CE623 MIDSEM. SOL. 2006 FALL. (1)

भारतीय प्रौद्योगिकी संस्थान मुंबई
INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

परिशिष्ट/Supplement - 8

रोल नं./Roll No.

पाठ्यक्रम नाम/Course Name

शाखा/प्रभाग/Branch/Div.

शिक्षण बैच/Tutorial Batch

अनुभाग/Section

पाठ्यक्रम सं./Course No.

तिथि/Date

P.1. $\underline{\underline{\sigma}} = \begin{pmatrix} a & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & b \end{pmatrix}$

(a) $\underline{\sigma} = 0$ on $\underline{n} = (n_1, n_2, n_3)$

$$\Rightarrow a n_1 + 2n_2 + n_3 = 0 \quad \text{— (1)}$$
$$2n_1 + n_3 = 0 \quad \text{— (2)}$$
$$n_1 + n_2 + b n_3 = 0 \quad \text{— (3)}$$

Pure shear $\Rightarrow \sigma_{ii} = a + b = 0$ — (4)

Also $n_1^2 + n_2^2 + n_3^2 = 1$ — (5)

(1)–(5) $\Rightarrow$ $n_3 = -2n_1$

$n_2 = -n_1 - b n_3 = n_1(-1 + 2b)$

$n_1(a - 2 + 4b - 2) = 0 \Rightarrow a + 4b = 4.$

$\Rightarrow 3b = 4, \; b = \frac{4}{3}$

$a = -\frac{4}{3}$

$n_1^2(1 + 4b^2 + 1 - 4b + 4) = 1$

$$\Rightarrow \left[n_1 = \pm\sqrt{\frac{9}{70}}, \; n_2 = \pm\frac{5}{3}\sqrt{\frac{9}{70}} = \pm\sqrt{\frac{25}{70}}, \; n_3 = \mp\sqrt{\frac{36}{70}} \right]$$

(b) Do rotation about y axis so $\sigma_{yy} = \sigma_{YY} = 0$, ie σ_{yy} remains zero. Then drive $\sigma_{XX} \to 0$ by choosing rotation angle θ. Then $\because$ $\sigma_{ii} = 0$ = invariant,

$\sigma_{z_1 z_1} \to 0$ automatically $\therefore$ σ_{yy} remains zero.

Transformation is,

$$\underset{\underline{\underline{a}}}{\begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}} \underset{\underline{\underline{\sigma}}_{xyz}}{\begin{bmatrix} a & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & b \end{bmatrix}} \underset{\underline{\underline{a}}^T}{\begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}}$$

$$= \begin{pmatrix} ac\theta + s\theta & 2c\theta + s\theta & c\theta + bs\theta \\ 2 & 0 & 1 \\ -as\theta + c\theta & -2s\theta + c\theta & -s\theta + bc\theta \end{pmatrix} \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix}$$

$$\Rightarrow \sigma_{\overline{x}\overline{x}} = ac^2\theta + 2s\theta c\theta + bs^2\theta = 0 \quad \text{--- (6)}$$

$$\sigma_{z_1 z_1} = as^2\theta - 2s\theta c\theta + bc^2\theta = 0 \quad \text{--- (7)}$$

Note (6) + (7) is same as (4). (so only use any one of them to get θ).

$$\Rightarrow a(c^2\theta - s^2\theta) = -2s\theta c\theta$$

$$\tan 2\theta = -a \Rightarrow \theta =$$

P.2 $dV^* = J\,dV \Rightarrow \dfrac{dV^* - dV}{dV} = J - 1.$

$$J = \det\left[\frac{\partial x^*_i}{\partial x_j}\right]$$

$$x^*_1 = u_x + x = \sqrt{2x_1}\cos x_2$$

$$x^*_2 = u_y + y = \sqrt{2x_1}\sin x_2$$

$$x^*_3 = u_z + z = x_3$$

$$J = \det\begin{bmatrix} \dfrac{\cos x_2}{\sqrt{2x_1}} & -\sqrt{2x_1}\sin x_2 & 0 \\ \dfrac{\sin x_2}{\sqrt{2x_1}} & \sqrt{2x_1}\cos x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \cos^2 x_2 + \sin^2 x_2 = 1$$

$$\Rightarrow J - 1 = 0 \Rightarrow \frac{dV^* - dV}{dV} = 0 = D.$$ (3)

P.3 $\varepsilon_{xx} = -\varepsilon_{yy} = -\varepsilon_{zz} = f(y,z)$, $\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$

Infinitesimal theory

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & -f \end{bmatrix}$$

Compatibility must be satisfied.

$i=j=1,\ k=l=2$: $\frac{\partial^2 f}{\partial y^2} = 0 \quad \Rightarrow f$ linear in $y \Rightarrow p = k_2 y$

$i=j=2,\ k=l=3$: $-\frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial y^2} = 0 \rightarrow$ identically satisfied $\because \frac{\partial^2 f}{\partial y^2} = 0$ & $\frac{\partial^2 f}{\partial z^2} = 0$

$i=j=3,\ k=l=1$: $\frac{\partial^2 f}{\partial z^2} = 0 \quad \Rightarrow f$ linear in $z \Rightarrow q = k_3 z$

$i=j=1,\ k=2,\ l=3$: $\frac{\partial^2 f}{\partial y \partial z} = 0 \quad \Rightarrow f = p(y) + q(z) + k_1$

other two eqns : $0=0$, $0=0$.

$$\Rightarrow f = k_1 + k_2 y + k_3 z.$$

Engg ext strain along $\underline{n}$ is $\underline{n}^T \underline{\underline{\varepsilon}}\, \underline{n}$.

$$\varepsilon(1)\Big|_{(2,1,1)} = (1 \ 0 \ 0) \begin{bmatrix} f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & f \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f\Big|_{(2,1,1)} = 0.01$$

$$\Rightarrow k_1 + k_2 + k_3 = 0.01$$

$$\varepsilon(2)\Big|_{(-2,-1,-1)} = \left(0 \ \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right) \begin{pmatrix} f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = -f\Big|_{(-2,-1,-1)} = 0.01$$

$$\Rightarrow -(k_1 - k_2 - k_3) = 0.01$$

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$$\varepsilon(3)\Big|_{(0,-2,0)} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & -f \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = -f\Big|_{(0,-2,0)} = 0.01$$

$$\Rightarrow -(-2k_2) = 0.01 \Rightarrow k_2 = 0.005$$

$$k_1 = 0$$

$$k_3 = 0.005$$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 0.005(y+z) & 0 & 0 \\ 0 & -0.005(y+z) & 0 \\ 0 & 0 & -0.005(y+z) \end{pmatrix}$$

P.4 $\sigma_{xy}\Big|_{y=h} = kx$. is given loading.

So, Assume $\sigma_{xy} = x f(y)$

$$\Rightarrow -\phi_{,xy} = x f(y)$$

$$-\phi_{,y} = \frac{x^2}{2} f(y) + g(y)$$

$$-\phi = \frac{x^2}{2} h(y) + k(y) + p(x)$$

where $h(y) = \int f(y)\,dy$, $k(y) = \int g(y)\,dy$.

$\nabla^4\phi = 0$ must be satisfied by ϕ.

$$\Rightarrow \nabla^4\phi = \underbrace{p^{IV}(x)}_{\text{fn of } x} + \underbrace{k^{IV}(y) + 2f'(y)}_{\text{fn of } y} + \underbrace{\frac{x^2}{2} h^{IV}(y)}_{\text{fn of } (x,y)} = 0$$

$$\text{fn of } x + \text{fn of } y + \text{fn of } (x,y) = 0.$$

Only way this can be true is iff

$$h^{IV}(y) = c_1$$

in which case,

$$p^{IV}(x) + \frac{x^2}{2} c_1 = c_6$$

$$k^{IV}(y) + 2f'(y) = -c_6$$

Integrating,

$$h(y) = \frac{c_1}{24} y^4 + c_2 y^3 + c_3 y^2 + c_4 y + c_5$$

$$P(x) = -\frac{c_1}{2*360} x^6 + \frac{c_6}{24} x^4 + c_7 x^3 + c_8 x^2 + \underbrace{\cancel{c_9} x}_{\text{drop}} + \underbrace{\cancel{c_{10}}}_{\text{drop}}$$

(∵ they won't affect stresses).

$$k^{IV}(y) = -2\left(\frac{c_1}{2} y^2 + 6 c_2 y + 2 c_3\right) - c_6$$

$$k(y) = -\frac{c_1}{360} y^6 - \frac{12 c_2}{120} y^5 - \frac{(4c_3 + c_6)}{24} y^4 + c_9 y^3 + c_{10} y^2 + \cancel{c_{11}} y + \cancel{c_{12}}$$

(drop).

Apply boundary conditions:

stresses are:

$$\sigma_{xy} = -\phi_{,xy} = x\left(\frac{c_1}{6} y^3 + 3c_2 y^2 + 2c_3 y + c_4\right)$$

$$-\sigma_{xx} = -\phi_{,yy} = \frac{x^2}{2} f' + g' = \frac{x^2}{2} h'' + k''$$

$$= \frac{x^2}{2}\left(\frac{c_1}{2} y^2 + 6c_2 y + 2c_3\right) - \frac{c_1}{12} y^4 - 2c_2 y^3 - \frac{(4c_3 + c_6)}{2} y^2 + 6c_9 y + 2c_{10}$$

$$-\sigma_{yy} = -\phi_{,xx} = h(y) + p''$$

$$= \frac{c_1}{24} y^4 + c_2 y^3 + c_3 y^2 + c_4 y + c_5 - \frac{c_1}{24} x^4 + \frac{c_6}{2} x^2 + 6c_7 x + 2c_8$$

$$\sigma_{xy}\Big|_{y=0} = 0 \Rightarrow \boxed{c_4 = 0}$$

$$\sigma_{xy}\Big|_{y=h} = kx \Rightarrow \boxed{\frac{c_1}{6} h^3 + 3c_2 h^2 + 2c_3 h = k}$$

drop it.

$$\boxed{\sigma_{xx}\Big|_{x=0} = 0 \Rightarrow c_1 = c_2 = c_9 = c_{10} = \underline{\underline{4c_3 + c_6}} = 0}$$

can't satisfy

$$\sigma_{xy}\Big|_{x=0} \Rightarrow 0 = 0 \quad \text{(identically satisfied).}$$

$$\sigma_{yy}\Big|_{y=0} \Rightarrow \boxed{c_1 = \underline{\underline{c_6}} = c_7 = c_5 + 2c_8 = 0}$$

Underlined equations give $c_3 = 0 \Rightarrow \sigma_{xy} = 0$

So drop $\sigma_{xx}\Big|_{x=0} = 0$ and satisfy it in an $\int$ sense.

$$\sigma_{yy}\Big|_{y=h} = 0 \Rightarrow \cancelto{0}{\frac{c_1}{24}} h^4 + c_2 h^3 + c_3 h^2 + \cancelto{0}{c_4} h + \cancelto{0}{c_5 + 2c_8} = 0$$

$$\Rightarrow \boxed{c_2 h^3 + c_3 h^2 = 0}$$

$$\int_0^h \sigma_{xx}\,dy\Big|_{x=0} = \int_0^h \left(2c_2 y^3 + \left(\frac{\cancelto{0}{c_6} + 4c_3}{2}\right) y^2 - 6c_9 y - 2c_{10}\right) dy = 0$$

$$= \boxed{\frac{c_2}{2} h^4 + \frac{2}{3} c_3 h^3 - 3c_9 h^2 - 2c_{10} h = 0}$$

$$\int_0^h \sigma_{xx}\, y\,dy\Big|_{x=0} = \boxed{\frac{2}{5} c_2 h^5 + \frac{4c_3}{8} h^4 - 2c_9 h^3 - c_{10} h^2 = 0}$$

solution of c_2, c_3, c_9, c_{10}: (using boxed equations).

$$\left.\begin{aligned} 3c_2 h^2 + 2c_3 h &= k \\ c_2 h^3 + c_3 h^2 &= 0 \end{aligned}\right\} \Rightarrow \underline{c_3 = -k/h}, \quad \underline{c_2 = k/h^2}$$

$$\left.\begin{aligned} 3c_9 h^2 + 2c_{10} h &= \frac{k}{h^2}\frac{h^4}{2} - \frac{4}{6}\frac{k}{h} h^3 = -\frac{kh^2}{6} \\ 2c_9 h^3 + c_{10} h^2 &= \frac{2}{5}\frac{k}{h^2} h^5 - \frac{1}{2}\frac{k}{h} h^4 = -\frac{kh^3}{10} \end{aligned}\right\}$$

$$\Rightarrow \underline{c_{10} = -\frac{1}{30} kh}; \quad \underline{c_9 = -\frac{1}{30} k}$$

$$\boxed{\begin{aligned} \sigma_{xy} &= x\left[3\frac{k}{h^2}y^2 - 2\frac{k}{h}y\right] \\ -\sigma_{xx} &= \frac{x^2}{2}\left(\frac{6k}{h^2}y - \frac{2k}{h}\right) - \frac{2k}{h^2}y^3 + \frac{2k}{h}y^2 - \frac{1}{5}ky - \frac{1}{15}kh \\ -\sigma_{yy} &= \frac{k}{h^2}y^3 - \frac{k}{h}y^2 \end{aligned}}$$

X

Check NOT REQUIRED IN EXAM.

BC's: $\sigma_{xy}\big|_{y=0} = 0$ ✓ , $\sigma_{xy}\big|_{y=h} = kx$ ✓

$\sigma_{yy}\big|_{y=0} = 0$ ✓ , $\sigma_{yy}\big|_{y=h} = 0$ ✓

$$\int_0^h \sigma_{xx}\,dy\Big|_{x=0} = \frac{2}{4}\frac{k}{h^2}h^4 - \frac{2}{3}\frac{k}{h}h^3 + \frac{1}{5*2}kh^2 + \frac{1}{15}kh.h$$
$$= 0 \checkmark$$

$$\int_0^h \sigma_{xx}\,y\,dy\Big|_{x=0} = \left(\frac{2}{5} - \frac{2}{4} + \frac{1}{5*3} + \frac{1}{15*2}\right)kh^3 = 0 \checkmark$$

Compatibility: $\nabla^4\phi = 0$.

$$-\phi = \frac{x^2}{2}h(y) + k(y) + p(x)$$

not the same 'k'

$$= \frac{x^2}{2}\left(\frac{k}{h^2}y^3 - \frac{k}{h}y^2\right) - \frac{1}{10}\frac{k}{h^2}y^5 + \frac{1}{6}\frac{k}{h}y^4 - \frac{1}{30}ky^3 - \frac{1}{30}khy^2$$

$$\nabla^4\phi = 2\left(\frac{k}{h^2}6y - 2\frac{k}{h}\right) - 12\frac{k}{h^2}y + 4\frac{k}{h} = 0 \checkmark$$

Equilibrium: $\sigma_{xx,x} + \sigma_{xy,y} = -x\left(\cancel{\frac{6k}{h^2}y - \frac{2k}{h}}\right) + x\left(\cancel{\frac{6k}{h^2}y - \frac{2k}{h}}\right) = 0$ ✓

$$\sigma_{xy,x} + \sigma_{yy,y} = \left(\frac{3k}{h^2}y^2 - \frac{2k}{h}y\right) - \left(\frac{3k}{h^2}y^2 - \frac{2k}{h}y\right) = 0 \checkmark$$

All checks work out.