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A C^0 CONTINUOUS LINEAR BEAM/BILINEAR PLATE FLEXURE ELEMENT

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Abstract—Design of a simple C^0-continuous beam/plate flexure element based on a shear-deformable theory is attempted. Emphasis is placed on development of a low order linear/bilinear element. However, past experience shows that such elements become very "stiff" especially when thickness is reduced (conforming to Kirchhoff mode). This phenomenon is called "locking." Attempts have been made in the last several years by a few investigators to overcome this problem.

The locking problem is resolved here through a new approach. The total strain energy is split into bending and shear energies and an antiparameter in the shear energy term is introduced to avoid locking. By numerical experimentation on beam and plate problems it is shown that the present approach gives good results in the thin limit. It is also shown that the additional/spurious zero energy modes do not arise here because reduced/selective integration is avoided.

NOTATION

- A_s Area in shear for a section = $b \cdot t$
- a, b Plate dimensions
- $\mathbf{B}$ Matrix relating strains and nodal displacements of element
- $\mathbf{B}_b$ Matrix relating bending strains and nodal displacements
- $\mathbf{B}_s$ Matrix relating shear strains and nodal displacements
- $\mathbf{D}_b$ Elasticity matrix relating moments and bending strains
- $\mathbf{D}_s$ Elasticity matrix relating shear forces and shear strains
- D Plate constant = $Et^3/12\,(1 - \gamma^2)$
- E Young's modulus
- G Shear rigidity
- h Element length
- $\mathbf{K}$ Stiffness matrix
- $\mathbf{K}^e$ Element stiffness matrix
- $\mathbf{K}_b$ Bending stiffness matrix
- $\chi, \chi_x, \chi_y, \chi_{xy}$ Curvatures, bending strains
- $\mathbf{K}_s$ Shear stiffness matrix
- k Shear correction factor
- L Length of the beam
- M_x, M_y, M_{xy} Moments
- N_1, N_2 Shape functions
- $\mathbf{N}$ Matrix of shape functions
- p A free parameter used in the formulation
- Q Concentrated load
- Q_x, Q_y Shear forces
- q Uniformly distributed load, UDL
- t thickness
- u Displacement along X axis
- U Strain energy
- U_b Bending strain energy
- U_s Shear strain energy
- v Displacement along Y axis
- w Displacement along Z axis
- ϵ Strain vector
- ϵ_b Bending strain vector
- ϵ_s Shear strain vector
- σ Stress vector
- δ_e Nodal displacement vector for element
- δ Displacement vector at any point in the element
- θ_x, θ_y Rotations about Y and X axes
- $\phi, \phi_{xz}, \phi_{yz}$ Shear strains
- γ Poisson's ratio
- α Antiparameter used in the present formulation

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INTRODUCTION

The early displacement based finite element formulations for flexure problems relying on the "Kirchhoff hypothesis" (for beams, plates, etc.) were plagued with difficulties because of the requirement of slope continuity between adjacent element, i.e. C^1 continuity in shape functions. The main assumption in Kirchhoff's hypothesis is that the transverse normals to the reference middle plane remain so during bending, implying transverse shear strain becomes zero. Thus bending rotation becomes a first derivative of the transverse displacement w and hence requires the transverse displacement field C^1 continuous. Both compatible and incompatible and complicated higher order C^1 continuous elements have been derived in the past[1–6].

In recent years C^0 continuous elements based on shear deformable theories which use independent interpolation of slopes and displacements have been developed. This is mainly due to the ease in the development and the formulation of computer programmes. In the last 10 years or so a number of elements have been developed using a shear deformable theory such as that of Mindlin[7–11].

The recent trend has been towards using linear/bilinear elements, i.e. 2-noded beam/4-noded plate elements. However, such shear flexible elements using low order (linear/bilinear) interpolation for all components of nodal displacement vector become very stiff especially when the thickness is reduced;

i.e. when the system conforms to the Kirchhoff mode. This phenomenon, called "locking," arises from linear/bilinear element discretization. The trouble occurs due to the existence of spurious shear energy which turns out to be of the order $(h/t)^2$ of flexural energy, h and t being effective element size and thickness, respectively, and as $t \to 0$, $h/t \to \infty$. This situation poses an enormous problem in numerical analysis. Thus, the spurious shear energy in the discrete model causes the "shear-locking" phenomenon, which is a *numerical problem*. Attempts have been made in the last several years by a few investigators to overcome this problem. Remedial measures suggested to date to remove such difficulties are:

1. Use of the discrete Kirchhoff procedure, in which the element matrix equation is stabilized by tying together the two independent degrees of freedom w, θ at discrete points such that $\theta_{x_\alpha} = dw/dx_\alpha$. Since this method is complicated to implement, it is not widely used.
2. Use of selective/reduced integration procedure, in which the shear energy term is underintegrated[7, 8, 12–14]. This technique is viewed by many analysts as mere tricks rather than methods. "Heuristic" justification of these procedures has recently been provided[15]. Although this technique is effective, it creates unwanted spurious zero energy modes other than the rigid body modes. This poses many problems, e.g. use of these elements gives oscillatory results in the case of corner supported plates.

Recently a stabilization matrix with a free parameter has been developed by Belytschko *et al.*[16, 17] for the Hughes *et al.*[7] element and also for the Mukhopadhyay and Dinker element[8]. This "stabilization" matrix is developed by combining reduced and fully integrated stiffness matrices. Although this method is effective, the efficiency of the original element is lost and the choice of free parameter requires ones judgement.

Thus a simple, effective and efficient element is yet to be developed which could be free from locking and spurious zero energy modes so that the element could be safely used for wide (L/t) ranges and various boundary conditions.

In this paper a C^0 continuous linear beam/biolinear plate bending element is developed based on a shear deformable theory. Emphasis is given on resolving the locking problem by trying out a new approach. In this formulation total strain energy is split into bending and shear energies and an anti-parameter in the shear energy terms is introduced to avoid locking. By numerical experimentations on beam and plate problems it is shown that the present approach gives good results in the thin limit. Since reduced/selective integration is avoided, spurious zero energy modes do not arise. The pathological problem of the corner suppported plate is solved safely.

THEORETICAL FORMULATION

The following presents a brief account of the 2-noded beam element and the 4-noded plate element: (see Fig. 1):

nodal displacement vector,

$$\delta_e = (w_1, \theta_1, w_2, \theta_2)^T, \quad (1)$$

shape functions,

$$N_1 = (1 - x/h); \qquad N_2 = x/h, \quad (2)$$

displacement vector, $\delta = \mathbf{N}\,\delta_e$, (3)

strain vector,

$$\epsilon = (\chi, \phi)^T = (d\theta/dx,\ dw/dx + \theta)^T. \quad (4)$$

Here, we introduce an antiparameter α such that

$$\phi = \alpha\,(dw/dx + \theta). \quad (5)$$

The strain energy expression U is written as follows:

$$U = U_{\text{bending}} + U_{\text{shear}} = \frac{1}{2}\int_h EI\left(\frac{d\theta}{dx}\right)^2 dx + \frac{1}{2}\int_h kGA_s\,\alpha^2\,(dw/dx + \theta)^2\,dx. \quad (6)$$

For a rectangular beam cross section,

$$k = 5/6; \qquad A_s = b\,t; \qquad G = E/2\,(1 + \gamma), \quad (7)$$

and we set $\alpha = p\,(t/L)$ intuitively, where p is a free parameter, t is thickness and L is the total beam length. Finally, we get stiffness matrices

$$\mathbf{K}_{\text{bending}} = EI\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/h & 0 & -1/h \\ 0 & 0 & 0 & 0 \\ 0 & -1/h & 0 & 1/h \end{bmatrix} \quad (8)$$

and

$$\mathbf{K}_{\text{shear}} = \frac{5EI}{(1+\gamma)}\,\alpha^2(h/t)^2 \times \begin{bmatrix} 1/h^3 & -1/2h^2 & -1/h^3 & -1/2h^2 \\ -1/2h^2 & 1/3h & 1/2h^2 & 1/6h \\ -1/h^3 & 1/2h^2 & 1/h^3 & 1/2h^2 \\ -1/2h^2 & 1/6h & 1/2h^2 & 1/3h \end{bmatrix}. \quad (9)$$

If $\alpha = p(t/L)$ is substituted in, say (9), the multiplier to the matrix becomes

$$5EI/(1 + \gamma)\cdot(h/L)^2\cdot p^2.$$

By adding the above two matrices, we get, in the usual manner,

$$\mathbf{K}_{\text{element}} = \mathbf{K}_b + \mathbf{K}_s. \quad (10)$$

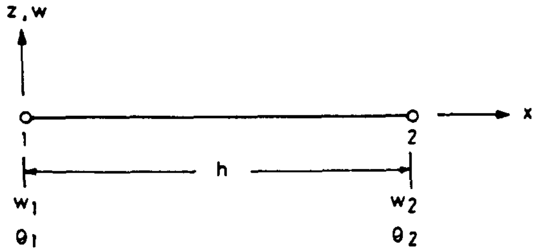

Fig. 1. 2-Noded linear element.

Using this $\mathbf{K}_{\text{element}}$, one and two element analyses were carried out for a cantilever beam of rectangular cross section for a tip load Q for which the results are

$$\delta_{\text{tip}} = \frac{QL^3}{3EI}\left[\frac{12(1+\gamma)}{5p^2}\cdot\frac{5p^2+3(1+\gamma)}{5p^2+12(1+\gamma)}\right] \quad (11)$$

and

$$\delta_{\text{tip}} = \frac{QL^3}{3EI}\left[\frac{16(1+\gamma)}{5p^2}\cdot\frac{15p^2+9(1+\gamma)}{5p^2+48(1+\gamma)}\right], \quad (12)$$

respectively.

The idea behind the above analyses was to assess a theoretical bound for the parameter "p" in the limit when the number of elements are increased to infinity. However, at the present moment such an analytical estimate appears infeasible. But the nature of the multipliers appearing in eqns (11) and (12) suggests that a numerical estimate can be obtained. In this paper, an estimate for the parameter p is obtained to ensure lower-bound monotonic convergence to true solution by subjecting this formulation to numerical experimentation.

A similar formulation is done for a bilinear plate element. For the sake of brevity the details are presented in the Appendix 1.

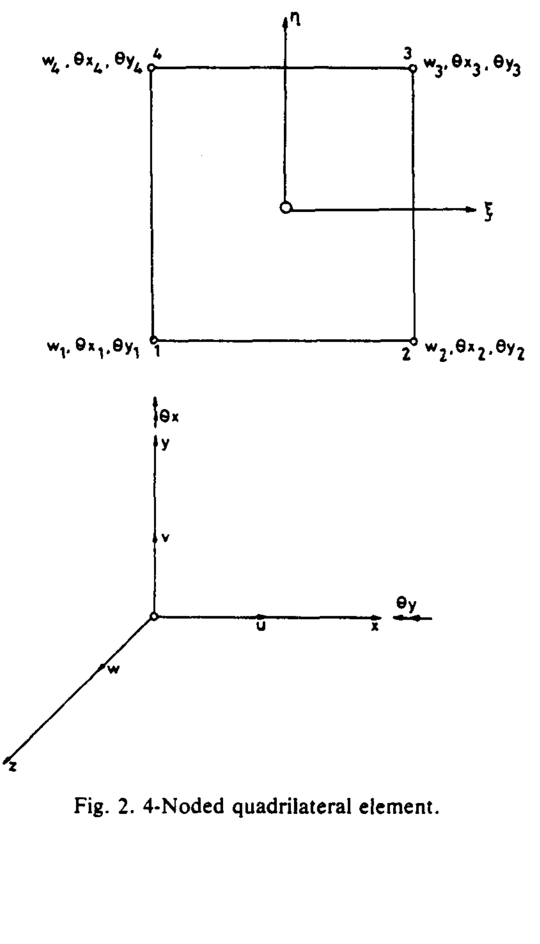

Fig. 2. 4-Noded quadrilateral element.

NUMERICAL EXPERIMENTS

The new formulation was tested with a few beam flexure problems with various load/boundary conditions such as (1) cantilever beam under tip load, (2) cantilever beam under UDL, (3) simply supported beam under concentrated load at the centre, (4) simply supported beam under UDL, (5) clamped beam under concentrated load at the centre and (6) clamped beam under UDL. All these problems were solved for the tip displacement (cantilever problems) or the centre displacements (simply supported and clamped beams) for different values of the parameter p. The results are presented in Tables 1–6 and Figs. 1–6.

Similar experiments were also performed for plate flexure problems such as (1) simply supported square plate under concentrated load at the centre, (2) simply supported square plate under UDL, (3) clamped square plate under concentrated load at the centre, (4) clamped square plate under UDL and (5) corner just supported plate under UDL.

The results are presented in Tables 1–11 and Figs. 3–13.

Finally the element was checked for spurious zero energy modes by finding out the eigenvalues of the element stiffness matrix (Table 12). It is seen that no extra spurious zero energy modes exist for the present element.

CONCLUSIONS

For each beam/plate problem we can see that (Tables 1–11, Figs. 3–13) the finite element displacement solution converges monotonically to the correct value for a particular value of parameter p. It is thus proposed that p lies in a certain range, that is, 6–12 for beams and 9–15 for plates, for which the FEM solution gives reasonable answers and thus avoids locking. We are also reasonably correct in proposing that the choice of p is independent of the (L/t) ratio as well as the nature of the boundary conditions and loading conditions. Finally it is also confirmed that the present formulation is free from spurious zero energy modes since reduced/selective integration is avoided. This assertion is proved by our getting an extremely good solution for the corner supported plate, which is otherwise not possible with a reduced/selective integrated element[7, 16].

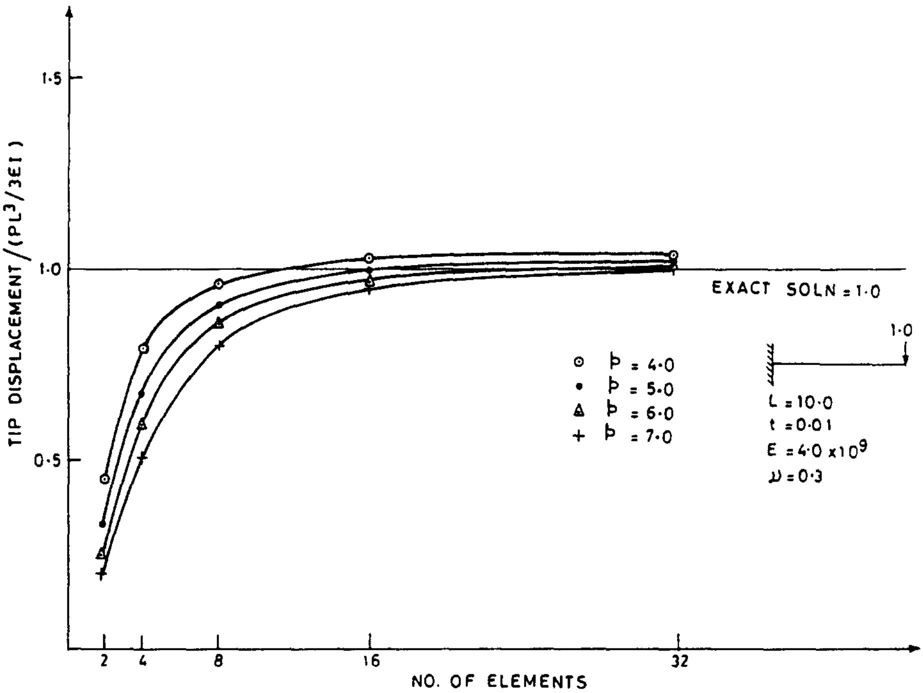

Fig. 3. Cantilever under tip load (see Table 1).

Table 1 (see Fig. 3). Tip displacement for a cantilever beam under tip load (L = 10.0, t = 0.01, $E = 4.0 \times 10^9$, $\gamma = 0.3$)

	p								
Elements	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
2	0.460	0.343	0.263	0.206	0.165	0.135	0.112	0.094	0.080
4	0.794	0.687	0.594	0.513	0.444	0.385	0.336	0.294	0.259
8	0.971	0.917	0.866	0.816	0.766	0.718	0.672	0.627	0.584
16	1.030	1.000	0.977	0.958	0.938	0.917	0.901	0.878	0.853
32	1.040	1.020	1.010	1.000	1.010	0.995	0.991	0.964	0.959

Exact: 1.0.

Table 2 (see Fig. 4). Tip displacement for a cantilever beam under UDL (L = 10.0, t = 0.01, $E = 1.5 \times 10^9$, $\gamma = 0.3$)

	p								
Elements	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
2	0.503	0.375	0.286	0.224	0.179	0.146	0.121	0.102	0.087
4	0.822	0.708	0.610	0.526	0.454	0.394	0.343	0.301	0.265
8	0.991	0.930	0.876	0.825	0.774	0.724	0.677	0.631	0.589
16	1.046	1.013	0.985	0.963	0.942	0.924	0.898	0.878	0.856
32	1.063	1.032	1.017	1.005	0.995	0.988	0.978	0.972	0.967

Exact: 1.0.

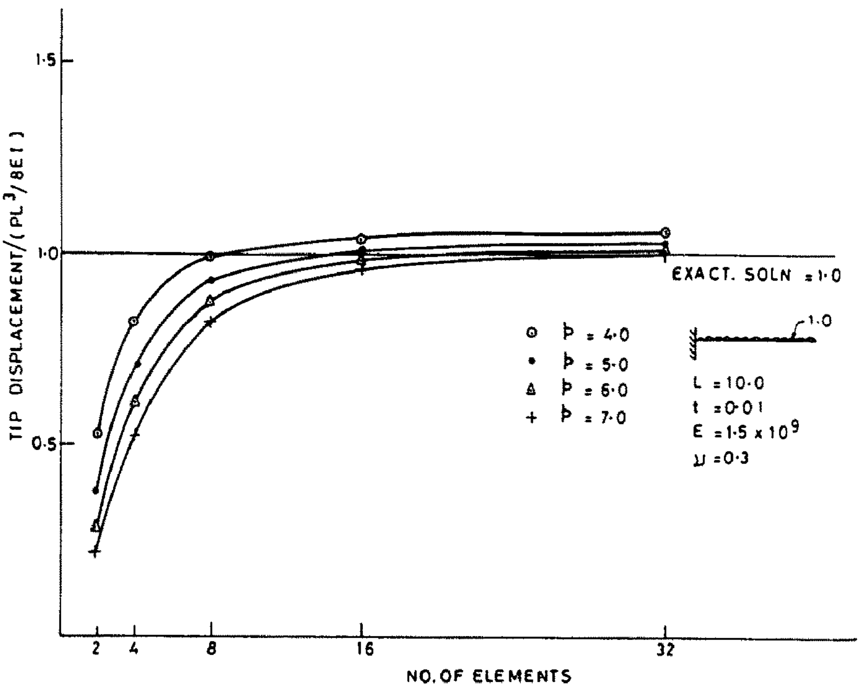

Fig. 4. Cantilever under UDL (see Table 2).

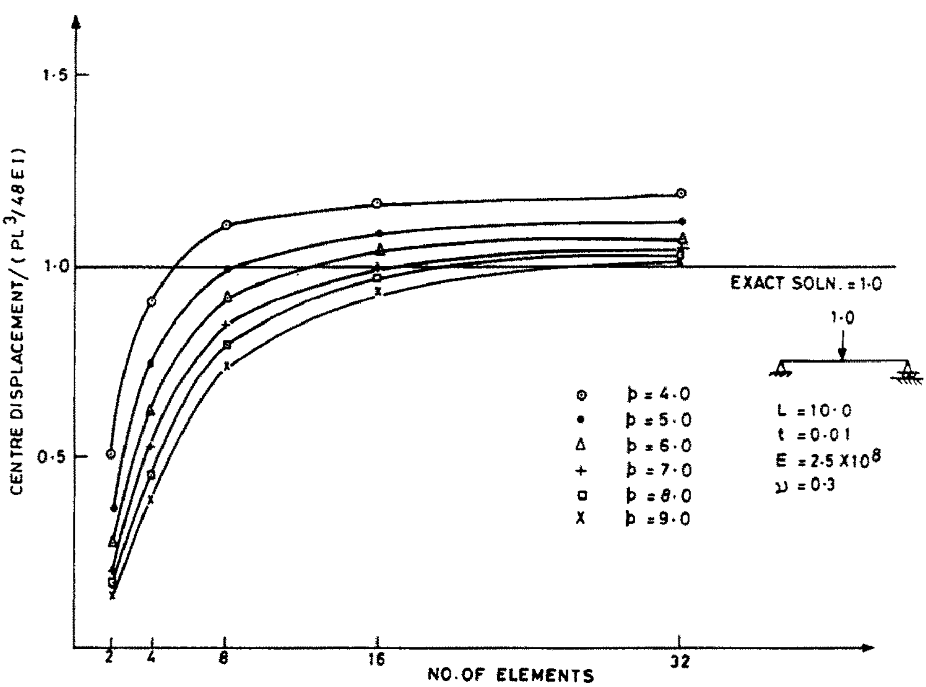

Fig. 5. Simply supported beam under concentrated load at centre (see Table 3).

Table 3 (see Fig. 5). Centre displacement for a simply supported beam under concentrated load at centre (L = 10.0, t = 0.01, E = 2.5×10^8, γ = 0.3)

	p								
Elements	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
2	0.524	0.375	0.280	0.216	0.171	0.139	0.114	0.0959	0.0815
4	0.905	0.749	0.631	0.537	0.460	0.396	0.343	0.300	0.263
8	1.110	1.000	0.921	0.854	0.794	0.739	0.687	0.639	0.594
16	1.170	1.090	1.040	1.000	0.971	0.943	0.917	0.891	0.866
32	1.190	1.120	1.070	1.050	1.030	1.010	1.000	0.988	0.977

Exact: 1.0.

Table 4 (see Fig. 6). Centre displacement for a simply supported beam under UDL (L = 10.0, t = 0.01, E = 1.5625×10^8, γ = 0.3)

	p								
Elements	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
2	0.419	0.300	0.224	0.173	0.137	0.111	0.092	0.077	0.065
4	0.838	0.700	0.592	0.505	0.433	0.374	0.325	0.283	0.249
8	1.059	0.966	0.895	0.834	0.777	0.724	0.675	0.628	0.584
16	1.130	1.063	1.020	0.987	0.969	0.933	0.908	0.884	0.859
32	1.149	1.091	1.056	1.034	1.018	1.007	0.997	0.987	0.976

Exact: 1.0.

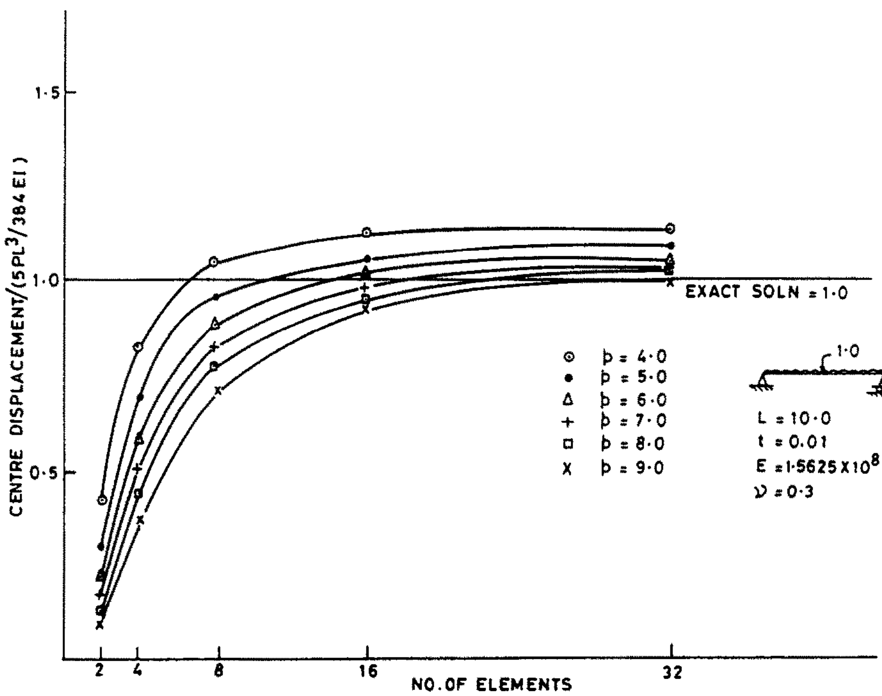

Fig. 6. Simply supported beam under UDL (see Table 4).

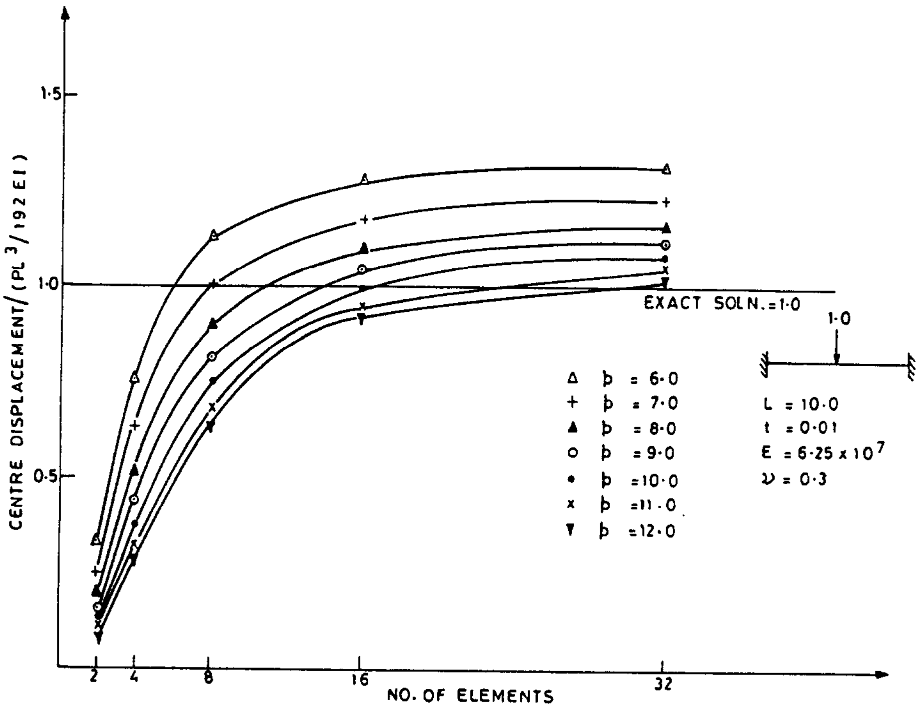

Fig. 7. Clamped beam under concentrated load at centre (see Table 5).

Table 5 (see Fig. 7). Centre displacement for a clamped beam under concentrated load at centre (L = 10.0, t = 0.01, E = 6.25 × 10^7, γ = 0.3)

	p								
Elements	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
2	0.780	0.499	0.347	0.255	0.195	0.154	0.125	0.103	0.0867
4	1.350	0.999	0.782	0.633	0.524	0.440	0.375	0.322	0.280
8	1.650	1.330	1.140	1.010	0.905	0.821	0.750	0.687	0.631
16	1.750	1.450	1.290	1.180	1.110	1.050	1.000	0.958	0.921
32	1.770	1.490	1.330	1.240	1.170	1.130	1.090	1.060	1.040

Exact: 1.0.

Table 6 (see Fig. 8). Centre displacement for a clamped beam under UDL (L = 10.0, t = 0.01, E = 3.125 × 10^7, γ = 0.3)

	p								
Elements	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
2	0.520	0.333	0.231	0.170	0.130	0.103	0.083	0.069	0.058
4	1.080	0.799	0.626	0.507	0.419	0.352	0.300	0.258	0.224
8	1.461	1.180	1.011	0.896	0.804	0.731	0.666	0.611	0.561
16	1.643	1.370	1.206	1.112	1.040	0.986	0.941	0.902	0.867
32	1.720	1.444	1.291	1.203	1.140	1.091	1.062	1.032	1.010

Exact: 1.0.

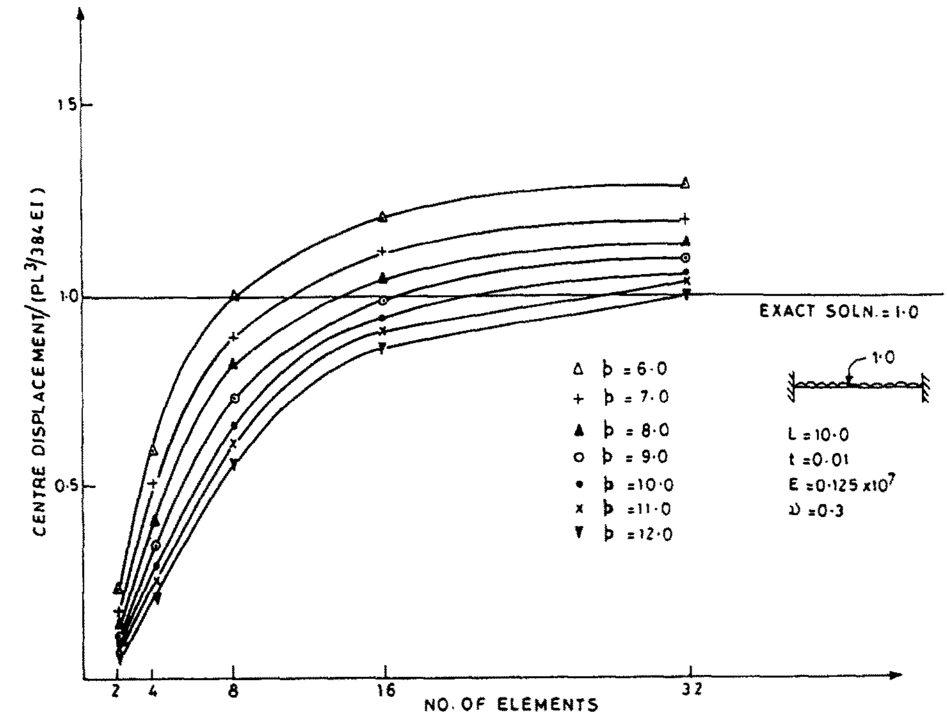

Fig. 8. Clamped beam under UDL (see Table 6).

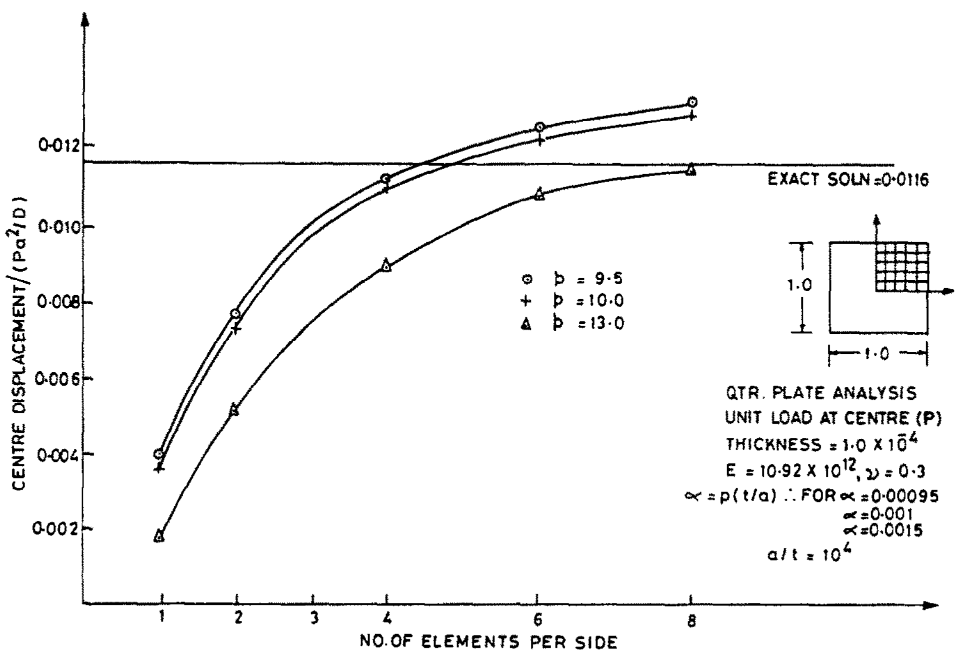

Fig. 9. Simply supported plate under concentrated load at centre (see Table 7).

Table 7 (see Fig. 9). Centre displacement for a simply supported square plate under concentrated load at centre (a = 1.0, t = 0.0001, E = 10.92 × 10^{12}, γ = 0.3)

	p		
Mesh	9.5	10.0	13.0
1 × 1	0.00397	0.0036	0.0018
2 × 2	0.00771	0.0073	0.0052
4 × 4	0.0113	0.0109	0.0089
6 × 6	0.0126	0.0123	0.0108
8 × 8	0.0132	0.0128	0.0113

Exact: 0.0116.

Table 8 (see Fig. 10). Centre displacement for a simply supported square plate under UDL (a = 1.0, t = 0.0001, E = 10.92 × 10^{12}, γ = 0.3)

	p	
Mesh	9.0	9.5
1 × 1	0.00108	0.00099
2 × 2	0.00254	0.0024
3 × 3	0.00329	0.00318
4 × 4	0.00367	0.00359
6 × 6	0.00401	0.00392
8 × 8	0.00414	0.00409
9 × 9	0.00417	0.00413

Exact: 0.00406.

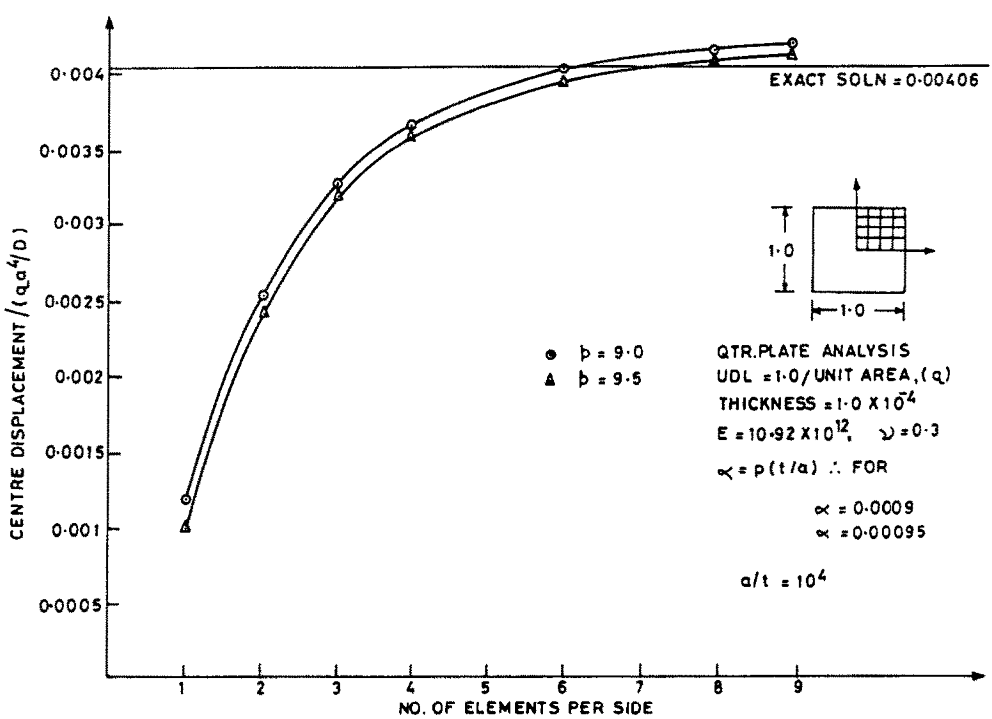

Fig. 10. Simply supported plate under UDL (see Table 8).

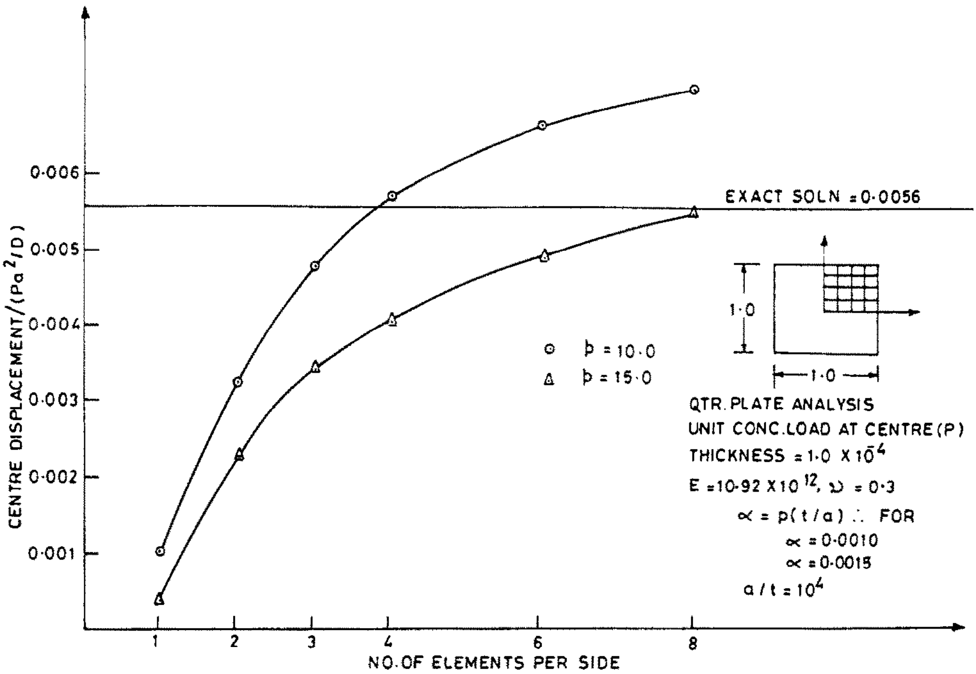

Fig. 11. Clamped plate under concentrated load at centre (see Table 9).

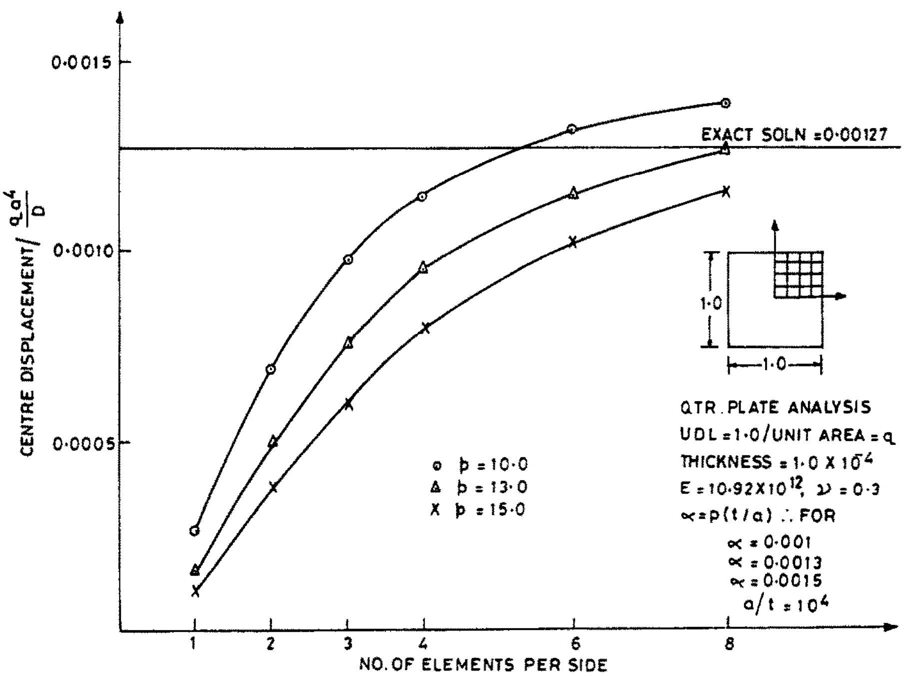

Fig. 12. Clamped plate under UDL (see Table 10).

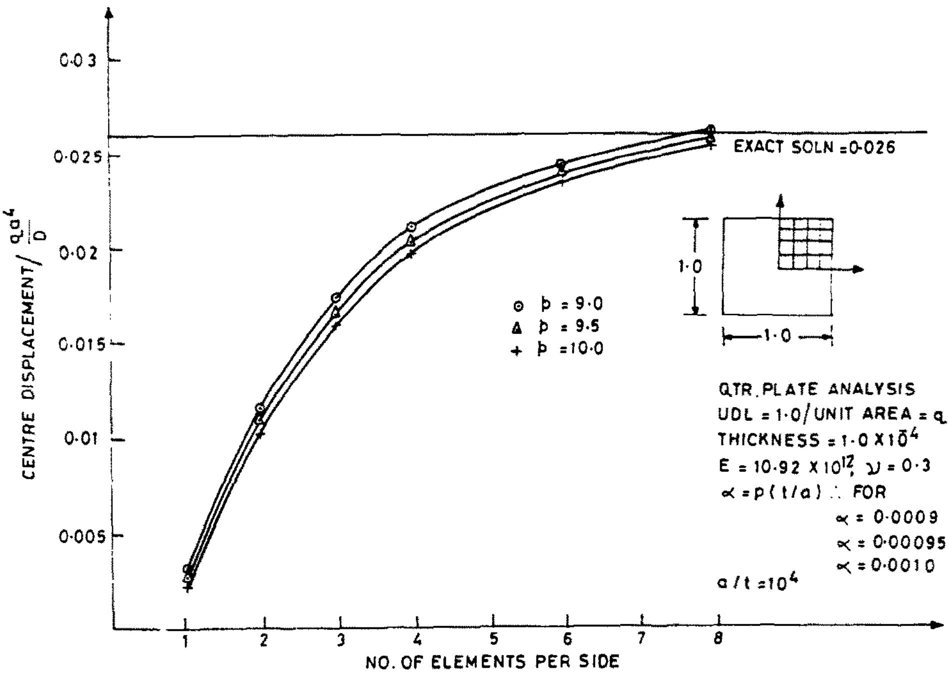

Fig. 13. Corner just supported plate under UDL (see Table 11).

Table 9 (see Fig. 11). Centre displacement for clamped square plate under concentrated load at centre (a = 1.0, t = 0.0001, E = 10.92 × 10^{12}, γ = 0.3)

	p	
Mesh	10.0	15.0
1 × 1	0.00107	0.00039
2 × 2	0.00329	0.0024
3 × 3	0.00478	0.00396
4 × 4	0.00571	0.00415
6 × 6	0.0067	0.00495
8 × 8	0.00714	0.00556

Exact: 0.0056.

Table 11 (see Fig. 13). Centre displacement for a corner just supported square plate under UDL (a = 1.0, t = 0.0001, E = 10.92 × 10^{12}, γ = 0.3)

	p		
Mesh	9.0	9.5	10.0
1 × 1	0.00305	0.00277	0.00253
2 × 2	0.0117	0.0110	0.0104
3 × 3	0.0176	0.0167	0.0161
4 × 4	0.0212	0.0205	0.0198
6 × 6	0.0248	0.0242	0.0237
8 × 8	0.0264	0.0259	0.0256

Exact: 0.026.

Table 10 (see Fig. 12). Centre displacement for clamped square plate under UDL (a = 1.0, t = 0.0001, E = 10.92 × 10^{12}, γ = 0.3)

	p		
Mesh	10.0	13.0	15.0
1 × 1	0.000267	0.000171	0.000119
2 × 2	0.000698	0.000498	0.000371
3 × 3	0.000981	0.000775	0.000608
4 × 4	0.00115	0.000963	0.000797
6 × 6	0.00133	0.00114	0.00106
8 × 8	0.00139	0.00123	0.00116

Exact: 0.00127.

Table 12. Eigenvalues of a plate element using present formulation (element length = 0.0625, t = 0.0001, E = 10.92 × 10^{12}, γ = 0.3)

p	9.5	10.0	15.0
Rigid body modes	3 zero	3 zero	3 zero
Elastic modes	0.100	0.110	0.530
	0.490	0.480	0.530
	0.490	0.480	0.700
	0.680	0.690	0.960
	0.820	0.830	0.150 E1
	0.140 E1	0.140 E1	0.100 E2
	0.210 E3	0.230 E3	0.530 E3
	0.320 E3	0.350 E3	0.790 E3
	0.320 E3	0.350 E3	0.790 E3

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APPENDIX 1

4 Noded Mindlin plate element formulation

Displacement field

$$u = z\theta_x, \qquad v = z\theta_y, \qquad W = w. \tag{1}$$

Nodal displacement vector

$$\delta_e = (w_1, \theta_{x1}, \theta_{y1}, \ldots, w_4, \theta_{x4}, \theta_{y4})^T. \tag{2}$$

Shape functions

$$N_1 = 1/4\,(1-\epsilon)(1-\eta); \qquad N_3 = 1/4\,(1+\epsilon)(1+\eta),$$
$$N_2 = 1/4\,(1+\epsilon)(1-\eta); \qquad N_4 = 1/4\,(1-\epsilon)(1+\eta). \tag{3}$$

Displacement components

$$w = \sum_1^4 N_i w_i,$$
$$\theta_x = \sum_1^4 N_i \theta_{xi}, \tag{4}$$
$$\theta_y = \sum_1^4 N_i \theta_{yi}.$$

If we define

$$\delta = (w, \theta_x, \theta_y)^T, \tag{5}$$

then

$$\delta = \mathbf{N}\,\delta_e \tag{6}$$

in which,

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_4 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_4 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_4 \end{bmatrix}. \tag{7}$$

Strain vector

$$\epsilon = (\epsilon_b, \epsilon_s)^T, \tag{8}$$

in which the bending strain,

$$\epsilon_b = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{bmatrix} \partial\theta_x/\partial x \\ \partial\theta_y/\partial y \\ \partial\theta_x/\partial y + \partial\theta_y/\partial x \end{bmatrix} \tag{9}$$

and the shear strain,

$$\epsilon_s = \begin{Bmatrix} \phi_{xz} \\ \phi_{yz} \end{Bmatrix} = \alpha \begin{bmatrix} \partial w/\partial x + \theta_x \\ \partial w/\partial y + \theta_y \end{bmatrix}, \tag{10}$$

in which α is an antiparameter we have introduced in the present formulation to avoid shear locking. We now write

$$\epsilon = \mathbf{B}\,\delta_e, \tag{11}$$

in which

$$\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4]$$

and

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{bi} \\ \mathbf{B}_{si} \end{bmatrix} \begin{bmatrix} 0 & \partial N_i/\partial x & 0 \\ 0 & 0 & \partial N_i/\partial y \\ 0 & \partial N_i/\partial y & \partial N_i/\partial x \\ \hline \alpha\partial N_i/\partial x & \alpha N_i & 0 \\ \alpha\partial N_i/\partial y & 0 & \alpha N_i \end{bmatrix}. \tag{12}$$

Elasticity relations

We define stress vector

$$\sigma = (\mathbf{M}, \mathbf{Q})^T = (M_x, M_y, M_{xy}, Q_x, Q_y)^T \tag{13}$$

such that

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon}. \tag{14}$$

in which

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_b & \mathbf{0} \\ 0 & \mathbf{D}_s \end{bmatrix}, \tag{15}$$

where

$$\mathbf{D}_b = \frac{Et^3}{12(1-\gamma^2)} \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix} \tag{16}$$

and

$$\mathbf{D}_s = kGA_s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad k = 5/6. \tag{17}$$

Strain energy

The total strain energy expression is written as

$$\begin{aligned} U &= U_b + U_s \\ &= \frac{1}{2}\iint \boldsymbol{\epsilon}_b^T \mathbf{D}_b \boldsymbol{\epsilon}_b \, \mathrm{d}x\, \mathrm{d}y + \frac{1}{2}\iint \boldsymbol{\epsilon}_s^T \mathbf{D}_s \boldsymbol{\epsilon}_s \, \mathrm{d}x\, \mathrm{d}y \\ &= \frac{1}{2}[\boldsymbol{\delta}_e^T \mathbf{K}_b^e \boldsymbol{\delta}_e + \boldsymbol{\delta}_e^T \mathbf{K}_s^e \boldsymbol{\delta}_e], \end{aligned} \tag{18}$$

in which

$$\mathbf{K}_b^e = \iint \mathbf{B}_b^T \mathbf{D}_b \mathbf{B}_b \, \mathrm{d}x\, \mathrm{d}y$$

and

$$\mathbf{K}_s^e = \alpha^2 \iint \mathbf{B}_s^T \mathbf{D}_s \mathbf{B}_s \, \mathrm{d}x\, \mathrm{d}y.$$

We set $\alpha = p\,(t/a)$, in which p is a free parameter whose estimate is provided based on numerical experimentation, t is the plate thickness and a is the plate dimension.