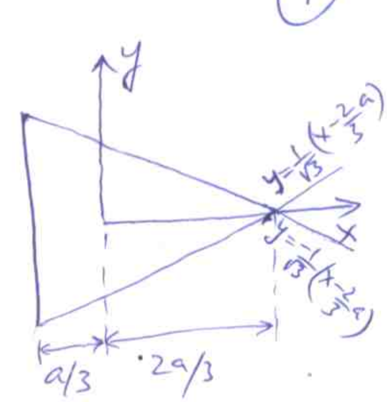
CE 623 — Homework # 4.

P-1 $\phi = m\left[x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right]$



(a) $\nabla^2\phi = -4am = -2G\alpha$

$\Rightarrow m = \frac{G\alpha}{2a} \longrightarrow$ 1

(b) $\phi = \frac{G\alpha}{2a}\left[x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right]$

$\tau_{xz} = \frac{\partial\phi}{\partial y} = -\frac{G\alpha y}{a}(3x+a) \longrightarrow$ 2

$\tau_{yz} = -\frac{\partial\phi}{\partial x} = -\frac{G\alpha}{2a}\left[3x^2 - 2ax - 3y^2\right] \longrightarrow$ 3

$\boxed{\tau_{xy} = \tau_{yz} = 0 \text{ at corners}}$ (can see directly from fact that $\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$ = tangential to boundary — ie $\phi = \overset{\text{const}}{\phi}$ line, so $\tau = 0$ at corners since otherwise τ would have two directions at the corners $\Rightarrow \tau_{xz} = \tau_{yz} = 0$). Can also get by substituting coordinates of corners into 2 & 3.

$\boxed{\tau_{xy} = \tau_{yz} = 0 \text{ at centroid}}$ ($x = y = 0$).

(c) In order to find the maximum, it suffices to consider the side $x = -a/3$ (since all sides are symmetrically located wrt centroid). Put $x = -a/3$,

$\tau_{xz} = 0, \quad \tau_{yz} = -\frac{G\alpha}{2a}\left[3\left(\frac{a^2}{9}\right) + \frac{2}{3}a^2 - 3y^2\right]$

Max magnitude obviously occurs at $y = 0$

$\Rightarrow |\tau|_{max} = \frac{G\alpha a}{2}$ at $x = -\frac{a}{3}, y = 0$

Max mag, $|\tau|_{max}$, will also occur at the middle of the other two sides.

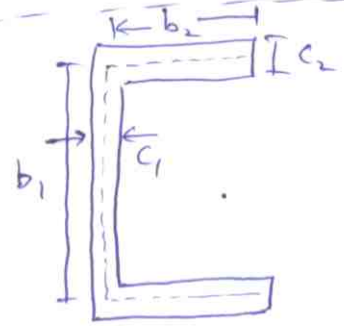
(d) $M = \frac{2G\alpha}{a}\int_{-a/3}^{2a/3}\int_{0}^{1/\sqrt{3}(x-\frac{2}{3}a)}\left(x^3 - ax^2 - 3y^2x + \frac{4}{27}a^3 - ay^2\right)dy\,dx$

$= \frac{2G\alpha}{a\sqrt{3}}\int_{-a/3}^{2a/3}\left[\left(x^3 - ax^2 + \frac{4}{27}a^3\right) - \frac{1}{3}\left(x^3 - ax^2 + \frac{4}{27}a^3\right)\right]\left(x - \frac{2}{3}a\right)dx$

$$= \frac{4G\alpha}{3a\sqrt{3}} \int_{-a/3}^{2a/3} \left(x^4 - \frac{5}{3} a x^3 + \frac{2}{3} a^2 x^2 + \frac{4}{27} a^3 x - \frac{8}{81} a^4 \right) dx$$

$$= \frac{4G\alpha a^4}{\sqrt{3}} \left(\frac{81}{4860} \right) = \frac{G\alpha a^4}{15\sqrt{3}}$$

$$C = \frac{M}{\alpha} = \frac{Ga^4}{15\sqrt{3}}$$



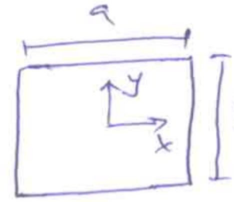
P.2 $\alpha \simeq \dfrac{3M}{(b_1 c_1^3 + 2 b_2 c_2^3) G} = \dfrac{3 * 600}{(0.191 * 0.005^3 + 2 * 0.0975 * 0.009^3) * 77.5 * 10^6}$

$= 139.9$ rad/m

$$\tau_{max} = \frac{3M(b_i)_{max}}{\sum a_i b_i^3} = \frac{3 * 600 * 0.009}{(\;)} = 97.57 \text{ MPa}$$

Note: this should have actually been GPa, the $\alpha = 0.1399$ (more realistic).

P.3. Note: Trying $\phi(x,y)$ as eqn of boundary



ie $\phi(x,y) = m\left(x^2 - \frac{a^2}{4}\right)\left(y^2 - \frac{b^2}{4}\right)$

satisfies $\phi = 0$ on boundary, but $\nabla^2 \phi = \text{const}$ not satisfied.

(a) The given double Fourier series satisfies $\phi = 0$ on boundary.

$$-\nabla^2 \phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 A_{mn} \sin \frac{m\pi(x + a/2)}{a} \sin \frac{n\pi(y + b/2)}{b} = +2G\alpha$$

Multiplying both sides by $\sin \dfrac{p\pi(x + a/2)}{a} \sin \dfrac{q\pi(y + b/2)}{b}$

and integrating between the limits $\left(-\frac{a}{2}, \frac{a}{2}\right)$ for x and $\left(-\frac{b}{2}, \frac{b}{2}\right)$ for y, we get, (using orthogonality property of sine function).

$$-\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 \left(\frac{ab}{4} \right) A_{mn} = \frac{-2G\alpha}{\left(\frac{mn\pi^2}{ab}\right)} \left[1 - (-1)^m\right]\left[1 - (-1)^n\right]$$

$$\Rightarrow A_{mn} = \frac{8G\alpha}{mn\pi^4} \frac{\left[1-(-1)^m\right]\left[1-(-1)^n\right]}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin \frac{m\pi(x+a/2)}{a} \cos \frac{n\pi(y+b/2)}{b} \left(n\pi/b\right)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \frac{m\pi}{a} \cos \frac{m\pi(x+a/2)}{a} \sin \frac{n\pi(y+b/2)}{b}$$

$$\tau_{xz}\Big|_{x=\pm a/2} = 0, \quad \tau_{yz}\Big|_{y=\pm b/2} = 0$$ (traction free b.c.'s verified)

(b) $$M = 2\int_A \phi \, dA = 2\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi(x+a/2)}{a} \sin \frac{n\pi(y+b/2)}{b} \, dx\,dy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2A_{mn} \frac{ab}{mn\pi^2}\left[1-(-1)^m\right]\left[1-(-1)^n\right]$$ → (subst A_{mn} from above)

$$M = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16G\alpha(ab)}{m^2n^2\pi^6} \frac{\left[1-(-1)^m\right]^2\left[1-(-1)^n\right]^2}{\left[(m/a)^2 + (n/b)^2\right]}$$

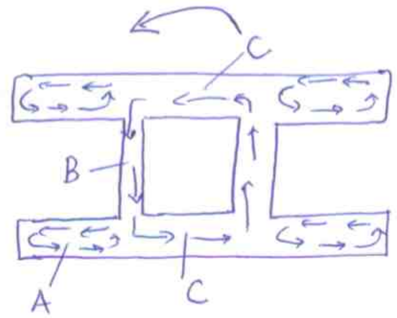
$$\Rightarrow \alpha = \frac{M\pi^6}{16G\,ab} \frac{1}{\displaystyle\sum_{m,n=1}^{\infty} \frac{\left[1-(-1)^m\right]^2\left[1-(-1)^n\right]^2}{m^2n^2\left[(m/a)^2+(n/b)^2\right]}}$$ subst in A_{mn}.

$$\Rightarrow A_{mn} = \frac{M\pi^2}{2\,ab\,mn} \frac{\left[1-(-1)^m\right]\left[1-(-1)^n\right]}{\left[(m/a)^2 + (n/b)^2\right]} \frac{1}{\displaystyle\sum_{m,n=1}^{\infty} \frac{\left[1-(-1)^m\right]^2\left[1-(-1)^n\right]^2}{m^2n^2\left[(m/a)^2+(n/b)^2\right]}}$$

subst in τ_{xz}, τ_{yz} to get them in terms of M.

(c) $$C = \frac{M}{\alpha} = \frac{16Gab}{\pi^6} \sum_{m,n=1}^{\infty} \frac{\left[1-(-1)^m\right]^2\left[1-(-1)^n\right]^2}{m^2n^2\left[(m/a)^2+(n/b)^2\right]}$$

P.4



Since the four limbs are identical they contribute equal torsional moments

Let M_L = torsional moments in all 4 limbs.

M_B = torsional moment in closed loop (box).

Note: 68950 kPa = 703.6 kgf/cm²

Case I: If τ_{max} occurs in limbs,

$$M_L = \frac{\overset{\tau_{max}}{\tau_i} \sum a_i b_i^3}{3 b_i} = \frac{703.6 * (4 * 25.4 * 0.63^3)}{3 * 0.63} = 9455.70 \text{ kgf-cm}$$

$$\alpha_1 = \frac{3 M_L}{G \sum a_i b_i^3} = \frac{3 * 9456}{G * 25.4} = \frac{1116.85}{G}$$

Case II: If τ_{max} occurs in box, then it must occur in lower & upper horizontal legs since 't' (thickness) is smaller there.

$$\alpha_2 = \frac{a q}{2 G A} = \frac{\overbrace{2 * \left(\frac{50.8}{0.63} + \frac{25.4}{1.27}\right)}^{a} \overbrace{\left(703.6 * 0.63\right)}^{q}}{2 * G * 50.8 * 25.4} = \frac{34.57}{G}$$

$\because \alpha_2 < \alpha_1 \Rightarrow \tau_{max}$ occurs as per Case II, ie in part C.

$\tau_c = 703.6$ kgf/cm² = 69023 kPa

$\tau_B = \frac{703.6}{2}$ ($\because$ B has twice thickness of C) = 351.8 kgf/cm² = 34511 kPa

$$\tau_A = \frac{3 M_i}{a_i b_i^2} = \frac{\alpha a_i b_i^3 G}{a_i b_i^2} = 34.57 * b_i = 21.7 \text{ kgf/cm}^2 = 2129 \text{ kPa}$$

$$M = M_L + M_B = \frac{\alpha G}{3} \sum a_i b_i^3 + 2 q A$$

$$= \frac{34.57}{3} * (4 * 25.4 * 0.63^3) + 2 * (703.6 * 0.63) * (50.8 * 25.4)$$

$$= 1.144207 * 10^6 \text{ kgf.cm.} = 112.247 \text{ kN.m}$$

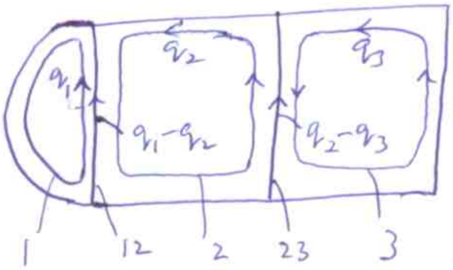
P.5 $M = 2q_1A_1 + 2q_2A_2 + 2q_3A_3 \longrightarrow (1)$

$A_1 = \frac{\pi}{2}(0.127)^2 = 0.02533$, $A_2 = A_3 = (0.254)^2 = 0.064516$

$a_1 = \frac{\pi(0.127)}{0.0006} + \frac{0.254}{0.0013} = 860.35$, $a_{12} = \frac{0.254}{0.0013} = 195.38$

$a_2 = \frac{3*0.254}{0.0008} + \frac{0.254}{0.0013} = 1147.88$, $a_{23} = \frac{0.254}{0.0008} = 317.5$

$a_3 = \frac{3*0.254}{0.0008} + \frac{0.254}{0.0010} = 1206.5$



$a = \int \frac{ds}{t}$ for loop or leg as indicated.

from $2G\alpha = \frac{1}{A_i}\oint_{c_i} \frac{q}{t} ds$, c_i ith closed contour.

$$2G\alpha = \frac{1}{A_1}\left[a_1q_1 - a_{12}q_2\right] \longrightarrow (2)$$

$$2G\alpha = \frac{1}{A_2}\left[a_2q_2 - a_{12}q_1 - a_{23}q_3\right] \longrightarrow (3)$$

$$2G\alpha = \frac{1}{A_3}\left[a_3q_3 - a_{23}q_2\right] \longrightarrow (4)$$

$$(2,3) \longrightarrow 2G\alpha \underbrace{(A_1a_2 + A_2a_{12})}_{=K_{1\alpha} = 41.68} = q_1 \underbrace{(a_1a_2 - a_{12}^2)}_{=K_{11} = 949413.19} + q_3 \underbrace{(-a_{12}a_{23})}_{=K_{13} = -62034.61} \longrightarrow (\mathrm{I})$$

$$(3,4) \longrightarrow 2G\alpha \underbrace{(A_2a_{23} + A_3a_2)}_{=K_{2\alpha} = 94.54} = q_1 \underbrace{(-a_{12}a_{23})}_{=K_{21} = -62034.61} + q_3 \underbrace{(a_3a_2 - a_{23}^2)}_{=K_{23} = 1284116.54} \longrightarrow (\mathrm{II})$$

$$(\mathrm{I},\mathrm{II}) \longrightarrow q_1 = 2G\alpha \underbrace{\frac{(K_{1\alpha}K_{23} - K_{2\alpha}K_{13})}{(K_{11}K_{23} - K_{21}K_{13})}}_{=C_1} = 2G\alpha\, C_1, \quad C_1 = 4.887353*10^{-5}$$

$$q_3 = 2G\alpha \underbrace{\frac{(K_{1\alpha}K_{21} - K_{2\alpha}K_{11})}{(K_{13}K_{21} - K_{23}K_{11})}}_{=C_3} = 2G\alpha\, C_3, \quad C_3 = 7.598423*10^{-5}$$

$$q_2 = 2G\alpha \underbrace{\frac{(-A_3 + a_3C_3)}{a_{23}}}_{=C_2} = 2G\alpha\, C_2, \quad C_2 = 8.554007*10^{-5}$$

$$M = 113000 = 4G\alpha(C_1A_1 + C_2A_2 + C_3A_3)$$

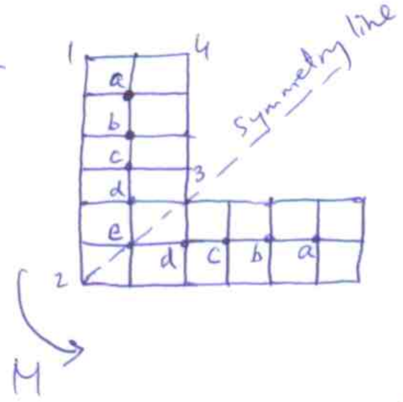
$\Rightarrow G\alpha = 2422993697$

$\Rightarrow q_1 = 236840.5$, $q_2 = 414526.1$, $q_3 = 368218.6$

$\tau_1 = \frac{q_1}{0.0006} = 394734*10^3\ N/m^2$, $\tau_2 = \frac{q_2}{0.0008} = 518157*10^3\ N/m^2$, $\tau_3 = \frac{q_3}{0.0008} = 460273*10^3\ N/m^2$

$\tau_4 = \frac{q_1 - q_2}{0.0013} = -136681*10^3\ N/m^2$, $\tau_5 = \frac{q_2 - q_3}{0.0008} = 57884*10^3\ N/m^2$, $\tau_6 = \frac{q_3}{0.0010} = 368218*10^3\ N/m^2$

P.6. (6)



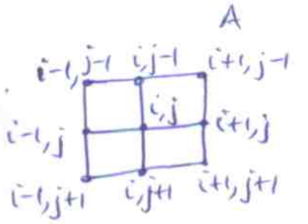
$\nabla^2\phi = -2G\alpha$ — Central difference

$$\phi_b - 4\phi_a = -2G\alpha h^2 \longrightarrow (1)$$
$$\phi_a + \phi_c - 4\phi_b = -2G\alpha h^2 \longrightarrow (2)$$
$$\phi_b + \phi_d - 4\phi_c = -2G\alpha h^2 \longrightarrow (3)$$
$$\phi_e + \phi_c - 4\phi_d = -2G\alpha h^2 \longrightarrow (4)$$
$$2\phi_d - 4\phi_e = -2G\alpha h^2 \longrightarrow (5)$$

We have used Central diff approx of th.3, ie,

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -2G\alpha h^2$$ to get (1)-(5).

$M = 2\iint\limits_A \phi\, dx\, dy \rightarrow$ use Simpsons formula, ie,



$$\int_{x_i-h}^{x_i+h}\int_{y_j-h}^{y_j+h} \phi\, dx\, dy = \frac{h^2}{9}\Big[16\phi_{i,j} + 4(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j-1} + \phi_{i,j+1}) + \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\Big]$$

and add integrals over all such 4-square areas.

$$M = \frac{h^2}{9}\Big[\underbrace{32\phi_a + 8\phi_b + 32\phi_c + 8\phi_b + 8\phi_d}_{\text{includes contribution of symmetric counterpart of 1-2-3-4}} + 16\phi_e + 8\phi_d\Big]$$

$$= \frac{16h^2}{9}\left[2\phi_a + \phi_b + 2\phi_c + \phi_d + \phi_e\right] \longrightarrow (6)$$

$(1,2) \rightarrow \phi_c - 15\phi_a = -10G\alpha h^2 \longrightarrow (A)$

$(2,3,A) \rightarrow \phi_a + \phi_c + 4\phi_d - 16\phi_c = -10G\alpha h^2$

$\Rightarrow \phi_a + 4\phi_d + 15(10G\alpha h^2 - 15\phi_a) = -10G\alpha h^2$

$\Rightarrow -224\phi_a + 4\phi_d = -160G\alpha h^2 \longrightarrow (B)$

$(3,4,1,A) \rightarrow 4\phi_b - 16\phi_c + \phi_e + \phi_c = -10G\alpha h^2$

$\Rightarrow 4(4\phi_a - 2G\alpha h^2) + 15(10G\alpha h^2 - 15\phi_a) + \phi_e = -10G\alpha h^2$

$\Rightarrow -209\phi_a + \phi_e = -152G\alpha h^2 \longrightarrow (C)$

$(4,5,A) \longrightarrow \phi_e + \phi_c - 8\phi_e = -6G\alpha h^2$

$15\phi_a - 10G\alpha h^2 - 7\phi_e = -6G\alpha h^2$

$15\phi_a - 7\phi_e = 4G\alpha h^2 \longrightarrow (D)$

$$(C, D) \longrightarrow (-7 * 209 + 15)^{-1} * (-7 * 152 + 4)\, G\alpha h^2 = \phi_a$$

$$\phi_a = 0.732044 \; G\alpha h^2$$

$$\phi_b = 0.928177 \; G\alpha h^2$$

$$\phi_c = 0.980663 \; G\alpha h^2$$

$$\phi_d = 0.994475 \; G\alpha h^2$$

$$\phi_e = 0.997238 \; G\alpha h^2$$

$$\alpha = 0.0886482 \; M/Gh^4$$