CE623. HW#5 Solutions.

The aim is to eliminate all σ_i's and e_i's in favor of σ & e.

P-1.

$$e = e_3 = e_1 + e_2 \longrightarrow (a) \quad (a_1, a_2)$$
$$e_1 = e_4 \longrightarrow (b)$$
$$\sigma_2 = \sigma_1 + \sigma_4 \longrightarrow (c)$$
$$\sigma = \sigma_3 + \sigma_2 \longrightarrow (d)$$

$$\sigma_1 = k_1 e_1 \longrightarrow (e)$$
$$\sigma_2 = c_2 \dot{e}_2 \longrightarrow (f)$$
$$\sigma_3 = k_3 e_3 \longrightarrow (g)$$
$$\sigma_4 = c_4 \dot{e}_4 \longrightarrow (h)$$

Using (d, f, g, a_1, a_2)

$$\sigma = \sigma_3 + \sigma_2 = k_3 e_3 + c_2 \dot{e}_2 = k_3 e + c_2(\dot{e} - \dot{e}_1) \longrightarrow ①$$

Using $(a_2, b, f, h, c, d)\, g, a_1)$
Also,

$$\dot{e} = \dot{e}_1 + \dot{e}_2 = \dot{e}_4 + \dot{e}_2 = \frac{\sigma_4}{c_4} + \frac{\sigma_2}{c_2} = \frac{\sigma_2 - \sigma_1}{c_4} + \frac{\sigma - \sigma_3}{c_2}$$

$$= \frac{\sigma - \sigma_3 - \sigma_1}{c_4} + \frac{\sigma - \sigma_3}{c_2} = (\sigma - \underbrace{\sigma_3}_{= k_3 e_3 = k_3 e})\left(\frac{1}{c_4} + \frac{1}{c_2}\right) - \frac{\sigma_1}{c_4}$$

$$\dot{e} = (\sigma - k_3 e)\left(\frac{1}{c_4} + \frac{1}{c_2}\right) - \frac{\sigma_1}{c_4} \longrightarrow ②$$

Substitute (2), (e) in (1):

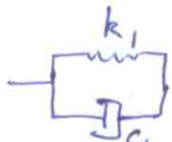
$$\sigma = k_3 e + c_2\left(\dot{e} - \frac{\dot{\sigma}_1}{k_1}\right) = k_3 e + c_2\left(\dot{e} - \frac{1}{k_1}\left\{c_4\left[(\dot{\sigma} - k_3 \dot{e})\left(\frac{c_2 + c_4}{c_2 c_4}\right) - \ddot{e}\right]\right\}\right)$$

$$= k_3 e + c_2\left(\dot{e} + \frac{c_4}{k_1}\ddot{e} - \frac{\cancel{c_4}}{k_1}\left(\frac{c_2 + c_4}{c_2 \cancel{c_4}}\right)\dot{\sigma} + \frac{\cancel{c_4}}{k_1} k_3 \left(\frac{c_2 + c_4}{c_2 \cancel{c_4}}\right)\dot{e}\right)$$

$$\sigma + \left(\frac{c_2 + c_4}{k_1}\right)\dot{\sigma} = k_3 e + \left[c_2 + \frac{k_3}{k_1}(c_2 + c_4)\right]\dot{e} + \left(\frac{c_2 c_4}{k_1}\right)\ddot{e}$$

Alternative - I

Use Voight element



We have, $\sigma_v = \sigma_1 + \sigma_4$, $e_v = e_1 = e_4$, $\sigma_1 = k_1 e_1$, $\sigma_4 = c_4 \dot{e}_4$

$$\Rightarrow \sigma_v = k_1 e_v + c_4 \dot{e}_v$$

Also, $\sigma_v = \sigma_2$, $e = e_3 = e_v + e_2$, $\sigma = \sigma_2 + \sigma_3$, $\sigma_2 = c_2 \dot{e}_2$, $\sigma_3 = k_3 e_3$

$$\Rightarrow \dot{\sigma} = \dot{\sigma}_2 + \dot{\sigma}_3 = c_2 \ddot{e}_2 + k_3 \dot{e}_3 = c_2(\ddot{e} - \ddot{e}_v) + k_3 \dot{e}$$

$$= c_2\left(\ddot{e} - \frac{1}{c_4}\left[\dot{\sigma}_v - k_1 \dot{e}_v\right]\right) + k_3 \dot{e} =$$

$$= c_2\left(\ddot{e} - \frac{1}{c_4}\left[\dot{\sigma}_2 - k_1(\dot{e} - \dot{e}_2)\right]\right) + k_3 \dot{e} = c_2\left(\ddot{e} - \frac{1}{c_4}\left[\dot{\sigma} - \dot{\sigma}_3 - k_1\left(\dot{e} - \frac{\sigma_2}{c_2}\right)\right]\right) + k_3 \dot{e}$$

$$= c_2\left(\ddot{e} - \frac{1}{c_4}\left[\dot{\sigma} - k_3 \dot{e}_3 - k_1\left(\dot{e} - \frac{\sigma - \sigma_3}{c_2}\right)\right]\right) + k_3 \dot{e}$$

$$= c_2\left(\ddot{e} - \frac{1}{c_4}\left[\dot{\sigma} - k_3 \dot{e} - k_1\left(\dot{e} - \frac{\sigma - k_3 e_3}{c_2}\right)\right]\right) + k_3 \dot{e}$$

$$= c_2 \ddot{e} - \frac{c_2}{c_4}\dot{\sigma} + \frac{c_2}{c_4} k_3 \dot{e} + \frac{c_2}{c_4} k_1 \dot{e} - \frac{\cancel{c_2}}{c_4}\frac{k_1}{\cancel{c_2}}\sigma + \frac{\cancel{c_2}}{c_4} k_1 \frac{k_3}{\cancel{c_2}} e + k_3 \dot{e}$$

$$\Rightarrow \sigma + \left(\frac{c_2 + c_4}{\cancel{c_4}}\right)\left(\frac{\cancel{c_4}}{k_1}\right)\dot{\sigma} = \frac{c_2 c_4}{k_1}\ddot{e} + \left(c_2 \frac{k_3}{k_1} + c_2 + c_4 \frac{k_3}{k_1}\right)\dot{e} + k_3 e$$

checks out with previous result.

Alternative – II

Use (a–h) but in L.T. (Laplace transform domain). Then eliminate E_i, Σ_i in favor of E & Σ_1.

Let $\mathcal{L}(\sigma_i) = \Sigma_i(s)$, $\mathcal{L}(e_i) = E_i(s)$.

$$\Sigma_1 = \Sigma_{13} + \Sigma_{12} = \Sigma_{13} + \Sigma_{11} + \Sigma_{14} = k_3 E_3 + k_1 E_1 + s c_4 E_4$$

$$= k_3 E + k_1(E - E_2) + s c_4 (E - E_2)$$

$$= (k_3 + k_1)E + s c_4 E - (k_1 + s c_4)E_2$$

$$= (k_3 + k_1)E + s c_4 E - (k_1 + s c_4)\frac{\Sigma_{12}}{s c_2}$$

$$= (k_3 + k_1)E + s c_4 E - (k_1 + s c_4)\frac{(\Sigma_1 - \Sigma_{13})}{s c_2} \quad \Sigma_{13} \to k_3 E_3 \to E$$

$$\Rightarrow s c_2 \Sigma_1 = s c_2 (k_3 + k_1)E + s^2 c_2 c_4 E - (k_1 + s c_4)(\Sigma_1 - k_3 E)$$

$\mathcal{L}^{-1}$ gives,

$$c_2 \dot{\sigma} = c_2(k_3 + k_1)\dot{e} + c_2 c_4 \ddot{e} - k_1 \sigma - c_4 \dot{\sigma} + k_1 k_3 e + c_4 k_3 \dot{e}$$

$$\Rightarrow \sigma + \frac{c_2 + c_4}{k_1}\dot{\sigma} = k_3 e + \left[c_2 + \frac{k_3}{k_1}(c_2 + c_4)\right]\dot{e} + \frac{c_2 c_4}{k_1}\ddot{e} \longrightarrow \text{same as before}$$

$E_1 + E_2 = E$ ✓
$E_3 = E$ ✓
$E_1 - E_4 = 0$ ✓
$\Sigma_{11} + \Sigma_{14} - \Sigma_{12} = 0$ ✓
$\Sigma_{13} + \Sigma_{12} - \Sigma_1 = 0$
$\Sigma_{11} - k_1 E_1 = 0$ ✓
$\Sigma_{12} - s c_2 E_2 = 0$ ✓
$\Sigma_{13} - k_3 E_3 = 0$ ✓
$\Sigma_{14} - s c_4 E_4 = 0$ ✓

Creep response: Take Laplace transform with zero initial conditions.

$$\Sigma_1 (1+As) = E\left(B + Cs + Ds^2 \right)$$

$$A = \frac{c_2+c_4}{k_1}, \quad B = k_3, \quad C = c_2 + \frac{k_3}{k_1}(c_2+c_4), \quad D = \frac{c_2 c_4}{k_1}$$

Now $\Sigma_1 = \frac{\sigma_0}{s}$

$$\Rightarrow E = \sigma_0 \frac{1}{s} \frac{(1+As)}{(B+Cs+Ds^2)}$$

Examine roots of $Ds^2 + Cs + B = 0$

$$C^2 - 4DB = c_2^2 + \frac{k_3^2}{k_1^2}(c_2+c_4)^2 + 2c_2\frac{k_3}{k_1}(c_2+c_4) - 4\frac{k_3}{k_1}c_2c_4$$

$$= c_2^2 + \frac{k_3^2}{k_1^2}(c_2-c_4)^2 + 2c_2\frac{k_3}{k_1}(c_2-c_4) + 4\frac{k_3^2}{k_1^2}c_2c_4$$

$$= \left[c_2 + \frac{k_3}{k_1}(c_2-c_4)\right]^2 + \frac{4k_3^2}{k_1^2}c_2c_4 > 0$$

Thus $\sqrt{C^2-4DB}$ is real and $< C$.

$\Rightarrow$ roots of $Ds^2 + Cs + B = 0$ are <u>real and negative</u>.

i.e, $Ds^2 + Cs + B = D(s+a)(s+b)$ where $(a,b) > 0$.

ie, $a, b = \dfrac{-C \pm \sqrt{C^2-4BD}}{2D}$

$\Rightarrow$ from L.T. tables, $e(t) = \mathcal{L}^{-1}\left\{ \frac{\sigma_0}{D}\left(\frac{1}{s(s+a)(s+b)} + \frac{A}{(s+a)(s+b)} \right) \right\}$

$$e(t) = \frac{\sigma_0}{D}\left(\frac{1}{ab}\left[1 + \frac{1}{a-b}\left(be^{-at} - ae^{-bt}\right)\right] + \frac{A}{b-a}\left[e^{-at} - e^{-bt}\right] \right)$$

Creep response $\rightarrow$ $e(t) = \dfrac{\sigma_0}{D}\left(\dfrac{1}{ab} + \dfrac{1}{a-b}\left(e^{-at}\left[\dfrac{1}{a} - A\right] + e^{-bt}\left[A - \dfrac{1}{b}\right] \right) \right)$ ◀

$$e(0^+) = \frac{\sigma_0}{D}\left(\frac{1}{ab} + \frac{1}{a-b}\left(\frac{1}{a} - \cancel{A} + \cancel{A} - \frac{1}{b} \right) \right) = \frac{\sigma_0}{D}\left(\frac{1}{ab} - \frac{1}{ab} \right) = 0$$

<u>check</u>: (i) Initial value theorem gives,

$$e(0) = \lim_{s\to\infty} sE(s) = \lim_{s\to\infty} \sigma_0 \frac{(1+As)}{(B+Cs+Ds^2)} = 0$$

(ii) Physically, we see that $\because$ instantaneous strain in dashpots is zero, the instantaneous overall strain is zero.

P-2 Displacements:

$$\left.\begin{aligned}
\varepsilon_x &= -\nu\frac{\sigma_z}{E} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} = \frac{\partial u}{\partial x} \\
\varepsilon_y &= -\nu\frac{\sigma_z}{E} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} = \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\sigma_z}{E} = -\frac{x}{R_x} + \frac{y}{R_y} = \frac{\partial w}{\partial z} \\
\gamma_{xz} &= 0 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
\gamma_{yx} &= 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{zy} &= 0 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{aligned}\right\} \rightarrow (10)$$

$$(10c) \rightarrow \quad w = -\frac{xz}{R_x} + \frac{yz}{R_y} + g(x,y) \longrightarrow \text{(i)}$$

$$(10d, i) \rightarrow \quad \frac{\partial u}{\partial z} = \frac{z}{R_x} - \frac{\partial g}{\partial x} \longrightarrow u = \frac{z^2}{2R_x} - z\frac{\partial g}{\partial x} + f(x,y) \longrightarrow \text{(ii)}$$

$$(10f, i) \rightarrow \quad \frac{\partial v}{\partial z} = -\frac{z}{R_y} - \frac{\partial g}{\partial y} \rightarrow v = -\frac{z^2}{2R_y} - z\frac{\partial g}{\partial y} + h(x,y) \rightarrow \text{(iii)}$$

$$(10e, ii, iii) \rightarrow \quad -2z\frac{\partial^2 g}{\partial x \partial y} + \underbrace{\frac{\partial f}{\partial y} + \frac{\partial h}{\partial x}}_{\text{fn of } (x,y) \text{ only}} = 0$$

$$\Rightarrow \frac{\partial^2 g}{\partial x \partial y} = 0, \quad \frac{\partial f}{\partial y} = -\frac{\partial h}{\partial x} \longrightarrow \text{(iv, v)}$$

$$(10a, ii, iii) \rightarrow \quad -z\frac{\partial^2 g}{\partial x^2} + \underbrace{\frac{\partial f}{\partial x} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y}}_{\text{fn of } (x,y) \text{ only}}$$

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} = 0 \longrightarrow \text{(vi)}$$

$$\frac{\partial f}{\partial x} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} \Rightarrow f = \frac{\nu x^2}{2R_x} - \nu\frac{xy}{R_y} + m(y) \rightarrow \text{(vii)}$$

$$(10b, ii, iii) \rightarrow \quad -z\frac{\partial^2 g}{\partial y^2} + \underbrace{\frac{\partial h}{\partial y} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y}}_{\text{fn of } (x,y) \text{ only}}$$

$$\Rightarrow \frac{\partial^2 g}{\partial y^2} = 0 \longrightarrow \text{(viii)}$$

$$\frac{\partial h}{\partial y} = \frac{\nu x}{R_x} - \frac{\nu y}{R_y} \Rightarrow h = \frac{\nu xy}{R_x} - \frac{\nu y^2}{2R_y} + n(x) \rightarrow \text{(ix)}$$

$$(v, vii, ix) \rightarrow \quad -\nu\frac{x}{R_y} + \underbrace{\frac{dm}{dy} = -\nu\frac{y}{R_x}}_{\text{fn of } y} - \frac{dn}{dx}$$

fn of x

$$\Rightarrow m = -\frac{\nu y^2}{2R_x} + c_1 y + c_2$$

$$n = +\frac{\nu x^2}{2R_y} - c_1 x + c_3$$

Apply bc's $u = v = w = 0$ at $(x,y,z) = (0,0,0)$, and rotations $\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$, $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$ are also zero at $(x,y,z) = (0,0,0)$ (to get rid of R B motions)

Now (iv, vi, viii) $\longrightarrow$ $g = c_4 x + c_5 y$

$$\Rightarrow u = \frac{z^2}{2R_x} - c_4 z + \frac{\nu}{2R_x} x^2 - \frac{\nu}{R_y} xy - \frac{\nu}{2R_x} y^2 + c_1 y + c_2$$

$$v = -\frac{z^2}{2R_y} - z c_5 + \frac{\nu}{R_x} xy - \frac{\nu}{2R_y} y^2 + \frac{\nu x^2}{2R_y} - c_1 x + c_3$$

$$w = -\frac{xz}{R_x} + \frac{yz}{R_y} + c_4 x + c_5 y$$

BC'S $\Rightarrow$ $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ (could have got it by noting that const & linear terms in displ represent RB translation & rotation, so drop them).

$$\Rightarrow \left.\begin{aligned} u &= \frac{1}{2R_x}\left[z^2 + \nu(x^2 - y^2)\right] - \frac{\nu}{R_y} xy \\ v &= \frac{1}{2R_y}\left[-z^2 + \nu(x^2 - y^2)\right] + \frac{\nu}{R_x} xy \\ w &= -\frac{xz}{R_x} + \frac{yz}{R_y} \end{aligned}\right\} \longrightarrow (11)$$

HW#5, P.3. Solution.

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P.3 Determination of Shear Center.

Use (25) to determine (x_{CF}, y_{CF}) as follows. So (25), (19), (22b), (23) yield,

$$x_{CF} W_y - y_{CF} W_x = \int_A (x\tau_{yz} - y\tau_{xz})\, dA$$

$$= \int_A \left[x\left(-\frac{\partial\phi}{\partial x} - g(x) - \frac{1}{2}EK_y y^2\right) - y\left(\frac{\partial\phi}{\partial y} + f(y) - \frac{1}{2}EK_x x^2\right)\right] dA$$

$$= \int_A \left[2\phi + \frac{\partial}{\partial x}\left[-x\phi - xy f(y) + \frac{1}{6}EK_x x^3 y\right] - \frac{\partial}{\partial y}\left[y\phi + xy g(x) + \frac{1}{6}EK_y xy^3\right]\right] dA$$

$$= \int_A 2\phi\, dA + \oint_C \left[y\phi + xy g(x) + \frac{1}{6}EK_y xy^3\right] dx + \oint_C \left[-x\phi - xy f(y) + \frac{1}{6}EK_x x^3 y\right] dy$$

$$= \int_A 2\phi\, dA - \frac{1}{3}E\left[K_y \oint_C xy^3\, dx + K_x \oint_C x^3 y\, dy\right] \qquad \text{(used (22b), (23))}$$

$$= \int_A (2\phi - EK_x x^2 y + EK_y xy^2)\, dA$$

where ϕ is determined from (20), (22b), (23), setting $\alpha = 0$. Inserting (18) in the above & equating coefficients of W_x, W_y ($\because$ they are independent), we obtain

$$x_{CF} = \frac{1}{\cancel{W_y}} \int_A \left[2\cancel{W_y}\phi_2 + \frac{I_{xy}}{F}\cancel{W_y} x^2 y + \frac{I_y}{F}\cancel{W_y} xy^2\right] dA$$

where $F = I_x I_y - I_{xy}^2$

and
$$y_{CF} = -\frac{1}{\cancel{W_x}} \int_A \left[2\cancel{W_x}\phi_1 - \frac{I_x}{F}\cancel{W_x} x^2 y - \frac{I_{xy}}{F}\cancel{W_x} xy^2\right] dA$$

$\rightarrow$ (26).

In the above we have used the fact that since (20) is linear in ϕ, its solution is $\phi = W_y\phi_2 + W_x\phi_1$ where ϕ_2 and ϕ_1 are solutions of (20), (22b), (23) with $(W_y, W_x) = (1,0)$ and $(0,1)$, respectively. Thus ϕ_1 and ϕ_2 depend on the geometry of the cross-section only, and consequently so does (x_{CF}, y_{CF}).