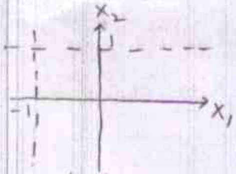
CE 623 HW1 Solutions

P.1



$$\left.\begin{aligned}\sigma_{11,1} + \sigma_{12,2} &= 0\\ \sigma_{12,1} + \sigma_{22,2} &= 0\end{aligned}\right\}\begin{array}{l}\text{when}\\ m_i \,\&\, f_i\\ \text{are zero}\end{array}$$

$\Rightarrow \sigma_{12} = \int \sigma_{11,1}\,dx_2 + f(x_1) = -cx_2 + f(x_1) + K_1$

$\Rightarrow \sigma_{12} = \int -\sigma_{22,2}\,dx_1 + g(x_2) = -bx_1 + g(x_2) + K_2$) equate

$\therefore \sigma_{12} = -bx_1 - cx_2 + K \quad (K_1 = K_2 = K)$ const

On the (edge) face having unit normal in $\underline{e}_1$ dir.: (given $a=b=0$)

$\sigma_{13} = 0, \quad \sigma_{12} = -cx_2 + K, \quad \sigma_{11} = cx_1 + dx_2$

in particular for $x_1 = -1$ face, $\sigma_{11} = -c + dx_2$

On (edge) face having unit normal in $\underline{e}_2$ dir: (for $a=b=0$, given)

$\sigma_{23} = 0, \quad \sigma_{12} = -cx_2 + K, \quad \sigma_{22} = 0$

On top and bottom face, i.e, having $\underline{e}_3$ unit normal:

$\sigma_{13} = \sigma_{23} = 0$

Thus the b.c.'s imply that:

(i) plate unloaded on top & bottom face.

(ii) On edges $x_1 =$ const, plate has normal loading that varies linearly (with x_2) along the edge. This loading may change from tensile to compressive, depending on values of c, d.
On edges $x_1 =$ const, plate has linearly varying (in-plane) shear loading that changes direction at $x_2 = K/c$ (i.e., it is zero at $x_2 = K/c$). If $c=0$ then in-plane shear loading is const. $(=K)$

(iii) On edges $x_2 =$ const there is only a constant in-plane shear loading whose magnitude depends c, K & the location of the edge (i.e, x_2 [$\equiv$ const] value)

P.2 When m_i & f_i are zero, the equil. eqns are,

$\sigma_{11,1} + \sigma_{12,2} = 0 \Rightarrow \exists\, F(x_1, x_2) : \sigma_{11} = F_{,2}$ & $\sigma_{12} = -F_{,1}$ (This is a standard result from Calculus)

$\sigma_{12,1} + \sigma_{22,2} = 0 \Rightarrow \exists\, G(x_1, x_2) : \sigma_{12} = -G_{,2}$ & $\sigma_{22} = G_{,1}$

$\therefore \sigma_{12} = -F_{,1} = -G_{,2} \Rightarrow \exists\, \phi(x_1, x_2) : F = \phi_{,2}$ & $G = \phi_{,1}$

$\therefore \sigma_{11} = F_{,2} = \phi_{,22}\ ; \quad \sigma_{22} = G_{,1} = \phi_{,11}, \quad \sigma_{12} = -F_{,1} = -G_{,2} = -\phi_{,12}$

P-3

(Note: $n_1(2)=0$ given)

$$\left.\begin{aligned} n_i(1)\,n_i(2)=0 &\Rightarrow \tfrac{1}{2}n_2(2)+\sqrt{\tfrac{1}{2}}\,n_3(2)=0 \\ n_i(2)\,n_i(2)=1 &\Rightarrow n_2^2(2)+n_3^2(2)=1 \end{aligned}\right\} \Rightarrow n_3(2)=\pm\sqrt{\tfrac{1}{3}},\ n_2(2)=\mp\sqrt{\tfrac{2}{3}}, \text{ so } n_i(2)\Rightarrow\left(0,-\sqrt{\tfrac{2}{3}},+\sqrt{\tfrac{1}{3}}\right)$$

$$n_i(3)=\epsilon_{ijk}\,n_j(1)\,n_k(2) \Rightarrow n_1(3)=n_2(1)n_3(2)-n_3(1)n_2(2)=\tfrac{1}{2}\left(\sqrt{\tfrac{1}{3}}\right)-\sqrt{\tfrac{1}{2}}\left(-\sqrt{\tfrac{2}{3}}\right)=\sqrt{\tfrac{1}{12}}+\sqrt{\tfrac{1}{3}}$$

$$n_2(3)=n_3(1)n_1(2)-n_1(1)n_3(2)=-\tfrac{1}{2}\sqrt{\tfrac{1}{3}}=-\sqrt{\tfrac{1}{12}}$$

$$n_3(3)=n_1(1)n_2(2)-n_2(1)n_1(2)=\tfrac{1}{2}\left(-\sqrt{\tfrac{2}{3}}\right)=-\sqrt{\tfrac{1}{6}}$$

$$\therefore n_i(3)\Rightarrow\left(\sqrt{\tfrac{1}{12}}+\sqrt{\tfrac{1}{3}},\ -\sqrt{\tfrac{1}{12}},\ -\sqrt{\tfrac{1}{6}}\right)$$

Let x_i' be the principal axes system.

Then $\sigma_{11}'=\sigma(1)=1$, $\sigma_{22}'=\sigma(2)=4$, $\sigma_{33}'=\sigma(3)=-2$, $\sigma_{ij}'=0$ for $i\neq j$

Now $\sigma_{ij}=a_{mi}\,a_{nj}\,\sigma_{mn}'$ where $a_{mn}\Rightarrow\begin{pmatrix} n_1(1) & n_2(1) & n_3(1)\\ n_1(2) & n_2(2) & n_3(2)\\ n_1(3) & n_2(3) & n_3(3)\end{pmatrix}$

$$\left.\begin{aligned}
\therefore \sigma_{11}&=n_1^2(1)\sigma(1)+n_1^2(2)\sigma(2)+n_1^2(3)\sigma(3)=-1.25\\
\sigma_{22}&=n_2^2(1)\sigma(1)+n_2^2(2)\sigma(2)+n_2^2(3)\sigma(3)=2.75\\
\sigma_{33}&=n_3^2(1)\sigma(1)+n_3^2(2)\sigma(2)+n_3^2(3)\sigma(3)=1.5\\
\sigma_{12}&=n_1(1)n_2(1)\sigma(1)+n_1(2)n_2(2)\sigma(2)+n_1(3)n_2(3)\sigma(3)=0.75\\
\sigma_{13}&=n_1(1)n_3(1)\sigma(1)+n_1(2)n_3(2)\sigma(2)+n_1(3)n_3(3)\sigma(3)=1.061\\
\sigma_{23}&=n_2(1)n_3(1)\sigma(1)+n_2(2)n_3(2)\sigma(2)+n_2(3)n_3(3)\sigma(3)=-1.768
\end{aligned}\right\}\Rightarrow \sigma_{ij}\Rightarrow\begin{pmatrix}-1.25 & 0.75 & 1.061\\ 0.75 & 2.75 & -1.768\\ 1.061 & -1.768 & 1.5\end{pmatrix}$$

P-4 (a)(i) $\phi=2x_1+x_2+x_3-6=0$, $n_i=\dfrac{\phi_{,i}}{(\phi_{,j}\phi_{,j})^{1/2}}=\dfrac{2\underline{e}_1+\underline{e}_2+\underline{e}_3}{\sqrt{6}}\Rightarrow\left(\frac{2}{\sqrt6},\frac{1}{\sqrt6},\frac{1}{\sqrt6}\right)$

$$\sigma_i=\sigma_{ji}n_j=\sigma_{ij}n_j\Rightarrow\left(-ax_2\tfrac{1}{\sqrt6},\ ax_1\tfrac{1}{\sqrt6},\ \left[-ax_2\tfrac{2}{\sqrt6}+ax_1\tfrac{1}{\sqrt6}\right]\right)$$

$$\therefore \sigma_i \text{ @ } (1,2,2)\Rightarrow\left(-\frac{2a}{\sqrt6},\ \frac{a}{\sqrt6},\ -\frac{3a}{\sqrt6}\right)$$

$$N=\sigma_{ij}n_in_j=-a$$

$$S=(\sigma_i\sigma_i-N^2)^{1/2}=\frac{2}{\sqrt3}a$$

(ii) $\phi=x_ix_i-9=0$, $n_j=\dfrac{\phi_{,j}}{(\phi_{,k}\phi_{,k})^{1/2}}=\dfrac{\cancel{2}x_j}{\cancel{2}(x_1^2+x_2^2+x_3^2)^{1/2}}$

$$\therefore \sigma_i=\sigma_{ij}n_j\Rightarrow\left(\frac{-ax_2x_3}{(x_1^2+x_2^2+x_3^2)^{1/2}},\ \frac{ax_1x_3}{(x_1^2+x_2^2+x_3^2)^{1/2}},\ 0\right)$$

$$\sigma_i \text{ @ } (1,2,2)\Rightarrow\left(-\frac{4}{3}a,\ \frac{2a}{3},\ 0\right)$$

$$N=\sigma_{ij}n_in_j=0$$

$$S=(\sigma_i\sigma_i-N^2)^{1/2}=\frac{2\sqrt5}{3}a$$

P4(b) $\sigma_{ij} \Rightarrow \begin{pmatrix} 0 & 0 & -2a \\ 0 & 0 & a \\ -2a & a & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\sigma & 0 & -2a \\ 0 & -\sigma & a \\ -2a & a & -\sigma \end{vmatrix} = -\sigma(\sigma^2 - a^2) - 2a(-2a\sigma) = 0$

$\Rightarrow \sigma^3 - 5\sigma a^2 = 0$

$\Rightarrow \sigma(1) = a\sqrt{5}, \ \sigma(2) = 0, \ \sigma(3) = -a\sqrt{5}$ ◀

$\left.\begin{array}{l} -\sigma n_1 - 2a n_3 = 0 \\ -\sigma n_2 + a n_3 = 0 \end{array}\right\}$ so, $n_i \Rightarrow \left(-\frac{2a}{\sigma} n_3, \frac{a}{\sigma} n_3, n_3\right)$ when $\sigma \neq 0$

$n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow n_3 = \dfrac{1}{\left(\frac{4a^2}{\sigma^2} + \frac{a^2}{\sigma^2} + 1\right)^{1/2}}$ when $\sigma \neq 0$

So, $n_i(1) \Rightarrow \left(\mp\frac{2}{\sqrt{10}}, \pm\frac{1}{\sqrt{10}}, \pm\frac{1}{\sqrt{2}}\right)$, $n_i(3) \Rightarrow \left(\pm\frac{2}{\sqrt{10}}, \mp\frac{1}{\sqrt{10}}, \pm\frac{1}{\sqrt{2}}\right)$ ◀

When $\sigma = 0$, $n_3 = 0$, we must use 3rd eqn, i.e, $-2an_1 + an_2 - \sigma n_3 = 0$ or else use $n_i(2) = \epsilon_{ijk} n_j(1) n_k(3)$.

Thus $n_i(2) \Rightarrow (n_1, 2n_1, 0)$

$n_i(2) n_i(2) = 1 \Rightarrow n_1 = \frac{1}{\sqrt{5}} \Rightarrow n_i(2) \Rightarrow \left(\pm\frac{1}{\sqrt{5}}, \pm\frac{2}{\sqrt{5}}, 0\right)$ ◀

$S(1) = \left|\frac{\sigma(2) - \sigma(3)}{2}\right|$, $S(2) = \left|\frac{\sigma(3) - \sigma(1)}{2}\right|$, $S(3) = \left|\frac{\sigma(1) - \sigma(2)}{2}\right|$

So max shearing stress is $S = S(2) = a\sqrt{5}$ ◀

P.5 Principal stresses : $|\sigma_{ij} - \sigma\delta_{ij}| = 0 \Rightarrow \sigma^3 - 6\sigma^2 - 4\sigma + 24 = 0$

$\Rightarrow (\sigma - 6)(\sigma^2 - 4) = 0 \Rightarrow \sigma(1) = 6, \ \sigma(2) = 2, \ \sigma(3) = -2$

(a) Principal Deviator Stresses

$\hat{\sigma}(1) = \sigma(1) - \frac{1}{3}\sigma_{kk} = \sigma(1) - \frac{6}{3} = 4$

$\hat{\sigma}(2) = \sigma(2) - \frac{1}{3}\sigma_{kk} = 0$ ◀

$\hat{\sigma}(3) = \sigma(3) - \frac{1}{3}\sigma_{kk} = -4$

(b) Let x_i' = principal directions & $n_i' \Rightarrow \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is the octahedral plane

In general (for any n_i'), $N = \sigma(1) n_1'^2 + \sigma(2) n_2'^2 + \sigma(3) n_3'^2$

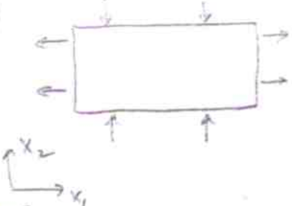
$\therefore N_{oct} = \frac{1}{3}(\sigma(1) + \sigma(2) + \sigma(3)) = \frac{1}{3}\sigma_{ii} = 2$ ◀

$(\sigma_i')_{oct} \Rightarrow (\sigma(1) n_1', \sigma(2) n_2', \sigma(3) n_3') \Rightarrow \frac{1}{\sqrt{3}}(6, 2, -2)$

$\therefore S_{oct} = \left[(\sigma_i'\sigma_i')_{oct} - N_{oct}^2\right]^{1/2} = \left(\frac{44}{3} - 4\right)^{1/2} = \left(\frac{32}{3}\right)^{1/2} = \frac{4}{3}\sqrt{6}$ ◀

(c) $S_{max} = \left|\frac{\sigma(3) - \sigma(1)}{2}\right| = 4$ ◀

P.6



B.C.'s : edges $x_1 = \text{const}$: $\sigma_{11} = \sigma_{11}^*$, $\sigma_{12} = 0$, $\sigma_{13} = 0$

edges $x_2 = \text{const}$: $\sigma_{22} = -\sigma_{22}^*$, $\sigma_{12} = 0$, $\sigma_{23} = 0$

face $x_3 = \text{const}$: $\sigma_{i3} = 0$

} Plane State of Stress

← (see note)

Equil: $\sigma_{ij,j} = 0 \Rightarrow \left.\begin{matrix}\sigma_{11,1} + \sigma_{12,2} = 0 \\ \sigma_{12,1} + \sigma_{22,2} = 0\end{matrix}\right\}$ $\left\{\begin{matrix}\sigma_{11} = \sigma_{11}^*, \ \sigma_{22} = -\sigma_{22}^* \\ \sigma_{12} = 0 \ \text{ is sol}^{n} \text{ that} \\ \text{satisfies equil \& BC's}\end{matrix}\right\}$

Thus $\sigma_{ij} \Rightarrow \begin{pmatrix} \sigma_{11}^* & 0 & 0 \\ 0 & -\sigma_{22}^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$ → principal stresses are $\sigma(1) = \sigma_{11}^*$, $\sigma(2) = -\sigma_{22}^*$, $\sigma(3) = 0$

& p-axes are $n_i(1) \Rightarrow (1,0,0)$, $n_i(2) \Rightarrow (0,1,0)$, $n_i(3) = (0,0,1)$

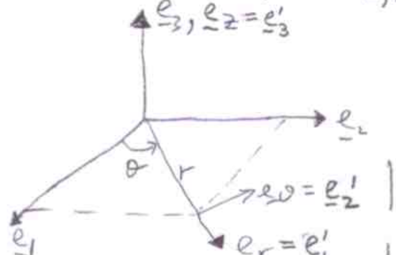
Thus max shearing stress is $\frac{1}{2}|\sigma(1) - \sigma(2)| = \frac{\sigma_{11}^* + \sigma_{22}^*}{2}$ which acts on plane having unit normal $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$.

But this is precisely the plane containing the flaw so the shearing stress in the flaw plane is also the max shearing stress.

$\therefore$ at $S_{max} > \sigma_{cr}$ the plate will fail.

so $\boxed{\sigma_{11}^* + \sigma_{22}^* > 2\sigma_{cr}}$ → is failure condition

P.7 $\underline{i=1}$ $\quad \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 = 0$



$$a_{ij} = \begin{pmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

let, $\sigma_{ij} \equiv \sigma_{ij}[x_1, x_2, x_3] \equiv \sigma_{ij}[r, \theta, z]$

$\sigma'_{ij} \equiv \sigma'_{ij}[r, \theta, z] \equiv \sigma'_{ij}[x_1, x_2, x_3]$

$f_i \equiv f_i[x_1, x_2, x_3] \equiv f_i[r, \theta, z]$

Now $\sigma_{ij} = a_{mi} a_{nj} \sigma'_{mn}$

$$\sigma_{11} = c^2\theta\, \sigma'_{11} - 2c\theta s\theta\, \sigma'_{12} + s^2\theta\, \sigma'_{22}$$

$$\sigma_{12} = c\theta s\theta\, \sigma'_{11} + c^2\theta\, \sigma'_{12} - s^2\theta\, \sigma'_{21} - s\theta c\theta\, \sigma'_{22}$$

$$= c\theta s\theta\,(\sigma'_{11} - \sigma'_{22}) + (c^2\theta - s^2\theta)\,\sigma'_{12}$$

$$\sigma_{13} = c\theta\, \sigma'_{13} - s\theta\, \sigma'_{23}$$

$r^2 = x_1^2 + x_2^2 \Rightarrow 2r \frac{\partial r}{\partial x_1} = 2x_1 \Rightarrow \frac{\partial r}{\partial x_1} = x_1/r = \frac{\cancel{r}\cos\theta}{\cancel{r}}$

$\tan\theta = \frac{x_2}{x_1} \Rightarrow (1 + t^2\theta)\frac{\partial \theta}{\partial x_1} = -\frac{x_2}{x_1^2} = -\frac{t\theta}{x_1}$

$\therefore \frac{\partial\theta}{\partial x_1} = \frac{-c^2\theta\, t\theta}{x_1} = -\frac{c\theta s\theta}{r c\theta} = -\frac{s\theta}{r}$

$///^{ly}\ \frac{\partial\theta}{\partial x_2} = \frac{c^2\theta}{x_1} = \frac{c^2\theta}{r c\theta} = \frac{c\theta}{r}$

Now,

$$\sigma_{11,1} = \frac{\partial\sigma_{11}}{\partial x_1} = \frac{\partial\sigma_{11}}{\partial r}\underbrace{\frac{\partial r}{\partial x_1}}_{c\theta} + \frac{\partial\sigma_{11}}{\partial\theta}\underbrace{\frac{\partial\theta}{\partial x_1}}_{-\frac{s\theta}{r}} + \frac{\partial\sigma_{11}}{\partial z}\underbrace{\frac{\partial z}{\partial x_1}}_{0}$$

$$\sigma_{12,2} = \frac{\partial\sigma_{12}}{\partial x_2} = \frac{\partial\sigma_{12}}{\partial r}\underbrace{\frac{\partial r}{\partial x_2}}_{s\theta} + \frac{\partial\sigma_{12}}{\partial\theta}\underbrace{\frac{\partial\theta}{\partial x_2}}_{\frac{c\theta}{r}} + \frac{\partial\sigma_{12}}{\partial z}\underbrace{\frac{\partial z}{\partial x_2}}_{0}$$

$$\sigma_{13,3} = \frac{\partial\sigma_{13}}{\partial x_3} = \frac{\partial\sigma_{13}}{\partial r}\underbrace{\frac{\partial r}{\partial x_3}}_{0} + \frac{\partial\sigma_{13}}{\partial\theta}\underbrace{\frac{\partial\theta}{\partial x_3}}_{0} + \frac{\partial\sigma_{13}}{\partial z}\underbrace{\frac{\partial z}{\partial x_3}}_{1}$$

$$\therefore \sigma_{11,1} = c\theta\,\sigma_{11,r} - \frac{s\theta}{r}\sigma_{11,\theta} = c^3\theta\,\sigma'_{11,r} - 2c^2\theta s\theta\,\sigma'_{12,r} + c\theta s^2\theta\,\sigma'_{22,r}$$

$$- \frac{s\theta}{r}\left(-2c\theta s\theta\,\sigma'_{11} + c^2\theta\,\sigma'_{11,\theta} + 2[s^2\theta - c^2\theta]\sigma'_{12} - 2c\theta s\theta\,\sigma'_{12,\theta} + 2s\theta c\theta\,\sigma'_{22} + s^2\theta\,\sigma'_{22,\theta}\right)$$

$$\therefore {}_{12,2} = s^2\theta\, c\theta\left(\sigma'_{11,r} - \sigma'_{22,r}\right) + s\theta\left(c^2\theta - s^2\theta\right)\sigma'_{12,r} + \frac{c\theta}{r}\Big\{\left[c^2\theta - s^2\theta\right]\left(\sigma'_{11} - \sigma'_{22}\right) + c\theta s\theta\left(\sigma'_{11,\theta} - \sigma'_{22,\theta}\right)$$
$$+\left[-2c\theta s\theta - 2s\theta c\theta\right]\sigma'_{12} + \left[c^2\theta - s^2\theta\right]\sigma'_{12,\theta}\Big\}$$

Now put $\sigma'_{11} = \sigma_{rr}$, $\sigma'_{22} = \sigma_{\theta\theta}$, $\sigma'_{12} = \sigma_{r\theta}$, $\sigma'_{13} = \sigma_{rz}$, $\sigma'_{23} = \sigma_{\theta z}$

$$\therefore \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1$$
$$= \sigma_{rr,r}\, c\theta - \sigma_{r\theta,r}\, s\theta + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} c\theta - 2\frac{\sigma_{r\theta}}{r} s\theta - \frac{\sigma_{\theta\theta,\theta}}{r} s\theta + \frac{\sigma_{r\theta,\theta}}{r} c\theta + \sigma_{rz,z}\, c\theta - \sigma_{\theta z,z}\, s\text{i}$$
$$+ f_r\cos\theta - f_\theta \sin\theta = 0$$

$$\therefore c\theta\left(\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\sigma_{r\theta,\theta}}{r} + \sigma_{rz,z} + f_r\right) - s\theta\left(\sigma_{r\theta,r} + 2\frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \sigma_{\theta z,z} + f_\theta\right) = (\quad \to (1)$$

<u>Extension of problem:</u>

<u>$i=2$</u> $\quad \sigma_{12,1} + \sigma_{22,2} + \sigma_{23,3} = 0$

$$\sigma_{22} = s^2\theta\, \sigma'_{11} + 2s\theta c\theta\, \sigma'_{12} + c^2\theta\, \sigma'_{22} \quad ; \quad \sigma_{23} = s\theta\, \sigma'_{13} + c\theta\, \sigma'_{23}$$

$$\sigma_{12,1} = c\theta\, \sigma_{12,r} - \frac{s\theta}{r}\sigma_{12,\theta} \quad ; \quad \sigma_{22,2} = s\theta\, \sigma_{22,r} + \frac{c\theta}{r}\sigma_{22,\theta} \quad ; \quad \sigma_{23,3} = \sigma_{23,z}$$

$$\sigma_{12,1} = c^2\theta s\theta\left(\sigma'_{11,r} - \sigma'_{22,r}\right) + \left(c^3\theta - c\theta s^2\theta\right)\sigma'_{12,r}$$
$$-\frac{s\theta}{r}\Big\{\left[c^2\theta - s^2\theta\right]\left(\sigma'_{11} - \sigma'_{22}\right) + c\theta s\theta\left(\sigma'_{11,\theta} - \sigma'_{22,\theta}\right) + \left[-4c\theta s\theta\right]\sigma'_{12} + \left[c^2\theta - s^2\theta\right]\sigma'_{12,\theta}\Big\}$$

$$\sigma_{22,2} = s^3\theta\, \sigma'_{11,r} + 2s^2\theta c\theta\, \sigma'_{12,r} + s\theta c^2\theta\, \sigma'_{22,r}$$
$$+ \frac{c\theta}{r}\Big\{2s\theta c\theta\, \sigma'_{11} + s^2\theta\, \sigma'_{11,\theta} + 2\left[c^2\theta - s^2\theta\right]\sigma'_{12} + 2s\theta c\theta\, \sigma'_{12,\theta} - 2c\theta s\theta\, \sigma'_{22} + c^2\theta\, \sigma'_{22,\theta}\Big\}$$

$$\therefore \sigma_{12,1} + \sigma_{22,2} + \sigma_{23,3} + f_2$$
$$= \sigma_{rr,r}\, s\theta + \sigma_{r\theta,r}\, c\theta + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} s\theta + 2\frac{\sigma_{r\theta}}{r} c\theta + \frac{\sigma_{r\theta,\theta}}{r} s\theta + \frac{\sigma_{\theta\theta,\theta}}{r} c\theta + f_r \sin\theta + f_\theta\cos\theta$$
$$+ s\theta\, \sigma_{rz,z} + c\theta\, \sigma_{\theta z,z} = 0$$

$$\therefore s\theta\left(\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\sigma_{r\theta,\theta}}{r} + \sigma_{rz,z} + f_r\right) + c\theta\left(\sigma_{r\theta,r} + 2\frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \sigma_{\theta z,z} + f_\theta\right) = 0 \quad \to (2)$$

From (1) & (2) we get,

$$\left.\begin{aligned} &\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\sigma_{r\theta,\theta}}{r} + \sigma_{rz,z} + f_r = 0 \\ &\sigma_{r\theta,r} + 2\frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta\theta,\theta}}{r} + \sigma_{\theta z,z} + f_\theta = 0 \end{aligned}\right\} \text{two of the eqns in cylindrical coords}$$

P.8 $\sigma_{11,1} + \sigma_{12,2} = 0 \Rightarrow \sigma_{12,2} = -\frac{q}{2I}[2x_1x_2] \Rightarrow \sigma_{12} = -\frac{q}{2I}x_1x_2^2 + f(x_1) + c_1$

$\sigma_{12,1} + \sigma_{22,2} = 0 \Rightarrow \sigma_{12,1} = -\frac{q}{2I}[x_2^2 - c^2] \Rightarrow \sigma_{12} = -\frac{q}{2I}[x_1x_2^2 - c^2x_1] + g(x_2) + c_2$

$\Rightarrow f(x_1) = \frac{q}{2I}c^2x_1, \quad g(x_2) = 0, \quad c_1 = c_2 = k$

$\therefore \boxed{\sigma_{12} = \frac{q}{2I}x_1(c^2 - x_2^2) + \underbrace{k}_{\text{(const)}}}$

B.C's Let applied load @ boundary be $\sigma_{11}^*, \sigma_{22}^*, \sigma_{12}^*$.

at $x_1 = 0$, $\sigma_{11}^* = \frac{q}{I}x_2\left(\frac{c^2}{5} - \frac{x_2^2}{3}\right)$, $\sigma_{12}^* = k$, $\sigma_{13}^* = 0$

at $x_1 = 2b$, $\sigma_{11}^* = \frac{q}{I}x_2\left(2b^2 - \frac{x_2^2}{3} + \frac{c^2}{5}\right)$, $\sigma_{12}^* = \frac{q}{I}b(c^2 - x_2^2) + k$, $\sigma_{13}^* = 0$

at $x_2 = +c$, $\sigma_{22}^* = 0$, $\sigma_{12}^* = k$, $\sigma_{23}^* = 0$

at $x_2 = -c$, $\sigma_{22}^* = \frac{q}{2I}\frac{4}{3}c^3 = q$, $\sigma_{12}^* = k$, $\sigma_{23}^* = 0$

at $x_3 = 0, t$, $\sigma_{33}^* = \sigma_{31}^* = \sigma_{32}^* = 0$ (ie top and bottom faces unloaded)

Q(9)(a) For a plane equally inclined to the 3-principal axes we have the stress vector $= \sigma_{oct}$.

Now, $\sigma_1 = \sigma(1)n_1$; $\sigma_2 = \sigma(2)n_2$; $\sigma(3) = \sigma_3 n_3$. (by letting the y_i-axes coincide with the principal axes and using the Cauchy equation).

Thus the stress tensor has the form :- $\sigma_{ij} = \begin{pmatrix} \sigma(1) & 0 & 0 \\ 0 & \sigma(2) & 0 \\ 0 & 0 & \sigma(3) \end{pmatrix}$

$\therefore$ for the octahaedral plane,

$$n_1 = n_2 = n_3 = \pm\frac{1}{\sqrt{3}}.$$

$\therefore \sigma_i\sigma_i = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \sigma(1)^2n_1^2 + \sigma^2(2)n_2^2 + \sigma^2(3)n_3^2$

$\therefore \sigma_{oct} = \sqrt{\frac{1}{3}\left(\sigma(1)^2 + \sigma^2(2) + \sigma^2(3)\right)}$.

(b). $\cancel{N^2 \ldots} \cancel{S^2}$ $N = \sigma_i n_i = \sigma_{ji} n_j n_i$

Now the shearing components of the stress tensor are zero, from our choice of coordinate axes.

$$\therefore N_{oct} = \sigma_{11} n_1 n_1 + \sigma_{22} n_2 n_2 + \sigma_{33} n_3 n_3$$

$$= \sigma_{11} \frac{1}{3} + \sigma_{22} \frac{1}{3} + \sigma_{33} \frac{1}{3} \quad \left(\text{Since } n_1 = n_2 = n_3 = \pm \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{3}(\sigma_{kk}) = \frac{1}{3}\Theta$$

$$\therefore N_{oct} = \frac{1}{3}\Theta$$

(c) $S^2 = \sigma_i \sigma_i - N^2$

$$= \sigma(1)^2 n_1^2 + \sigma(2)^2 n_2^2 + \sigma(3)^2 n_3^2 - \left(\sigma(1) n_1^2 + \sigma(2) n_2^2 + \sigma(3) n_3^2\right)^2$$

$$= \sigma_{11}^2 n_1^2 + \sigma_{22}^2 n_2^2 + \sigma_{33}^2 n_3^2 - \left(\sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2\right)^2$$

$$\therefore S_{oct}^2 = \frac{1}{3}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \frac{1}{3}\left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{11}\sigma_{33} + 2\sigma_{22}\sigma_{33}\right]\right)$$

(I) $\leftarrow$ $\therefore S_{oct} = \left[\quad '' \quad\right]^{1/2} = \left[\frac{1}{3}\left(\frac{2}{3}\sigma_{11}^2 + \frac{2}{3}\sigma_{22}^2 + \frac{2}{3}\sigma_{33}^2 - \frac{2}{3}\sigma_{11}\sigma_{22} - \frac{2}{3}\sigma_{22}\sigma_{33} - \frac{2}{3}\sigma_{11}\sigma_{33}\right)\right]^{1/2}$

Now let us examine the expression $\left(-\frac{2}{3}\bar{\Phi}\right)^{1/2}$

$$\bar{\sigma}_{ij} = \begin{pmatrix} \left(\frac{2}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{33}\right) & 0 & 0 \\ 0 & \left(\frac{2}{3}\sigma_{22} - \frac{1}{3}\sigma_{11} - \frac{1}{3}\sigma_{33}\right) & 0 \\ 0 & 0 & \left(\frac{2}{3}\sigma_{33} - \frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{11}\right) \end{pmatrix}$$

$$\therefore -\frac{2}{3}\bar{\Phi} = -\frac{2}{3}\Big[-\frac{2}{9}\sigma_{11}^2 - \frac{2}{9}\sigma_{22}^2 + \frac{1}{9}\sigma_{33}^2 + \frac{5}{9}\sigma_{11}\sigma_{22} - \frac{1}{9}\sigma_{11}\sigma_{33} - \frac{1}{9}\sigma_{22}\sigma_{33}$$
$$- \frac{2}{9}\sigma_{11}^2 - \frac{2}{9}\sigma_{33}^2 + \frac{1}{9}\sigma_{22}^2 + \frac{5}{9}\sigma_{11}\sigma_{33} - \frac{1}{9}\sigma_{11}\sigma_{22} - \frac{1}{9}\sigma_{33}\sigma_{22}$$
$$- \frac{2}{9}\sigma_{22}^2 - \frac{2}{9}\sigma_{33}^2 + \frac{1}{9}\sigma_{11}^2 + \frac{5}{9}\sigma_{22}\sigma_{33} - \frac{1}{9}\sigma_{22}\sigma_{11} - \frac{1}{9}\sigma_{33}\sigma_{11}\Big]$$

$$= -\frac{2}{3}\left[-\frac{3}{9}\sigma_{11}^2 - \frac{3}{9}\sigma_{22}^2 - \frac{3}{9}\sigma_{33}^2 + \frac{3}{9}\sigma_{11}\sigma_{22} + \frac{3}{9}\sigma_{11}\sigma_{33} + \frac{3}{9}\sigma_{22}\sigma_{33}\right]$$

(II) $\leftarrow$ $\therefore \left[-\frac{2}{3}\bar{\Phi}\right]^{1/2} = \left\{\frac{1}{3}\left[\frac{2}{3}\sigma_{11}^2 + \frac{2}{3}\sigma_{22}^2 + \frac{2}{3}\sigma_{33}^2 - \frac{2}{3}\sigma_{11}\sigma_{22} - \frac{2}{3}\sigma_{11}\sigma_{33} - \frac{2}{3}\sigma_{22}\sigma_{33}\right]\right\}^{1/2}$

So (I) = (II) $\longrightarrow$ hence proved.

P10 The state of stress satisfying all equilibrium eqns and b.c.'s is $\sigma_{11} = P_{max}/A$, other components are zero.

Case I : $S_{max} = \left|\frac{\sigma(1)}{2}\right| = \frac{\sigma_{11}}{2} = \frac{P_{max}}{2A} \le 300 \times 10^3$

also $\sigma_{11} = \frac{P_{max}}{A} \le 10^5$

$\Rightarrow P_{max} = 10^5 A = 10N$.

i.e., tensile failure along plane $\perp^{ar}$ to x_1 axis.

Case II : $S_{max} = \left|\frac{\sigma(1)}{2}\right| = \frac{\sigma_{11}}{2} = \frac{P_{max}}{2A} \le 450 \times 10^3$

also $\sigma_{11} = \frac{P_{max}}{A} \le 10^6$

$\Rightarrow P_{max} = 0.9 \times 10^6 A = 90N$

ie shear failure along plane inclined at 45° to x_1 axis ($\because$ p-axes for $\sigma(2), \sigma(3)$ are any two orthogonal axes in $x_2 - x_3$ plane.

(11)

$$\begin{vmatrix} (80-\sigma) & 45 & 0 \\ 45 & (-35-\sigma) & 0 \\ 0 & 0 & (-50-\sigma) \end{vmatrix} = 0$$

$$-\sigma^3 - 5\sigma^2 + 7075\sigma + 241250 = 0$$

$\sigma_1 = +95.52$ MPa

$\sigma_2 = -50.52$ MPa

$\sigma_3 = -50.0$ MPa.

$\tau_{max} = \pm\left(\frac{\sigma_1 - \sigma_2}{2}\right)$

$= \pm 73.02$ MPa.

$S_{oct} = \frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2}$

$= 68.72$ MPa

(12)

$$\begin{vmatrix} (-150-\sigma) & -40 & 50 \\ -40 & -\sigma & 0 \\ 50 & 0 & 80-\sigma \end{vmatrix} = 0$$

$$-\sigma^3 - 70\sigma^2 + 16100\sigma - 128000 = 0$$

$\sigma_1 = +91.18$ MPa

$\sigma_2 = +8.28$ MPa

$\sigma_3 = -169.46$ MPa.

$\tau_{max} = 130.32$ MPa.

(13)

$\Sigma F_y = 0$

$$40 \times 5 + \sigma_{xy} \times 12 = 80 \times 13 \times \sin\theta + 60 \times 13 \times \cos\theta$$

$$\sigma_{xy} = 76.66 \text{ Mpa.}$$

$\Sigma F_x = 0$

$$80 \times 13 \times \cos\theta = 60 \times 13 \times \sin\theta + \sigma_{xy} \times 5 + \sigma_{xx} \times 12$$

$$\sigma_{xx} = 23.03 \text{ MPa.}$$