P.1 $\varepsilon_{i3} = 0 \Rightarrow$ plane strain.

$$\varepsilon_E = \varepsilon_{ij}\, n_i n_j$$

(4)

$$n_i \Rightarrow (1, 0, 0) \;:\; \varepsilon_{EA} = \varepsilon_{11} = 200 \times 10^{-6}$$

$$n_i \Rightarrow \left(0.5, \tfrac{\sqrt{3}}{2}, 0\right) \;:\; \varepsilon_{EB} = \varepsilon_{11} n_1 n_1 + \varepsilon_{22} n_2 n_2 + 2\varepsilon_{12} n_1 n_2$$

$$\Rightarrow 0.75\,\varepsilon_{22} + 0.866\,\varepsilon_{12} = -150 \times 10^{-6}$$

$$n_i \Rightarrow \left(-0.5, \tfrac{\sqrt{3}}{2}, 0\right) \;:\; \varepsilon_{EC} \Rightarrow 0.75\,\varepsilon_{22} - 0.866\,\varepsilon_{12} = 350 \times 10^{-6}$$

$$\Rightarrow \varepsilon_{12} = -288.68 \times 10^{-6}, \quad \varepsilon_{22} = 133.33 \times 10^{-6}$$

Principal strains: (3)

$$10^{-6} \begin{vmatrix} (200-\varepsilon) & -288.68 & 0 \\ -288.68 & (133.33-\varepsilon) & 0 \\ 0 & 0 & (0-\varepsilon) \end{vmatrix} = 0 \Rightarrow \varepsilon(3) = 0$$

$$(200-\varepsilon)(133.33-\varepsilon) - 288.68^2 = 0$$

$$\varepsilon(1) = 457.26 \times 10^{-6}$$

$$\varepsilon(2) = -123.94 \times 10^{-6}$$

Principal stresses: (3)

$$\sigma(1) = \frac{E}{1+\nu}\left[\varepsilon(1) + \frac{\nu}{1-2\nu}\left(\varepsilon(1) + \varepsilon(2) + \varepsilon(3)\right)\right] = 104.8 \times 10^{6}\ N/m^2$$

$$\sigma(2) = \frac{E}{1+\nu}\left[\varepsilon(2) + \frac{\nu}{1-2\nu}\left(\varepsilon(1) + \varepsilon(2) + \varepsilon(3)\right)\right] = 7.18 \times 10^{6}\ N/m^2$$

$$\sigma(3) = \frac{E}{1+\nu}\left[\varepsilon(3) + \frac{\nu}{1-2\nu}\left(\varepsilon(1) + \varepsilon(2) + \varepsilon(3)\right)\right] = 28 \times 10^{6}\ N/m^2$$

P.2(a) (2) $n_i \Rightarrow (n_1, n_2, 0)$

(3) — $N = \sigma_{ij} n_i n_j = \sigma_{11} n_1 n_1 + 2\sigma_{12} n_1 n_2 + \sigma_{22} n_2 n_2 = -a n_1^2 + b n_2^2 = 0$

(2) — $n_2^2 + n_1^2 = 1$

(2) — $\Rightarrow n_2 = \pm\left(\frac{a}{a+b}\right)^{1/2}, \quad n_1 = \pm\left(\frac{b}{a+b}\right)^{1/2}$

P.2(b) (6)

$$\left.\begin{aligned} F_{,22} + F_{,11} &= 0 \\ F_{,11} + F_{,33} &= 0 \\ F_{,33} + F_{,22} &= 0 \\ F_{,23} &= 0 \\ F_{,13} &= 0 \\ F_{,12} &= 0 \end{aligned}\right\} \text{compatibility eqns.}$$

$$\Rightarrow F_{,11} = F_{,22} = F_{,33} = 0 \quad (2)$$

$$\Rightarrow F = a x_1 + b x_2 + c x_3 \quad (2)$$

P3. $\phi = C\theta$

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0$$

(3) $$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{C}{r^2}$$

So this distribution fits with the bc's of the problem.

(3) Now $\frac{C}{b^2} \cdot 2\pi b \cdot b = M \Rightarrow C = \frac{M}{2\pi}$.

$$\epsilon_r = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) = 0 \longrightarrow (1)$$

(3) $$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) = 0 \longrightarrow (2)$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \frac{1+\nu}{E}\frac{M}{2\pi r^2} \longrightarrow (3)$$

(3) $(1) \rightarrow u_r = f(\theta)$

symmetry $\Rightarrow f(\theta) = \text{const}$, $\text{b.c's}\big|_{r=a} \Rightarrow \text{const} = 0$.

$\Rightarrow u_r = 0$. ◀

(3) $(2) \rightarrow u_\theta = g(r)$

$(3) \rightarrow g' - \frac{g}{r} = \frac{M}{2\pi G}\frac{1}{r^2}$

(3) $$\left(\frac{g}{r}\right)' = \frac{M}{2\pi G}\frac{1}{r^3}$$

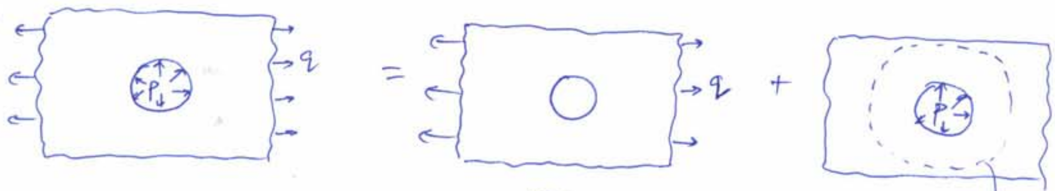
$$\frac{g}{r} = -\frac{M}{4\pi G r^2} + K$$

b.c : $g = u_\theta = 0$ at $r = a \Rightarrow K = M/4\pi G a^2$

(2) $$\Rightarrow u_\theta = \frac{M}{4\pi G}\left[-\frac{1}{r} + \frac{r}{a^2}\right]$$ ◀

P. 4

(a)



(I) plate w/hole under uniaxial tension only + (II) Thick walled cyl. with $p_i = p$, $r_0 = \infty$, $p_0 = 0$

(II) $\rightarrow$ put $p_0 = 0$, $r_0 = \infty$, $p_i = p$ in formulae,

$$\sigma_{r/\theta} = \pm \frac{a^2}{r^2}(-p) \qquad , \quad \tau_{r\theta} = 0 .$$

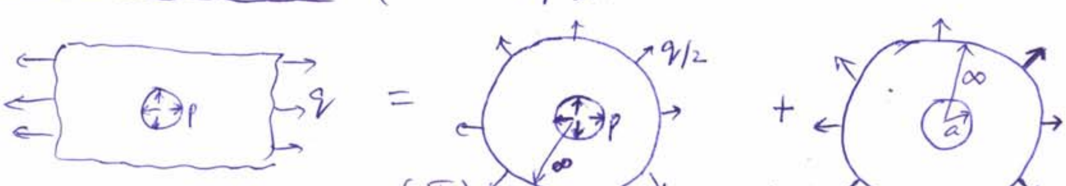
(I) $\rightarrow$ directly from formulae (class notes p. 51).

(I) + (II) $\rightarrow$

$$\boxed{\begin{aligned} \sigma_r &= \frac{q}{2}\left(1-\frac{a^2}{r^2}\right) - p\frac{a^2}{r^2} + \frac{q}{2}\cos 2\theta\left(1-\frac{a^2}{r^2}\right)\left(1-3\frac{a^2}{r^2}\right) \\ \sigma_\theta &= \frac{q}{2}\left(1+\frac{a^2}{r^2}\right) + p\frac{a^2}{r^2} - \frac{q}{2}\cos 2\theta\left(1+3\frac{a^4}{r^4}\right) \\ \tau_{r\theta} &= -\frac{q}{2}\sin 2\theta\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right) \end{aligned}}$$

(*) $\leftarrow$

Alternative method: (but longer).



(I) Thick walled cyl, $r_0 = \infty$, $p_0 = -\frac{q}{2}$, $p_i = p$

(II) loading $\begin{cases} \sigma_r \big|_{r=\infty} = \frac{q}{2}\cos 2\theta \\ \tau_{r\theta}\big|_{r=\infty} = -\frac{q}{2}\sin 2\theta \end{cases}$

So you can add stress functions for these two basic problems as done in class, ie,

$$\phi = \underbrace{\left(A \ln r + Br^2\right)}_{\text{I}} + \underbrace{\left(Cr^2 + \frac{D}{r^2} + E\right)}_{\text{(II)}}\cos 2\theta$$

$$\sigma_\theta = \phi_{,\theta\theta} = -\frac{A}{r^2} + 2B + 2C\cos 2\theta + \frac{6D}{r^4}\cos 2\theta$$

$$\sigma_r = \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} = \frac{A}{r^2} + 2B + 2C\cos 2\theta - \frac{2D}{r^4}\cos 2\theta - \frac{4}{r^2}\left(Cr^2 + \frac{D}{r^2} + E\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 2\sin 2\theta\left(C - \frac{3D}{r^4} - \frac{E}{r^2}\right)$$

BC's $r=a$: $\sigma_r = -p \Rightarrow \frac{A}{a^2} + 2B = -p \longrightarrow$ (i)

$$-2C - \frac{6D}{a^4} - \frac{4E}{a^2} = 0 \longrightarrow \text{(ii)}$$

$$\sigma_{r\theta} = 0 \Rightarrow C - \frac{3D}{a^4} - \frac{E}{a^2} = 0 \longrightarrow \text{(iii)}$$

$r=\infty$: $\sigma_r = \frac{q}{2}(\cos 2\theta + 1) \Rightarrow -2C = q/2, \quad 2B = q/2 \longrightarrow$ (iv)

$\sigma_{r\theta} = -\frac{q}{2}\sin 2\theta \Rightarrow 2C = -q/2$ — repeated

(also if you do $\sigma_{\theta\theta} = \frac{q}{2}(1-\cos 2\theta)$ you get repeat of (iv)).

(i)–(iv) $\rightarrow$. $A = \left(-p - \frac{q}{2}\right)a^2, \; B = \frac{q}{4}, \; C = -\frac{q}{4}, \; D = -\frac{q}{4}a^4, \; E = \frac{q}{2}a^2$

$$\Rightarrow \boxed{\phi = \left(-p - \frac{q}{2}\right)a^2 \ln r + \frac{q}{4}r^2 + \left(-\frac{q}{4}r^2 - \frac{q}{4}\frac{a^4}{r^2} + \frac{q}{2}a^2\right)\cos 2\theta}$$

use this ϕ to get σ_r, σ_θ, $\sigma_{r\theta}$ as in (*).

4(b) $\sigma_\theta\big|_{r=a} = q + p - 2q\cos 2\theta$

least value of $p = q$ which makes $\sigma_\theta\big|_{r=a} \geq 0$

(i.e., zero for $\theta = 0, \pi$).

P.5 (a) $\sigma_{xy} = f_1(x)$ (ie replace given $f(x)$ by $f_1(x)$).

$$\Rightarrow -\phi_{,xy} = f_1(x) \rightarrow \phi_{,y} = f(x) + g_1(y) \rightarrow \phi = y f(x) + g(y) + \overset{\text{drop}}{\cancel{h}(x)}$$

(∵ $\sigma_{yy}|_{y=0} = 0 \Rightarrow h''(x) = 0 \Rightarrow h(x)$ will not affect stresses.)

$$\nabla^4 \phi = 0 \Rightarrow y f^{IV} + g^{IV} = 0$$

$$\Rightarrow f^{IV} = k, \quad g^{IV} = -ky$$

$$\Rightarrow f = k\frac{x^4}{24} + Ax^3 + Bx^2 + Cx + \cancel{D}$$

$$g = -k\frac{y^5}{120} + Ey^4 + Fy^3 + Gy^2 + \cancel{H}y + \cancel{I}$$

} cancelled terms wont affect stresses so drop them. w/o loss of generality

$$\Rightarrow \sigma_{xx} = -\frac{ky^3}{6} + 12Ey^2 + 6Fy + 2G - \rho g y$$

$$\sigma_{xy} = -\left(\frac{kx^3}{6} + 3Ax^2 + 2Bx + C\right)$$

$$\sigma_{yy} = y\left(\frac{kx^2}{2} + 6Ax + 2B\right) - \rho g y$$

BC's : $y=0$: $\sigma_{yy} = 0 \rightarrow$ i.s.

$\sigma_{xy} = 0 \Rightarrow k = A = B = C = 0 \longrightarrow$ ①

$x=h$: $\sigma_{xx} = 0 \rightarrow k = E = (6F - \rho g) = G = 0 \longrightarrow$ ②

$\sigma_{xy} = q \rightarrow -\left(\frac{kh^3}{6} + 3Ah^2 + 2Bh + C\right) = q \longrightarrow$ ③

$x=0$: $\sigma_{xx} = 0 \rightarrow k = E = (6F - \rho g) = G = 0 \longrightarrow$ ④

$\sigma_{xy} = 0 \rightarrow C = 0 \longrightarrow$ ⑤

From physical considerations (ie complimentarity of σ_{xy} at $(x,y) = (0,h)$) you see that $\sigma_{xy}|_{y=0} = 0$ is not satisfiable. Moreover, satisfying this would result in all coeffs being zero, ie $\phi = 0$, & hence $\sigma_{xy}|_{x=h} = q$ not being satisfiable. Hence we relax ①, and instead satisfy,

$$\int_0^h \sigma_{xy}\, dx = 0 \Rightarrow -\left(f(h) - f(0)\right) = 0$$

$$\Rightarrow \frac{kh^4}{24} + Ah^3 + Bh^2 + Ch = 0 \longrightarrow ⑥$$

② – ⑥ $\rightarrow A = -\frac{q}{h^2}, \quad B = \frac{q}{h}, \quad k = C = E = (6F - \rho g) = G = 0$

(contd on p.3 reverse)

(a) (contd)

$$\Rightarrow \boxed{\begin{aligned} \sigma_{xx} &= 0 \\ \sigma_{yy} &= y\left(-\frac{6q}{h^2}x + \frac{2q}{h}\right) - \rho g y \\ \sigma_{xy} &= -\left(-\frac{3q}{h^2}x^2 + \frac{2q}{h}x\right) \end{aligned}}$$

(b) $\varepsilon_{zz} = 0 \rightarrow \sigma_{zz} = \nu\sigma_{yy} \quad (\because \sigma_{xx} = 0)$.

$$\varepsilon_{xx} = \frac{\cancelto{0}{\sigma_{xx}}}{E} - \frac{\nu}{E}\left(\sigma_{yy} + \nu\sigma_{zz}\right) = -\frac{\nu}{E}(1+\nu)\sigma_{yy} = -\frac{\nu}{E}(1+\nu)\left(y\left[-\frac{6q}{h^2}x + \frac{2q}{h}\right] - \rho g y\right)$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\left(\cancelto{0}{\sigma_{xx}} + \sigma_{zz}\right) = \frac{1-\nu^2}{E}\sigma_{yy} = \frac{1-\nu^2}{E}\left(\quad\right)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \varepsilon_{xx} \xrightarrow{\int dx} u = -\frac{\nu(1+\nu)}{E}x\left[y\left(-\frac{6q}{2h^2}x + \frac{2q}{h}\right) - \rho g y\right] + f(y) \quad (7)$$

$$\frac{\partial v}{\partial y} = \varepsilon_{yy} \xrightarrow{\int dy} v = \frac{1-\nu^2}{E}\left[\frac{y^2}{2}\left(-\frac{6q}{h^2}x + \frac{2q}{h}\right) - \rho g\frac{y^2}{2}\right] + g(x) \quad (8)$$

Subst (7), (8) into ε_{xy}:

$$(v_{,x} + u_{,y}) = \underline{\frac{1-\nu^2}{E}\left[-\frac{3q}{h^2}y^2\right]} + \underline{\underline{g'}} - \underline{\underline{\frac{\nu(1+\nu)}{E}x\left[\frac{-6q}{2h^2}x + \frac{2q}{h} - \rho g\right]}} + \underline{f'}$$

$$\underset{\frac{2(1+\nu)}{E}\sigma_{xy}}{=} \underline{\underline{\frac{2(1+\nu)}{E}\left[3q\frac{x^2}{h^2} - \frac{2q}{h}x\right]}}$$

$\Rightarrow$ single underlined terms = $-$(double underlined terms) = k (const).
fn of y — fn of x

Thus upon integrating single-underlined wrt y,
double-underlined " x,

$$\frac{E}{1+\nu}f(y) = ky + \frac{3q}{h^2}(1-\nu)\frac{y^3}{3} + C_1 \longrightarrow (9)$$

$$\frac{E}{1+\nu}g(x) = -kx + 2\left(q\frac{x^3}{h^2} - q\frac{x^2}{h}\right) + \nu\left(-q\frac{x^3}{h^2} + \frac{qx^2}{h} - \frac{\rho g}{2}x^2\right) + C_2$$

$$\hookrightarrow = -kx + (2-\nu)\left(q\frac{x^3}{h^2} - q\frac{x^2}{h}\right) - \frac{\nu\rho g}{2}x^2 + C_2 \rightarrow (10)$$

$u = v = u_{,y} - v_{,x} = 0$ at $x = 0 \Rightarrow k = C_1 = C_2 = 0. \rightarrow (11)$

$$\Rightarrow u = -\frac{\nu(1+\nu)}{E}x\left[y-\left(-\frac{6q}{2h^2}x + \frac{2q}{h}\right) - \rho g y\right] + \frac{(1-\nu^2)}{E}\frac{q y^3}{h^2}$$

$$\Rightarrow v = \frac{1-\nu^2}{E}\left[\frac{y^2}{2}\left(-\frac{6q}{h^2}x + \frac{2q}{h}\right) - \rho g\frac{y^2}{2}\right] + \frac{(1+\nu)(2-\nu)}{E}\left(q\frac{x^3}{h^2} - q\frac{x^2}{h}\right) - \frac{\nu(1+\nu)}{E}\frac{\rho g}{2}x^2$$