CE623 Advanced Mechanics of Solids **Endsem** 21/11/06

- Marks: **Q1=15, Q2=20, Q3=20, Q4=20 Q5=25.**
- Show all working.
- Attempt all parts of a question in a contiguous manner, i.e., dont scatter parts of the same question all over the answerbook.
- **Only one attempt per question will be graded**. So cancel out any attempt you do not want graded. The first not-cancelled attempt will be graded by default.
- **Open book, open notes exam**

1. The uniform cross section rod is subjected to uniform axial stress N at its ends, as shown (Fig. 1). The origin is restrained from rigid body motion. Determine the displacements at point (x,y,z).

2. The infinite wedge is loaded with a uniform load P Nm^{-2}. Obtain the stresses $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta}$ as a function of r, θ, P, α. (Fig. 2)

3. The cross-section of a prismatical member is shown in Fig. 3. Using the finite difference method with step size a, find the torsional rigidity (M/α) in terms of the shear modulus G, and a.

4. The link in Fig. 4 has a circular cross section and is made of steel having a yield strength of 250 MPa. Determine the magnitude of P that will initiate yielding. For a circular section (refer Fig. 4a) $A_m = 2\pi(R - \sqrt{R^2 - b^2}$.

5. A thin-walled cantilevered beam having cross-section as shown (Fig. 5) is loaded at its free end as shown. Determine (a) The shear center (b) The total maximum shear stress at section G.

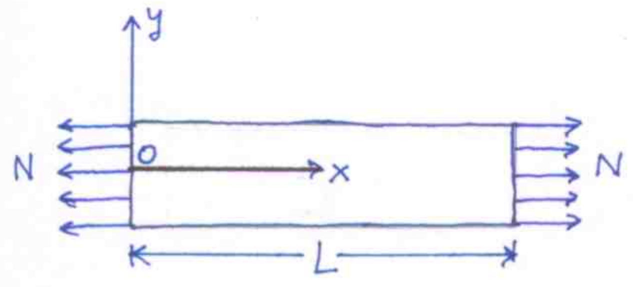


Fig. 1

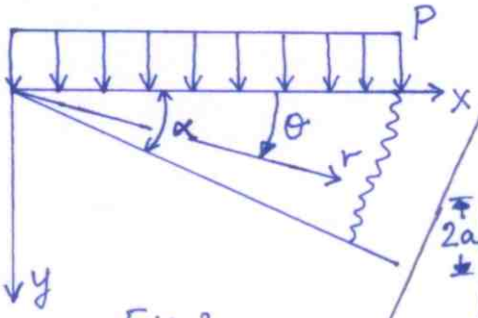


Fig. 2

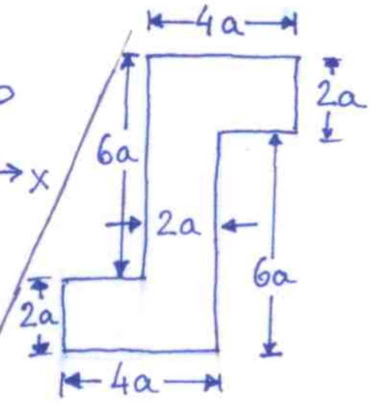


Fig. 3

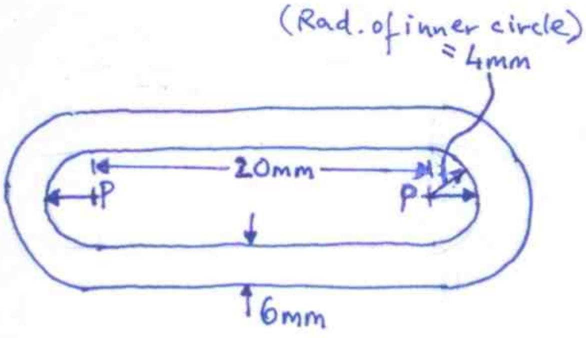


Fig. 4

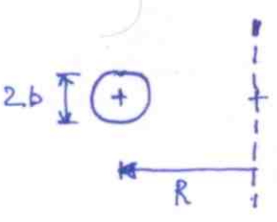


Fig. 4a

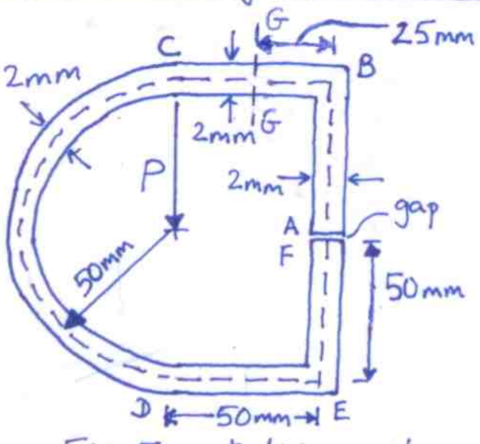


Fig. 5 ℄ distances shown

P.1 BC's : $\sigma_{xx} = N$, $\sigma_{xy} = \sigma_{xz} = 0$ for $x = 0, L$

$\sigma_{ij} n_j = 0$ on lateral face

Assume $\sigma_{xx} = N$, other stresses zero. This satisfies BC's, Equil, & (strain) (or stress) compatibility eqns.

Constitutive : $e_{xx} = \frac{1}{E} N$, $e_{yy} = e_{zz} = -\frac{\nu}{E} N$

$e_{xy} = e_{xz} = e_{yz} = 0$

Integrate normal strain displ eqns :

$$u = \frac{N}{E} x + f(y,z), \quad v = -\frac{\nu}{E} N y + g(x,z)$$

$$w = -\frac{\nu}{E} N z + h(x,y)$$

Subst in shear strain – displ relations :

$\frac{\partial f}{\partial y}(y,z) + \frac{\partial g}{\partial x}(x,z) = 0 \Rightarrow$ f is linear in y, g is linear in x

$\frac{\partial f}{\partial z}(y,z) + \frac{\partial h}{\partial x}(x,y) = 0 \Rightarrow$ f is linear in z, h is linear in x

$\frac{\partial g}{\partial z}(x,z) + \frac{\partial h}{\partial y}(x,y) = 0 \Rightarrow$ g is linear in z, h is linear in y.

$\Rightarrow f = a_1 y + a_2 z + a_3 yz + a_4$

$g = a_5 x + a_6 z + a_7 xz + a_8$

$h = a_9 x + a_{10} y + a_{11} xy + a_{12}$

$e_{xy} = 0$: $a_1 + a_3 z + a_5 + a_7 z = 0$ $\Rightarrow a_1 = -a_5$, $a_3 = -a_7$

$\Rightarrow e_{yz} = a_6 + a_7 x + a_{10} + a_{11} x = 0$ $\Rightarrow a_6 = -a_{10}$, $a_7 = -a_{11}$

$e_{xz} = a_2 + a_3 y + a_9 + a_{11} y = 0 \Rightarrow a_2 = -a_9$, $a_3 = -a_{11}$

$a_3 = -a_7$, $a_7 = -a_{11}$, $a_3 = -a_{11} \Rightarrow a_3 = a_7 = a_{11} = 0$

$\Rightarrow f = a_1 y + a_2 z + a_4$, $g = -a_1 x + a_6 z + a_8$, $h = -a_2 x + a_{12} - a_6 y$

origin: u, v, w at $(0,0,0)$ is zero $\Rightarrow$ $a_4 = a_8 = a_{12} = 0$

$u_{,y} - v_{,x} = 0$, $v_{,z} - w_{,y} = 0$, $u_{,z} - w_{,x} = 0$ at $(0,0,0)$

$\Downarrow$ $a_1 + a_1 = 0$ $\quad$ $\Downarrow$ $a_2 + a_2 = 0$ $\quad$ $\Downarrow$ $a_6 + a_6 = 0$

$$\Rightarrow \boxed{u = \frac{N}{E} x, \quad v = -\frac{\nu}{E} N y, \quad w = -\frac{\nu}{E} N z.}$$

P.2. $P \equiv Nm^{-2}$

Let $Q = P\, g(r) f(\theta)$

$Nm^{-2} \equiv \sigma_{\theta\theta} = \phi_{,rr} = P g'' f \Rightarrow g'' = \text{const} \Rightarrow g \propto r^2$

So take $\phi = r^2 f(\theta)$ where P absorbed back in $f(\theta)$.

$$\nabla^4 \phi = 0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(2f + 2f + f'')$$

$$= \frac{1}{r^2}\left(4f'' + f^{IV}\right) = 0$$

$f = e^{s\theta} \Rightarrow 4s^2 + s^4 = 0 \Rightarrow s = 0, 0, 2i, -2i$

$f = A\cos 2\theta + B\sin 2\theta + C\theta + D$ — ①

BC's $\sigma_{r\theta} = 0$ at $\theta = 0, \alpha$

$\sigma_{\theta\theta} = -P$ at $\theta = 0$

$\sigma_{\theta\theta} = 0$ at $\theta = \alpha$

$\sigma_{\theta\theta} = \phi_{,rr} = 2f(\theta) \rightarrow$ ⓐ

$\Rightarrow 2A + 2D = -P \rightarrow$ ②

$2(A\cos 2\alpha + B\sin 2\alpha + C\alpha + D) = 0 \rightarrow$ ③

$\sigma_{r\theta} = -\left(\frac{1}{r}\phi_{,\theta}\right)_{,r} = -\left(\frac{r^2}{r}\left\{-2A\sin 2\theta + 2B\cos 2\theta + C\right\}\right)_{,r}$

$= 2A\sin 2\theta - 2B\cos 2\theta - C \rightarrow$ ⓑ

$\Rightarrow -2B - C = 0 \rightarrow$ ④

$2A\sin 2\alpha - C = 0 \rightarrow$ ⑤

Soln of ② – ⑤ :

$2A = P / \{\cos 2\alpha + \sin^2 2\alpha + \alpha \sin 2\alpha - 2\} \Delta$

$C = P\sin 2\alpha / \Delta$

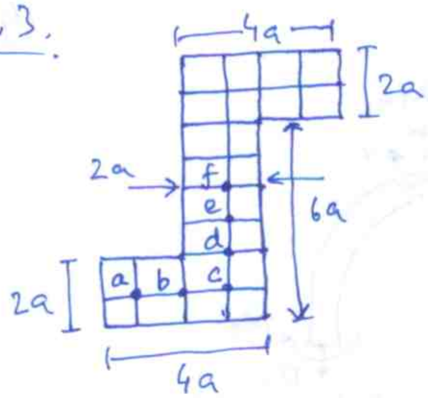
$2B = -P\sin 2\alpha / \Delta$

$2D = -2A$

Assemble soln for $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ via ①, ⓐ, ⓑ

$\sigma_{rr} = \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} = 2f(\theta) + f''(\theta)$

P.3.



$h = a$

$$\nabla^2\phi = \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}$$

$$\frac{M}{2} = \frac{h^2}{9}\Big\{16\,\phi_{i,j} + 4\left(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}\right) + \left[\phi_{i+1,j+1} + \phi_{i-1,j-1} + \phi_{i+1,j-1} + \phi_{i-1,j+1}\right]\Big\}$$

Using above formulae and symmetry: (let $a = \phi_a$, $b = \phi_b$, etc).

$\phi_b - 4\phi_a = -2G\alpha h^2 = -2K \Rightarrow b = 4a - 2K$

$\phi_a + \phi_c - 4\phi_b = -2K \quad \Rightarrow c = -a + 4b - 2K = 15a - 10K$

$\phi_b + \phi_d - 4\phi_c = -2K \quad \Rightarrow d = 4c - b - 2K = 60a - 40K - 4a + 2K - 2K = 56a - 40K.$

$\phi_e + \phi_c - 4\phi_d = -2K \Rightarrow e = 4d - c - 2K = 224a - 160K - 15a + 10K - 2K = 209a - 152K.$

$\phi_f + \phi_d - 4\phi_e = -2K \Rightarrow f = 4e - d - 2K = 836a - 608K - 56a + 40K - 2K = 780a - 570K$

$2\phi_e - 4\phi_f = -2K \Rightarrow 2e - 4f = -2K \Rightarrow 418a - 304K - 3120a + 2280K = -2K$

$\Rightarrow a = \dfrac{-1978}{-2702}K = pK$

$$\frac{M}{2} = 2 * \frac{h^2}{9}\left[16a + 4b + 16c + 4d + 16e + 4f + 4b + 4d\right]$$

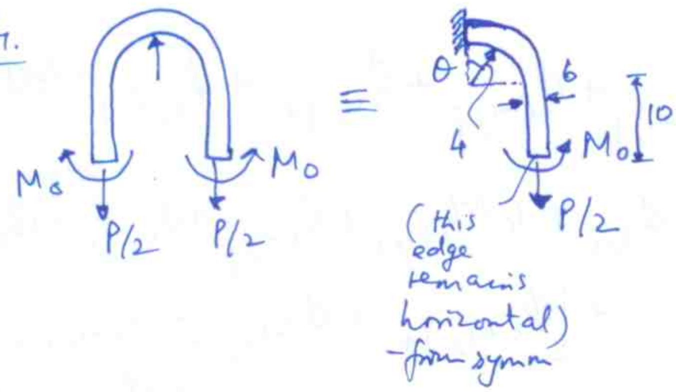
$$= 2 * \frac{h^2}{9}\left[16a + 32a + 240a + 448a + 3344a + 3120a - 16K - 160K - 320K - 2432K - 2280K\right]$$

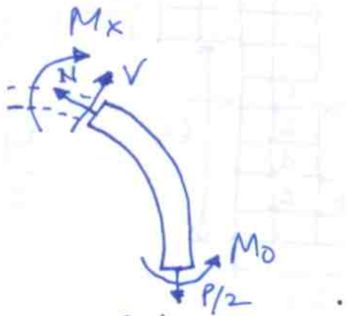
$$= \frac{2h^2}{9}\left[7200a - 5208K\right] = \frac{2h^2}{9}(7200p - 5208)K =$$

$$= 13.9472\,h^2 G\alpha h^2$$

$$\Rightarrow \boxed{\frac{M}{\alpha} = 13.9472\,Ga^4}$$

P.4.





$$V = \frac{P}{2}\cos\theta, \quad N = \frac{P}{2}\sin\theta, \quad M_x = M_0 - \frac{P}{2}R(1-\sin\theta)$$

$$U = \int_0^{\pi/2}\left(\frac{kV^2R}{2AG} + \frac{N^2R}{2AE} + \frac{A_m M_x^2}{2A(RA_m - A)E} - \frac{M_x N}{EA}\right)d\theta + \int_0^{10}\left(\frac{M_x^2}{2EI} + \frac{N^2}{2EA}\right)dx$$

$$\frac{\partial U}{\partial M_0} = \int_0^{\pi/2}\left(\frac{A_m M_x(1)}{A(RA_m - A)E} - \frac{N(1)}{EA}\right)d\theta + \int_0^L \frac{M_x(1)}{EI}dx$$

$$= \frac{A_m\left(M_0 - \frac{PR}{2}\right)}{A(RA_m - A)E}\frac{\pi}{2} + \left(\frac{A_m}{A(RA_m - A)E}\frac{PR}{2} - \frac{P}{2EA}\right)\underbrace{\left|-\cos\theta\right|_0^{\pi/2}}_{=1} + \frac{LM_0}{EI}$$

$$\frac{\partial U}{\partial M_0} = 0 \Rightarrow \frac{\frac{P}{2A\cancel{E}}\left[-\frac{A_m R}{RA_m - A}\left(1 - \frac{\pi}{2}\right) + 1\right]}{\frac{L}{\cancel{E}I} + \frac{A_m}{\cancel{E}A(RA_m - A)}\frac{\pi}{2}} = M_0$$

$$A = \pi\frac{6^2}{4} = 9\pi\ \text{mm}^2, \quad R = 7\ \text{mm}, \quad A_m = 2\pi\left(R - \sqrt{R^2 - b^2}\right) = 1.3509\pi$$

$$\Rightarrow \boxed{M_0 = 0.7053P}$$

$$\sigma_{\theta\theta} = \frac{P}{2}\frac{\sin\theta}{A} + P\left(0.7053 - \frac{R}{2}(1 - \sin\theta)\right)\frac{A - rA_m}{Ar(RA_m - A)}$$

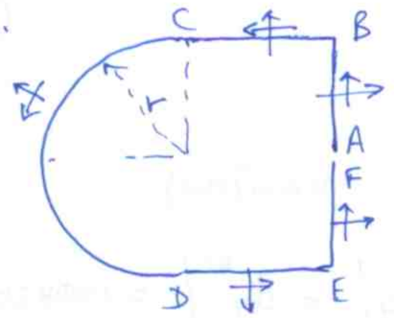
$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0 \Rightarrow \frac{P}{2A}\left(\cos\theta + \frac{R}{r}\frac{A - rA_m}{(RA_m - A)}\cos\theta\right) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ.$$

So check stresses at root ($\theta = 0^\circ$) & tip ($\theta = 90^\circ$) for maxima

$$\sigma_{\theta\theta}\Big|_{90^\circ = \theta,\ r = 4\text{mm}} = 0.06683554P\ (T)$$

$$\sigma_{\theta\theta}\Big|_{\theta = 0^\circ,\ r = 4} = -0.1947598P\ (C) = 250 \Rightarrow \boxed{P_{max} = \frac{250}{0.1947} = 1283.6N}$$

$$\sigma_{\theta\theta}\Big|_{\theta = 0^\circ,\ r = 10} = 0.09767P\ (T)$$

5. 

$r = 50\,mm, \quad t = 2\,mm$  $CG = \frac{2r}{\pi}$

$x_{CF} = 0, \quad \bar{x} = 0$

$$y_{CF} = \frac{-\left(\frac{2*50}{\pi}\right)(\pi*50*2) + (25)(50*2*2) + (50)*(50*2*2)}{\pi*50*2 + 50*2*2*2}$$

$$= 7.001239 \simeq 7\,mm$$

$$I_y = \frac{1}{12}\left[100^3*2 + 100*2^3\right] + (2)(50*2)(50^2) + (\pi*50*2)\left(\frac{50^2}{2}\right)$$

$$= 1059432.415$$

$$Q_y^{AB} = \int_x^0 \int_{42}^{44} -x\,dx\,dy = \frac{x^2}{2}(2) = x^2 = f(x)$$

$$Q_y^{BC} = Q_y^{AB}\Big|_{x=-50} + \int_{-49}^{-51}\int_y^{43} x\,dx\,dy = \left(\frac{51^2-49^2}{2}\right)(43-y) + 50^2$$

$$= 100(43-y) + 2500 = g(y)$$

$$Q_y^{CD} = Q_y^{BC}\Big|_{y=-7} + \int_\theta^0 \underset{(-r\cos\theta)}{\cancel{x}}\,\cancel{dA} \rightarrow r\,d\theta\,t = 100*50 + 2500 + (50^2)(2)\sin\theta$$

$$= 7500 + 5000\sin\theta = h(\theta)$$

$Q_y^{FE} = Q_y^{AB} = x^2$ (by inspection).

$$Q_y^{DE} = Q_y^{FE}\Big|_{x=50} + \int_{49}^{51}\int_y^{43} x\,dx\,dy = Q_y^{BC} = g(y)$$

$$-I_y y_{CF} = \int_0^{-50} -f(x)\,\overset{43}{\cancel{y}}\,dx + \int_{43}^{-7} g(y)\,\overset{-50}{\cancel{x}}\,dy + \int_0^{\pi}(x\,dy - y\,dx)h(\theta)$$

$$+ \int_{-7}^{43} g(y)\,\underset{50}{\cancel{x}}\,dy + \int_{50}^{0} -f(x)\,\underset{43}{\cancel{y}}\,dx$$

For circular part, $x = -r\cos\theta, \; y = -7 - r\sin\theta$

$dx = r\sin\theta\,d\theta, \; dy = -r\cos\theta\,d\theta$

$$\Rightarrow (x\,dy - y\,dx) = (-r\cos\theta)(-r\cos\theta)d\theta - (-7 - r\sin\theta)(r\sin\theta\,d\theta)$$

$$= r^2 d\theta + 7r\sin\theta\,d\theta = 2500\,d\theta + 350\sin\theta\,d\theta$$

$$-I_y y_{CF} = 2\left[-\frac{(-50)^3}{3}*(43) + \left\{100\left(43*50 - \frac{43^2}{2} + \frac{7^2}{2}\right) + 50^2*50\right\}*50\right]$$

$$+ (7500)(2500)\pi + (2500)(5000)\left|-\cos\theta\right|_0^\pi + (7500)(350)\left|-\cos\theta\right|_0^\pi + (350)(5000)\overbrace{\int_0^\pi\left(\frac{1-\cos 2\theta}{2}\right)d\theta}^{\pi/2}$$

$-I_y\, y_{CF} = 2*14291666.67 + 91903755.83$

$y_{CF} = -113.728\ mm$

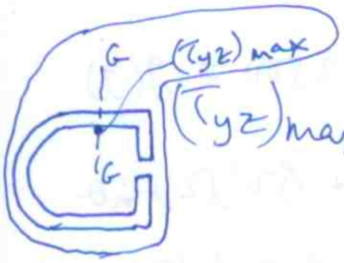
For P applied center of half circle (P in Newtons)

$\Rightarrow (\tau_{yz})_{Bending} = \dfrac{P\, \cancel{I_x}\, Q_y\, \cancelto{-1}{\eta_x}}{\cancelto{I_y}{\Delta}\, t}$, $Q_y\big|_G = Q_y^{BC}\big|_{y=18} = 100*25 + 2500 = 5000$

$= \dfrac{P\,(5000)(-1)}{I_y\,(2)} = P\,(-2.3597*10^{-3})\ N/mm^2$

$M = P\,(113.728 - 7.0012) = 106.7268\,P\ N.mm$

$(\tau_{yz})_{torsion} = \dfrac{3M}{ab^2} = \dfrac{3*106.7268}{(50*4+\pi*50)(2)^2} = 0.224166\ N/mm^2$



$(\tau_{yz})_{max}$ at section G occurs on the inside ($x = -49mm$)

$= 2.3597*10^{-3} + 0.224166 = 0.226526\ (\leftarrow)$

left ward