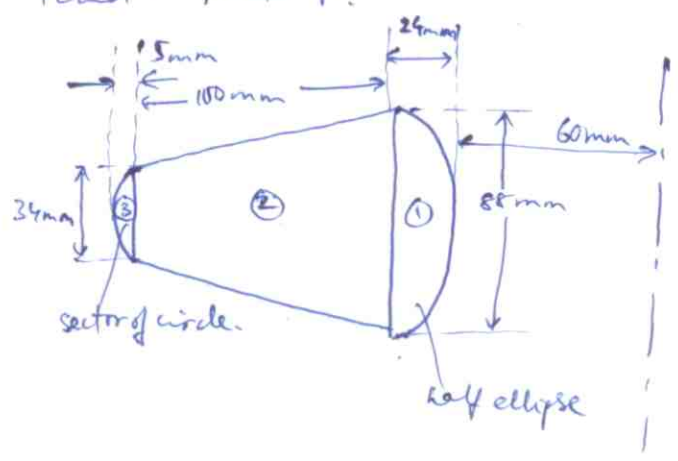
From Table 8-2-2 (Boresi), A_i, R_i, A_{m_i} for $i = 1, 2, 3$ can be obtained. Then obtain A, R, A_m for composite area as discussed on bot of p. 126.

Since N and M_x are maximum for section BC, it is the critical section. Use eqn ⑧ to find $\sigma_{\theta\theta}$ at B and C and decide max load P based on the larger magnitude of $|(\sigma_{\theta\theta})_B|$ or $|(\sigma_{\theta\theta})_C|$, based on $SF = 2$. Use $N = P$, $M_x = PR$ in the eqn ⑧.

Radial Stresses in Curved Beams

So far we assumed $\sigma_{rr} \simeq 0$. For sections with thin webs (H, T, I sections) the $(\sigma_{rr})_{max}$ may exceed $(\sigma_{\theta\theta})_{max}$. Consider the curved beam with section as shown:

Curved beam

Section

Stress distribution on slice FBGD ($\sigma_{\theta\theta}$, σ_{rr} shown).
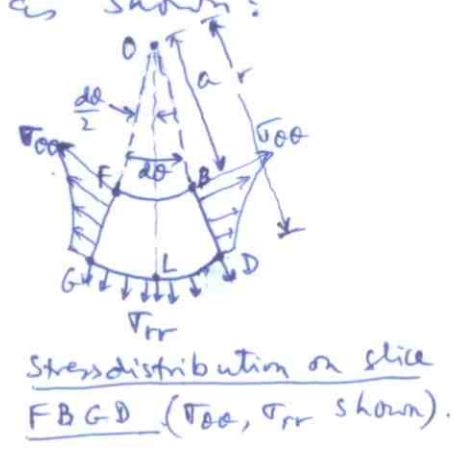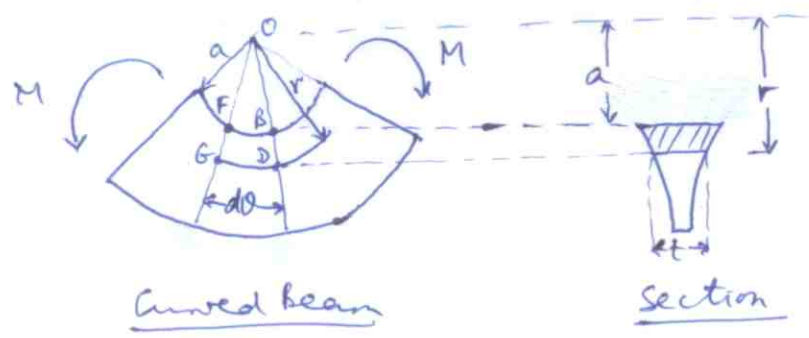
The radial stress, σ_{rr}, occurring at radius r is shown.
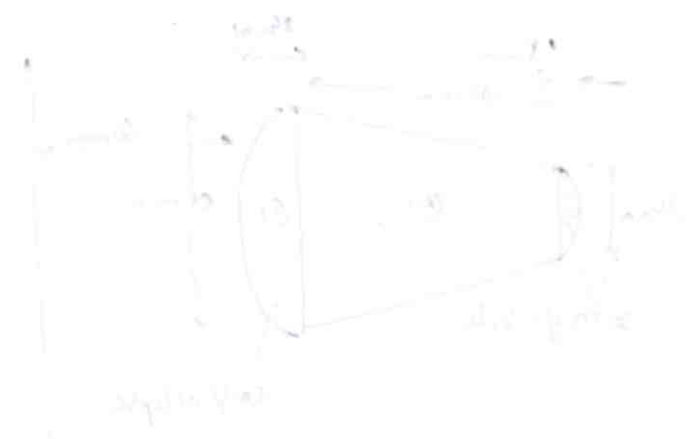
(a)   (b)   (c)
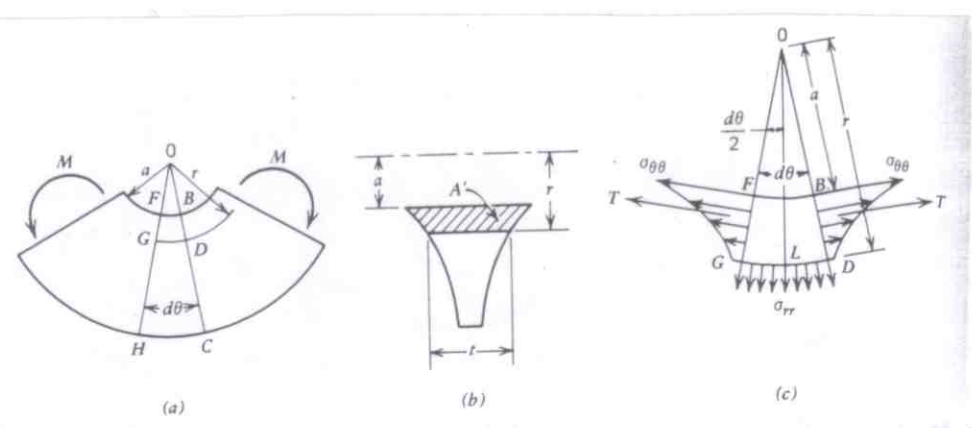
Consider equilibrium of element BDGF. Resultant circumferential force on face BD or FG is,

$$T = \int_a^r \sigma_{\theta\theta} \, dA \longrightarrow \textcircled{11}$$

Let r direction be OL. $\Sigma F_\theta = 0$ ensured since θ-component of T on faces BD & FG cancel out. $\Sigma F_r = 0$ requires,

$$\sigma_{rr} \, t r \, d\theta - 2T \sin(d\theta/2) = 0 \quad \Rightarrow \quad \boxed{\sigma_{rr} = \frac{T}{tr}} \longrightarrow \textcircled{12}$$

From $\textcircled{8}, \textcircled{11}$,

$$T = \frac{N}{A} \int_a^r dA + \frac{M_x}{(RA_m - A)} \int_a^r \frac{1}{r} \, dA - \frac{M_x A_m}{A(RA_m - A)} \int_a^r dA$$

$$\boxed{T = \frac{A'}{A} N + \frac{AA'_m - A' A_m}{A(RA_m - A)} M_x} \longrightarrow \textcircled{13}$$

where $\boxed{A'_m \triangleq \int_a^r \frac{dA}{r} \quad , \quad A' \triangleq \int_a^r dA}$

- For only shear loading, in case of a rectangular ⊗-section beam, the above approximate solution provides a conservative estimate when compared to plane-stress elasticity solution. The differences remain to within 6% for $R/h > 1.0$ even if the axial force term (N) is neglected in $\textcircled{13}$.

(eg) (Boresi, Ex 8.3-1)
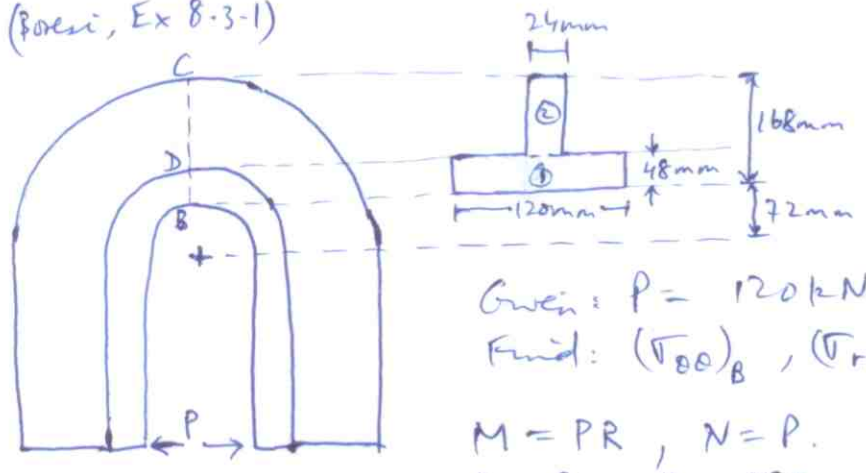
Given: $P = 120 \, kN$.

Find: $(\sigma_{\theta\theta})_B$, $(\sigma_{rr})_D$.

$M = PR$, $N = P$.

$R_1 = 96$, $R_2 = 180$

Find A, R, A_m as discussed on p.125 bottom, using Table 8.2.2 for A_m. Then use $\textcircled{8}, \textcircled{12}, \textcircled{13}$ to find respective stresses. For $(\sigma_{rr})_D$, A', A'_m based on area $\textcircled{1}$ only.

Here the effect of stress concentration at the fillet joining web & flange is neglected. These need to be considered for brittle materials or even ductiles under cyclic loading.

Even the $\sigma_{\theta\theta}$ need to be corrected, especially in case of I or T sections where effect of σ_{rr} is non-negligible and it changes $\sigma_{\theta\theta}$. We use Bleich's correction factors (ref. Boresi).

DEFLECTIONS IN CURVED BEAMS.

We use Castigliano's theorem.

Strain energy:
$$U = \frac{1}{2}\int_V \sigma_{ij}\, \varepsilon_{ij}\, dV = \int_V U_o\, dV.$$

Symmetry of loading wrt yz plane, coupled with complimentarity of shear stresses, gives $\tau_{xy} = \tau_{xz} = 0$

Neglecting $\sigma_{rr} (\equiv \sigma_{yy})$ and σ_{xx}, ie,

retaining only $\sigma_{\theta\theta} (\equiv \sigma_{zz})$ and $\sigma_{r\theta} (\equiv \sigma_{yz})$ we have

$$U_o = \frac{1}{2E}\sigma_{\theta\theta}^2 + \frac{1}{2G}\sigma_{r\theta}^2$$

$\sigma_{\theta\theta} \rightarrow$ contributes to U_N (axial / due to N), U_M (bending) and U_{MN} (coupling).

$\sigma_{r\theta} \rightarrow$ contributes to U_S (shear).

$$U_S = \int \frac{k V^2}{2AG}\, dz = \int \frac{k V^2}{2AG}\, R\, d\theta \quad \text{(based on straight beam approximation with V passing thru centroid).}$$

$$U_N = \int \frac{N^2}{2AE}\, dz = \int \frac{N^2}{2AE}\, R\, d\theta \quad \text{(based on straight beam approx with N thru centroid).}$$

For U_M, U_{MN} we dont use straight beam approx.

For U_M: Let V and N be applied first and then the moment be increased from zero to M_x. Since material is linearly elastic, the order of loading does not affect strain energy. Incremental bending strain energy is,

$$dU_M = \frac{1}{2} M_x (\Delta d\theta) = \frac{1}{2} M_x \omega \, d\theta$$

where $\Delta(d\theta)$ and ω arise due to M_x alone. Thus, Using $N=0$

$\left(\because \Delta(d\theta), \omega \atop \text{are due to} \atop M_x \text{ alone}\right)$

in ⑦ to obtain ω, we get,

$$dU_M = \frac{A_m M_x^2}{2A(RA_m - A)E} \, d\theta$$

During application of M_x, the centroidal surface stretches an amount $d\bar{\varepsilon}_{\theta\theta}$ and hence N does additional work given by,

$$dU_{MN} = N \, d\bar{\varepsilon}_{\theta\theta} = N \bar{\varepsilon}_{\theta\theta} R \, d\theta \quad \text{(from ① with } r=R\text{)}.$$

Using $r=R$, ①, ⑤ for obtaining $R_n \omega$, and ⑦ with $N=0$ to obtain ω, we have,

$$\bar{\varepsilon}_{\theta\theta} = \left(\frac{R_n}{R} - 1\right)\omega = \frac{M_x}{ER(RA_m - A)} - \frac{A_m M_x}{EA(RA_m - A)}$$

$$\Rightarrow dU_{MN} = \frac{N}{E}\left[\frac{M_x}{(RA_m - A)} - \frac{R A_m M_x}{A(RA_m - A)}\right] d\theta = -\frac{M_x N}{EA} \, d\theta$$

Thus,
$$\boxed{U = \int \left(\frac{k V^2 R}{2AG} + \frac{N^2 R}{2AE} + \frac{A_m M_x^2}{2A(RA_m - A)E} - \frac{M_x N}{EA}\right) d\theta} \quad ⑭$$

- ⑭ based on assumptions that σ_{rr} negligible & plane sections remaining plane.

- If M_x and N are same sign then $U_{MN} < 0$. Usually U_{MN} is small and negative. If it is negative, neglect it to compensate somewhat for loss of strain energy due to $\sigma_{rr} \simeq 0$ assumption.

- For shear-only loading $\delta_P = \delta_u = \frac{\partial U}{\partial P}$ and for moment-only loading (ie pure bending) (Boresi) $\delta_M = \delta_u = \frac{\partial U}{\partial M}$. These results are shown in Table 8-5-1, for a rectangular ⊗ section beam with $k=1.5$ and $\nu = 0.30$ used. Approximate results from Castigliano's theorem and plane-stress elasticity solution are compared.

Table 8-5.1

Ratios of Deflections in Rectangular Section Curved Beams as
Computed by Elasticity Theory and by Approximate Strain
Energy Solution

| | Neglecting U_{MN} | | Including U_{MN} | |
| | Pure Bending | Shear Loading | Pure Bending | Shear Loading |
$\left(\dfrac{R}{h}\right)$	$\left(\dfrac{\delta_U}{\delta_{elast}}\right)$	$\left(\dfrac{\delta_U}{\delta_{elast}}\right)$	$\left(\dfrac{\delta_U}{\delta_{elast}}\right)$	$\left(\dfrac{\delta_U}{\delta_{elast}}\right)$
0.65	0.923	1.563	0.697	1.215
0.75	0.974	1.381	0.807	1.123
1.0	1.004	1.197	0.914	1.048
1.5	1.006	1.085	0.968	1.016
2.0	1.004	1.048	0.983	1.008
3.0	1.002	1.021	0.993	1.003
5.0	1.000	1.007	0.997	1.001

- Deflections in curved beams are much less influenced
by curvature than are the $\sigma_{\theta\theta}$ stresses. Thus for $R/h > 2.0$
the $(U_M + U_{MN})$ terms can be replaced by straight beam
version of U_M alone, ie by $\int \frac{M_x^2}{2EI_x} Rd\theta$. The deflection results are
conservative when $R/h = 2$ and the straight beam U_M is
used, with straight beam approximation result being 7.7% higher
than curved beam ones.

(eg)

(Boresi
Ex 8-5-1)

100 mm   150 mm

M   P=0   M
                 P=0
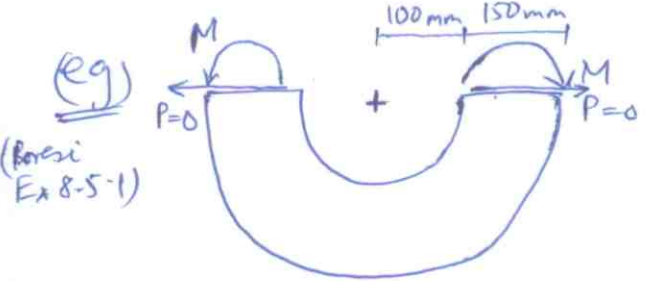
Given : $M = 24\,kN \cdot m$ applied,
rectangular cross section with
$60mm$ depth, $E = 72\,GPa$

Find : a) angle change between two
horizontal faces where M
applied
(b) Relative displ of centroids
of horizontal face.

First find A, A_m, R from Table
$8 \cdot 2 \cdot 2$ ($A = 150 \times 60\,mm^2$, $R = 175mm$ by observation).

(a) $\because V = N = 0$, only U_M term
present, ie $U = U_M$. From ⑭ and Castigliano's theorem,

$$\theta = \frac{\partial U}{\partial M} = \int_0^{\bar{\pi}} \frac{A_m M}{A(RA_m - A)E} d\theta = \frac{A_m M \pi}{A(RA_m - A)E}$$

(b) Vertical displ of centroids will be identical so the
relative vertical displ is zero. For relative horizontal
displ, apply fictitious horizontal force P thru centroids as
shown and finally put $P = 0$.

$$V = P\cos\theta, \quad N = P\sin\theta, \quad M_x = M + PR\sin\theta$$

From ⑭, and $\delta_p = \left. \frac{\partial U}{\partial P} \right|_{P=0}$,

$$\delta_p = \int_0^{\bar{\pi}} \left\{ \frac{k\,P\cos^2\theta\,R}{AG} + \frac{P\sin^2\theta\,R}{AE} + \frac{A_m(MR\sin\theta + PR^2\sin^2\theta)}{A\,(RA_m - A)\,E} \right.$$

$$\left. - \frac{M\sin\theta + 2PR\sin^2\theta}{EA} \right\}d\theta \Bigg|_{P=0}$$

$$= \int_0^{\bar{\pi}} \left(\frac{A_m\,MR\sin\theta}{A\,(RA_m - A)\,E} - \frac{M\sin\theta}{E A} \right)d\theta$$

neglect since it deducts from displ
(ie neglect U_{Nm})

(eg) (Boresi, Ex 8·5·2)
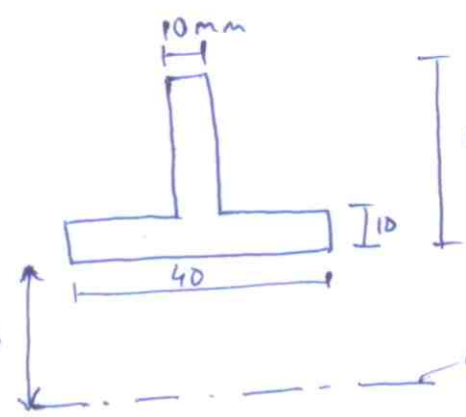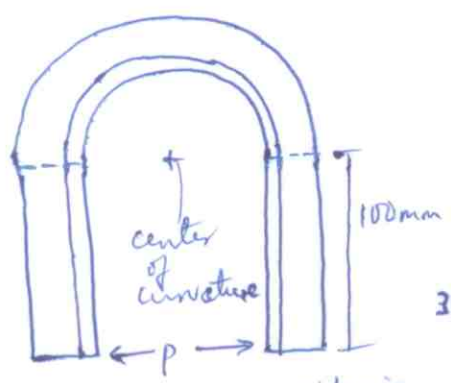
Given: $P = 11·2\,kN$
$E = 200\,GPa,\ \nu = 0·3$
Find: separation of jaws.

Done without using Bleich's correction for T sections.
(considering)

$V = P$, $N = 0$, $M_x = Pz$ in straight beam portion

$V = P\cos\theta$, $N = P\sin\theta$, $M_x = P(100) + PR\sin\theta$ for curved beam portion.

Use table 8·2·2 to get A, A_m, R. Note I_x is about centroidal section.

From ⑭,

$$\delta_p = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left[2\int_0^{100} \left\{ \frac{kP^2}{2AG} + \frac{P^2 z^2}{2EI_x} \right\} + \int_0^{\bar{\pi}} \left\{ \frac{kP^2\cos^2\theta\,R}{2AG} + \frac{P^2\sin^2\theta\,R}{2AE} \right. \right.$$

↳straight beam portion

$$\left. + \frac{A_m\,(100P + PR\sin\theta)^2}{2A\,(RA_m - A)\,E} - \frac{100P^2\sin\theta + P^2 R\sin^2\theta}{EA} \right\}d\theta \Bigg]$$

$$= 2\int_0^{100} \left\{ \frac{kP}{AG} + \frac{Pz^2}{EI_x} \right\}dz$$

$$+ \int_0^{\bar{\pi}} \left\{ \frac{kP\cos^2\theta\,R}{AG} + \frac{P\sin^2\theta\,R}{AE} + \frac{A_m\,P(100 + R\sin\theta)^2}{A\,(RA_m - A)\,E} \right\}d\theta$$ curved beam portion

where the last term (ie U_{Nm}) in curved beam portion is neglected since it has negative contribution to δ_p.

BEAMS ON ELASTIC FOUNDATIONS.

Assumptions:

1) Foundation stronger than beam so failure occurs first in beam (since that is often the case).

2) Linear elastic foundation. [Note that nonlinear foundations are generally of hardening type, ie, $f = k_1 x + k_2 x^3$, $(k_1, k_2) > 0$. Hardening nonlinearity reduces stresses/displacements when compared to linear case. So linear analysis is conservative].

   [Could also have viscoelastic foundation].

3) Beam **attached** to foundation (so foundation undergoes compression as well as tension).

(I) Infinite Beam with Point load.
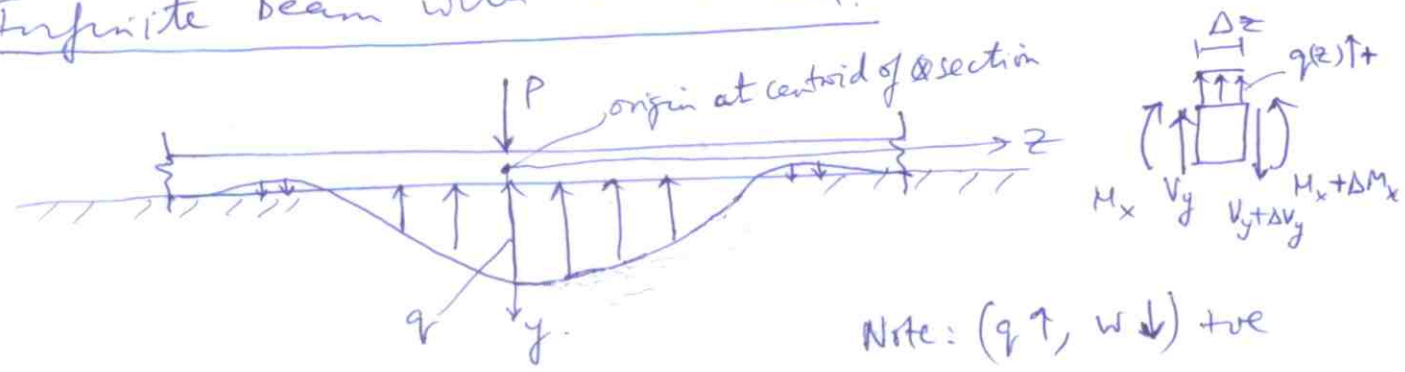
Note: $(q \uparrow, w \downarrow)$ +ve

For Euler-Bernoulli beam,

$$\frac{dw}{dz} = \theta \quad , \quad EI_x w'' = -M_x \quad , \quad EI_x w''' = -V_y \left.\right\} \to \text{①}$$

$$EI_x w^{IV} = -q(z)$$

For elastic foundation,

$$q = kw \quad , \quad k = b k_0$$

where b is width of beam, k is N·m^{-2}, k_0 is foundation constant ie N·m^{-3}. Let $I_x \equiv I$, $V_\cdot \equiv V$, $M_x \equiv M$.

$$EI w^{IV} = -kw$$

$$w = e^{sz} \Rightarrow s^4 + 4\beta^4 = 0 \quad , \quad \boxed{\beta^4 \triangleq \frac{k}{4EI}} \to \text{②}$$

$$(s^2 + 2i\beta^4)(s^2 - 2i\beta^4) = 0 \quad \Rightarrow s = \pm\sqrt{2}\beta\sqrt{\pm i} = \pm\sqrt{2}\beta e^{\frac{\pm i\pi}{4}}$$
$$= \pm\sqrt{2}\beta(1 \pm i)$$
$$\sqrt{2}$$

$$\Rightarrow w = \sum_{i=1}^{4} A_i e^{s_i z} = A_1 e^{\beta z} e^{i\beta z} + A_2 e^{\beta z} e^{-i\beta z} + A_3 e^{-\beta z} e^{+i\beta z} + A_4 e^{-\beta z} e^{-i\beta z}$$

$$= e^{\beta z}\left([A_1 + A_2]\cos\beta z + [A_1 - A_2]i\sin\beta z\right) + e^{-\beta z}\left([A_3 + A_4]\cos\beta z + i[A_3 - A_4]\sin\beta z\right)$$

$W = \text{real} \Rightarrow A_2 = \bar{A_1}$, $A_4 = \bar{A_3}$, let $A_1 + A_2 = C_1$, $i(A_1 - A_2) = C_2$, $(A_3 + A_4) = C_3$,

$\Rightarrow W = e^{\beta z}(C_1 \cos\beta z + C_2 \sin\beta z) + e^{-\beta z}(C_3 \cos\beta z + C_4 \sin\beta z)$ $i(A_3 - A_4) = C_4$

$\therefore W = \text{finite} \Rightarrow C_1 = C_2 = 0$ for $z \geq 0$, $C_3 = C_4 = 0$ for $z \leq 0$.

$$\Rightarrow \boxed{\begin{array}{l} W = e^{-\beta z}(K_1 \cos\beta z + K_2 \sin\beta z) \quad \text{for } z \geq 0 \\ W[-z] = W[z] \quad (\text{from symmetry}) \end{array}} \longrightarrow ③.$$

BC's: $\left. W' \right|_{z=0} = 0$ (from symmetry) and $\dfrac{P}{2} = \displaystyle\int_0^\infty kw\,dz$ (from equilibrium)

$\Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$

$-\beta K_1 + \beta K_2 = 0 \Rightarrow K_1 = K_2 = C \quad\Longrightarrow \text{use} \int e^{-\beta z}(\sin\beta z + \cos\beta z)\,dz$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\frac{1}{\beta} e^{-\beta z}\cos\beta z$

$$\Rightarrow C = \frac{P\beta}{2k}$$

$$\Rightarrow \boxed{\begin{array}{l} W = \dfrac{P\beta}{2k}\, \underbrace{e^{-\beta z}(\cos\beta z + \sin\beta z)}_{A_{\beta z}} = \dfrac{P\beta}{2k} A_{\beta z} \\[4mm] \theta = -\dfrac{P\beta^2}{k}\, \underbrace{e^{-\beta z}\sin\beta z}_{B_{\beta z}} = -\dfrac{P\beta^2}{k} B_{\beta z} \\[4mm] M = \dfrac{EI\,P\beta^3}{k}\, \underbrace{(\cos\beta z - \sin\beta z)e^{-\beta z}}_{C_{\beta z}} = \dfrac{P}{4\beta} C_{\beta z} \\[4mm] V = \dfrac{P}{4\beta}\beta\Big(-2\, \underbrace{e^{-\beta z}\cos\beta z}_{D_{\beta z}}\Big) = -\dfrac{P}{2} D_{\beta z} \end{array}}$$

use ① $\Rightarrow$ (left of box)

$\longrightarrow$ see Table 9.2-1 Boresi for $A_{\beta z}, B_{\beta z}, C_{\beta z}, D_{\beta z}$

$\longrightarrow ④$

$\longrightarrow \boxed{\text{for } z \geq 0}$

$\longrightarrow$ (see Fig 9.1.1, Boresi).

i.e., $\boxed{\begin{array}{ll} \int A_{\beta z}\,dz = -\dfrac{1}{\beta} D_{\beta z}, & \int B_{\beta z}\,dz = -\dfrac{1}{2\beta} A_{\beta z} \\[4mm] \int C_{\beta z}\,dz = \dfrac{1}{\beta} B_{\beta z}, & \int D_{\beta z}\,dz = -\dfrac{1}{2\beta} C_{\beta z} \end{array}} \longrightarrow ⑤$

Note symmetry of displ and BM and antisymmetry of slope and SF — these arise due to symmetry of problem and the sign convention adopted. (see fig 9.1.1 Boresi).

$$\Rightarrow \boxed{\begin{array}{ll} W(-z) = W(z), & M(-z) = M(z) \\ V(-z) = -V(z), & \theta(-z) = -\theta(z) \end{array}} \longrightarrow ⑥$$

Table 9-2.1

βz	$A_{\beta z}$	$B_{\beta z}$	$C_{\beta z}$	$D_{\beta z}$
0	1	0	1	1
0.001	1.0000	0.0010	0.9980	0.9990
0.002	1.0000	0.0020	0.9960	0.9980
0.003	1.0000	0.0030	0.9940	0.9970
0.004	1.0000	0.0040	0.9920	0.9960
0.005	1.0000	0.0050	0.9900	0.9950
0.006	1.0000	0.0060	0.9880	0.9940
0.007	0.9999	0.0070	0.9861	0.9930
0.008	0.9999	0.0080	0.9841	0.9920
0.009	0.9999	0.0087	0.9821	0.9910
0.010	0.9999	0.0099	0.9801	0.9900
0.011	0.9999	0.0109	0.9781	0.9890
0.012	0.9999	0.0119	0.9761	0.9880
0.013	0.9998	0.0129	0.9742	0.9870
0.014	0.9998	0.0138	0.9722	0.9860
0.015	0.9998	0.0148	0.9702	0.9850
0.016	0.9997	0.0158	0.9683	0.9840
0.017	0.9997	0.0167	0.9663	0.9830
0.018	0.9997	0.0177	0.9643	0.9820
0.019	0.9996	0.0187	0.9624	0.9810
0.02	0.9996	0.0196	0.9604	0.9800
0.03	0.9991	0.0291	0.9409	0.9700
0.04	0.9984	0.0384	0.9216	0.9600
0.05	0.9976	0.0476	0.9025	0.9501
0.10	0.9906	0.0903	0.8100	0.9003
0.15	0.9796	0.1283	0.7224	0.8510
0.20	0.9651	0.1627	0.6398	0.8024
0.25	0.9472	0.1927	0.5619	0.7546
0.30	0.9267	0.2189	0.4888	0.7078
0.35	0.9036	0.2416	0.4204	0.6620
0.40	0.8784	0.2610	0.3564	0.6174
0.45	0.8515	0.2774	0.2968	0.5742
0.50	0.8231	0.2908	0.2414	0.5323
0.55	0.7934	0.3016	0.1902	0.4918
0.60	0.7628	0.3099	0.1430	0.4529
0.65	0.7315	0.3160	0.0996	0.4156
0.70	0.6997	0.3199	0.0599	0.3798
0.75	0.6676	0.3220	0.0237	0.3456
$\frac{1}{4}\pi$	0.6448	0.3224	0	0.3224
0.80	0.6353	0.3223	−0.0093	0.3131
0.85	0.6032	0.3212	−0.0391	0.2821
0.90	0.5712	0.3185	−0.0658	0.2527
0.95	0.5396	0.3146	−0.0896	0.2250
1.00	0.5083	0.3096	−0.1109	0.1987

(a) Beam loading

(b)

(c) Deflection, y

(d) Slope, $\theta = \dfrac{dy}{dz}$

(e) Bending moment, $M_x = -EI_x \dfrac{d^2 y}{dz^2}$

(f) Shear, $V_y = -EI_x \dfrac{d^3 y}{dz^3}$
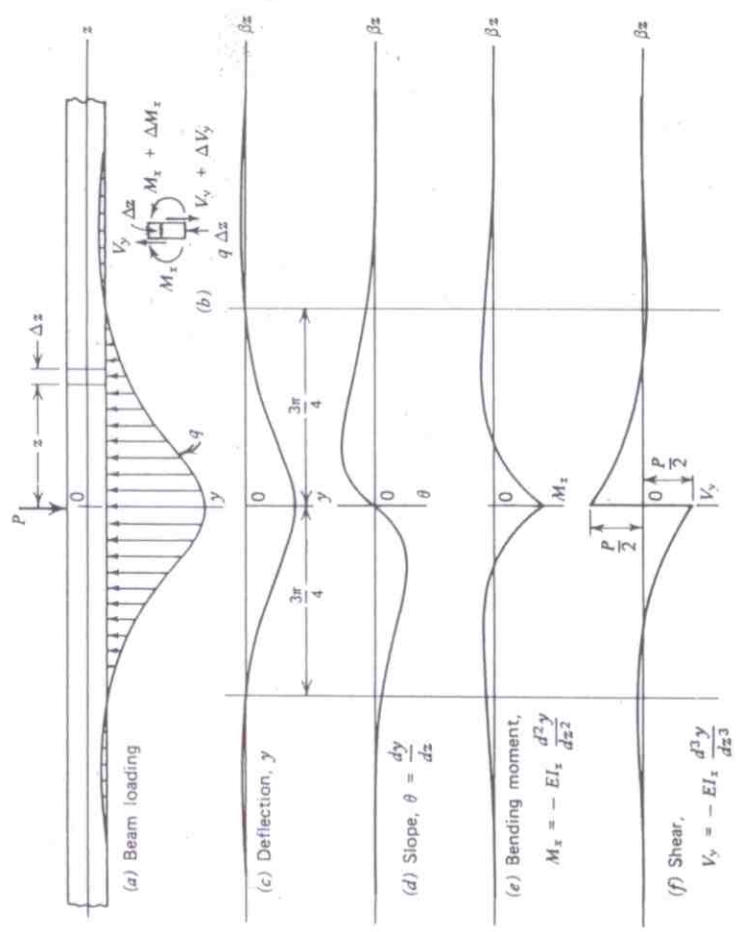
Fig. 9-1.1/Infinite beam on an elastic foundation and loaded at origin.

Table 9-2.1
Continued

1.05	0.4778	0.3036	−0.1294	0.1742
1.10	0.4476	0.2967	−0.1458	0.1509
1.15	0.4183	0.2890	−0.1597	0.1293
1.20	0.3898	0.2807	−0.1716	0.1091
1.25	0.3623	0.2719	−0.1815	0.0904
1.30	0.3355	0.2626	−0.1897	0.0729
1.35	0.3098	0.2530	−0.1962	0.0568
1.40	0.2849	0.2430	−0.2011	0.0419
1.45	0.2611	0.2329	−0.2045	0.0283
1.50	0.2384	0.2226	−0.2068	0.0158
1.55	0.2166	0.2122	−0.2078	0.0044
$\frac{1}{2}\pi$	0.2079	0.2079	−0.2079	0
1.60	0.1960	0.2018	−0.2077	−0.0059
1.65	0.1763	0.1915	−0.2067	−0.0152
1.70	0.1576	0.1812	−0.2046	−0.0236
1.75	0.1400	0.1720	−0.2020	−0.0310
1.80	0.1234	0.1610	−0.1985	−0.0376
1.85	0.1078	0.1512	−0.1945	−0.0434
1.90	0.0932	0.1415	−0.1899	−0.0484
1.95	0.0795	0.1322	−0.1849	−0.0527
2.00	0.0667	0.1230	−0.1793	−0.0563
2.05	0.0549	0.1143	−0.1737	−0.0594
2.10	0.0438	0.1057	−0.1676	−0.0619
2.15	0.0337	0.0975	−0.1613	−0.0638
2.20	0.0244	0.0895	−0.1547	−0.0652
2.25	0.0157	0.0820	−0.1482	−0.0663
2.30	0.0080	0.0748	−0.1416	−0.0668
2.35	0.0008	0.0679	−0.1349	−0.0671
$\frac{3}{4}\pi$	0	0.0671	−0.1342	−0.0671
2.40	−0.0056	0.0613	−0.1282	−0.0669
2.45	−0.0114	0.0550	−0.1215	−0.0665
2.50	−0.0166	0.0492	−0.1149	−0.0658
2.55	−0.0213	0.0435	−0.1083	−0.0648
2.60	−0.0254	0.0383	−0.1020	−0.0637
2.65	−0.0289	0.0334	−0.0956	−0.0623
2.70	−0.0320	0.0287	−0.0895	−0.0608
2.75	−0.0347	0.0244	−0.0835	−0.0591
2.80	−0.0369	0.0204	−0.0777	−0.0573
2.85	−0.0388	0.0167	−0.0721	−0.0554
2.90	−0.0403	0.0132	−0.0666	−0.0534
2.95	−0.0415	0.0100	−0.0614	−0.0514
3.00	−0.0422	0.0071	−0.0563	−0.0493
3.05	−0.0427	0.0043	−0.0515	−0.0472
3.10	−0.0431	0.0019	−0.0469	−0.0450
π	−0.0432	0	−0.0432	−0.0432

Table 9-2.1
Continued

3.15	−0.0432	−0.0004	−0.0424	−0.0428
3.20	−0.0431	−0.0024	−0.0383	−0.0407
3.25	−0.0427	−0.0042	−0.0343	−0.0385
3.30	−0.0422	−0.0058	−0.0306	−0.0365
3.35	−0.0417	−0.0073	−0.0271	−0.0344
3.40	−0.0408	−0.0085	−0.0238	−0.0323
3.45	−0.0399	−0.0097	−0.0206	−0.0303
3.50	−0.0388	−0.0106	−0.0177	−0.0283
3.55	−0.0378	−0.0114	−0.0149	−0.0264
3.60	−0.0366	−0.0121	−0.0124	−0.0245
3.65	−0.0354	−0.0126	−0.0101	−0.0227
3.70	−0.0341	−0.0131	−0.0079	−0.0210
3.75	−0.0327	−0.0134	−0.0059	−0.0193
3.80	−0.0314	−0.0137	−0.0040	−0.0177
3.85	−0.0300	−0.0139	−0.0023	−0.0162
3.90	−0.0286	−0.0140	−0.0008	−0.0147
$\frac{5}{4}\pi$	−0.0278	−0.0140	0	−0.0139
3.95	−0.0272	−0.0139	0.0005	−0.0133
4.00	−0.0258	−0.0139	0.0019	−0.0120
4.50	−0.0132	−0.0108	0.0085	−0.0023
$\frac{3}{2}\pi$	−0.0090	−0.0090	0.0090	0
5.00	−0.0046	−0.0065	0.0084	0.0019
$\frac{7}{4}\pi$	0	−0.0029	0.0058	0.0029
5.50	0.0000	−0.0029	0.0058	0.0029
6.00	0.0017	−0.0007	0.0031	0.0024
2π	0.0019	0	0.0019	0.0019
6.50	0.0018	0.0003	0.0012	0.0018
7.00	0.0013	0.0006	0.0001	0.0007
$\frac{9}{4}\pi$	0.0012	0.0006	0	0.0006
7.50	0.0007	0.0005	−0.0003	0.0002
$\frac{5}{2}\pi$	0.0004	0.0004	−0.0004	0

$A_{\beta z}, B_{\beta z}, C_{\beta z}, D_{\beta z}$ tabulated in Table (9.2.1) — Boresi & Sidebottom.

From this table, note that

(1) · $(A_{\beta z}, B_{\beta z}, C_{\beta z}, D_{\beta z}) \rightarrow 0$ as $\beta z \rightarrow \infty$.

(2) · $|A_{\beta z}, C_{\beta z}, D_{\beta z}|_{max} = 1$ at $\beta z = 0$.

(3) · $|B_{\beta z}|_{max} = 0.3224$ at $\beta z = \pi/4$.

(4) · $A_{\beta z} = 0$ at $\beta z = \frac{3\pi}{4}$

\# Thus ∞ beam theory can be used for long finite beam (length L) (due to (1) above). Further, due to (4) above, as a guideline we use ∞ theory for finite beam if distance from load to nearer end is $\geq 3\pi/4\beta$ (in which case $L \geq 3\pi/2\beta$).

Although the error of this approx is nonconservative (i.e, ∞ theory applied to finite beam, yields smaller w, BM, etc than if we used the more exact finite theory) the error is very small.

\# Since $A_{\beta z}$ and hence w are +ve for $\beta z < \frac{3\pi}{4}$ (i.e (4) above), a beam having $L \leq 3\pi/2\beta$ need not be attached to foundation & still the ∞ theory herein will give exact solution.

\# For an infinite beam, supported on discrete springs, equally spaced (l mts apart), having identical stiffness (K Nm^{-1}), we smear the springs over L, i.e $l/2$ to left & right, i.e, use $\boxed{k = K/l}$. The solution is quite accurate when $\boxed{l \leq \pi/\beta}$.

\# For finite beam, on discrete springs (stiffness K, spacing l) (actual length L) use effective length $\boxed{L'' = ml}$, where m = nos of equally spaced springs (i.e, extend beam length from L to L'' so that each spring is smeared over $l/2$ to its left and right). Then use ∞ beam approx method as above. This is fairly accurate if $\boxed{\beta L'' \geq 3\pi/2}$.

(II) Infinite beam subject to distributed$^{(f)}$ load segment.
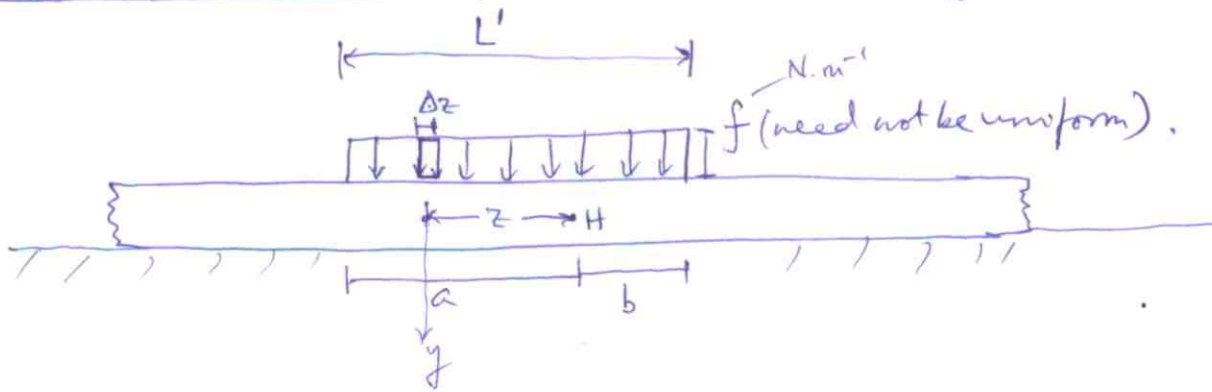
We concentrate on portion under f, since max values of W, BM, SF usually occur under f region.

For $\Delta P = f \Delta z$,

$$\Delta W_H = \frac{f \Delta z \beta}{2k} \underbrace{e^{-\beta z}(\cos \beta z + \sin \beta z)}_{A_{\beta z}}$$

$$\Rightarrow \quad W_H = \int_0^a \frac{f\beta}{2k} A_{\beta z} \, dz + \int_0^b \frac{f\beta}{2k} A_{\beta z} \, dz = \frac{f\beta}{2k}\left(\frac{-1}{\beta}\right)\left(D_{\beta z}\Big|_0^a + D_{\beta z}\Big|_0^b\right)$$

$$\boxed{W_H = \frac{f}{2k}\left(2 - D_{\beta a} - D_{\beta b}\right)} \longrightarrow \text{(7a)}$$

④,⑤ $\rightarrow$

$$\theta = -\int_0^a \frac{f\beta^2}{k} B_{\beta z} \, dz \;\oplus\; \int_0^b \frac{f\beta^2}{k} B_{\beta z} \, dz = \frac{f\beta}{2k}\left(A_{\beta z}\Big|_0^a - A_{\beta z}\Big|_0^b\right)$$

$$(\because \theta(-z) = -\theta(z)).$$
for region 0-b.

$$\boxed{\theta = \frac{f\beta}{2k}\left(A_{\beta a} - A_{\beta b}\right)} \longrightarrow \text{(7b)}$$

$$M = \frac{f}{4\beta}\left(\int_0^a C_{\beta z} \, dz + \int_0^b C_{\beta z} \, dz\right) = \frac{f}{4\beta^2}\left(B_{\beta z}\Big|_0^a + B_{\beta z}\Big|_0^b\right)$$

$$\boxed{M = \frac{f}{4\beta^2}\left(B_{\beta a} + B_{\beta b}\right)} \longrightarrow \text{(7c)}$$

$$(\because V(-z) = -V(z))$$

$$V = -\frac{f}{2}\left[\int_0^a D_{\beta z} \;\ominus\; \int_0^b D_{\beta z}\right] = \frac{f}{4\beta}\left(C_{\beta z}\Big|_0^a - C_{\beta z}\Big|_0^b\right)$$

$$\boxed{V = \frac{f}{4\beta}\left(C_{\beta a} - C_{\beta b}\right)} \longrightarrow \text{(7d)}$$

\# Max deflection occurs at center of segment L' (can see it form physical symmetry or from Table 9.2.1).

\# <u>Max BM</u>:

(i) <u>$\beta L' \leq \pi$</u> : From Table 9.2.1, see that max BM occurs at center of L' segment.

(ii) <u>$\beta L' \to \infty$</u> : $\quad \beta L' \to \infty \Rightarrow (\beta a, \beta b) \to \infty$ except for points 'H' located near ends of L' segment.

$\Rightarrow$ (from (7)) $(\theta, M, V) \to 0$ and $w \to \frac{f}{R}$ except near ends of L' segment.

Then, Table 9.2.1 shows that max BM occurs when either $\overset{\text{(ie at pt. H for which)}}{}$

$\{\beta a = \frac{\pi}{4}, \text{ ie } \beta b \to \infty \ (\because \beta L' \to \infty)\}$ or $\{\beta b = \frac{\pi}{4}, \text{ ie } \beta a \to \infty \ (\because \beta L' \to \infty)\}$

(iii) <u>$\beta L' > \pi$</u> (ie intermediate values of β) :

In this case max BM may lie outside load segment L'. However, sufficient accuracy can still be obtained by assuming max BM location as in (ii) above (ie at $\frac{\pi}{4\beta}$ from either end).

\# Note: Eqs (7a, c) valid only for points under distributed load. [See Boresi Prob 9-3-3 p. 419].

(<u>III</u>) <u>Semi-infinite beam with end loads</u>

i.e, $z \geq 0$
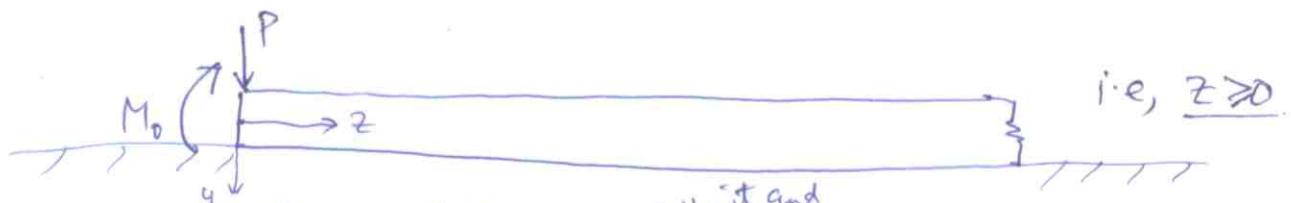
Note that ③ is still valid. Use it and Apply bc's,

$EI\,w''\big|_{z=0} = -M_0 \quad , \quad EI\,w'''\big|_{z=0} = -V = P$

and get,

$$K_2 = \frac{2\beta^2 M_0}{k} \quad , \quad K_1 = \frac{2\beta P}{k} - K_2$$

$$\Rightarrow \quad W = \frac{2\beta e^{-\beta z}}{R}\left[P\cos\beta z - \beta M_0(\cos\beta z - \sin\beta z)\right]$$

$$\boxed{W = \frac{2P\beta}{R} D_{\beta z} - \frac{2\beta^2 M_0}{R} C_{\beta z}} \longrightarrow \text{(8a)}$$

$$\text{①,⑤,⑧ⓐ} \rightarrow \boxed{\begin{array}{l} Q = -\frac{2P\beta^2}{R} A_{\beta z} + \frac{4\beta^3 M_0}{R} D_{\beta z} \longrightarrow \text{(8b)} \\[4mm] M = -\frac{P}{\beta} B_{\beta z} + M_0 A_{\beta z} \longrightarrow \text{(8c)} \\[4mm] V = -P C_{\beta z} - 2M_0 \beta B_{\beta z} \longrightarrow \text{(8d)} \end{array}} \quad\Bigg\}\; z \geqslant 0$$

(IV) Semi-infinite beam with concentrated load near its end.

Use superposition as follows:

(a : given problem)

$\parallel$

(b : finite beam with P & M_x, V_y based on ∞ beam BM and SF)
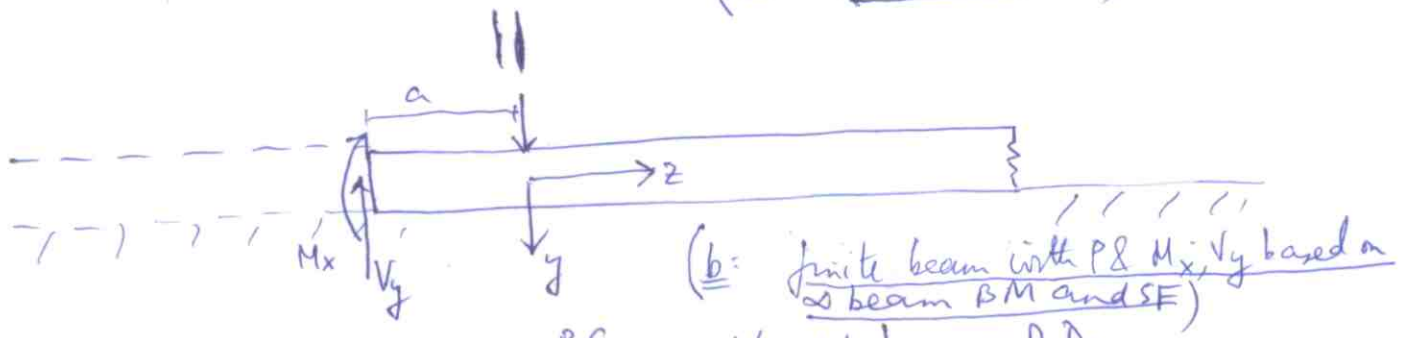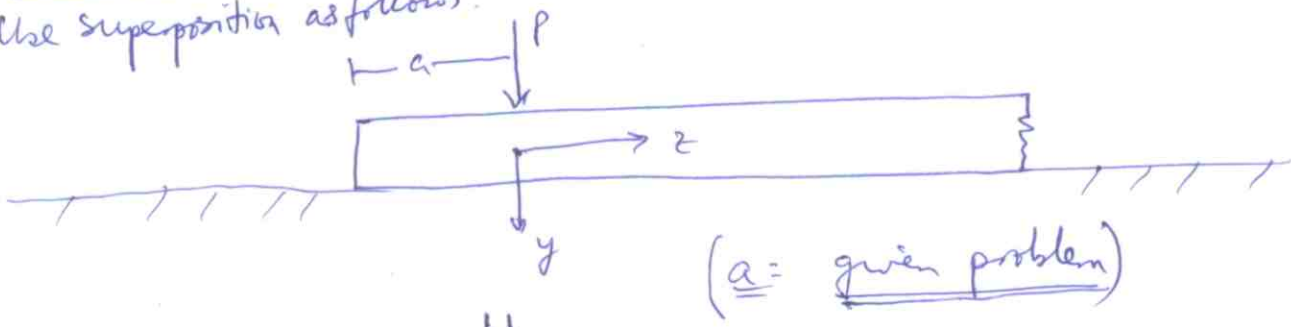
$$M_x = M_x\big|_{z=-a} = \frac{P C_{\beta a}}{4\beta}, \quad V_y = V_y\big|_{z=-a} = \frac{P D_{\beta a}}{2}$$

as obtained from ∞ beam (eqns 4(c,d))

$+$

(c : finite beam with end load M, Q as shown)

$$\boxed{M = \frac{P C_{\beta a}}{4\beta}, \quad Q = \frac{P D_{\beta a}}{2}} \longrightarrow \text{chosen to negate end loads in probl (b)}$$

⑨ $\leftarrow$

Thus Prob (a) = Prob (b) + Prob (c)
   Sol   Sol   Sol.

Since origin is, 'a' dist, to the right of end of beam, use
$z \rightarrow (a+z)$ in eqns (8) when superposing. Using (4a), 4(c) and
(with $M_0 \equiv -M$, $P \equiv Q$) and ⑨
shifted version of 8(a) and 8(c), and adding them, respectively,

$$W = \frac{P\beta}{2k}\left[A_{\beta z} + 2 D_{\beta a} D_{\beta(z+a)} + C_{\beta a} C_{\beta(z+a)} \right]$$

$$M = \frac{P}{4\beta}\left[C_{\beta z} - 2 D_{\beta a} B_{\beta(z+a)} - C_{\beta a} A_{\beta(z+a)} \right] \longrightarrow ⑩$$

(Ex) (9.2-1 Boresi) ^{example} — Infinite beam, multiple point loads
  Given: Railroad with steel rails ($E = 200\,GPa$) having depth 184mm,
     and distance from top of rail to centroid is 99.1mm,
     and $I = 36.9 \times 10^6\,mm^4$. Also $k = 14\,N/mm^2$
  Find: (a) Max deflection, max BM, max bending stress
       for case of single wheel with $P = 170\,kN$.
       (b) Do (a) if locomotive has three equally spaced wheels, spacing $= 1.70m$,
   with $P = 170\,kN$ for each wheel

(a) For single wheel, max defl, max BM occur at $\beta z = 0$,
  ie., under the wheel ($\because A_{\beta z}$ and $C_{\beta z}$ are max value of unity
                                                    at $\beta z = 0$).
  Now β is found by ②, W_{max}, M_{max} by 4(a,c) with $A_{\beta z} = C_{\beta z} = 1$.
  From given section, y_{max} occurs at top ($= 99.1mm$) which gives
  max bending stress. Thus $W_{max} = \frac{P\beta}{2k}$, $M_{max} = \frac{P}{4\beta}$, $\sigma_{max} = \frac{M_{max} \, y_{max}}{I}$

(b) Need to use superposition. In general max values depend
  on inter-load spacing and value of β. For large
  inter-load spacing and/or large β value (representing
  steep exponential decay, it is evident that the max values will be
  under the loads (or near about that). Table (9.2.1) reveals that
  to be the case here, but this is not generally true. In general
  we just use superposition to find max values. Thus for the

present case, let $z_0=0$, $z_1 = 1\cdot7\times10^3$ mm, $z_2 = 3\cdot4\times10^3$ mm. Then,

$$W_1 = W_3 = \frac{P\beta}{2k}\left(A_{\beta z_0} + A_{\beta z_1} + A_{\beta z_2}\right)$$

$$M_1 = M_3 = \frac{P}{4\beta}\left(C_{\beta z_0} + C_{\beta z_1} + C_{\beta z_2}\right)$$
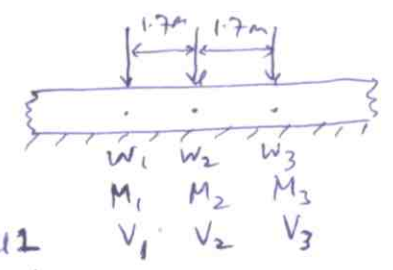
$$\begin{cases} V_1 = -\frac{P}{2}\left(\pm D_{\beta z_0} - D_{\beta z_1} - D_{\beta z_2}\right), \begin{array}{l}\text{+ for just to right of wheel 1}\\ \text{− for just to left of wheel 1}\end{array} \\[2mm] V_3 = -\frac{P}{2}\left(\mp D_{\beta z_0} + D_{\beta z_1} + D_{\beta z_2}\right), \begin{array}{l}\text{− for just to left of wheel 3}\\ \text{+ for just to right of wheel 3}\end{array} \end{cases}$$

consistent with the physical intuition that the shear forces have a numerically antisymmetric distribution about middle wheel, as per sign convention, ie, a physically symmetric distribution about middle wheel.

$$W_2 = \frac{P\beta}{2k}\left(2A_{\beta z_1} + A_{\beta z_0}\right)$$

$$M_2 = \frac{P}{4\beta}\left(C_{\beta z_0} + 2\,C_{\beta z_1}\right)$$

$$V_2 = -\frac{P}{2}\left(\pm D_{\beta z_0} + D_{\beta z_1} - D_{\beta z_1}\right) = -\frac{P}{2}\left(\pm D_{\beta z_0}\right), \begin{array}{l}\text{+ for just to right of wheel 2}\\ \text{− " " " left " " "}\end{array}$$

The shear force analysis is not required.

Choose $\text{max} = \max(W_1, W_2)$

$$M_{max} = \max(M_1, M_2)$$

Then $\sigma_{max} = \dfrac{M_{max}\, y_{max}}{I}$

(Ex) (9.2.2, Boresi) example — Finite beam on discrete springs.

Given: I Beam with depth = 100 mm, $I_x = 2\cdot45\times10^6$ mm⁴, $E = 72\,GPa$,
L = 6·8 m, supported on 7 springs, K = 110 N/mm, spacing = 1·10 m, (l).
load P = 12 kN at center.

Find : load carried by each spring, deflection under the load, max bending moment and associated bending stress, based on approximation using smearing concept.

First check that $l \leq \pi/\beta$ so that smearing approx is good and that $L'' (= ml = 7l) \geq 3\pi/2\beta$ so that the infinite beam formula gives a good approx. These conditions are satisfied in the present case so we can proceed.

M_{max}, $W_{max} = W_4$ occur at $\beta z = 0$ (ie $A_{\beta z} = C_{\beta z} = 1 = max$), ie under load at center spring position. Also $\sigma_{max} = \dfrac{M_{max}(s_0)}{I}$.
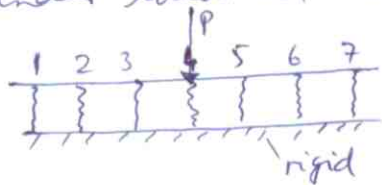
$\rightarrow$ Easy to find spring forces — first find deflections of w_5, w_6, w_7 and by K.

The spring forces in the leftmost & right most spring are in considerable error from this smearing type solution (as compared to the exact solution based on Castigliano's theorem using discrete springs. However, other spring forces, deflections, and BM compare very well (error $\approx 7\%$ when compared to exact solution).

(Ex) (Boresi Ex 9.4-1)

Given: I beam, $E = 200\,GPa$, depth $= 102\,mm$, width $= 68\,mm$, $I = 2.53 \times 10^6\,mm^4$, $L = 4m$, $k_0 = 0.350\,N/mm^3$, concentrated downward load $P = 30\,kN$ at end.

Find: Max defl and max bending stress, and their location.

$k = 0.350 \times 68\,N/mm^2$

Here $L (= 4000) > \dfrac{3\pi}{2\beta}$ so ∞ beam theory can be used.

$W = \dfrac{2P\beta}{k} D_{\beta z} \Rightarrow W_{max}$ at $\beta z = 0$, $D_{\beta z}\big|_0 = 1$, $z = 0$.

$M_x = -\dfrac{P}{\beta} B_{\beta z} \Rightarrow M_{max}$ at $\beta z = \pi/4$, $B_{\beta z}\big|_{\pi/4} = 0.3224$

$\sigma_{max} = M_{max}(s_0)/I$.

(Ex)

VISCOELASTICITY – Linear Theory.

Recall: $\sigma_{ij} = \overline{\sigma}_{ij} + \overline{\overline{\sigma}}_{ij}$, $\overline{\sigma}_{ij} = \overline{\sigma}_{ij}(A_{ij}, T) = $ conservative stress tensor, $A_{ij} \rightarrow$ generic strain tensor.

$\overline{\overline{\sigma}}_{ij} = $ dissipative stress tensor

For viscoelasticity, we do not have such a kinetic equation of state to relate the conservative part of the stress tensor to the strain tensor. Instead, we relate the stress tensor to the strain tensor thru a differential eqn as follows:

$$\cdots \overline{\overline{B}}_{ijkr} \ddot{\sigma}_{kr} + \widetilde{B}_{ijkr} \dot{\sigma}_{kr} + \sigma_{ij} = \overbrace{\mathcal{E}_{ij}}^{(=0 \text{ when no prestress is assumed})} + C_{ijkr} \, \mathcal{E}_{kr} + \widetilde{C}_{ijkr} \, \dot{\mathcal{E}}_{kr} + \cdots$$

Linear Viscoelasticity – Assumptions.

(i) Coeffs $\widetilde{B}_{ijkr}$, $\widetilde{C}_{ijkr}$, etc. are not functions of stress/strain tensors or their time rates.

(ii) Small displ. gradients assumed (i.e, linear strain tensors b_{kr}, $\mathcal{E}_{kr}$, considered)

NOTE: In viscoelasticity, we have dissipation of energy due to which the process is irreversible, in general.

Three Parameter Solid. – 1-D modelling

$\sigma_1 = R_1 e_1$, $\sigma_2 = C_2 \dot{e}_2$, $\sigma_1 = \sigma_2$

$\sigma_3 = R_3 e_3$, $e = e_3 = e_1 + e_2$, $\sigma = \sigma_1 + \sigma_3$
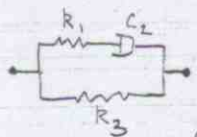
$\left.\begin{array}{l}\end{array}\right\}$ 7 eqns, 8 unknowns, so we can eliminate and get one eqn relating the two unknowns σ, e.

$\dot{e} = \dot{e}_1 + \dot{e}_2 = \dfrac{\dot{\sigma}_1}{R_1} + \dfrac{\sigma_2}{C_2} = \dfrac{\dot{\sigma}_1}{R_1} + \dfrac{\sigma_1}{C_2} = \dfrac{\dot{\sigma} - \dot{\sigma}_3}{R_1} + \dfrac{\sigma - \sigma_3}{C_2} = \dfrac{\dot{\sigma} - R_3 \dot{e}_3}{R_1} + \dfrac{\sigma - R_3 e_3}{C_2}$

$\dot{e} = \dfrac{\dot{\sigma}}{R_1} - \dfrac{R_3}{R_1}\dot{e} + \dfrac{\sigma}{C_2} - \dfrac{R_3}{C_2} e$ $\Rightarrow$ $\boxed{\dfrac{\dot{\sigma}}{R_1} + \dfrac{\sigma}{C_2} = \dfrac{R_3}{C_2} e + \left(1 + \dfrac{R_3}{R_1}\right)\dot{e}}$

Elements of Laplace Transform.

Definition of L.T.: $F(s) \triangleq \displaystyle\int_0^\infty f(t) e^{-st} \, dt$ provided the integral exists (i.e, it is finite).

This means that $f(t)$ must satisfy the condt. $\displaystyle\int_0^\infty |f(t) e^{-\sigma t}| \, dt < \infty$ where σ is real & finite, & $s = \sigma + i\omega$. It also implies that $f(t)$ is piecewise continuous.

Note: For a causal system wherein response (output) does not precede excitation (input), the field functions (response, excitation, etc), which we will call $f(t)$, are assumed to satisfy $f(t) = 0$ for $t < 0$.

Inverse L.T.: $f(t) = \mathcal{L}^{-1}[F(s)] = \dfrac{1}{2\pi i} \displaystyle\int_{c - i\infty}^{c + i\infty} F(s) e^{st} \, ds$ where c is a real constant that is greater than the real parts of all the singularities of $F(s)$. Thus, we need to determine all the singularities of $F(s)$ and then evaluate the above line integral in the s-plane. Generally, this definition is not used.

theorems on L.T.

(1) $\mathcal{L}[k f(t)] = k F(s)$, where k is a complex const.

(2) $\mathcal{L}\left[\dfrac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$.

(3) $\mathcal{L}\left[\displaystyle\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\int_0^{t_1} f(\tau)\,d\tau\, dt_1\, dt_2 \cdots dt_{n-1}\right] = \dfrac{F(s)}{s^n}$

(4) Shift in time : $\mathcal{L}[f(t-T)\, H(t-T)] = e^{-Ts} F(s)$, where $H(t-T)$ is shifted step f^n.

(5) Initial value theorem : $\displaystyle\lim_{t\to 0} f(t) = \lim_{s\to\infty} s F(s)$ if the limit exists

(6) Final value theorem : If $sF(s)$ is analytic on the imag. axis (ie, $s=i\omega$) and the right half complx plane (ie, $s = \sigma + i\omega$, $\sigma > 0$) then,
$\displaystyle\lim_{t\to\infty} f(t) = \lim_{s\to 0} s F(s) \to$ valid only if $sF(s)$ has no poles with $\sigma \geq 0$.

(7) Shift of axis : $\mathcal{L}\left[e^{\pm \alpha t} f(t)\right] = F(s \mp \alpha)$

(8) Convolution Theorem : Let $f_1(t) = 0$, $f_2(t) = 0$, for $t < 0$. Then,
$$\mathcal{L}\left[\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau\right] = \mathcal{L}\left[\int_0^\infty f_1(\tau)\, f_2(t-\tau)\, dt\right] = F_1(s)\, F_2(s) .$$
$$\underbrace{\qquad\qquad}_{\mathcal{L}[f_1 * f_2]}$$

Also, we have the dual relation, $\mathcal{L}[f_1(t)\, f_2(t)] = F_1(s) * F_2(s)$.

Some commonly used L.T.'s

(i) Unit impulse function $[\delta(t)]$.
Consider $\delta(t-a)$, $a > 0$.
$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st} \delta(t-a)\, dt = e^{-as}$, $(a>0), (s>0)$
Then for $a = 0$, we get $\mathcal{L}\{\delta(t)\} = 1$. We see that strictly speaking, L.T. should be defined as $\int_{0^-}^\infty e^{-st} f(t)\, dt$ and then we can directly get this result.

(ii) $\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt = \int_a^\infty e^{-s(t-a)} H(t-a)\, e^{-as}\, dt = e^{as}\int_0^\infty e^{-s\tau} H(\tau)\, d\tau = e^{as}\left[-\dfrac{1}{s} e^{-s\tau}\right]_0^\infty$
$= \dfrac{e^{-as}}{s}$, $s > 0$ $\left(\begin{array}{l}\text{can also do by Theorem \#4 above by putting } f(t-a) = H(t-a)\\ \text{and noting that } \mathcal{L}[H(t-a)\, H(t-a)] = \mathcal{L}[H(t-a)] \because (H(t-a))^n = H(t-a)\end{array}\right)$

(iii) $\mathcal{L}\{e^{-\alpha t}\} = \int_0^\infty e^{-\alpha t} e^{-st}\, dt = \left[-\dfrac{e^{-(s+\alpha)t}}{s+\alpha}\right]_0^\infty = \dfrac{1}{s+\alpha}$, $s+\alpha > 0$

coming back to 3-parameter solid, rewriting the differential constitutive law, we get,
$\dot{\sigma} + \dfrac{\sigma}{T_R} = E_\delta\left\{\dot{e} + \dfrac{e}{T_c}\right\}$ where, $T_R^{-1} = k_1/c_2$, $E_\delta = k_1 + k_3$, $T_c^{-1} = \dfrac{k_1 k_3}{c_2(k_1 + k_3)}$ $\left(\begin{array}{l}\text{see reverse for}\\ \text{reconciliation with}\\ \text{4CF notation}\end{array}\right)$

Taking the L.T. for zero I.C.'s (ie, $\sigma(0) = e(0) = 0$), we get,
$\sigma(s) = E(s)\, e(s)$, $E(s) = E_\delta\left(\dfrac{s + T_c^{-1}}{s + T_R^{-1}}\right) =$ transform modulus.

...ing in general (ie, taking L.T. of differential const. laws for a general viscoelastic material) we get (assuming zero IC's),

$$\bar{\sigma}(s) = \bar{E}(s)\,\bar{e}(s) \implies \sigma(t) = \int_0^t E(t-\tau)\,e(\tau)\,d\tau \quad \text{(from convolution integral)}$$

$\hookrightarrow$ shows that $\sigma(t)$ depends on history of e — ie, hereditary effect

Creep function

Consider 3-parameter solid with $\sigma(t) = \sigma_0 H(t)$; $H(t)$=step fⁿ $= 0, \ t<0$
$\qquad\qquad = \sigma_0, \ t>0$

Then, for zero I.C.'s, (ie, $\sigma(0)=e(0)=0$)

$$\bar{e}(s) = \frac{\sigma_0}{E_D}\,\frac{1}{s}\left(\frac{s+T_R^{-1}}{s+T_c^{-1}}\right) = \frac{\sigma_0}{E_D}\left[\frac{1}{s+T_c^{-1}} + \frac{T_R^{-1}}{T_c^{-1}}\left(\frac{1}{s} - \frac{1}{s+T_c^{-1}}\right)\right] =$$

$$\implies e(t) = \frac{\sigma_0}{E_D}\left[\frac{T_R^{-1}}{T_c^{-1}}H(t) + \left(1 - \frac{T_R^{-1}}{T_c^{-1}}\right)e^{-t/T_c}\right] = \frac{\sigma_0}{E_D}\left[\left(\frac{T_R^{-1}}{T_c^{-1}} + 1 - 1\right)H(t) + \left(1 - \frac{T_R^{-1}}{T_c^{-1}}\right)e^{-t/T_c}H(t)\right]$$

$\uparrow$

$$c(t) = e(t) = \frac{\sigma_0}{E_D}\left[1 + \left(\frac{T_c}{T_R} - 1\right)\left(1 - e^{-t/T_c}\right)\right]H(t)$$

(creep fⁿ in YCF)

introduced ∵ we assume system to be causal ie, output $e(t)$ cannot precede input $\sigma(t)$.

$(\psi(t) \to$ Creep fⁿ in S.C. Hunter, defined: $\psi(0)=0)$

valid for $t>0$ / For $t=0$ use IC's $\to$

$$\boxed{e(t) = \frac{\sigma_0}{E_D}\left(1 + \psi(t)\right)H(t)}$$

$\to$ response to unit step $\sigma(t)$. The first term in ( ), ie, 1, represents the elastic component and second term in ( ), ie, $\psi(t)$, represents the visco-el. comp.

NOTE: This is a general form for unit step response for any viscoelastic material. For the case when the $\dot{\sigma}$ term does not appear in the differential form of the constitutive law, we can argue that $e(t) = \frac{\sigma_0}{E_D K}\left(1 + K\psi(t)\right)$ where $K \to \infty$, so that $e(0^+) = 0$. (see details below).

Voight element
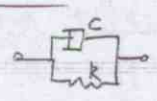
$\sigma_s = k e_s$, $\sigma_D = c\dot{e}_D$, $e = e_s = e_D$, $\sigma = \sigma_s + \sigma_D$

$\sigma = ke + c\dot{e} = c\left(\dot{e} + \frac{k}{c}e\right)$

not $\dot{\sigma}$ ∴ $\psi(t)=0$ elastic solid; but if $\psi(\infty) \to 0$ visco solid; $\frac{d(t)}{dt}$, $16/11/97$ if $\psi(\infty) \to 0$ visco liquid

For $\sigma(t) = \sigma_0 H(t)$ & zero IC's we get, $\bar{e}(s) = \frac{\sigma_0}{c}\,\frac{1}{s}\left(\frac{k}{c}+s\right)^{-1} = \frac{\sigma_0}{k}\left(\frac{1}{s} - \frac{1}{s+k/c}\right)$

$$e(t) = c(t) = \frac{\sigma_0}{k}\underbrace{\left(1 - e^{-k/ct}\right)}_{\psi(t)/K}H(t) = \frac{\sigma_0}{kK}\left(1 + \psi(t)\right) \text{ where } K \text{ is large (ie, } K\to\infty).$$

(cf YCF)

So we see that here the elastic comp. vanishes ∵ K is large.
Also, $e(0^+) = 0$

Maxwell element

$\sigma = \sigma_s = \sigma_D$; $e = e_s + e_D$; $\sigma_s = k e_s$, $\sigma_D = c\dot{e}_D$,

∴ $\dot{e} = \dot{e}_s + \dot{e}_D = \frac{\dot{\sigma}}{k} + \frac{\sigma}{c} \implies \dot{\sigma} + \frac{k}{c}\sigma = k\dot{e}$ (due to causality)
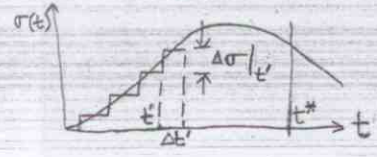
For $\sigma(t) = \sigma_0 H(t)$ & zero IC's, $\bar{e}(s) = \frac{1}{k}\,\frac{1}{s}\,\frac{\sigma_0}{s}\left(s + k/c\right) \implies e(t) = c(t) = \frac{\sigma_0}{k}\left(H(t) + \frac{k}{c}t\right) = \frac{\sigma_0}{k}\left(1 + \frac{k}{c}t\right)H(t)$

(cf YCF) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{}_{\psi(t)}$

Thus, $e(0^+) = \sigma_0/k$

Returning to 3-p model, $e(0^+) = \sigma_0/E_D = \sigma_0/(k_1 + k_3)$

General response for Viscoelastic material — Boltzmann's constitutive Law

We sum the responses ($\Delta e(t^*)$) due to step inputs $\Delta\sigma$ applied at t', where $t' = 0 \to t^*$

∴ $\Delta e(t^*) = \frac{\Delta\sigma|_{t'}}{E_D}\left(1 + \psi[t^* - t']\right)H(t^* - t')$

$$\text{or} \quad \Delta\sigma\Big|_{t'} = \frac{d\sigma(t)}{dt}\,dt\Big|_{t=t'} = \frac{d\sigma(t')\,dt'}{dt'}$$

$$\therefore \quad e(t^*) = \lim_{\Delta\sigma\to 0}\frac{1}{E_D}\sum \Delta\sigma\Big|_{t'}\big(1+\psi[t^*-t']\big)H(t^*-t') = \frac{1}{E_D}\int_0^{t^*}\frac{d\sigma(t')}{dt'}\big(1+\psi[t^*-t']\big)dt', \quad \underline{t^*>0}$$

Dropping the * notation and considering zero IC's ($\sigma(0)=0$), we get,

$$\boxed{e(t)=\frac{1}{E_D}\left[\sigma(t) + \int_0^t \frac{d\sigma(t')}{dt'}\psi[t-t']\,dt'\right]} \longrightarrow \text{Boltzmann's constitutive Law}$$

$\underbrace{\qquad\qquad}_{\substack{\text{instantaneous,}\\ \text{elastic part}}}$ $\underbrace{\qquad\qquad\qquad}_{\text{hereditary part}}$ $\qquad\qquad t>0$

Taking the L.T. of the above we get, for zero IC's, $\quad e(s) = \sigma(s)\dfrac{\big(1+s\psi(s)\big)}{E_D}$

Thus we get $\boxed{E(s) = E_D\big(1+s\psi(s)\big)^{-1}} \longrightarrow$ relationship between transform modulus and creep function ψ.

<u>NOTE:</u> In YCF he gets $e(t) = \int_0^t c(t-t')\dfrac{d\sigma(t')}{dt'}\,dt'$. This is equivalent to the above since in YCF, $c(t-t') = \dfrac{(1+\psi(t-t'))}{E_D}$.

In both YCF & Hunter, $e(0)=0$. Note that IC's for $\overset{\text{various visco models, ie,}}{\text{Maxwell, Voight, \&}}$ 3-p solid, given in YCF $_{\text{to the respective models}}$ are based on physical considerations. They are not unique, since any IC's can be considered for any visco model. Thus we should not use the step response (ie creep fn) to evaluate $e(0)$ since the step response is valid (strictly speaking) only for $t>0$. For $t=0$ we should just use the given IC's. That is to say, we should separate out the homogeneous sol. & particular sol. Thus,

$e(t) = e_H(t) + e_p(t)$, $t>0$, where

$e_H(t)$ is due to IC's ($\sigma(0)$) alone for which $\mathcal{L}\{\dot\sigma(t)\} = s\sigma(s) - \overset{=0}{\sigma(0)}$, and

$e_p(t)$ is due to forcing fn ($\sigma(t) = \sigma_0 H(t)$) alone for which $\mathcal{L}\{\dot\sigma(t)\} = s\underset{\sigma_0/s}{\sigma(s)} - \underset{0}{\sigma(0)}$.

$\quad$ Also, $e(0) = e_p(0)$.

<u>Relaxation function</u>

Input is $e(t) = e_0 H(t)$. Output is $\sigma(t)$.

In general we can write, $k(t) = \sigma(t) = E_D e_0\big(1-\phi(t)\big)$ $\overset{\text{(YCF notation for relaxation fn)}}{}$ for any visco model. This is analogous to the general expression for creep fn $c(t)$.

$\qquad\qquad\qquad\qquad$ (ref. fn of Hunter)

Maxwell: $k(t) = \sigma(t) = e_0 R\, e^{-R/c\, t}\, H(t)$

Voight: $k(t) = \sigma(t) = e_0\big(c\,\delta(t) + \mu\big)H(t)$

3-p solid: $k(t) = \sigma(t) = E_D e_0\left[1 - (1-T_R/T_c)(1-e^{-t/T_R})\right]$