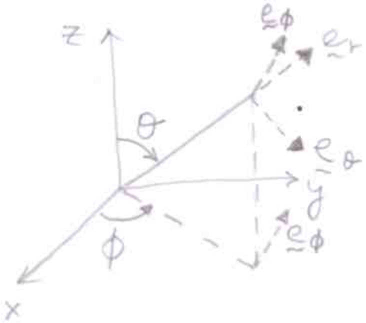
CE623 – HW2 – Solutions

1. At boundary, $\sigma_{xx} = \dfrac{x^2 z}{a^4}$, $\sigma_{yy} = \dfrac{y^2 z}{a^4}$, $\sigma_{zz} = \dfrac{z^3}{a^4}$

$\sigma_{xy} = \dfrac{xyz}{a^4}$, $\sigma_{yz} = \dfrac{yz^2}{a^4}$, $\sigma_{xz} = \dfrac{xz^2}{a^4}$.

$x = r s\theta c\phi$, $y = r s\theta s\phi$, $z = r c\theta$



$$\begin{Bmatrix} e_\theta \\ e_\phi \\ e_r \end{Bmatrix} = \underbrace{\begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -s\phi & c\phi & 0 \\ s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix}}_{= \underline{\underline{a}}} \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}$$

$$\left(\underline{\underline{\sigma}}\right)_{r,\theta,\phi} = \underline{\underline{a}}\left(\underline{\underline{\sigma}}\right)_{x,y,z} a^T$$

$$\begin{bmatrix} \sigma_{\theta\theta} & \sigma_{\theta\phi} & \sigma_{r\theta} \\ \sigma_{\theta\phi} & \sigma_{\phi\phi} & \sigma_{r\phi} \\ \sigma_{r\theta} & \sigma_{r\phi} & \sigma_{rr} \end{bmatrix} = \frac{1}{a^4} \begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -s\phi & c\phi & 0 \\ s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix} \begin{bmatrix} x^2 z & xyz & xz^2 \\ xyz & y^2 z & yz^2 \\ xz^2 & yz^2 & z^3 \end{bmatrix} \begin{bmatrix} c\theta c\phi & -s\phi & s\theta c\phi \\ c\theta s\phi & c\phi & s\theta s\phi \\ -s\theta & 0 & c\theta \end{bmatrix}$$

Loading on surface: only components involving r participate in b.c. Hence we need only σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\phi}$.

$$\sigma_{rr} = \Big[s\theta c\phi \overbrace{\left(s\theta c\phi \cdot s^2\theta c^2\phi c\theta + s\theta s\phi \cdot s\theta c\phi s\theta s\phi c\theta + c\theta \cdot s\theta c\phi c^2\theta \right)}^{\text{I}}$$
$$+ s\theta s\phi \underbrace{\left(s\theta c\phi \cdot s\theta c\phi s\theta s\phi c\theta + s\theta s\phi \cdot s^2\theta s^2\phi c\theta + c\theta \cdot s\theta s\phi c^2\theta \right)}_{\text{II}}$$
$$+ c\theta \underbrace{\left(s\theta c\phi \cdot s\theta c\phi c^2\theta + s\theta s\phi \cdot s\theta s\phi c^2\theta + c\theta \cdot c^3\theta \right)}_{\text{III}} \Big] * \frac{1}{a}$$

Here, and in what follows $\sigma_{rr}, \sigma_{r\theta}, \sigma_{r\phi}$ represent stress components on surface, hence they represent the loading

$$\sigma_{r\phi} = \frac{1}{a}\left[-s\phi\,(\text{I}) + c\phi\,(\text{II}) \right]$$

$$\sigma_{r\theta} = \frac{1}{a}\left[c\theta c\phi\,(\text{I}) + c\theta s\phi\,(\text{II}) - s\theta\,(\text{III}) \right]$$

$\text{I} = s^3\theta c\theta c^3\phi + s^3\theta c\theta s^2\phi c\phi + s\theta c^3\theta c\phi = s\theta c\theta c\phi$

$\text{II} = s^3\theta c\theta s\phi c^2\phi + s^3\theta c\theta s^3\phi + s\theta c^3\theta s\phi = s\theta c\theta s\phi$

$\text{III} = s^2\theta c^2\theta c^2\phi + s^2\theta c^2\theta s^2\phi + c^4\theta = c^2\theta$

$\sigma_{rr} = (s^2\theta c\theta c^2\phi + s^2\theta c\theta s^2\phi + c^3\theta) * \frac{1}{a} = c\theta / a$

$\sigma_{r\phi} = (-s\theta c\theta s\phi c\phi + s\theta c\theta s\phi c\phi)/a = 0$

$\sigma_{r\theta} = (s\theta c^2\theta c^2\phi + s\theta c^2\theta s^2\phi - s\theta c^2\theta)/a = 0$

∵ σ_{rr} is function of θ alone, the stress distribution σ_{rr} along every latitudinal band cancels out $\Rightarrow F_x = F_y = 0$

$$F_z = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{\sigma_{rr}}_{c\theta/a} \underbrace{(a\,d\theta\; a s\theta\, d\phi)}_{dA}\, c\theta = 2\pi a \int_0^{\pi/2} s\theta\, c^2\theta\, d\theta = -\frac{2\pi a}{3}\, c^3\theta \Big|_0^{\pi/2}$$

$$F_z = \frac{2\pi a}{3}$$

Extra - I

In general,

$$F_z = \iint (\sigma_{rr}\, dA\, c\theta - \sigma_{r\theta}\, dA\, s\theta) = \int_0^{2\pi}\int_0^{\pi/2} (\sigma_{rr} c\theta - \sigma_{r\theta} s\theta)(a^2 s\theta\, d\theta\, d\phi)$$

$$F_x = \iint (\sigma_{rr}\, dA\, s\theta c\phi + \sigma_{r\theta}\, dA\, c\theta c\phi - \sigma_{r\phi}\, dA\, s\phi)$$
$$= \int_0^{2\pi}\int_0^{\pi/2} (\sigma_{rr} s\theta c\phi + \sigma_{r\theta} c\theta c\phi - \sigma_{r\phi} s\phi)(a^2 s\theta\, d\theta\, d\phi)$$

$$F_y = \iint (\sigma_{rr}\, dA\, s\theta s\phi + \sigma_{r\theta}\, dA\, c\theta s\phi + \sigma_{r\phi}\, dA\, c\phi)$$
$$= \int_0^{2\pi}\int_0^{\pi/2} (\sigma_{rr} s\theta s\phi + \sigma_{r\theta} c\theta s\phi + \sigma_{r\phi} c\phi)(a^2 s\theta\, d\theta\, d\phi)$$

Extra-2. Alternative & much shorter method.

Find stress vector on surface, i.e., $\boxed{\underline{\sigma} = \underline{\underline{\sigma}}\,\underline{n}}$

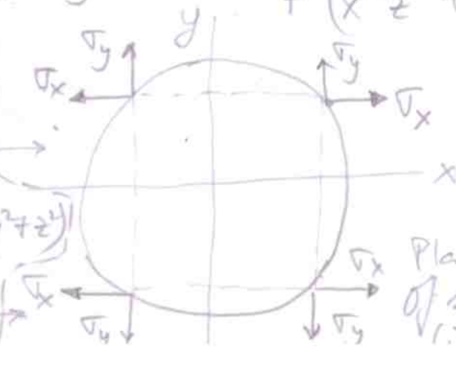
surface $\equiv f(x,y,z) = x^2+y^2+z^2-a^2 = 0$.

$$\underline{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\,\underline{i} + 2y\,\underline{j} + 2z\,\underline{k}}{2\underbrace{\sqrt{x^2+y^2+z^2}}_{=a}} = \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}$$

$$\Rightarrow \underline{\sigma} = \frac{1}{a^4}\begin{bmatrix} x^2 z & xyz & xz^2 \\ xyz & y^2 z & yz^2 \\ xz^2 & yz^2 & z^3 \end{bmatrix} \frac{1}{a}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \frac{1}{a^5}\Big[(x^3 z + xy^2 z + xz^3)\,\underline{i} + (x^2 yz + y^3 z + yz^3)\,\underline{j} + (x^2 z^2 + y^2 z^2 + z^4)\,\underline{k}\Big]$$

$$\Rightarrow \sigma_x = \frac{1}{a^5}(x^3 z + xy^2 z + xz^3) = \frac{xz}{a^5}\underbrace{(x^2+y^2+z^2)}_{a^2}$$
$$= \frac{xz}{a^3} = \text{odd in } x.$$

$$\sigma_y = \frac{1}{a^5}(x^2 yz + y^3 z + yz^3) = \frac{yz}{a^5}(x^2+y^2+z^2)$$
$$= \frac{yz}{a^3} = \text{odd in } y.$$



Plan of a frustrum of sphere at latitude (i.e. θ, z are const)

So we see that in every strip of annular area $2\pi a \sin\theta\, a\, d\theta$, the contributions from opposite sides cancel when summing, i.e.,

$$F_x = \int_A \sigma_x\, dA = 0, \qquad F_y = \int_A \sigma_y\, dA = 0$$

$$F_z = \int \sigma_z\, dA = \frac{1}{a^5}\int\!\!\int (\underbrace{x^2z^2 + y^2z^2}_{=z^2a^2} + z^4)\, dA = \frac{1}{a^5}\iint a^2 z^2\, dA$$

$$= \int\!\!\int \frac{\cancel{a^2} a^2 c^2\theta}{\cancel{a^5}} (\cancel{a}\, d\theta\, a s\theta\, d\phi) = 2\pi a \int_0^{\pi/2} c^2\theta\, s\theta = \frac{2\pi a}{3}$$

same as by longer method involving stress transf.

(2). It can be intuitively shown that

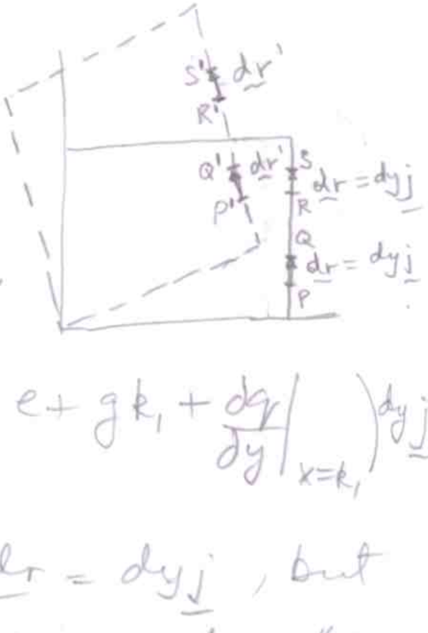
(a) $\left.\begin{aligned} u_x &= ax + by + cxy \\ u_y &= dx + ey + gxy \end{aligned}\right\} \rightarrow$ Result.

from the fact that straight edges remain straight. However, below is a rigorous proof yielding this result.

$$\underline{r}' = \underline{r} + \underline{u} = x\underline{i} + y\underline{j} + u_x\underline{i} + u_y\underline{j}$$

$$\underline{dr} = (dx + du_x)\underline{i} + (dy + du_y)\underline{j}$$

Assume $\left.\begin{aligned} u_x &= ax + by + cxy + p(x,y) \\ u_y &= dx + ey + gxy + q(x,y) \end{aligned}\right\} \rightarrow (1)$



Corresponding to an initially vertical edge, we have,

$$\underline{dr}\Big|_{x=k_1} = \left(b + ck_1 + \frac{\partial p}{\partial y}\Big|_{x=k_1}\right) dy\, \underline{i} + \left(1 + e + gk_1 + \frac{\partial q}{\partial y}\Big|_{x=k_1}\right) dy\, \underline{j}$$

Now $\underline{PQ}$ and $\underline{RS}$ are same vector, i.e. $\underline{dr} = dy\underline{j}$, but corresponding to points at different locations y, along the $x = k_1$ edge. These deform to $\underline{P'Q'} = (\underline{dr})_{PQ}$ and $\underline{R'S'} = (\underline{dr})_{RS}$, which need not be same vector, since stretching can take place. However the orientations of $(\underline{dr}')_{PQ}$ and $(\underline{dr}')_{RS}$ must be same if deformed edge remains straight line. This means,

$$\frac{1 + e + gk_1 + \left.\frac{\partial q}{\partial y}\right|_{x=k_1}}{b + ck_1 + \left.\frac{\partial p}{\partial y}\right|_{x=k_1}} = \text{function of } x \text{ only.}$$

$\Rightarrow$ q and p are linear in y.

Considering initially horizontal edge, we have (using similar procedure),

$$\left.\underline{dr}\right|_{y=k_2} = \left(1 + a + ck_2 + \left.\frac{\partial p}{\partial x}\right|_{y=k_2}\right) dx\,\underline{i} + \left(d + gk_2 + \left.\frac{\partial q}{\partial x}\right|_{y=k_2}\right) dx\,\underline{j}$$

For line $y = k_2$ to remain straight after deformation, we require,

$$\frac{\left(1 + a + ck_2 + \left.\frac{\partial p}{\partial x}\right|_{y=k_2}\right)}{\left(d + gk_2 + \left.\frac{\partial q}{\partial x}\right|_{y=k_2}\right)} = \text{function of } y \text{ only}$$

$\Rightarrow$ q and p are linear in x

$\Rightarrow$ q, p have bi-linear form (xy) which is already included in u_x, u_y. Hence $p(x,y)$, $q(x,y)$ are discarded from (1) $\rightarrow$ QED $\rightarrow$ contd on pg (1e)

P.3. You must use finite (large) strain theory, with given assumptions.

So, $\gamma_{xz} = 2E_{xz} = (1+\varepsilon_x)(1+\varepsilon_z)\cos\theta$

where θ = angle between A^*C^* & A^*B^* (ie between two line elements originally $\perp$ ar) and ε_x, ε_z are engg ext. strains of elements originally along x & z directions (ie along AB & AC directions).

$\Rightarrow \gamma_{xz} = (1+0)(1 + \{AC[1/\cos\alpha - 1]\}/AC)\cos\theta = \frac{1}{\cos\alpha}\cos\theta = \frac{1}{\cos\alpha}\sin\alpha = \tan\alpha$

$$u_x[1,0] = -0.002 = a$$
$$u_x[1,1] = -0.005 = a+b+c$$
$$u_x[0,1] = -0.003 = b$$
$\Rightarrow a=-0.002,\ b=-0.003,\ c=0$

$u_x = -0.002x - 0.003y$

$$u_y[1,0] = 0.001 = d$$
$$u_y[1,1] = d+e+g = 0.0035$$
$$u_y[0,1] = e = 0.0025$$
$\Rightarrow d=0.001,\ e=0.0025,\ g=0$

$u_y = 0.001x + 0.0025y$

Note: The above expressions give u_x, u_y in metres if x, y are in metres

(b) $\because (a, b, c, d) \ll 1$, you can use infinitesmal displ. gradient theory (ie linear theory). However we'll use nonlinear theory to start with.

$$E_{xx} = u_{x,x} + \tfrac{1}{2}u_{x,x}^2 + \tfrac{1}{2}u_{y,x}^2 = -0.002 + \frac{0.002^2 + 0.001^2}{2} = -0.0019975$$

$$E_{yy} = u_{y,y} + \tfrac{1}{2}u_{x,y}^2 + \tfrac{1}{2}u_{y,y}^2 = 0.0025 + \frac{0.003^2 + 0.0025^2}{2} = 0.002507625$$

$$E_{xy} = \tfrac{1}{2}\left(u_{x,y} + u_{y,x} + u_{x,x}u_{x,y} + u_{y,x}u_{y,y}\right)$$
$$= \tfrac{1}{2}\left(-0.003 + 0.001 + [-0.002][-0.003] + [0.001][0.0025]\right)$$
$$= -0.00099575$$

$$(\underline{\underline{E}})_{x,y} = \begin{pmatrix} -0.0019975 & -0.00099575 \\ -0.00099575 & 0.002507625 \end{pmatrix} \simeq \begin{pmatrix} -0.002 & -0.001 \\ -0.001 & 0.0025 \end{pmatrix} = (\underline{\underline{e}})_{xy}$$

linear strain tensor

SO FROM HERE ON I USE LINEAR THEORY ($\underline{\underline{e}}$) FOR CONVENIENCE

(c) $$(\underline{\underline{e}})_{X,Y} = \underline{\underline{a}}\,(\underline{\underline{e}})_{x,y}\,\underline{\underline{a}}^T = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}\begin{pmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{pmatrix}\begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix}$$

$$= \begin{pmatrix} (e_{xx}c^2\theta + e_{yy}s^2\theta + 2e_{xy}\cos\theta) & (\cos\theta[-e_{xx}+e_{yy}] + e_{xy}[c^2\theta - s^2\theta]) \\ \text{symmetric} & (e_{xx}s^2\theta + e_{yy}c^2\theta - 2e_{xy}\cos\theta) \end{pmatrix}$$

$\theta = 30^\circ$

$$(\underline{\underline{e}})_{XY} = \begin{pmatrix} -0.001741 & 0.001449 \\ 0.001449 & 0.002241 \end{pmatrix}$$

(1f)

(d) $|\underline{\underline{e}} - \lambda \underline{\underline{I}}| = 0$, for convenience, scale up $\underline{\underline{e}}$ by 10^3

$\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2.5-\lambda \end{pmatrix} = 0$, so actual p-strains will be $10^{-3}\lambda$.

$\lambda^2 - 4.5\lambda + 4 = 0 \Rightarrow \lambda = \dfrac{4.5 \pm \sqrt{4.5^2 - 16}}{2}$.

$\lambda(1) = 3.2808, \; 1.2192 = \lambda(2)$

p-strains are $e(1) = 3.2808 * 10^{-3}$, $e(2) = 1.2192 * 10^{-3}$

p-axes : $(2-\lambda(1))n_1(1) - n_2(1) = 0 \rightarrow$ (i)

$n_1^2(1) + n_2^2(1) = 1 \longrightarrow$ (ii)

$\Rightarrow n_1^2(1)\left[1 + (2-\lambda(1))^2\right] = 1$

$\Rightarrow n_1(1) = 0.6154, \; n_2(1) = -0.7882$

$\underset{\sim}{n}(1) = (0.6154, -0.7882)^T$

↳ p-axis corresponding to $e(1)$

$(2-\lambda(2))\, n_1(2) - n_2(2) = 0$

$n_1^2(2) + n_2^2(2) = 1$

$\Rightarrow n_1^2(2)\left[1 + (2-\lambda(2))^2\right] = 1$

$\Rightarrow n_1(2) = 0.7882, \; n_2(2) = 0.6154$

$\underset{\sim}{n}(2) = (0.7882, 0.6154)^T$

Observe that $\underset{\sim}{n}(1) \cdot \underset{\sim}{n}(2) = 0$; ie p-axes are orthogonal.

Problem 5

$\because$ strains are linear in a_1, a_2, a_3 and compat eqns involve double differentiation of strain components w.r.t. a_i's, compat eqns are satisfied identically. Hence it is a possible strain distribution ◀

$$l_{11} = 2a_1 = \frac{\partial u_1}{\partial a_1} \Rightarrow u_1 = a_1^2 + f[a_2, a_3]$$

$$l_{22} = 2a_1 = \frac{\partial u_2}{\partial a_2} \Rightarrow u_2 = 2a_1 a_2 + g[a_1, a_3]$$

$$l_{12} = a_1 + 2a_2 = \frac{1}{2}\left[\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right] = \frac{1}{2}\left[\frac{\partial f}{\partial a_2} + 2a_2 + \frac{\partial g}{\partial a_1}\right]$$

$$\Rightarrow \frac{\partial f[a_2, a_3]}{\partial a_2} + \frac{\partial g[a_1, a_3]}{\partial a_1} = 2a_1 + 2a_2 \longrightarrow ①$$

$$l_{33} = 2a_3 = \frac{\partial u_3}{\partial a_3} \Rightarrow u_3 = a_3^2 + h[a_1, a_2]$$

$$l_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_3} + \frac{\partial u_3}{\partial a_1}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial a_3} + \frac{\partial h}{\partial a_1}\right) = 0 \Rightarrow \frac{\partial f[a_2, a_3]}{\partial a_3} + \frac{\partial h[a_1, a_2]}{\partial a_1} = 0 \rightarrow ②$$

$$l_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_3} + \frac{\partial u_3}{\partial a_2}\right) = \frac{1}{2}\left(\frac{\partial g}{\partial a_3} + \frac{\partial h}{\partial a_2}\right) = 0 \Rightarrow \frac{\partial g[a_1, a_3]}{\partial a_3} + \frac{\partial h[a_1, a_2]}{\partial a_2} = 0 \rightarrow ③$$

Now $\dfrac{\partial ①}{\partial a_3} = \dfrac{\partial^2 f}{\partial a_2 \partial a_3} + \dfrac{\partial^2 g}{\partial a_1 \partial a_3} = 0$

From $\dfrac{\partial ②}{\partial a_2} + \dfrac{\partial ③}{\partial a_1} = 0$ and the above we get $\dfrac{\partial^2 h}{\partial a_1 \partial a_2} = 0 \Rightarrow h = A + p[a_1] + q[a_2] \rightarrow ④$

From ② & ④, $\dfrac{\partial f[a_2, a_3]}{\partial a_3} + p'[a_1] = 0 \Rightarrow \because f = f[a_2, a_3]$ that $p[a_1] = Ba_1$

From ③ & ④, $\dfrac{\partial g[a_1, a_3]}{\partial a_3} + q'[a_2] = 0 \Rightarrow \because g = g[a_1, a_3]$ that $q[a_2] = Ca_2$

constants omitted $\because$ 'A' already present

Thus, $h = A + Ba_1 + Ca_2 \longrightarrow ⑤$

From ② & ⑤, $f = -Ba_3 + r[a_2] \longrightarrow ⑥$

From ③ & ⑤, $g = -Ca_3 + s[a_1] \longrightarrow ⑦$

$\rightarrow$ from ①, $r' + s' = 2a_1 + 2a_2$

$\Rightarrow r = a_2^2 + D + Ka_2$, $s = a_1^2 + E - Ka_1$

So $f = -Ba_3 + a_2^2 + D + Ka_2$, $g = -Ca_3 + a_1^2 + E - Ka_1$

Thus
$$u_1 = a_1^2 + a_2^2 - Ba_3 + D + Ka_2$$
$$u_2 = 2a_1 a_2 + a_1^2 - Ca_3 + E - Ka_1$$
$$u_3 = a_3^2 + Ba_1 + Ca_2 + A$$

If origin has zero displ ($u_1 = u_2 = u_3 = 0$ @ origin) $\Rightarrow A = D = E = 0$

Further, if an (arbitrary) infinitesimal line element @ origin has zero rotation, $\Rightarrow \tilde{w}_i$ and hence $\tilde{w}_{ij}$ vanish @ origin.

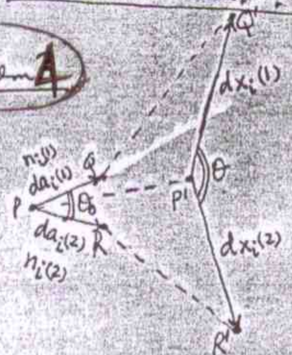
$$\tilde{\omega}_{13} = \frac{1}{2}(u_{1,3} - u_{3,1})\Big|_{a_1=a_2=a_3=0} = -B = 0 \quad , \quad \tilde{\omega}_{23} = \frac{1}{2}(u_{2,3} - u_{3,2})\Big|_{a_1=a_2=a_3=0} = -C = 0$$

$$\tilde{\omega}_{12} = \frac{1}{2}(u_{1,2} - u_{2,1})\Big|_{a_1=a_2=a_3=0} = K = 0$$

$$\Rightarrow B = C = 0$$

So $\boxed{u_1 = a_1^2 + a_2^2 \,,\quad u_2 = 2a_1 a_2 + a_1^2 \,,\quad u_3 = a_3^2}$ ◄

Problem 4



Consider $\underline{dx}(1) \cdot \underline{dx}(2)$. We have,

$$\underbrace{dx_i(1)\, dx_i(2)}_{LHS} = \underbrace{\sqrt{dx_i(1)\, dx_i(1)}\, \sqrt{dx_j(2)\, dx_j(2)}\, \cos\theta}_{RHS}$$

$$LHS = \left(\frac{\partial x_i}{\partial a_j}\, da_j\right)_{(1)} \left(\frac{\partial x_i}{\partial a_k}\, da_k\right)_{(2)} = \left(\frac{\partial(a_i + u_i)}{\partial a_j}\, da_j\right)_{(1)} \left(\frac{\partial(a_i + u_i)}{\partial a_k}\, da_k\right)_{(2)}$$

$$= \left(da_i + \frac{\partial u_i}{\partial a_j}\, da_j\right)_{(1)} \left(da_i + \frac{\partial u_i}{\partial a_k}\, da_k\right)_{(2)}$$

$$LHS = da_i(1)\, da_i(2) + da_i(1)\, da_k(2)\, u_{i,k} + da_j(1)\, da_i(2)\, u_{i,j} + da_j(1)\, da_k(2)\, u_{i,j}\, u_{i,k}$$

$$= da_i(1)\, da_i(2) + da_i(1)\, da_j(2) \left[u_{i,j} + u_{j,i} + u_{m,i}\, u_{m,j}\right]$$

$$RHS = \sqrt{da_i(1)\, da_i(1) + 2L_{ik}\, da_i(1)\, da_k(1)}\; \sqrt{da_j(2)\, da_j(2) + 2L_{jm}\, da_j(2)\, da_m(2)}\; \cos\theta$$

(used $(dx)^2 - (da)^2$ formula given).

Now $\dfrac{LHS}{da(1)\, da(2)} = \dfrac{RHS}{da(1)\, da(2)}$ (Note $da(1) = |\underline{da}(1)|$ etc.).

$$\Rightarrow \underbrace{n_i(1)\, n_i(2)}_{\cos\theta_0} + n_i(1)\, n_j(2)\left[2L_{ij}\right] = \sqrt{\underbrace{n_i(1)\, n_i(1)}_{1} + 2L_{ik}\, n_i(1)\, n_k(1)} \;*\; \sqrt{\underbrace{n_j(2)\, n_j(2)}_{1} + 2L_{jm}\, n_j(2)\, n_m(2)}\; \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\cos\theta_0 + 2L_{ij}\, n_i(1)\, n_j(2)}{\sqrt{1 + 2L_{ik}\, n_i(1)\, n_k(1)}\; \sqrt{1 + 2L_{jm}\, n_j(2)\, n_m(2)}}$$ ◄

P-6

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i} u_{m,j}) \longrightarrow$ nonlinear

$$\varepsilon_{ij} = \frac{1}{2}\begin{pmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{nonlinear.}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) \longrightarrow \text{linear}$$

$$\therefore \varepsilon_{ij} = \begin{pmatrix} 0 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(dy)^2 - (da)^2 = 2\varepsilon_{rs}\, da_r\, da_s$$
$$= dy_i\, dy_i - da_i\, da_i$$

Now $dy_1 = \frac{\partial y_1}{\partial a_1} da_1 + \frac{\partial y_1}{\partial a_2} da_2 + \frac{\partial y_1}{\partial a_3} da_3$

$$= da_1 + k\, da_2 + 0$$

$$dy_1 = da_1 + k\, da_2 + 0$$

$$dy_2 = da_2 \;;\; dy_3 = da_3.$$

For AB :-

$$da_1 = dL \; ; \; da_2 = da_3 = 0.$$

$$\therefore \; dy_1 = da_1 = dL.$$

$$dy_2 = 0 \; ; \; dy_3 = 0.$$

$$\therefore (dy)^2 - (da)^2 = da_1 \, da_1 - da_1 \, da_1 = 0.$$

$$\therefore (dy)^2 - (da)^2 = 0.$$

For AD :- $da_1 = 0 \; ; \; da_2 = dL \; ; \; da_3 = 0.$

$$\therefore (dy)^2 - (da)^2 = dy_1 \, dy_1 - da_1 \, da_1$$

$$= K^2 (dL)^2 - \cancel{(dL)^2} \; 0$$

$$(dy)^2 - (da)^2 = \cancel{(k^2 - 1)(dL)^2} \; k^2 (dL)^2$$

$$\therefore (dy)^2 - (da)^2 = k^2 (dL)^2$$

For AC :- $da_1 = dL \; ; \; da_2 = dL \; ; \; da_3 = 0.$

$$(dy)^2 - (da)^2 = dy_1 \, dy_1 - da_1 \, da_1$$

$$= (dL + k\,dL)^2 - (dL)^2$$

$$= (dL)^2 (1 + k^2 + 2k - 1)$$

$$= (k^2 + 2k)(dL)^2$$

For DB :- $da_1 = dL \; ; \; da_2 = -dL \; ; \; da_3 = 0.$

$$\therefore (dy)^2 - (da)^2 = dy_1 \, dy_1 - da_1 \, da_1$$

$$= (dL - k\,dL)^2 - (dL)^2$$

$$= (k^2 - 2k)(dL)^2$$

$$\lambda = \frac{dY}{da}$$

$$\therefore \frac{(dY)^2 - (da)^2}{(da)^2} = \lambda^2 - 1.$$

$$\therefore \lambda = \left[1 + \frac{(dY)^2 - (da)^2}{(da)^2}\right]^{1/2}$$

$$\therefore \mathcal{E} = \frac{dY - da}{da} = \lambda - 1. = \text{unit extension.}$$

in linear theory $\mathcal{E} = \varepsilon_{ij} n_i n_j$

For AB :-

$$\lambda_{AB} = (1+0)^{1/2} = 1.$$

$$\therefore \lambda_{AB} = 1$$

$\mathcal{E}_{AB} = 0.$ by both linear & nonlinear theories.

For AD :-

$$\lambda_{AD} = \left[1 + \frac{K^2 (dL)^2}{(dL)^2}\right]^{1/2}$$

$$\therefore \lambda_{AD} = (1+K^2)^{1/2}$$

$\mathcal{E}_{AD} = (1+K^2)^{1/2} - 1. \longrightarrow$ nonlinear theory

$\varepsilon_{AD} = \varepsilon_{22} n_2 n_2 = 0 \longrightarrow$ linear theory

equal if $K \ll 1$

For AC :-

$$\lambda_{AC} = \left[1 + \frac{(K^2+2K)(dL)^2}{2(dL)^2}\right]^{1/2}$$

$$\lambda_{AC} = \left(1 + K + \frac{K^2}{2}\right)^{1/2}$$

again they are same if $K \ll 1$ we put linear or nonlinear ε_{AC}

$\mathcal{E}_{AC} = \left(1 + K + \frac{K^2}{2}\right)^{1/2} - 1. \longrightarrow$ nonlinear.

$$\varepsilon_{AC} = \underbrace{\varepsilon_{11}}_{0} n_1 n_1 + \varepsilon_{12} n_1 n_2 + \varepsilon_{21} n_1 n_2 + \underbrace{\varepsilon_{22}}_{0} n_2 n_2, \quad n_1 = \frac{1}{\sqrt{2}}, \; n_2 = \frac{1}{\sqrt{2}}$$

$$= 2 \times \frac{K}{2} \times \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{K}{2}$$

For DB

$$\lambda_{DB} = \left(1 + \frac{(K^2-2K)(dL)^2}{2(dL)^2}\right)^{1/2}$$

$$\lambda_{DB} = \left(1 - K + \frac{K^2}{2}\right)^{1/2}$$

$\mathcal{E}_{DB} = \left(1 - K + \frac{K^2}{2}\right)^{1/2} - 1. \longrightarrow$ nonlinear.

$$\varepsilon_{DB} = \varepsilon_{ij} n_i n_j = 2 \times \frac{K}{2} \times \frac{1}{\sqrt{2}} \left(\frac{-1}{\sqrt{2}}\right) = -\frac{K}{2}$$

again same if we put $K \ll 1$ in nonlinear ε_{AC}.

$dV(0) = da_1\, da_2\, da_3$

P7 $dV(t) = J[\underset{\sim}{a}, t]\, dV(0)$

$J[\underset{\sim}{a},t] = \det \begin{vmatrix} \lambda & k & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad \Rightarrow \quad dV(t) = dV(0) \Rightarrow$ incompressibility.

Extra: Lagrangian continuity states, $\rho[\underset{\sim}{a}, 0] = J[\underset{\sim}{a}, t]\, \rho[\underset{\sim}{a}, t]$

Thus we see that $\rho[\underset{\sim}{a}, 0] = \rho[\underset{\sim}{a}, t] \Rightarrow$ incompressibility.

For a differential parallelepiped of sides da_1, da_2, da_3, lying along the x_1, x_2, x_3 axes, we have,

$dx_i^{(1)} = \dfrac{\partial x_i}{\partial a_1} da_1 \Rightarrow (\lambda, 0, 0)\, da_1 \quad ; \quad dx_i^{(2)} = \dfrac{\partial x_i}{\partial a_2} da_2 \Rightarrow (k, \lambda^{-1}, 0)\, da_2$

$dx_i^{(3)} = \dfrac{\partial x_i}{\partial a_3} da_3 \Rightarrow (0, 0, 1)\, da_3$

Since λ is continuous, and $\lambda[0] > 0$, thus if λ @ any t is < 0 it must pass thru zero at $t = t^*$, i.e, $\lambda[t^*] = 0$. This means that @ $t = t^*$, $dx_i^{(1)} = (0, 0, 0)$ which means two pts of a continuum collapse on each other, ie, the side which was initially length da_1 shrinks to a pt. This is physically impossible. Thus $\lambda[t] > 0 \ \forall\, t$.

P(8) $\varepsilon_{11} = u_{1,1} = \frac{a}{4}(x_2+x_3)^2 = a$; $\varepsilon_{22} = a$; $\varepsilon_{33} = a$

$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{\cancel{2}}\frac{a}{4}\left(\cancel{2}x_1(x_2+x_3) + \cancel{2}x_2(x_1+x_3)\right) = a$

$\varepsilon_{13} = \varepsilon_{31} = \varepsilon_{23} = \varepsilon_{32} = a$

Principal strains.

$$a^3 \det\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(1-\lambda)^2-1\right] - 1\left[(1-\lambda)-1\right] + 1\left[1-(1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)^3 - (1-\lambda) - (1-\lambda) + 1 + 1 - (1-\lambda) = 0$$

$$\Rightarrow \cancel{1} - \cancel{3\lambda} + 3\lambda^2 - \lambda^3 - \cancel{3} + \cancel{3\lambda} + \cancel{2} = 0 \Rightarrow \lambda^2(3-\lambda) = 0 \Rightarrow \lambda = 0, 0, 3$$

$$\Rightarrow \epsilon(1) = 3a,\ \epsilon(2) = 0,\ \epsilon(3) = 0$$

When referred to the principal system, $\epsilon'_{11} = \epsilon(1)$, $\epsilon'_{22} = \epsilon(2)$, $\epsilon'_{33} = \epsilon(3)$, all other components are zero.

Then $\varepsilon = \epsilon_{ij} n_i n_j = \epsilon'_{ij} n'_i n'_j = \epsilon(1) n_1'^2 + \epsilon(2) n_2'^2 + \epsilon(3) n_3'^2$

Thus $(\varepsilon)_{max} = \epsilon(1)\, 1^2 = 3a \rightarrow$ occurs along p-direction corresponding to $\epsilon(1)$

$(\varepsilon)_{min} = \epsilon(2)\, 1^2$ or $\epsilon(3)\, 1^2 = 0 \rightarrow$ " " " " " $\epsilon(2)$ or $\epsilon(3)$

END of P.3(b)

extra

So there is no need to find p-directions. Anyway we will do it.

For $\epsilon(1)$: $\left.\begin{array}{l} -2n_1(1) + n_2(1) + n_3(1) = 0 \\ n_1(1) - 2n_2(1) + n_3(1) = 0 \end{array}\right\} \Rightarrow -3n_2(1) + 3n_3(1) = 0 \Rightarrow n_2(1) = n_3(1)$
$\Rightarrow n_1(1) = n_2(1)$

$n_1^2(1) + n_2^2(1) + n_3^2(1) = 1 \Rightarrow n_i(1) \Rightarrow \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

For $\epsilon(2)$: $\left.\begin{array}{l} n_1(2) + n_2(2) + n_3(2) = 1 \\ n_1^2(2) + n_2^2(2) + n_3^2(2) = 1 \end{array}\right\}$ So $n_i(2)$ has some arbitrariness, as expected. Choose as any direction orthogonal to $n_i(1)$. Thus $n_i(2) \Rightarrow \left(0, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$

For $\epsilon(3)$: Similarly choose $n_i(3)$ to be orthogonal to $n_i(1)$ & $n_i(2)$

So $n_i(3) \Rightarrow \left(\mp\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}\right)$

Now
$$\left.\begin{array}{l} \varepsilon_{max} = \epsilon_{ij} n_i(1) n_j(1) = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) * 3 = 3 \\ \varepsilon_{min} = \epsilon_{ij} n_i(2) n_j(2) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0 \\ \text{or } \varepsilon_{min} = \epsilon_{ij} n_i(3) n_j(3) = \left(\frac{-2}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right)^2 = 0 \end{array}\right\} \text{as expected}$$

P9) Since u_θ is multi-valued (ie, for the same physical point we can have $\theta = \theta^*, \theta^* + 2n\pi$), the displ field is not compatible for arbitrary constants.

- In order to ensure finite displacements, $A = 0$ if the origin is part of the continuum.
- In order " " single valued displ.'s, $B = 0$

P.10 a) $\tilde{w}_i = \frac{1}{2}\epsilon_{ijk}\tilde{w}_{kj} = \frac{1}{2}\frac{1}{2}\epsilon_{ijk}(u_{k,j} - u_{j,k}) = \frac{1}{4}(\epsilon_{ijk}u_{k,j} - \epsilon_{ijk}u_{j,k}) = \frac{1}{4}(\epsilon_{ijk}u_{k,j} + \epsilon_{ikj}u_{j,k})$

$= \frac{1}{4}2\epsilon_{ijk}u_{k,j} = \frac{1}{2}\epsilon_{ijk}u_{k,j}$ ◄ (where ',' is $\frac{\partial}{\partial a}$)

(b). $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$e_{11} = 0,\ e_{22} = 0,\ e_{33} = 0,\ e_{12} = CX_3,\ e_{23} = cx_1,\ e_{13} = cx_2$

$w_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \longrightarrow$ get $w_{ij} = 0$

So this displacement field represents a case of pure straining.

(c) $du_i = \tilde{w}_{ij}\, da_j$, $\tilde{w}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)$

$$\tilde{w}_{ij} \Rightarrow \begin{pmatrix} 0 & \frac{c}{2}(a_1 - a_2) & -ca_3 \\ \frac{c}{2}(a_2 - a_1) & 0 & -ca_3 \\ ca_3 & ca_3 & 0 \end{pmatrix}$$

The two pts PQ lie on the line (in $x_1 x_2$ plane) making equal acute ∠'s with positive x_1 & x_2-axes.

Hence $a_1 = a_2$, $a_3 = 0 \Rightarrow \tilde{w}_{ij} = 0$ @ P (also @ Q)

So $du_i = 0$ ◄

P-11 For infinitesimal strains, $\tilde{D} = \frac{\Delta V}{V} = l_{ii} = u_{1,1} + u_{2,2} + u_{3,3}$

For incompressibility, $\Delta V = 0 \Rightarrow \tilde{D} = 0$

$\therefore l_{ii} = u_{1,1} + u_{2,2} + u_{3,3} = (1 - x_2^2)(b + 2cx_1) + u_{2,2} + 0 = 0$

$\Rightarrow u_2 = \left(\frac{x_2^3}{3} - x_2\right)(b + 2cx_1) + f(x_1, x_3)$

Now $u_2|_{x_2 = \pm\sqrt{3}} = f(x_1, x_3) = 0$ (given) $\Rightarrow u_2 = x_2\left(\frac{x_2^2}{3} - 1\right)(b + 2cx_1)$ ◄

P.VI3 $\phi = -\frac{A}{6}x_2^3 - \frac{B}{6}x_1^2x_2^3 + \frac{C}{2}x_1^2x_2 - \frac{q}{4}x_1^2 + \frac{B}{30}x_2^5$ (Shames & Dym, pp64, Prob 1.26)

$\sigma_{11} = \phi_{,22} = -Ax_2 - Bx_1^2x_2 + \frac{2B}{3}x_2^3$

$\sigma_{22} = \phi_{,11} = -\frac{B}{3}x_2^3 + Cx_2 - \frac{q}{2}$

$\sigma_{12} = -(-Bx_1x_2^2 + Cx_1)$

(a) $\nabla^4\phi = 0 \Rightarrow$ Inplane compatibility eqn satisfied. (i.e., the $i=j=1$, $i=j=2$ B-M compat. eqns)
$i=1, j=2$ B-M compat. eqn not satisfied
$i=3, j=3$ " " satisfied ($\because f_i = 0$)

(b) B.C's

$(\sigma_{12} = \sigma_{21})_{x_2=\pm h/2} = 0 \Rightarrow -x_1\left(-\frac{Bh^2}{4} + C\right) = 0 \longrightarrow ①$

$(\sigma_{22})_{x_2=+h/2} = -q \Rightarrow -\frac{Bh^3}{24} + \frac{Ch}{2} - \frac{q}{2} = -q$

$(\sigma_{22})_{x_2=-h/2} = 0 \Rightarrow -\left(-\frac{Bh^3}{24} + \frac{Ch}{2}\right) - \frac{q}{2} = 0$

$\longrightarrow ②$ (Both eqns identical)

$\left(\int_{-h/2}^{h/2}\sigma_{11}x_2\,dx_2\right)_{x_1=\pm\frac{L}{2}} = 0$ (ie zero applied moment at left/right faces)

$\Rightarrow -\frac{Ah^3}{12} - \frac{BL^2}{4}\frac{h^3}{12} + \frac{2B}{3}\frac{h^5}{80} = 0 \longrightarrow ③$

①, ②, ③ $\Rightarrow A = \frac{6q}{h^3}\left(\frac{L^2}{4} - \frac{h^2}{10}\right)$, $B = -\frac{6q}{h^3}$, $C = -\frac{3}{2}\frac{q}{h}$

NOTE: $\left(\int_{-h/2}^{h/2}\sigma_{11}\,dx_2\right)_{x_1=\pm\frac{L}{2}} = 0$ (i.e, zero axial force condt, identically satisfied $\because \sigma_{11}$ has only odd powers of x_2)

$\left(\int_{-h/2}^{h/2}\sigma_{12}\,dx_2\right)_{x_1=\pm\frac{L}{2}} = \pm q\frac{L}{2}$ (i.e, shear force is $\pm\frac{qL}{2}$ at left/right faces, identically satisfied
$\longrightarrow$ (in fact this condt. gives eqn identical to ②)

(c) Stress Distribution (subst A, B, C into σ_{11}, σ_{22}, σ_{12} above)

$\sigma_{11} = -\frac{q}{2I}\left[\left(\frac{L}{2}\right)^2 - \frac{h^2}{10} - x_1^2 + \frac{2}{3}x_2^2\right]x_2$

$\sigma_{22} = \frac{q}{2I}\left[\frac{x_2^3}{3} - \left(\frac{h}{2}\right)^2x_2 - \frac{h^3}{12}\right]$

$\sigma_{12} = -\frac{q}{2I}\left[x_1x_2^2 - \frac{h^2}{4}x_1\right]$

where, $I = \frac{th^3}{12} = \frac{h^3}{12}$

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ (plane stress assumed)

Strain Distribution (Hookes law used, $l_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$

$l_{11} = -\frac{q}{2EI}\left[(2+\nu)\frac{x_2^3}{3} + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_2 - x_1^2x_2 - \frac{\nu h^3}{12}\right]$

$$l_{22} = -\frac{q}{2EI}\left[-(1+2\nu)\frac{x_2^3}{3} + \left(-\nu\frac{L^2}{4} + \frac{\nu h^2}{10} + \frac{h^2}{4}\right)x_2 + \nu x_1^2 x_2 + \frac{h^3}{12}\right]$$
$$l_{12} = -\left(\frac{1+\nu}{E}\right)\frac{q}{2I}\left(x_1x_2^2 - \frac{h^2}{4}x_1\right)$$

stress distribution not valid on left/right faces ∵ σ_{11}, σ_{12} non-zero on these faces.

(d) Displacements

$$l_{11} = \frac{\partial u_1}{\partial x_1} \Rightarrow u_1 = -\frac{q}{2EI}\left[(2+\nu)\frac{x_1x_2^3}{3} + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_1x_2 - \frac{x_1^3x_2}{3} - \frac{\nu h^3}{12}x_1 + f[x_2]\right]$$
$$l_{22} = \frac{\partial u_2}{\partial x_2} \Rightarrow u_2 = -\frac{q}{2EI}\left[-(1+2\nu)\frac{x_2^4}{12} + \left(-\frac{\nu L^2}{4} + \frac{\nu h^2}{10} + \frac{h^2}{4}\right)\frac{x_2^2}{2} + \nu\frac{x_1^2x_2^2}{2} + \frac{h^3}{12}x_2 + g[x_1]\right]$$

subst u_1, u_2 into l_{12}:

$$2l_{12} = (u_{1,2} + u_{2,1}) \Rightarrow \cancel{-\frac{q}{2EI}}\left[2(1+\nu)\left\{\cancel{x_1x_2^2} - \frac{h^2}{4}x_1\right\}\right] = \cancel{-\frac{q}{2EI}}\left[\cancel{(2+\nu)x_1x_2^2} + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_1 - \frac{x_1^3}{3} + f' + \cancel{\nu x_1x_2^2} + g'\right]$$
$$\Rightarrow g' + \left(\frac{L^2}{4} - \frac{h^2}{10} - \frac{\nu h^2}{4}\right)x_1 - \frac{x_1^3}{3} + 2(1+\nu)\frac{h^2}{4}x_1 = -f' = K \text{ (const)}$$
$$\Rightarrow f = -Kx_2 + C_1, \qquad g = Kx_1 + \frac{x_1^4}{12} - \left(\frac{L^2}{4} + \frac{2}{5}h^2 + \frac{1}{4}\nu h^2\right)\frac{x_1^2}{2} + C_2$$

Displacement BC's

$u_2 = 0$ at $x_1 = \frac{L}{2}, x_2 = -\frac{h}{2}$
$u_2 = 0$ at $x_1 = -\frac{L}{2}, x_2 = -\frac{h}{2}$

→ prevent R.B. transl in x_2 dir. and RB rot. in x_1x_2 plane.
Further to prevent RB transl in x_1 dir, put $u_1|_{x_1=\frac{L}{2}, x_2=-\frac{h}{2}} = 0$, find C_1

$\Rightarrow K = 0$

$$C_2 = \frac{5}{192}L^4 + \left(\frac{1}{20} + \frac{\nu}{32}\right)h^2L^2 + h^4\left(\frac{1}{64} - \frac{\nu}{480}\right)$$

$$\Rightarrow u_2[x_1, 0] = -\frac{qL^4}{24EI}\left[\left(\frac{x_1}{L}\right)^4 - \frac{3}{2}\left(\frac{x_1}{L}\right)^2 + \frac{5}{16} + \left(\frac{h}{L}\right)^2\left\{\left(\frac{12}{5} + \frac{3\nu}{2}\right)\left(\frac{1}{4} - \left\{\frac{x_1}{L}\right\}^2\right) + \left(\frac{3}{16} - \frac{\nu}{40}\right)\left(\frac{h}{L}\right)^2\right\}\right]$$

$24u_2^*[x_1, 0]$

(e) Slender-beam

put $\frac{h}{L} \simeq 0$ in $u_2^*[x_1, 0]$ and get,

$$u_2^*[x_1] = -\frac{1}{24}\left[\left(\frac{x_1}{L}\right)^4 - \frac{3}{2}\left(\frac{x_1}{L}\right)^2 + \frac{5}{16}\right]$$

P. 12

Hookes law after inverting → $l_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$ ⇒ $l_{kk} = \frac{1-2\nu}{E}\sigma_{kk}$

$\Rightarrow l_{kk} = 1.869 * 10^{-4} = \tilde{D}$

Now $\tilde{D} = \lim_{V \to 0}\frac{\Delta V}{V}$ but stress/strain distr uniform throughout cube ⇒ $\tilde{D}$ constant throughout cube. ⇒ $\Delta V = \tilde{D}V = 2.336 * 10^{-8} m^3$

P17

$\tilde{f}_1 = \tilde{f}_1[x_1, x_2]$, $\tilde{f}_2 = \tilde{f}_2[x_1, x_2]$, $\tilde{f}_3 = 0$

ext loading $\perp^{ar}$ to longitudinal axis & indep of x_3 $\Rightarrow \frac{\partial}{\partial x_3} = 0$ $\Bigg\} \Rightarrow u_i = u_i[x_1, x_2]$. Assume : $u_3 = 0$ $\Bigg\} \Rightarrow e_{i3} = 0$

Hookes Law (linear case) $\rightarrow$ $\sigma_{ij} = \frac{E}{1+\nu}\left[e_{ij} + \frac{\nu}{1-2\nu} e_{kk}\delta_{ij}\right] \Rightarrow \sigma_{13} = \sigma_{23} = 0$

B-M compatibility (includes equilibrium eqn) :

$i=j=1$: $\nabla^2\sigma_{11} + \frac{1}{1+\nu}(\sigma_{11,11} + \sigma_{22,11} + \sigma_{33,11}) + \tilde{f}_{1,1} + \tilde{f}_{1,1} + \frac{\nu}{1-\nu}(\tilde{f}_{1,1} + \tilde{f}_{2,2} + \cancel{\tilde{f}_{3,3}}^{\,0}) = 0 \longrightarrow$ 1

$i=j=2$: $\nabla^2\sigma_{22} + \frac{1}{1+\nu}(\sigma_{11,22} + \sigma_{22,22} + \sigma_{33,22}) + \tilde{f}_{2,2} + \tilde{f}_{2,2} + \frac{\nu}{1-\nu}(\tilde{f}_{1,1} + \tilde{f}_{2,2} + \cancel{\tilde{f}_{3,3}}^{\,0}) = 0 \rightarrow$ 2

$i=1, j=2$: $\nabla^2\sigma_{12} + \frac{1}{1+\nu}(\sigma_{11,12} + \sigma_{22,12} + \sigma_{33,12}) + \tilde{f}_{1,2} + \tilde{f}_{2,1} = 0 \longrightarrow$ 3

$i=j=3$: $\nabla^2\sigma_{33} + \frac{1}{1+\nu}(\cancel{\sigma_{11,33}}^{\,0} + \cancel{\sigma_{22,33}}^{\,0} + \cancel{\sigma_{33,33}}^{\,0}) + \cancel{f_{3,3}}^{\,0} + \cancel{f_{3,3}}^{\,0} + \frac{\nu}{1-\nu}(f_{1,1} + f_{2,2} + \cancel{f_{3,3}}^{\,0}) = 0 \rightarrow$ 4

$i=1, j=3$:, $i=2, j=3$: $\Big\}$ identically satisfied $(0=0)$

Hookes law $\rightarrow$ $\sigma_{33} = \frac{E}{1+\nu}\left[\cancel{e_{33}}^{\,0} + \frac{\nu}{1-2\nu}(e_{11} + e_{22} + \cancel{e_{33}}^{\,0})\right] = \frac{\cancel{E}\nu}{(1+\nu)(1-2\nu)}\left[\frac{1+\nu}{\cancel{E}}\{\sigma_{11}+\sigma_{22}\} - \frac{2\nu}{E}\{\sigma_{11}+\sigma_{22}+\sigma_{33}\}\right]$

$\Rightarrow \sigma_{33}\left(1 + \frac{2\nu^2}{(1+\nu)(1-2\nu)}\right) = \frac{(1-\nu)\nu}{(1+\nu)(1-2\nu)}\{\sigma_{11} + \sigma_{22}\}$

$\Rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$

Subst. σ_{33} into eqn 3, and get,

$$\nabla^2\sigma_{12} + (\sigma_{11} + \sigma_{22})_{,12} + \tilde{f}_{1,2} + \tilde{f}_{2,1} = 0 \longrightarrow 3^*$$

EOM's are, $\sigma_{11,1} + \sigma_{12,2} + \tilde{f}_1 = 0 \rightarrow$ A
$\sigma_{12,1} + \sigma_{22,2} + \tilde{f}_2 = 0 \rightarrow$ B
$0 = 0$
$\Bigg\}$ Thus $3^* \equiv \frac{\partial A}{\partial x_2} + \frac{\partial B}{\partial x_1}$

So we can replace 3 hence 3^* by the EOM's, ie, A, B.

Adding 1, 2 & 4, use Hookes Law for σ_{33} and get,

$$(1+\nu)\nabla^2(\sigma_{11}+\sigma_{22}) + \frac{1}{1+\nu}\left[(1+\nu)\nabla^2(\sigma_{11}+\sigma_{22})\right] + 2\left(1 + \frac{3\nu}{2(1-\nu)}\right)(\tilde{f}_{1,1} + \tilde{f}_{2,2}) = 0$$

$$\Rightarrow \nabla^2(\sigma_{11}+\sigma_{22}) = -\frac{1}{1-\nu}(\tilde{f}_{1,1} + \tilde{f}_{2,2}) \longrightarrow C$$

Thus A, B, C give exact sol. of p-strain problem.

Now for conservative body forces, the solution which satisfies equil eqns A, B is $\longrightarrow$ $\sigma_{11} = \phi_{,22} - \psi$, $\sigma_{22} = \phi_{,11} - \psi$, $\sigma_{12} = -\phi_{,12}$,
where $f_3 = 0$, $f_i = \psi[x_1, x_2]_{,i}$ for $i = 1, 2$ (Mid-term test Result)

This solution when substituted into (c) yields,

$$\nabla^4\phi = \left(\frac{1-2\nu}{1-\nu}\right)\nabla^2 p$$

NOTE: only ' ' need to be solved in conjunction with appropriate BC's on ϕ, for the p-strain problem. Thus p-strain problems exact sol. reduces to solving this eqn.

P.14 For p-stresses & corresponding p-axes we solve,

$$(\sigma_{ij} - \sigma\delta_{ij})n_j = 0 \longrightarrow (1)$$

Subst. constitutive law for isotropic body, we get,

$$\left(\frac{E}{1+\nu}\left[l_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}l_{mm}\right] - \sigma\delta_{ij}\right)n_j = 0$$

$$\Rightarrow \left(l_{ij} - \left\{\frac{1+\nu}{E}\sigma - \frac{\nu}{1-2\nu}l_{mm}\right\}\delta_{ij}\right)n_j = 0 \longrightarrow (2)$$

Now for p-strains & corresponding p-axes we solve,

$$(l_{ij} - l\delta_{ij})n_j^* = 0 \rightarrow (3)$$

Now (3) & (2) are of the same form. Their solution is obtained by solving either evalue problem (1) or (3). Hence e-vectors n_j & n_j^* must coincide (ie, p-axes of stress (n_j) coincide with p-axes of strain (n_j^*)).

P.15 Thin plate, inplane edge loads only. Since plate is thin it implies that edge loads do not vary in the thickness direction. Furthermore $\bar{f}_3 = 0$ (infact $\bar{f}_1 = \bar{f}_2 = 0$ is given). Thus this can be approximated as a plane stress problem.

Hence $\sigma_{i3} = 0$, $\frac{\partial}{\partial x_3} = 0$

$l_A = -100 * 10^{-6} = l_{ij}n_in_j \quad (n_i = [1,0,0]) \Rightarrow l_A = -100*10^{-6} = l_{11}$

$l_C = 400*10^{-6} = l_{ij}n_in_j \quad (n_i = [0,0,1]) \Rightarrow l_C = 400*10^{-6} = l_{22}$

$l_B = -200*10^{-6} = l_{ij}n_in_j \quad (n_i = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]) \Rightarrow l_B = -200*10^{-6} = \frac{l_{11}}{2} + \frac{l_{22}}{2} + l_{12}$

$\Rightarrow l_{12} = -350*10^{-6}$

From constitutive law $l_{13} = l_{23} = 0$ ($\because \sigma_{13} = \sigma_{23} = 0$)

Also, $\sigma_{33} = 0 = \frac{E}{1+\nu}\left[l_{33} + \frac{\nu}{1-2\nu}(l_{11} + l_{22} + l_{33})\right] \Rightarrow l_{33} = -128.6*10^{-6}$

Thus $\sigma_{11} = \frac{E}{1+\nu}\left[l_{11} + \frac{\nu}{1-2\nu}(l_{11} + l_{22} + l_{33})\right] = 658.85$ psi

$\sigma_{22} = \frac{E}{1+\nu}\left[l_{22} + \frac{\nu}{1-2\nu}(l_{11} + l_{22} + l_{33})\right] = 12197.3$ psi

$\sigma_{12} = \frac{E}{1+\nu}l_{12} = -8076.9$ psi.

For p-stresses,

$$|\sigma_{ij} - \sigma\,\delta_{ij}| = 0 \quad \Rightarrow \quad -\sigma\left[(\sigma-\sigma_{11})(\sigma-\sigma_{22}) - \sigma_{12}^2\right] = 0$$

$$\Rightarrow \sigma(3) = 0, \ \& \ \sigma\begin{smallmatrix}(1)\\(2)\end{smallmatrix} = \frac{(\sigma_{11}+\sigma_{22}) \pm \sqrt{(\sigma_{11}+\sigma_{22})^2 - 4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)}}{2}$$

$$\Rightarrow \sigma(1) = 16353.8 \text{ psi}, \quad \sigma(2) = -3497.7 \text{ psi}, \quad \sigma(3) = 0$$

P.16 (a) $l_{11} = c$, $l_{22} = -c\nu$, $l_{33} = -c\nu$, $l_{12} = l_{13} = l_{23} = 0$

$\Rightarrow l_{ij}$ are const $\Rightarrow \sigma_{ij}$ are const $\Rightarrow \tilde{f}_i = 0$ (from equil eqns).

(b) $\mu = \dfrac{E}{2(1+\nu)}$ & $K = \dfrac{E}{3(1-2\nu)}$

$$\left.\begin{array}{l} \text{For } E>0, \ \& \ \mu>0, \ \nu \geq -1 \\ \text{For } E>0, \ \& \ K>0, \ \nu \leq \frac{1}{2} \end{array}\right\} \Rightarrow -1 \leq \nu \leq 1/2$$