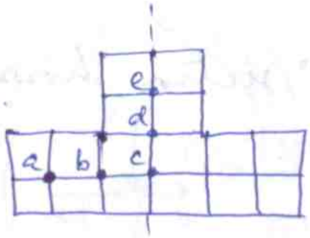
P.1

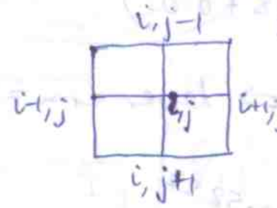


line of symmetry

$h = a/2$

$$\nabla^2\phi = \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}$$

$$M = 2\int_A \phi\, dx\, dy.$$



$$\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}} \phi\, dx\, dy = \Big[16\phi_{i,j} + 4(\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i,j-1} + \phi_{i-1,j}) + \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\Big]$$

Using above formulae & symmetry,

$$\phi_b - 4\phi_a = -2G\alpha h^2 \triangleq -2K \longrightarrow (1)$$

$$\phi_c + \phi_a - 4\phi_b = -2K \longrightarrow (2)$$

$$2\phi_b - 4\phi_c + \phi_d = -2K \longrightarrow (3)$$

$$\phi_c + \phi_e - 4\phi_d = -2K \longrightarrow (4)$$

$$\phi_d - 4\phi_e = -2K \longrightarrow (5)$$

$$M = 2\frac{h^2}{9}\left[32\phi_a + 8(\phi_b) + 16\phi_c + 4(2\phi_b + \phi_d) + 16\phi_e + 4(\phi_d)\right]$$

$$= 2\frac{h^2}{9}\left[32\phi_a + 16\phi_b + 16\phi_c + 8\phi_d + 16\phi_e\right] \longrightarrow (6)$$

$(1) \rightarrow \phi_b = 4\phi_a - 2K \longrightarrow (a)$

$(1),(2) \rightarrow \phi_c = -\phi_a - 2K + 16\phi_a - 8K = 15\phi_a - 10K \longrightarrow (b)$

$(3),(a) \rightarrow \phi_d = -8\phi_a + 4K + 60\phi_a - 40K - 2K = 52\phi_a - 38K \longrightarrow (c)$

$(5),(c) \rightarrow \phi_e = \dfrac{\phi_d + 2K}{4} = \dfrac{52\phi_a - 38K + 2K}{4} = 13\phi_a - 9K$

$(4),(a)-(c) \rightarrow 15\phi_a - 10K + 13\phi_a - 9K - 208\phi_a + 152K = -2K$

$\Rightarrow -180\phi_a = -135K \Rightarrow \phi_a = \underbrace{\dfrac{135}{180}}_{P} K = PK \longrightarrow (7)$

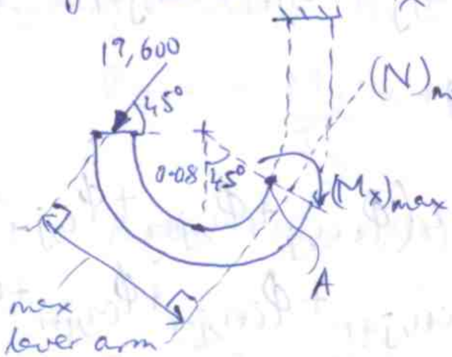
$(6),(7) \rightarrow M = 2\dfrac{h^2}{9}\big(32P + 16(4P-2) + 16(15P-10) + 8(52P-38) + 16(13P-9)\big)K$

$$= \frac{2h^2}{9}(960P - 640)K \longrightarrow (8)$$

Put $h^2 = \dfrac{a^2}{16}$, $P = 135/180$, $K = G\alpha h^2$, $(8) \Rightarrow$ $\boxed{\alpha = \dfrac{9}{10}\dfrac{M}{Ga^4}}$

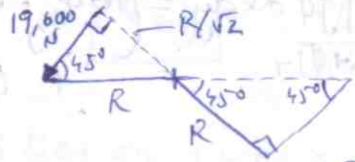
P.3.

Max tensile stress occurs at the 45° section shown, for which M_x and N are max.



$(N)_{max} = 19600$ N.

$R = 0.11\text{m}$ $(= 0.08 + 0.03)$



$\Rightarrow$ lever arm $= R\left(1 + \frac{1}{\sqrt{2}}\right)$

$$(M_x)_{max} = 19600\, R\left(1+\frac{1}{\sqrt{2}}\right)^{=3680.5222}, \quad (N)_{max} = 19600$$

$$A_m = 2\pi\left(R - \sqrt{R^2 - b^2}\right) = 2\pi\left(0.11 - \sqrt{0.11^2 - 0.03^2}\right) = \boxed{0.0262005}$$

$$A = \pi (0.03)^2 = 0.282743 \times 10^{-2}$$

$r = 0.08$ for max tensile stress (ie pt. A)

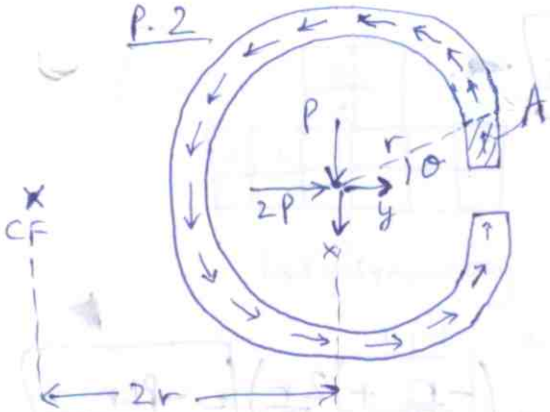
$(\sigma_z)_A$ = max tensile stress in curved part

$$= \frac{\overset{N_{max}}{19600}}{A} + \frac{(M_x)_{max}\,(A - r A_m)}{A r\,(R A_m - A)}$$

$$= \frac{19600}{\pi(0.3)^2} + \frac{3680.52 \times (0.00282743 - 0.08 \times 0.0262005)}{0.00282743 \times 0.08 \times (0.11 \times 0.0262005 - 0.00282743)}$$

$$= 224785568 \text{ N/m}^2 = \boxed{224785.6 \text{ kN/m}^2 \text{ (kPa)}}$$

P. 2



$I_x = I_y = \frac{2\pi r t r^2}{2} = \pi r^3 t$

Need to find shear center.

Method-1 (direct usage of formula) → more straight-forward.

$x_{CF} = 0$ (Symmetry)

$x = -r\sin\theta, \quad y = r\cos\theta$

$$y_{CF} = -\frac{1}{I_y}\int_C Q_y (x\,dy - y\,dx) = -\frac{1}{I_y}\int_0^{2\pi} Q_y\left([-r\sin\theta][-r\sin\theta] + [-r\cos\theta]*[-r\cos\theta]\right)d\theta$$

$$Q_y = \iint_A x\,ds\,dn = \int_\theta^0 (-r\sin\theta)(r\,d\theta)t = r^2 t(1-\cos\theta)$$

$$y_{CF} = -\frac{1}{I_y}\int_0^{2\pi} r^2 t(1-\cos\theta)r^2\,d\theta = -\frac{r^4 t\cdot 2\pi}{\pi r^3 t} = -2r$$

Method 2. (Physical meaning — ie, sum moment about 0 due to applied load & due to resulting shear stresses, & equate to zero)

Since CF independent of load, for convenience assume only $W_x = P$ acts, i.e $W_y = 0$.
V_x V_y

$$\tau_{sz} = \frac{V_x Q_y}{I_y t} = \frac{P Q_y}{I_y t}$$

$$CF: +Pe + \int_0^{2\pi} \tau_{sz} r^2\,d\theta\,t = 0 \Rightarrow \cancel{P}e = -\frac{\cancel{P}}{I_y t} r^4 t^2 \int_0^{2\pi}(1-\cos\theta)\,d\theta$$

$$\Rightarrow e = -\frac{r^4 t^2}{\pi r^3 t^2} 2\pi = -2r$$

So $M = M$ about $CF = 2Pr$

$$\boxed{\alpha} = \frac{3M}{G\sum a_i b_i^3} \text{ (rolled thin walled section)} = \frac{6Pr}{G 2\pi r t^3} \boxed{= \frac{3P}{\pi G t^3}}$$

Neutral Plane: $\frac{y}{x} = \tan\beta = \frac{M_y I_x}{M_x I_y}$ or $\frac{y}{x} = \tan\beta = -\frac{K_x}{K_y} = -\frac{W_x}{W_y}$

for $I_x = I_y$, $I_{xy} = 0$.

Now $M_y = PL$, $M_x = -2PL$

So either way we get, $\boxed{\dfrac{y}{x} = \tan\beta = -\dfrac{1}{2}}$

Bending stresses: For $I_{xy} = 0$, $I_x = I_y$

$$\sigma_z = -\frac{M_y}{I_y}x + \frac{M_x}{I_x}y$$

$$A \Rightarrow (-r/\sqrt{2},\ r/\sqrt{2}) \Rightarrow \boxed{(\sigma_z)_A} = -\frac{PL}{\pi r^3 t}\left(-\frac{r}{\sqrt{2}} + \frac{2r}{\sqrt{2}}\right) = \boxed{\frac{-PLr}{\sqrt{2}\pi r^3 t}}$$

$$B \Rightarrow \left(-\frac{2r}{\sqrt{5}}, \frac{r}{\sqrt{5}}\right) \Rightarrow B \text{ lies on } y = -\tfrac{1}{2}x \Rightarrow \text{neutral plane}$$

$$\Rightarrow \boxed{(\sigma_z)_B = 0}$$

p.4 Constitutive law

Direct way: $\sigma_1 = k_1 e_1$, $\sigma_2 = k_2 e_2$, $\sigma_3 = c_3 \dot{e}_3$, $\sigma_4 = c_4 \dot{e}_4$,

$$\sigma_2 = \sigma_3, \quad e_1 = e_4 = e_2 + e_3 = e$$

$$\sigma = \sigma_1 + \sigma_2 + \sigma_4$$

$$\Rightarrow \sigma = k_1 e_1 + k_2 e_2 + c_4 \dot{e}_4 = k_1 e + c_4 \dot{e} + k_2(e - e_3)$$

$$\Rightarrow \dot{\sigma} = k_1 \dot{e} + c_4 \ddot{e} + k_2(\dot{e} - \dot{e}_3)$$

$$\dot{e}_3 = \sigma_3/c_3, \quad \sigma_3 = \sigma_2 = \sigma - \sigma_1 - \sigma_4, \quad \sigma_1 = k_1 e, \quad \sigma_4 = c_4\dot{e}$$

$$\Rightarrow \dot{\sigma} = k_1 \dot{e} + c_4 \ddot{e} + k_2\left(\dot{e} - \frac{\sigma - k_1 e - c_4 \dot{e}}{c_3}\right)$$

$$\boxed{\begin{aligned} &\dot{\sigma} + \underbrace{\frac{k_2}{c_3}}_{A}\sigma = \dot{e}\underbrace{\left(k_2 + k_1 + k_2\frac{c_4}{c_3}\right)}_{C} + \underbrace{c_4}_{D}\ddot{e} + \underbrace{\frac{k_1 k_2}{c_3}}_{B} e \\ &\dot{\sigma} + A\sigma = Be + C\dot{e} + D\ddot{e} \end{aligned}}$$

Relaxation response: $\bar{e} = e_0/s$

Take $\mathcal{L}$(const law)

$$\Rightarrow \bar{\sigma}(s + A) = \frac{e_0}{s}(B + Cs + Ds^2)$$

$$\Rightarrow \sigma = \mathcal{L}^{-1}\left\{\frac{B}{s(s+A)} + \frac{C}{s+A} + \frac{Ds}{s+A}\right\} e_0$$

$$\sigma = e_0 \mathcal{L}^{-1}\left\{ \frac{B}{A}\left(\frac{1}{s} - \frac{1}{s+A}\right) + \frac{C}{s+A} + D\left(1 - \frac{A}{s+A}\right)\right\}$$

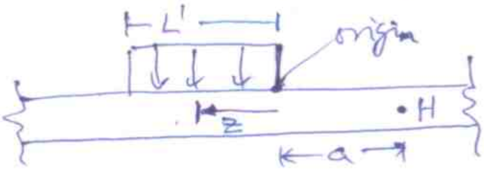
$$\boxed{\sigma(t) = e_0 \left\{ \frac{B}{A} u(t) + e^{-At}\left(C - \frac{B}{A} - AD\right) + D\delta(t)\right\},\ t \geq 0}$$

$$\boxed{\sigma(\infty) = \frac{B}{A} = k_1 e_0}$$

Physically: at ∞ dashpots relax completely (ie they carry zero stress) so total stress carried by uppermost spring (since middle spring stress = middle dashpot stress = 0). Thus $\boxed{\sigma(\infty) = k_1 e(\infty) = k_1 e_0}$

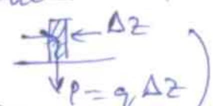
So they match (!!)

P.5 Use first principles and integrals from class notes.



$$\Delta w_H = \frac{q \Delta z \beta}{2k} A_{\beta(z+a)}$$

(using formula for point load from thin strip of distributed load



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superposition $\Rightarrow w_H = \frac{q\beta}{2k}\int_0^{L'} A_{\beta(z+a)}\,dz = \frac{q\beta}{2k}\int_a^{L'+a} A_{\beta\xi}\,d\xi$

using formulae for integrals of $A_{\beta z}, B_{\beta z}, C_{\beta z}, D_{\beta z}$ $\Rightarrow = \frac{q\beta}{2k}\left(-\frac{1}{\beta}\right)\left(-D_{\beta a} + D_{\beta(L'+a)}\right)$

$$\Rightarrow \boxed{w_H = \frac{q}{2k}\left(D_{\beta a} - D_{\beta(L'+a)}\right)}$$

Similarly for θ, M, V:

$$\theta = -\frac{q\beta^2}{k}\int_a^{L'+a} B_{\beta\xi}\,d\xi = -\frac{q\beta^2}{k}\left(-\frac{1}{2\beta}\right)\left(A_{\beta(L'+a)} - A_{\beta a}\right)$$

$$\boxed{\theta = \frac{q\beta}{2k}\left(A_{\beta(L'+a)} - A_{\beta a}\right)}$$

$$\boxed{M} = \frac{q}{4\beta}\int_a^{L'+a} C_{\beta\xi}\,d\xi = \frac{q}{4\beta}\frac{1}{\beta}\left(B_{\beta(L'+a)} - B_{\beta a}\right) \boxed{= \frac{q}{4\beta^2}\left(B_{\beta(L'+a)} - B_{\beta a}\right)}$$

$$\boxed{V} = -\frac{q}{2}\int_a^{L'+a} D_{\beta\xi}\,d\xi = -\frac{q}{2}\left(-\frac{1}{2\beta}\right)\left(C_{\beta(L'+a)} - C_{\beta a}\right) \boxed{= \frac{q}{4\beta}\left(C_{\beta(L'+a)} - C_{\beta a}\right)}$$