CE623 Advanced Mechanics of Solids Quiz-1 30/8/06

- Marks: **Q1=15, Q2=15, Q3=20.**
- Show all working.
- Show finally only one attempt per question (i.e., cancel out multiple attempts that you do not want to be graded).
- Open notes exam

1. The linear state of strain at a point in a solid is given as

$$e_{ij} = \begin{pmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{pmatrix}$$

Determine:

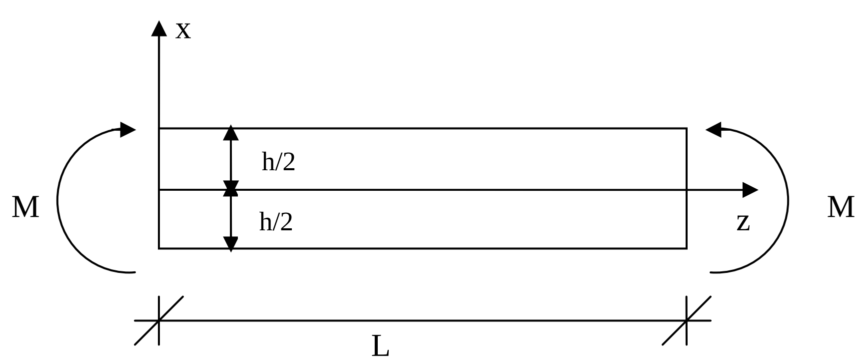
(a) The extensional strain in the direction $(1,\ -1,\ \sqrt{2})$

(b) The angle after deformation between two line elements originally along $(1,\ -1,\ \sqrt{2})$ and $(-1,\ 1,\ \sqrt{2})$.

2. Find the direction of the total force acting on the surface of a cylinder due to the stress distribution given below. The cylinder lies in the region $x > 0$, has radius $R = 1$ and length $L = 1$, and its longitudinal axis is the *x*-axis. The stress distribution is

$$\sigma_{ij} = \begin{pmatrix} 3xy & 5y^2 & 0 \\ 5y^2 & 0 & 2z \\ 0 & 2z & 0 \end{pmatrix}$$

3. Consider a beam of uniform cross-section subject to couple M at its ends, as shown. The couple is applied by means of a linear distribution of σ_{zz} along the boundaries $z = 0$ and $z = L$. Obtain the stresses, strains, and displacements. The beam is restrained from rigid body displacement.
(Hint: From basic solid mechanics, assume a solution of the stress tensor σ_{ij} having only one non-zero component, with the non-zero component varying linearly with the x coordinate only. Check that your stress distribution satisfies boundary condition, equilibrium, compatibility. Then, use this distribution to obtain strains and hence displacements.)



P.1.(a) Linear theory $\longrightarrow$ $\varepsilon_E = M = e_{ij} n_i n_j = \underline{n}^T \underline{\underline{e}}\, \underline{n}$

$$\varepsilon_E = \frac{1}{\sqrt{4}}\begin{pmatrix}1 & -1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}6\\ -6\\ 6\sqrt{2}\end{pmatrix} = \frac{24}{(\sqrt{4})^2} = 6$$

(If $\underline{\underline{e}}$ was given as $*$ by 10^{-6}, say, then $\varepsilon_E = 6*10^{-6}$).

(b) For direction $\dfrac{(-1, 1, \sqrt{2})}{\sqrt{4}} = \underline{n}(2)$

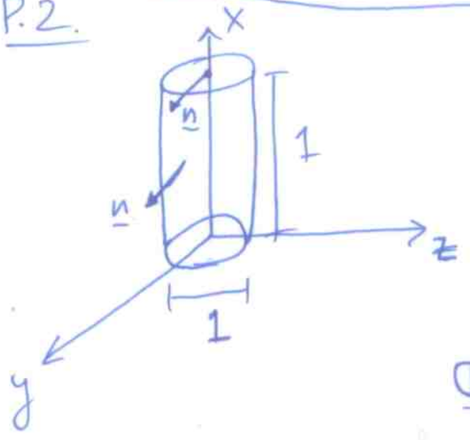
$$\underline{\underline{e}}\,\underline{n}(2) = \begin{pmatrix}1 & -3 & \sqrt{2}\\ -3 & 1 & -\sqrt{2}\\ \sqrt{2} & -\sqrt{2} & 4\end{pmatrix}\begin{pmatrix}-1\\ 1\\ \sqrt{2}\end{pmatrix} * \frac{1}{2} = \frac{1}{2}\begin{pmatrix}-2\\ 2\\ 2\sqrt{2}\end{pmatrix}$$

So $\underline{n}(1)^T \underline{\underline{e}}\,\underline{n}(2) = \frac{1}{2}\begin{pmatrix}1 & -1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}-2\\ 2\\ 2\sqrt{2}\end{pmatrix} * \frac{1}{2} = \frac{1}{4}(-4+4) = 0.$

Also $\cos\theta_0 = \underline{n}(1)^T \underline{n}(2) = \frac{1}{2}\begin{pmatrix}1 & -1 & \sqrt{2}\end{pmatrix}\begin{pmatrix}-1\\ 1\\ \sqrt{2}\end{pmatrix} * \frac{1}{2} = 0$

$$\Rightarrow \cos\theta = \left[\frac{\cos\theta_0 + 2\,\underline{n}(1)^T \underline{\underline{e}}\,\underline{n}(2)}{(1+2\,\underline{n}(1)^T\underline{\underline{e}}\,\underline{n}(1))^{1/2}\,(1+2\,\underline{n}(2)^T\underline{\underline{e}}\,\underline{n}(2))^{1/2}}\right] = 0 \Rightarrow \theta = 90^\circ$$

P.2.

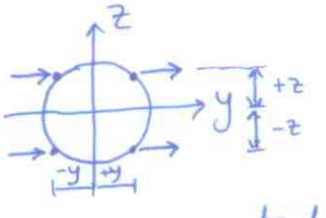


$\underline{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$, $\phi = y^2 + z^2 - 1 = 0.$

$$\underline{n} = \frac{2(y\underline{j} + z\underline{k})}{2(y^2+z^2)} = y\underline{j} + z\underline{k} = (0, y, z)$$

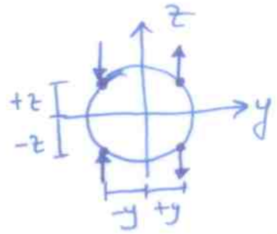
$(y^2+z^2) = 1$ on surface.

$$\underline{\sigma} = \underline{\underline{\sigma}}\,\underline{n} = \begin{pmatrix}3xy & 5y^2 & 0\\ 5y^2 & 0 & 2z\\ 0 & 2z & 0\end{pmatrix}\begin{pmatrix}0\\ y\\ z\end{pmatrix} = \begin{pmatrix}5y^3\\ 2z^2\\ 2yz\end{pmatrix}$$



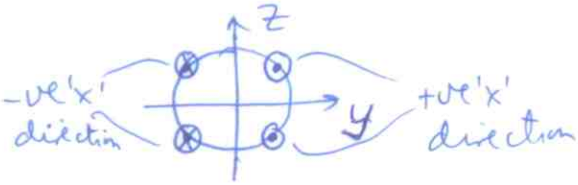
y-component of $\underline{\sigma}$

$\Rightarrow \Sigma F_y \neq 0$



z-component of $\underline{\sigma}$

$\Sigma F_z = 0$



x-component of $\underline{\sigma}$

$\Sigma F_x = 0$

So total force acts in y-direction.

P.3. Assume $\sigma_{zz} = ax$, $a = -\frac{M}{I}$, all other stresses zero.

This solution satisfies equilibrium ($\because$ all other stresses zero, body forces neglected, $\sigma_{zz,z} = 0$), compatibility ($\because$ stresses are linear functions of coordinates, no body forces, so BM compatibility eqns satisfied), and boundary conditions

$$\left\{\begin{array}{l} \sigma_{zz}\Big|_{x=0,L} = az, \text{ giving rise to } \underset{-BM}{-M} \overset{\text{NOTE -ve sign}\because\text{BM}}{=} \displaystyle\int_{-h/2}^{h/2} \sigma_{zz}\, x\, dx = \frac{ax^3}{3}\Big|_{-h/2}^{h/2} \\ \text{other stresses} = 0 \text{ satisfy } \sigma_{zx}\Big|_{x=0,L} = 0, \quad = \dfrac{ah^3}{12} = -\dfrac{M}{\cancel{I}}\dfrac{h^3}{\cancel{12}} \text{ for } t=1 \\ \sigma_{xx}\Big|_{x=\pm\frac{h}{2}} = 0, \quad \sigma_{zx}\Big|_{x=\pm\frac{h}{2}} = 0. \qquad I = \dfrac{th^3}{12} \end{array}\right\}$$

SO THIS IS THE ONLY POSSIBLE SOLUTION.

Constitutive equations (Hooke's law):

$$e_{xx} = e_{yy} = -\frac{\nu}{E}\sigma_{zz} = bx, \quad b = -\frac{\nu}{E}a,$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = cx, \quad c = \frac{a}{E}$$

$$e_{xy} = e_{yz} = e_{zx} = 0.$$

$$u_{x,x} = bx \Rightarrow u_x = \frac{bx^2}{2} + f_x(y,z) \longrightarrow ①$$

$$u_{y,y} = bx \Rightarrow u_y = bxy + f_y(x,z) \longrightarrow ②$$

$$u_{z,z} = cx \Rightarrow u_z = cxz + f_z(x,y) \longrightarrow ③$$

$$e_{xz} = 0 \Rightarrow \frac{\partial f_x}{\partial z} + \frac{\partial f_z}{\partial x} + cz = 0 \longrightarrow ④$$

$$e_{yz} = 0 \Rightarrow \frac{\partial f_y}{\partial z} + \frac{\partial f_z}{\partial y} = 0 \longrightarrow ⑤$$

$$e_{xy} = 0 \Rightarrow \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x} + by = 0 \longrightarrow ⑥$$

$$\frac{\partial ④}{\partial y} + \frac{\partial ⑥}{\partial z} = 0 \Rightarrow 2\frac{\partial^2 f_x}{\partial y \partial z} + \underbrace{\frac{\partial^2 f_z}{\partial x \partial y} + \frac{\partial f_y}{\partial x \partial z}}_{=0 \text{ from } \frac{\partial ⑤}{\partial x} = 0.} = 0$$

$$\Rightarrow f_x = p_y(y) + p_z(z) + k_1 \longrightarrow (7)$$

(7) in (4), (6) $\longrightarrow$ $p_z' + \frac{\partial f_z}{\partial x} + cz = 0 \Rightarrow \underbrace{p_z' + cz}_{\text{fn of } z} = \underbrace{-\frac{\partial f_z}{\partial x}}_{\text{fn of }(x,y)} = k_2$

$$p_y' + \frac{\partial f_y}{\partial x} + by = 0 \Rightarrow \underbrace{p_y' + by}_{\text{fn of } y} = \underbrace{-\frac{\partial f_y}{\partial x}}_{\text{fn of }(x,z)} = k_3$$

So $p_z' + cz = k_2 \Rightarrow p_z = k_2 z - \frac{c}{2}z^2 \longrightarrow (8)$

$p_y' + by = k_3 \Rightarrow p_y = k_3 y - \frac{b}{2}y^2 \longrightarrow (9)$

$-\frac{\partial f_z}{\partial x} = k_2 \Rightarrow f_z = -k_2 x + q_y(y) + k_4 \longrightarrow (10)$

$-\frac{\partial f_y}{\partial x} = k_3 \Rightarrow f_y = -k_3 x + q_z(z) + k_5 \longrightarrow (11)$

$\longrightarrow$ put in (5);

$$q_z' + q_y' = 0 \Rightarrow q_z' = -q_y' = k_6$$

$$q_z = k_6 z + k_7 \longrightarrow (12)$$

$$q_y = -k_6 y + k_8 \longrightarrow (13)$$

(1), (7), (8), (9) $\longrightarrow$ $u_x = b\frac{x^2}{2} - b\frac{y^2}{2} - c\frac{z^2}{2} + k_3 y + k_2 z$

(2), (11), (12) $\longrightarrow$ $u_y = bxy + k_6 z + k_7 - k_3 x + k_5$

(3), (10), (13) $\longrightarrow$ $u_z = cxz - k_6 y + k_8 - k_2 x + k_4$

$= 0$, see below

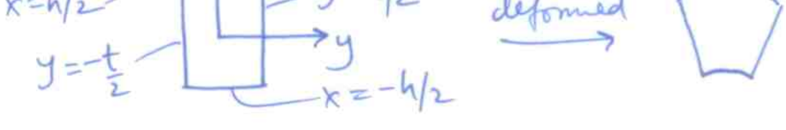
No rigid body translation $\Rightarrow u_x(0,0) = u_y(0,0) = u_z(0,0) = 0$

$$\Rightarrow k_7 + k_5 = k_8 + k_4 = 0$$

Remaining linear terms $= (-k_6 \underline{i} + k_2 \underline{j} - k_3 \underline{k}) \times (x\underline{i} + y\underline{j} + z\underline{k})$

$= \underline{\omega} \times \underline{r} \longrightarrow$ set to zero for no rigid body rotation.

Explanation: Consider deformation of edges of rectangular cross section at z = constant.



, ie curvature along span and along width in opposite dir. (see POPOV).