CE623 Advanced Mechanics of Solids Midsem 13/9/07

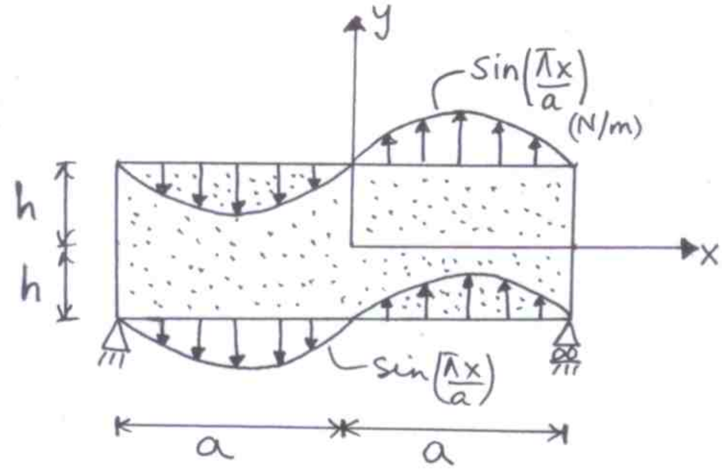
- Marks: **Q1=25, Q2=25, Q3=25, Q4=25.**
- Show all working.
- Show finally only one attempt per question (i.e., cancel out multiple attempts that you do not want to be graded).
- Open notes exam

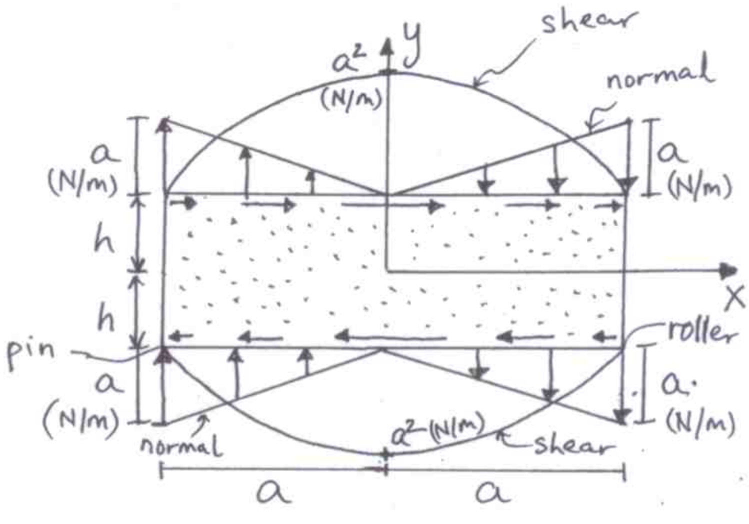
1. The infinitesimal strains in a solid are obtained as $e_{xx} = e_{yy} = e_{zz} = f(x,y,z)$, and all shear strains are zero. The engineering extensional strains are measured at the points $P \equiv (1,1,1)$ $Q \equiv (1,-1,-1)$ $R \equiv (-1,-1,1)$. Each measurement is made along the direction joining the origin to the point at which the measurement is made. All three measurements have value 0.01. The engineering extensional strains at the origin are zero along any direction.
(i) Using the fact that the strains must satisfy compatibility, obtain $f(x,y,z)$ in terms of undetermined constants.
(ii) Determine the constants in part (i) and hence the strain matrix at a point (x,y,z).

2. The principal stresses at a point P are distinct and equal T_1, T_2, T_3, with $2T_2 = T_1 + T_3$. Determine the direction (referred to the principal coordinate system at P) of the plane(s) on which the normal stress is T_2 and shear stress is $(T_1 - T_3)/4$.

3. For the simply supported beam with loading as shown in Fig. 3, use the Fourier series approach to determine the 2-D state of stress. The direction of applied loads is shown by the arrows in the figure and the magnitude is $\sin(\pi x/a)$ on each of the faces $y = \pm h$.

4. Consider the simply supported beam with loading as shown in Fig. 4. On each of the faces $y = \pm h$ the normal loading is linear with magnitude (x) and shear loading is parabolic with magnitude $(a^2 - x^2)$, and their directions are shown by the arrows in Fig. 4.
(i) Starting with a complete 6$^{\text{th}}$ degree polynomial in x, y for the stress function ϕ, eliminate terms in ϕ based on symmetry/antisymmetry considerations and write down the reduced polynomial for ϕ.
(ii) Using the boundary conditions on long edges, along with strong boundary conditions for normal stress on short edges, obtain the constants in the reduced polynomial. Hence obtain the 2-D state of stress.
(iii) Verify that the weak boundary condition on shear stresses is satisfied by your solution obtained in part (ii). (Be careful to include all loads when doing this part.)





P.1 (i) $e_{xx} = e_{yy} = e_{zz} = f(x,y,z)$, $e_{xy} = e_{yz} = e_{xz} = 0$

Compatibility equations become,

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2} = 0 \;;\; \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} = 0 \;;\; \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial y^2} = 0 \longrightarrow (1,2,3)$$

$$\frac{\partial^2 f}{\partial y \partial z} = 0 \;;\; \frac{\partial^2 f}{\partial x \partial z} = 0 \;;\; \frac{\partial^2 f}{\partial x \partial y} = 0 \longrightarrow (4,5,6)$$

Adding the first three (1+2+3) $\Rightarrow \cancel{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0 \longrightarrow (7)$

From this result (7) subtract each of the first three $\Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} = 0$ (8, 9, 10)

(4, 5, 6, 8, 9, 10) $\Rightarrow f = k_1 + k_2 x + k_3 y + k_4 z$.

(ii) No strain at origin $\Rightarrow k_1 = 0$

At P : $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = f\Big|_{(1,1,1)} = k_2 + k_3 + k_4 = 0.01$ (11)

At Q : $\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix} = f\Big|_{(1,-1,-1)} = k_2 - k_3 - k_4 = 0.01$ (12)

At R : $\begin{pmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = f\Big|_{(-1,-1,1)} = -k_2 - k_3 + k_4 = 0.01$ (13)

(11, 12, 13) $\Longrightarrow k_4 = k_2 = 0.01 = -k_3 \Rightarrow \underline{\underline{e}} = \begin{pmatrix} 0.01(x-y+z) & 0 & 0 \\ 0 & 0.01(x-y+z) & 0 \\ 0 & 0 & 0.01(x-y+z) \end{pmatrix}$

P.2 $\underline{\underline{T}} = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}$ in principal coordinate system.

$$N = T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2 = T_2 = (T_1 + T_3)/2 \longrightarrow (1)$$

$$S^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - T_2^2 = (T_1 - T_3)^2/16 \longrightarrow (2)$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \longrightarrow (3)$$

1, 3 $\Rightarrow$ eliminate T_2

$$\Rightarrow T_1\left(n_1^2-\frac{1}{2}\right)+T_3\left(n_3^2-\frac{1}{2}\right)+\left(\frac{T_1+T_3}{2}\right)\left(1-n_1^2-n_3^2\right)=0$$

$$\Rightarrow (T_1-T_3)\left(\frac{n_1^2}{2}-\frac{n_3^2}{2}\right)=0 \Rightarrow n_1=\pm n_3 \quad (\because T_1\neq T_3) \to 4.$$

4, 2 $\Rightarrow$ eliminate T_2

$$\Rightarrow T_1^2 n_1^2+\left(\frac{T_1^2}{4}+\frac{T_3^2}{4}+\frac{T_1T_3}{2}\right)\left(1-2n_1^2-1\right)+T_3^2n_1^2$$
$$-T_1^2/16-T_3^2/16+T_1T_3/8=0$$

$$\Rightarrow n_1^2\left(T_1^2+T_3^2-\frac{T_1^2}{2}-\frac{T_3^2}{2}-T_1T_3\right)-\frac{T_1^2}{16}-\frac{T_3^2}{16}+\frac{T_1T_3}{8}=0$$

$$\Rightarrow n_1^2=\frac{1}{8},\ n_3^2=\frac{1}{8},\ n_2^2=\frac{6}{8} \Rightarrow \underline{n}=\left(\pm\frac{1}{\sqrt{8}},\pm\sqrt{\frac{6}{8}},\pm\frac{1}{\sqrt{8}}\right)$$

4 distinct directions.

P.3. Loading is antisymmetric in x & y. So it is an f_4 problem.

$$\phi=\sum_{n=1}^{\infty}\{A_n y\cosh(\lambda_n y)+B_n\sinh(\lambda_n y)\}\sin\lambda_n x$$

BC's: $\sigma_{xy}=0$, $y=\pm h$ $\longrightarrow$ 1

$\sigma_{yy}=\pm f_4(x)$, $y=\pm h$ $\longrightarrow$ 2

$\sigma_{xx}=0$, $x=\pm a$. $\longrightarrow$ 3

3 $\Rightarrow$ $\lambda_n=n\pi/a$

$$f_1(x)=f_2(x)=f_3(x)=0,\quad p_1(x)=f_4(x)=-p_2(x)=\sin\frac{\pi x}{a}$$

$$A_m=\frac{\cosh(\lambda_m h)}{\lambda_m a\{\lambda_m h-\sinh(\lambda_m h)\cosh(\lambda_m h)\}}\int_{-a}^{a}-f_4(x)\sin(\lambda_m x)$$

$$B_m=\frac{-(\cosh(\lambda_m h)+\lambda_m h\sinh(\lambda_m h))}{\lambda_m^2 a\{\lambda_m h-\sinh(\lambda_m h)\cosh(\lambda_m h)\}}\int_{-a}^{a}-f_4(x)\sin(\lambda_m x)$$

$$A_1=\frac{-\cosh(\pi h/a)}{\left(\frac{\pi}{a}\right)\left\{\frac{\pi h}{a}-\sinh\left(\frac{\pi h}{a}\right)\cos\left(\frac{\pi h}{a}\right)\right\}},\quad B_1=\frac{\left(\cosh\left(\frac{\pi h}{a}\right)+\frac{\pi h}{a}\sin\left(\frac{\pi h}{a}\right)\right)}{\{\ \}}$$

$$A_m=B_m=0,\quad n=2,3,\cdots\infty$$

$$\sigma_{xx} = \left\{ 2A_1 \frac{\pi}{a} \sinh\left(\frac{\pi y}{a}\right) + A_1 \frac{\pi^2}{a^2} y \cosh\left(\frac{\pi}{a} y\right) + B_1 \frac{\pi^2}{a^2} \sinh\left(\frac{\pi}{a} y\right) \right\} \sin\left(\frac{\pi x}{a}\right)$$

$$\sigma_{xy} = \left\{ A_1 \frac{\pi}{a} \cosh\left(\frac{\pi}{a} y\right) + A_1 \frac{\pi^2}{a^2} y \sinh\left(\frac{\pi}{a} y\right) + B_1 \frac{\pi^2}{a^2} \cosh\left(\frac{\pi}{a} y\right) \right\} \left(-\cos\left(\frac{\pi}{a} x\right)\right)$$

$$\sigma_{yy} = -\left\{ A_1 \frac{\pi^2}{a^2} y \cos\left(\frac{\pi}{a} y\right) + B_1 \frac{\pi^2}{a^2} \sinh\left(\frac{\pi}{a} y\right) \right\} \sin\left(\frac{\pi}{a} x\right)$$

P.4 (i) $\sigma_{yy} = \phi_{,xx}$ = antisymm loading in x and y $\Rightarrow \phi$ = odd in x and y.

$\sigma_{xy} = -\phi_{,xy}$ = symmetric loading in x and y $\Rightarrow \phi$ = odd in x & y.

So from both loadings we have ϕ = odd in x and y.

$$\phi = b_2 xy + \frac{b_4}{6} x^3 y + \frac{d_4}{6} xy^3 + \frac{b_6}{20} x^5 y + \frac{d_6}{9} x^3 y^3 + \frac{f_6}{20} xy^5$$

(6 constants)

(ii) BC's: $\sigma_{yy}\big|_{y=\pm h} = \mp x$

$\sigma_{xy}\big|_{y=\pm h} = a^2 - x^2$

$\sigma_{xx}\big|_{x=\pm a} = 0$

Stresses: $\sigma_{yy} = b_4 xy + b_6 x^3 y + \frac{2}{3} d_6 xy^3$

$\sigma_{xy} = -b_2 - \frac{b_4}{2} x^2 - \frac{d_4}{2} y^2 - \frac{b_6}{4} x^4 - d_6 x^2 y^2 - \frac{f_6}{4} y^4$

$\sigma_{xx} = d_4 xy + \frac{2}{3} d_6 x^3 y + f_6 xy^3$

BC implementation:

$\sigma_{yy}\big|_{y=h} = -x = b_4 xh + b_6 x^3 h + \frac{2}{3} d_6 x h^3$

$\Rightarrow b_4 h + 1 + \frac{2}{3} d_6 h^3 = 0 \longrightarrow 1$

$b_6 = 0$

$\sigma_{yy}\big|_{y=-h} = x \longrightarrow$ gives no new information $\because \sigma_{yy}$ odd in y.

$$\sigma_{xy}\big|_{y=h} = a^2 - x^2 = -b_2 - \frac{b_4}{2}x^2 - \frac{d_4}{2}h^2 - \frac{b_6}{4}x^4 - d_6 x^2 h^2 - \frac{f_6}{4}h^4$$

$$\Rightarrow a^2 + b_2 + \frac{d_4}{2}h^2 + \frac{f_6}{4}h^4 = 0 \longrightarrow 2$$

$$-1 + \frac{b_4}{2} + d_6 h^2 = 0 \longrightarrow 3$$

$$b_6 = 0$$

$\sigma_{xy}\big|_{y=-h} = a^2 - x^2 \longrightarrow$ no new info $\because$ σ_{xy} even in y.

$$\sigma_{xx}\big|_{x=a} = 0 \Rightarrow \underline{\underline{f_6 = 0}} \longrightarrow 4$$

$$\underline{\underline{d_4 + \frac{2}{3}d_6 a^2 = 0}} \longrightarrow 5$$

} see below. Not possible to satisfy these

Now note that ϕ must satisfy $\nabla^4\phi = 0$.

$$\Rightarrow \underbrace{(6b_6 + 6f_6 + 4d_6)}_{\Rightarrow \text{ this } = 0 \longrightarrow 6} xy = 0$$

$$1a, 4, 6 \Rightarrow \underline{\underline{d_6 = 0}} \longrightarrow 5 \Rightarrow \underline{\underline{d_4 = 0}} \longrightarrow 2 \Rightarrow \underline{\underline{b_2 = -a^2}}$$

$$\hookrightarrow 3 \Rightarrow \underline{\underline{b_4 = 2}} \longrightarrow 1 \Rightarrow \underline{\underline{2h + 1 = 0}} \rightarrow \text{inconsistency}$$

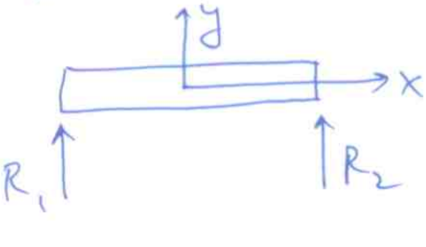
$\therefore$ This means that 4, 5 resulting from $\sigma_{xx}|_{x=a} = 0$ cannot be satisfied. So discard above double-underlined results.

Try to satisfy $\displaystyle\int_{-a}^{a} \sigma_{xx}\big|_{x=\pm a}\, dy = 0$ (weak BC).

This is identically satisfied $(0=0)$ since σ_{xx} is odd in y. So this weak BC is useless.

Hence we need to satisfy weak BC on σ_{xy}, i.e,

$$\int_{-h}^{h} \sigma_{xy}\big|_{x=a}\, dy = R_2$$ where R_2 is right hand reaction.



External equilibrium:

$$\sum F_y = 0 = R_1 + R_2$$

$$\sum M_0 = 0 = 4\int_0^a (x\,dx)x + 4\int_0^a (a^2 - x^2)\,dx\,h + R_1 a - R_2 a$$

$$\Rightarrow R_1 - R_2 = -\frac{4a^2}{3} - \frac{8ha^2}{3}$$

$$\Rightarrow R_2 = \frac{4a^2}{6} + \frac{8ha^2}{6} = -R_1$$

To implement weak BC on σ_{xy}, first solve for b_4, d_6, f_6.

$$①, ③ \Rightarrow b_4 = 2(1 - d_6h^2), \quad 2(1-d_6h^2)h + 1 + \frac{2}{3}d_6h^3 = 0$$

$$\Rightarrow \boxed{\begin{aligned} & d_6 = (2h+1)\frac{3}{4h^3} \longrightarrow (a) \\ & b_4 = 2\left(1 - (2h+1)\frac{3}{4h}\right) = \frac{2}{4h}(-2h-3) \longrightarrow (b) \\ ⑥ \Rightarrow\ & f_6 = -(2h+1)\frac{1}{2h^3} \longrightarrow (c) \end{aligned}}$$

$$\int_{-h}^{h} \sigma_{xy}\Big|_{x=a} dy = R_2$$

$$\Rightarrow -\left(2h\left[b_2 - \frac{a^2}{4h}(2h+3)\right] + \frac{2h^3}{3}\left[\frac{d_4}{2} + (2h+1)\frac{3a^2}{4h^3}\right] - \frac{2h^5}{5}\frac{(2h+1)}{8h^3}\right) = \frac{4a^2}{6} + \frac{8ha^2}{6} \longrightarrow ⑦$$

Solve ②, ⑦ for b_2, d_4.

$$2h\left(a^2 + \frac{d_4}{2}h^2 - \frac{(2h+1)}{2h^3}\frac{h^4}{4}\right) + \frac{a^2}{2}(2h+3)$$

$$-\frac{2h^3}{3}\left(\frac{d_4}{2} + (2h+1)\frac{3a^2}{4h^3}\right) + \frac{2h^5}{5}\frac{(2h+1)}{8h^3} = \frac{4a^2}{6} + \frac{8ha^2}{6}$$

$$\Rightarrow \boxed{\begin{aligned} & d_4 = \frac{3}{2h^3}\left[\frac{2}{5}h^3 - \frac{2}{3}ha^2 + \frac{h^2}{5} - \frac{1}{3}a^2\right] \\ ② \Rightarrow\ & b_2 = -\frac{a^2}{2} - \frac{1}{20}h^2 - \frac{1}{40}h + \frac{a^2}{4h} \end{aligned}}$$

All constants are solved. Substitute them in the stresses.