Tutorial 3(b) — Solution.

(1) $P \to Nm^{-1}$

Let $\phi = P\, g(r) f(\theta)$.

from 1^{st} term — from 2^{nd} term

Nm^{-2} $\sigma_{rr} = \frac{1}{r^2}\phi_{,\theta\theta} + \frac{1}{r}\phi_{,r} \Rightarrow g(r) = r$, or $g' = \text{const}$ (from dimensional analysis)

same thing

$\sigma_{\theta\theta} = \phi_{,rr} \Rightarrow g'' = \frac{1}{r}$, ie $g = r(\ln r - 1)$ (from dim analysis)

1^{st} term — 2^{nd} term

$\sigma_{r\theta} = -\left(\frac{1}{r}\phi_{,\theta}\right)_{,r} = \frac{1}{r^2}\phi_{,\theta} - \frac{1}{r}\phi_{,r\theta} \Rightarrow g(r) = r$ or $g' = \text{const}$

same thing

Notice that all these give a unique poly form for $g(r)$

We try only the polynomial forms, first.

So $g(r) = r$ chosen for semi-inverse method

If poly form doesn't work, we will have to try other functional forms...

So choose $\phi = r f(\theta)$ [P is absorbed back in $f(\theta)$ so it will appear in consts of integration later on]

Substitute in $\nabla^4\phi = 0$, get, $\rightarrow$ see bottom of page for 'routine' details.

$$\frac{1}{r^3}\left[f^{IV} + 2f'' + f\right] = 0$$

$$f = e^{st} \Rightarrow s^4 + 2s^2 + 1 = 0 \Rightarrow (s^2+1)^2 = 0, \quad s = \pm i, \pm i.$$

$$f = A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta.$$

$$\sigma_{rr} = \frac{1}{r^2} r\left[\cancel{-A\cos\theta} \cancel{- B\sin\theta} + C(\cos\theta - \theta\sin\theta)_{,\theta} + D(\sin\theta + \theta\cos\theta)_{,\theta}\right]$$
$$+ \frac{1}{r}\left[\cancel{A\cos\theta} + \cancel{B\sin\theta} + C\theta\cos\theta + D\theta\sin\theta\right]$$
$$= \frac{1}{r}\left[C(-\sin\theta - \sin\theta - \cancel{\theta\cos\theta}) + D(\cos\theta + \cos\theta - \cancel{\theta\sin\theta})\right.$$
$$\left. + \cancel{C\theta\cos\theta} + \cancel{D\theta\sin\theta}\right]$$

$$\sigma_{rr} = \frac{2}{r}(-C\sin\theta + D\cos\theta) \longrightarrow (1)$$

$$\sigma_{\theta\theta} = \phi_{,rr} = 0 \longrightarrow (2)$$

$$\sigma_{r\theta} = -\left(\frac{1}{r} r f'\right)_{,r} = 0 \longrightarrow (3)$$

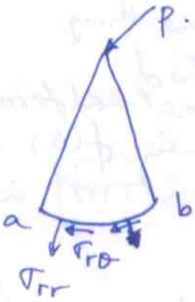
$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{1}{r}f + \frac{1}{r^2}rf''\right)$$
$$= \left(\frac{2}{r^3}f + \frac{2}{r^3}f'' - \frac{1}{r^3}f - \frac{1}{r^3}f'' + \frac{1}{r^3}f'' + \frac{1}{r^3}f^{IV}\right) = \frac{1}{r^3}\left(f^{IV} + 2f'' + f\right) = 0$$

Stress BC's.

$$\sigma_{\theta\theta}\Big|_{\theta=\pm\frac{\alpha}{2}} = 0 \longrightarrow \text{identically satisfied by } ②$$

$$\sigma_{r\theta}\Big|_{\theta=\pm\frac{\alpha}{2}} = 0 \longrightarrow \text{identically satisfied. by } ③$$

But we have to get load P into the solution. This is done via integral BC's along section a-b.



$$\sum F_x : \quad P\cos\beta + \int_{-\alpha/2}^{\alpha/2} \sigma_{rr} \underbrace{(r\,d\theta)}_{dA} \cos\theta = 0 \longrightarrow ④$$

$$\sum F_y : \quad \int_{-\alpha/2}^{\alpha/2} \sigma_{rr}\, r\, d\theta \sin\theta + P\sin\theta = 0 \longrightarrow ⑤$$

$$①, ④, ⑤ \Rightarrow \frac{2}{\cancel{r}} \int_{-\alpha/2}^{\alpha/2} (D\cos^2\theta - C\sin\theta\cos\theta)\cancel{r}\, d\theta + P\cos\beta = 0$$

$$\frac{2}{\cancel{r}} \int_{-\alpha/2}^{\alpha/2} (D\cos\theta\sin\theta - C\sin^2\theta)\cancel{r}\, d\theta + P\sin\beta = 0.$$

$$\text{Integrate} \Rightarrow \left.\begin{array}{l} D(\sin\alpha + \alpha) + P\cos\beta = 0 \\ C(\sin\alpha - \alpha) + P\sin\beta = 0 \end{array}\right\} \text{solve for } C, D$$

$$\Rightarrow \sigma_{rr} = -\frac{2P}{r}\left[\frac{\cos\beta\cos\theta}{\sin\alpha + \alpha} + \frac{\sin\beta\sin\theta}{\alpha - \sin\alpha}\right]$$

<u>Note:</u> A, B constants dont appear in solution. Notice that terms involving them, in ϕ, are $Ar\cos\theta = Ax$ and $Ar\sin\theta = Ay$. Since σ_{xx}, σ_{yy}, σ_{xy} involve double-diff so linear terms in ϕ dont affect stress components in Cartesian, and hence Polar, coordinates.

P.2 Put $\beta = 0$, $\alpha = \pi$ in result of P.1 and get the result (ie stress distribution of P.3 of HW# 3).

P.3. $M \to N m m^{-1} = N$.

Let $\phi = M\, g(r) f(\theta)$. Now do as in P.1, ie dimensional analysis.

$\sigma_{rr} \Rightarrow g(r) = \text{const}$ or $g(r) = \ln r$

$\sigma_{\theta\theta} \Rightarrow g(r) = \ln r$

$\sigma_{r\theta} \Rightarrow g(r) = \text{const}$ or $g(r) = \ln r$.

So try poly form first. Hopefully it will work.

Semi-inverse $\rightarrow$ $\phi = \underbrace{g(r)}_{\text{"const"}} f(\theta) = f(\theta)$ (where M and the "const" are absorbed in f).

So we expect consts of integration in f to contain M.

$$\nabla^4 \phi = 0 \Rightarrow \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{1}{r^2} f''\right)$$

$$= \frac{6f''}{r^4} - \frac{2f''}{r^4} + \frac{1}{r^4} f^{IV} = \frac{1}{r^4}\left(f^{IV} + 4f''\right) = 0$$

$$\Rightarrow s^4 + 4s^2 = 0, \quad s = 0, 0, \pm 2i$$

$$f(\theta) = A\cos 2\theta + B\sin 2\theta + C\theta + D.$$

$$\sigma_{rr} = -\frac{4}{r^2}\left(A\cos 2\theta + B\sin 2\theta\right) \longrightarrow (1)$$

$$\sigma_{\theta\theta} = 0 \longrightarrow (2)$$

$$\sigma_{r\theta} = +\frac{2}{r^2}\left[-A\sin 2\theta + B\cos 2\theta + \frac{C}{2}\right] \longrightarrow (3)$$

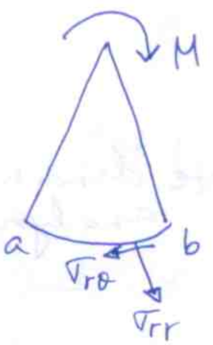
Stress BC's

$\sigma_{\theta\theta}\Big|_{\theta = \pm\frac{\alpha}{2}} = 0 \longrightarrow$ identically satisfied by (2)

$\sigma_{r\theta}\Big|_{\theta = \pm\frac{\alpha}{2}} = 0 \Rightarrow A = 0$ (directly observe that $A\sin 2\theta$ is an odd term, so all odd terms must vanish $\Rightarrow A = 0$).

$\frac{C}{2} = -B\cos\alpha$

Again, to get load into solution you need to use integral BC's along face a-b.



$$\Sigma M : \quad M + \int_{-\alpha/2}^{\alpha/2} \sigma_{r\theta} (r\,d\theta) r = 0$$

$$\Rightarrow \quad M + \int_{-\alpha/2}^{\alpha/2} \frac{2B}{\cancel{r^2}} (\cos 2\theta - \cos\alpha) \cancel{r^2}\, d\theta = 0$$

$$\text{Integration} \Rightarrow \quad 2B = - \frac{M}{\sin\alpha - \alpha\cos\alpha}$$

$$\sigma_{rr} = \frac{2M \sin 2\theta}{(\sin\alpha - \alpha\cos\alpha) r^2}$$

$$\sigma_{r\theta} = - \frac{M(\cos 2\theta - \cos\alpha)}{(\sin\alpha - \alpha\cos\alpha) r^2}$$

Extra:

You can check that this stress distribution also satisfies $\Sigma F_x = 0$, $\Sigma F_y = 0$ at face a-b, ie,

$$\left.\begin{aligned} \Sigma F_y &= \int_{-\alpha/2}^{\alpha/2} (\sigma_{rr} \sin\theta + \sigma_{r\theta} \cos\theta)\, r\,d\theta = 0 \\ \Sigma F_x &= \int_{-\alpha/2}^{\alpha/2} (\sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta)\, r\,d\theta = 0 \end{aligned}\right\}$$

Check it if interested.