CE623 Advanced Mechanics of Solids 3-0-0 Fall 2005 HW 2

10

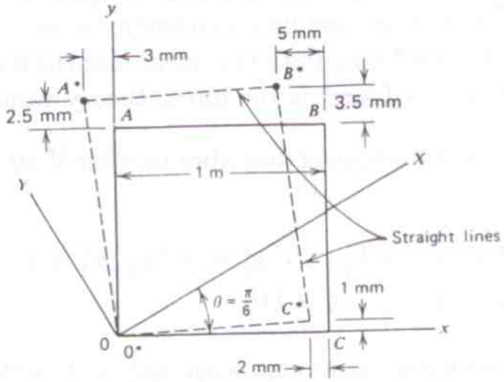
1. The stress components in the half-space $x_3 > 0$ are:

$$\sigma_{ij} = \frac{a x_i x_j}{r^5} x_3, \quad r \neq 0, \quad a > 0$$

Find the total force on the surface of the hemisphere $r = a$, $x_3 > 0$

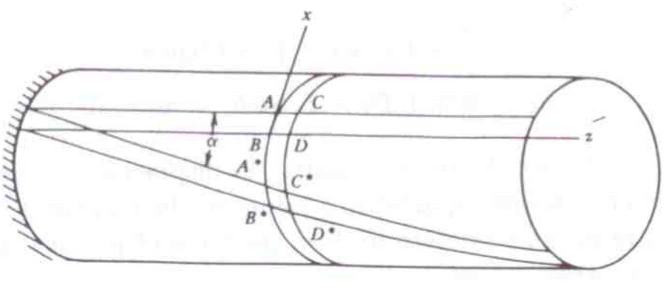
10

2. The square plate shown is loaded so that it is in a state of plane strain (i.e., $\epsilon_{zx} = \epsilon_{zy} = \epsilon_{zz} = 0$).
   (a) Determine the displacements $u_x \equiv u$, $u_y \equiv v$ for the plate for the deformations shown.
   (b) Determine the strain components in the (x, y) system.
   (c) Determine the strain components in the (X, Y) system.
   (d) Determine the principal axes and strains.



5

3. When solid circular torsion members are used to obtain material properties for finite strain applications, an expression for the engineering strain γ_{xz} is needed, where (x, z) is the tangent plane and the z-axes is along the axis of the member. Consider the element $ABCD$ for the undeformed member as shown. As an approximation, assume that the member deforms such that the volume and diameter remain constant. Thus for the deformed element we have $A^*B^* = AB$, $C^*D^* = CD$, and the distance (along the z-axis) between the parallel curved lines A^*B^* and C^*D^* remains unchanged. Show that the engineering shear strain $\gamma_{xz} = \tan\alpha$ where α is the angle between AC and A^*C^*.



$$\gamma_{xz} = 2\varepsilon_{xz} = (1+\varepsilon_1)(1+\varepsilon_2)\cos\theta$$
$$= (1+0)\left(1+\frac{AC\left[\frac{1}{\cos\alpha}-1\right]}{AC}\right)\cos\theta$$
$$= \frac{1}{\cos\alpha}\sin\alpha = \tan\alpha$$

(10) 4. Show that the final angle θ between two line elements whose initial directions $\vec{n}(1)$ and $\vec{n}(2)$ are inclined at θ_0, is given by

$$\cos\theta = \frac{2\epsilon_{ij}n_i(1)n_j(2) + \cos\theta_0}{(1 + 2\epsilon_{ij}n_i(1)n_j(1))^{1/2}(1 + 2\epsilon_{ij}n_i(2)n_j(2))^{1/2}}$$

(10) 5. Consider the infinitesimal strain distribution $\varepsilon_{11} = \varepsilon_{22} = 2x_1$, $\varepsilon_{12} = x_1 + 2x_2$, $\varepsilon_{33} = 2x_3$, $\varepsilon_{13} = \varepsilon_{23} = 0$. Show that this is a possible strain distribution. Then determine the displacements $u_i[x_1, x_2, x_3]$ if the origin has zero displacement and any infinitesimal line element at the origin has zero rotation. (Hint: integrate the strain displacement relations to obtain the displacements, and then determine the constants of integration by using the conditions given at the origin).

(5) 6. Consider the deformation field $x_1^* = x_1 + kx_2$, $x_2^* = x_2$, $x_3^* = x_3$. A differential square element $ABCD$ lies in the x_1-x_2 plane and has sides of length dL with sides AB and AD parallel to the x_1 and x_2 axis, respectively. Calculate the unit extension for sides AB, AD, and diagonals AC, DB considering both the linear and the nonlinear theory. Under what restriction (if any) is the linear theory valid.

(5) 7. Show that the following mapping describes motion of an incompressible solid:

$$x_1^* = \lambda x_1 + kx_2, \qquad x_2^* = \lambda^{-1}x_2,\ x_3^* = x_3$$
$$\lambda = \lambda[t], \qquad k = k[t]$$

If λ and k are continuous functions with $\lambda[0] = 1$, $k[0] = 0$, by considering the behaviour of an initially rectangular parallelepiped, show from physical arguments that λ remains necessarily positive.

(5) 8. The displacement field in a solid is given as:

$$u_1 = \frac{a}{4}x_1(x_2 + x_3)^2, \quad u_2 = \frac{a}{4}x_2(x_1 + x_3)^2, \quad u_3 = \frac{a}{4}x_3(x_1 + x_2)^2$$

where a is an infinitesimal constant. Find the maximum and minimum values of the elongation per unit length at the point $P \Rightarrow (1, 1, 1)$

(5) 9. The solution for the displacement field of a problem done in cylindrical coordinates is as follows:

$$u_r = \frac{A}{r} + Br(\ln r - 1) + C\sin\theta$$
$$u_\theta = Br\theta + Dr + E\cos\theta, \qquad u_z = 0$$

where A, B, C, D, E are constants. Is displacement compatibility assured when considering arbitrary values of the constants? Which of these constants can be determined on the basis of physical arguments, and what are their values?

10. (a) Based on the results obtained in class, show that the linear Lagrangian rotation vector equals one-half the curl of the displacement vector.

5*3

(b) Consider the displacement field $u_i \Rightarrow [cx_2x_3,\ cx_3x_1,\ cx_1x_2]$, where c is an infinitesimal constant. Does this displacement field represent a state of (a) pure straining, (b) pure rigid body rotation, or (c) general motion.

(c) Consider the displacement field $u_i \Rightarrow [cx_1x_2, cx_1x_2, 2c(x_1+x_2)x_3]$, where c is an infinitesimal constant. What is the rotational component of relative displacement between two neighbouring points originally lying on the line (in the x_1x_2 plane) that makes equal acute angles with positive x_1 and x_2 axes.

11. The displacement components of an incompressible continuum are:

(5)

$$u_1 = (1 - x_2^2)(a + bx_1 + cx_1^2), \quad u_2\Big|_{x_2=\pm\sqrt{3}} = 0, \quad u_3 = 0$$

where a, b, c are infinitesimal constants. Determine u_2

12. A 50 mm cube of an isotropic solid ($E = 210 \times 10^9\ \mathrm{Nm}^{-2}$, $\nu = 0.25$) is loaded in a manner that yields a uniform state of stress given as,

(5)

$$\sigma_{ij} \Rightarrow 10^5 \times \begin{pmatrix} 700 & 300 & 40 \\ 300 & -15 & 0 \\ 40 & 0 & 100 \end{pmatrix} \ \mathrm{Nm}^{-2}$$

Assuming that the loading (and resulting stresses) are within limits for which the assumption of linear elastic behaviour is valid, determine the total change in volume induced by this stress field.

13. The simply supported, rectangular cross-section thin plate has unit width ($t = 1$) such that $t \ll L$, $t \ll h$. It carries a uniform pressure load $q\ \mathrm{Nm}^{-2}$ as shown. Thus, we may treat this as a plane stress problem (i.e., $\sigma_{i3} = 0$, $\partial(\ )/\partial x_3 = 0$). The plate is light and I_{33}, E, and ν are constant. Consider the Airy stress function

$$\phi = -\frac{A}{6}x_2^3 - \frac{B}{6}x_1^2x_2^3 + \frac{C}{2}x_1^2x_2 - \frac{q}{4}x_1^2 + \frac{B}{30}x_2^5$$

(20)

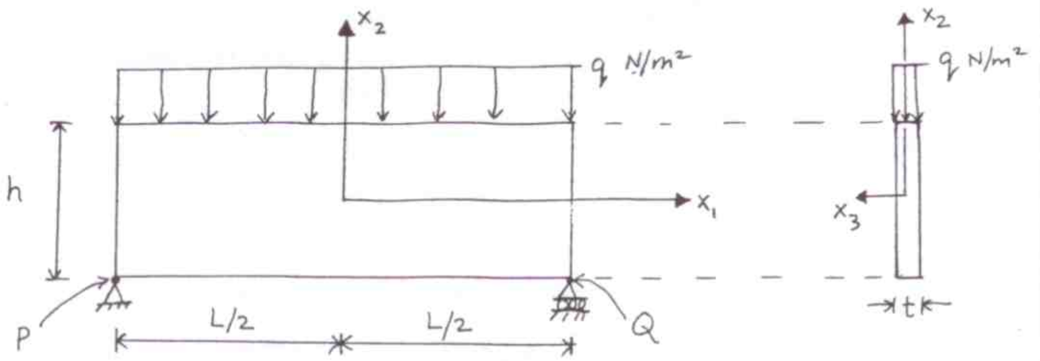
(a) Evaluate $\nabla^4\phi$ and comment on your result. Are all the Beltrami Mitchell compatibility equations satisfied by this plane stress solution ?

(b) By considering the boundary conditions on the upper and lower faces and also the zero applied moment condition at the left and right faces (i.e., $M_3 = 0$ on these faces), determine the constants A, B, C.

(c) Determine the stress and strain distributions. Is the stress distribution valid on the left and right faces.

(d) By considering the in-plane strain displacement relations in conjunction with the displacement boundary conditions at points P and Q (if required), determine the transverse displacement of the centreline, i.e., $u_2[x_1, 0]$. State your results for the nondimensional displacement $u_2^*[x_1, 0]$ defined as $\dfrac{EI_{33}}{qL^4}u_2[x_1, 0]$

(e) Simplify the thin-plate result for $u_2^*[x_1, 0]$ obtained in part (d) to obtain the nondimensional displacement $u_2^*[x_1]$ for a slender beam that is subjected to a uniform downward line load $q\,\mathrm{Nm}^{-1}$

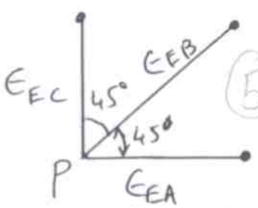


14. The constitutive law for a linear, elastic, isotropic solid is:

5

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{mm}\right]$$

Use this to prove that the principal axes of stress coincide with the principal axes of strain for the given material.



STRAIN ROSETTE.

5

15. A thin plate is made of a linear, isotropic, homogenous, elastic material. It is loaded along its edges with inplane loads, and body forces are negligible. The top and bottom faces are not loaded. A rectangular rosette (shown in the fig.) is attached to a point P on the upper face. The strain measurements recorded by the rosette are $\epsilon_{EA} = -100$, $\epsilon_{EB} = -200$, and $\epsilon_{EC} = 400$ microinches per inch. Find the principal stresses at the point P, if $E = 30 \times 10^6$ psi and $\nu = 0.3$

16. (a) The displacement field in an isotropic, homogenous, linear, elastic medium in equilibrium is given as:

$$u_1 = cx_1\ ,\quad u_2 = -c\nu x_2\ ,\quad u_3 = -c\nu x_3$$

5×2

where ν is the Poisson's ratio and c is an infinitesimal constant. Find the body forces in terms of ν and E.

(b) Given that for real elastic solids, E, K, and μ are all positive, show that $-1 \le \nu \le 0.5$.

17. Consider a long prismatic, linear, isotropic, homogenous, elastic body with the x_3 direction coinciding with the longitudinal axis of the body.

10

Assume that the body force is a function of only the x_1 and x_2 co-ordinates, and that its component $\tilde{B}_3 = 0$. Consider an external loading which is perpendicular to the longitudinal axis and does not vary along that axis. Thus, we may conclude that the displacement field is not a function of the x_3 co-ordinate. Further, if we assume that the displacement component $u_3 = 0$, we get a plane state of strain (i.e., $\epsilon_{i3} = 0$ and $\partial(\ )/\partial x_3 = 0$). Making use of the equilibrium equations and the Beltrami-Michell compatibility equations, show that for conservative body forces, the exact solution of the plane strain problem is obtained by solving the problem

$$\frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = \frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right)$$

where $\phi[x_1, x_2]$ is the Airy stress function and $\psi[x_1, x_2]$ is the body force potential function.