P.1 Refer everything to p-coord system. So,

$$\underline{\underline{\sigma}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$N = \underline{n}^T \underline{\underline{\sigma}}\, \underline{n} = 4n_1^2 + 5n_2^2 + 6n_3^2 = 5 \longrightarrow 1$$

$$S^2 = \sigma^2 - N^2 = (\sigma_{11} n_1)^2 + (\sigma_{22} n_2)^2 + (\sigma_{33} n_3)^2 - N^2$$

$$\Rightarrow \left(\tfrac{1}{2}\right)^2 = 4^2 n_1^2 + 5^2 n_2^2 + 6^2 n_3^2 - 5^2 \longrightarrow 2$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \longrightarrow 3$$

Sol. of 1 – 3 is $n_1 = \pm \frac{1}{2\sqrt{2}}$, $n_2 = \pm \frac{\sqrt{3}}{2}$, $n_3 = \pm \frac{1}{2\sqrt{2}}$

i.e, 8 planes exist (only 4 of them are distinct).

P.2 Use linear strain tensor, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\underline{\underline{e}} = k \begin{pmatrix} 2x & 2 & 1 \\ & 4y & 1/2 \\ \text{symm} & & 8z \end{pmatrix} \Bigg|_{\substack{x=2\\y=2\\z=3}} = k \begin{pmatrix} 4 & 2 & 1 \\ 2 & 8 & 0.5 \\ 1 & 0.5 & 24 \end{pmatrix}$$

(a) $$e\big|_{\underline{n}(1)} = \underline{n}^T(1)\, \underline{\underline{e}}\, \underline{n}(1) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \underline{\underline{e}}\, \underline{n}(1)$$

$$= \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{8.5}{\sqrt{2}} & \frac{24.5}{\sqrt{2}} \end{pmatrix} k\, \underline{n}(1) = \frac{33}{2} k$$

$$e\big|_{\underline{n}(2)} = \underline{n}^T(2)\, \underline{\underline{e}}\, \underline{n}(2) = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix} k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 4k$$

$$e\big|_{\underline{n}(3)} = \underline{n}^T(3)\, \underline{\underline{e}}\, \underline{n}(3) = \begin{pmatrix} 3.2 & 1.6 & 19.8 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0 \\ 0.8 \end{pmatrix} = 17.76k$$

(b) $$\cos\theta_0 = n_x(1) n_x(2) + n_y(1) n_y(2) + n_z(1) n_z(2) = 0, \quad \theta_0 = 90^\circ$$

$$\underline{n}^T(1)\, \underline{\underline{e}}\, \underline{n}(2) = \frac{3}{\sqrt{2}} k$$

$$\cos\theta = \frac{\cos\theta_0 + 2\, \underline{n}^T(1)\, \underline{\underline{e}}\, \underline{n}(2)}{(1 + 2\, \underline{n}^T(1)\, \underline{\underline{e}}\, \underline{n}(1))^{1/2} (1 + 2\, \underline{n}^T(2)\, \underline{\underline{e}}\, \underline{n}(2))^{1/2}}$$

$$\cos\theta = \frac{\cos\theta_0 + 2\,\underline{n}^T(1)\,\underline{\underline{e}}\,\underline{n}(2)}{(1+2\,\underline{n}^T(1)\,\underline{\underline{e}}\,\underline{n}(1))^{1/2}\,(1+2\,\underline{n}^T(2)\,\underline{\underline{e}}\,\underline{n}(2))^{1/2}} \quad \text{(repeated)}$$

$$= \frac{0 + 2 * \frac{3}{\sqrt{2}} * 0.001}{(1+2*\frac{33}{2}*0.001)^{1/2}\,(1+2*4*0.001)^{1/2}} = 4.1577*10^{-3}$$

$\theta = 89.7618^\circ$, $\Delta\theta = \theta_0 - \theta = 0.2382^\circ$

(c) $\cos\theta_0 = n_x(1)\,n_x(3) + n_y(1)\,n_y(3) + n_z(1)\,n_z(3) = \frac{0.8}{\sqrt{2}}$, $\theta_0 = 55.55^\circ$

$$\cos\theta = \frac{\cos\theta_0 + 2\,\underline{n}^T(1)\,\underline{\underline{e}}\,\underline{n}(3)}{(1+2\,\underline{n}^T(1)\,\underline{\underline{e}}\,\underline{n}(1))^{1/2}\,(1+2\,\underline{n}^T(3)\,\underline{\underline{e}}\,\underline{n}(3))}$$

$$= \frac{\frac{0.8}{\sqrt{2}} + 2*\frac{21.4}{\sqrt{2}}*0.001}{(1+2*\frac{33}{2}*0.001)^{1/2}\,(1+2*17.76*0.001)^{1/2}}$$

$= 0.576209 \Rightarrow \theta = 54.8156^\circ \Rightarrow \Delta\theta = \theta_0 - \theta$
$= 0.73445^\circ$

P.3 Note: stresses, hence strains are uniform (spatially constant).

$$e_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = \left(\frac{1-2\nu}{E}\right)(-p)$$

$$= e_{yy} = e_{zz}$$

$$e_{xx} = u_{x,x} \Rightarrow u_x = \frac{(2\nu-1)}{E}\,p\,x + f(y,z)$$

$$e_{yy} = u_{y,y} \Rightarrow u_y = \frac{(2\nu-1)}{E}\,p\,y + g(x,z)$$

$$e_{zz} = u_{z,z} \Rightarrow u_z = \frac{(2\nu-1)}{E}\,p\,z + h(x,y)$$

$$e_{xy} = \tfrac{1}{2}(u_{x,y} + u_{y,x}) = 0 = \tfrac{1}{2}(f_{,y} + g_{,x}) \rightarrow 1$$

$$e_{yz} = \tfrac{1}{2}(u_{y,z} + u_{z,y}) = 0 = \tfrac{1}{2}(g_{,z} + h_{,y}) \rightarrow 2$$

$$e_{zx} = \tfrac{1}{2}(u_{z,x} + u_{x,z}) = 0 = \tfrac{1}{2}(h_{,x} + f_{,z}) \rightarrow 3$$

$$\frac{\partial ①}{\partial z} + \frac{\partial ②}{\partial x} = 0 \text{ gives} \quad f_{,zy} + g_{,xz} + g_{,zx} + h_{,xy} = 0 \rightarrow ④$$

Insert ③ in ④, get $\underbrace{(f_{,z} + h_{,x})}_{=0 \text{ from } ③}{}_{,y} + 2g_{,xz} = 0$

$$\Rightarrow g_{,xz} = 0 \Rightarrow g = g_1(x) + g_2(z) \longrightarrow ⑤$$

Similarly you can obtain, $f_{,yz} = 0 \Rightarrow f = f_1(y) + f_2(z) \rightarrow ⑥$.

$\&\quad h_{,xy} = 0 \Rightarrow h = h_1(x) + h_2(y) \rightarrow ⑦$

⑤, ⑥ in ① gives $\rightarrow f_1' + g_1' = 0 \Rightarrow f_1' = -g_1' = c_1$ (const)

$\Rightarrow f_1 = c_1 y + k_1, \quad g_1 = -c_1 x + k_2$ (k_1, k_2, are consts).

⑤, ⑦ in ② gives $\rightarrow g_2' + h_2' = 0 \Rightarrow g_2' = -h_2' = c_2$

$\Rightarrow g_2 = c_2 z + k_3, \quad h_2 = -c_2 y + k_4$

⑥, ⑦ in ③ gives $\rightarrow f_2' + h_1' = 0 \Rightarrow f_2' = -h_1' = c_3$

$\Rightarrow f_2 = c_3 z + k_5, \quad h_1 = -c_3 x + k_6$

So,
$$u_x = cx + c_1 y + c_3 z + (k_1 + k_5)$$
$$u_y = cy + c_2 z - c_1 x + (k_3 + k_2)$$
$$u_z = cz - c_3 x - c_2 y + (k_6 + k_4)$$

From zero-displ at origin $\longrightarrow (k_1 + k_5) = (k_3 + k_2) = (k_6 + k_4) = 0$

From zero rotation at origin,

$2\omega_{xy} = u_{x,y} - u_{y,x} = 2c_1 = 0$ at origin $\Rightarrow c_1 = 0 \Rightarrow \omega_{xy} = 0$ everywhere

$2\omega_{yz} = u_{y,z} - u_{z,y} = 2c_2 = 0$ at origin $\Rightarrow c_2 = 0 \Rightarrow \omega_{yz} = 0$ everywhere

$2\omega_{zx} = u_{z,x} - u_{x,z} = -2c_3 = 0$ at origin $\Rightarrow c_3 = 0 \Rightarrow \omega_{zx} = 0$ everywhere.

$\Rightarrow$ rotation is zero everywhere.

$u_x = cx, \quad u_y = cy, \quad u_z = cz, \quad c = \left(\frac{2\nu - 1}{E}\right) p$ ◄