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Finite Element Analysis of Laminated Composite Plates using a Higher-Order Displacement Model

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ABSTRACT

A C^0 continuous displacement finite element formulation of a higher-order theory for flexure of thick arbitrary laminated composite plates under transverse loads is presented. The displacement model accounts for non-linear and constant variation of in-plane and transverse displacement respectively through the plate thickness. The assumed displacement model eliminates the use of shear correction coefficients. The discrete element chosen is a nine-noded quadrilateral with nine degrees-of-freedom per node. Results for plate deformations, internal stress-resultants and stresses for selected examples are shown to compare well with the closed-form, the theory of elasticity and the finite element solutions with another higher-order displacement model by the same authors. A computer program has been developed which incorporates the realistic prediction of interlaminar stresses from equilibrium equations.

1 INTRODUCTION

In early days, classical lamination theory[1] based on the Kirchhoff hypothesis was adopted for analysis of laminated composite plates. It was soon realised that this theory,[1] which neglects shear strains and transverse normal strain/stress is inadequate for analysis of laminated composite plates as transverse shear effects are more pronounced, even in thin composite plates/shells, by comparison with isotropic plates, because of the low

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transverse shear moduli relative to the in-plane Young's modulus. In addition, these interlaminar strains/stresses play a vital role in the delamination mode of laminated composite structures. This realisation was the starting point in the development of the first shear deformable plate theory. The credit for this development goes to Reissner[2] and Mindlin[3] who pioneered first-order shear deformable theories based on the assumed stress and displacement fields. However, both of these theories[2,3] neglected the effects of transverse normal strain/stress and were based on a non-realistic (constant) variation of the transverse shear strains/stresses through the plate thickness. This necessitated the introduction of a shear correction factor or factors. Later, these discrepancies were rectified by introducing higher-order functions in the displacement model leading to the higher-order plate theories.

Reissner[4] described an exact approach to the problem which reproduced the earlier[2] equations of two-dimensional plate theory and led to new supplementary information concerning certain three-dimensional aspects of the problem. Lo, Chistensen and Wu[5,6] presented closed-form solutions for isotropic[5] and laminated[6] plates with a higher-order displacement model. The displacement model assumed by them[5,6] incorporates the effects of transverse normal stress/strain and leads to the realistic (parabolic) variation of transverse shear stresses/strains. The theory, however, fails to eliminate the transverse shear stresses/strains on the bounding planes of the plate. Murthy[7] particularized the displacement model of Lo *et al.*[6] by neglecting the effects of transverse normal stress/strain, imposing conditions of zero transverse shear stresses/strains on the bounding planes of the plate and assuming averaged displacements as basic variables for closed-form solutions of laminated plates. Reddy[8] adopted the displacement model of Murthy[7] without following his averaged displacements concept[7] which leads to variationally inconsistent equilibrium equations. Reddy uses the principle of virtual displacements to derive the equilibrium equations appropriate for the assumed displacement fields leading to variationally consistent equations.[8] Later, Krishna Murty[9] pointed out that Reddy's displacement model[8] has no provision for considering the transverse shear strains at points in the plate where displacements are constrained to be zero (fixed edges). To overcome this limitation, he introduced an additional partial deflection (shear) as a variable in the transverse displacement expression and developed a new higher-order theory of laminated composite plates.[9] However, these solutions[4-9] are limited to few simple loading and boundary conditions and a need for a generalised solution technique arose. Kant, Owen and Zienkiewicz[10] were the first to realise this and presented a C^0 finite element formulation of the higher-order displacement model given by Lo *et al.*[5] for isotropic plates. Later, Reddy presented the displacement[11] and mixed[12] finite element formulation with the displacement model adopted earlier[8] for closed-form solutions. Pandya and Kant[13] have recently extended the work of Kant *et al.*[10] for symmetrically laminated plates. They have also introduced a novel approach of achieving zero transverse shear stress conditions on the bounding planes of the plate. Further, Kant and Pandya[14] have extended their earlier[13] work to unsymmetrically laminated composite plates. The present authors have also developed a simple isoparametric formulation[15] of the displacement model given by Lo *et al.*[6] for laminated plates.

This paper deals for the first time with a higher-order displacement model hitherto not considered for a simple isoparametric formulation. This displacement model is chosen to bring out the effects of neglecting transverse normal stress/strain but at the same time retaining the higher-order in-plane degrees-of-freedom in the formulation. The present solutions are compared with other finite element solutions[15] by the same authors, elasticity[16] and closed-form[17] solutions.

2 THEORY

The present higher-order shear deformation theory is developed with the assumption of the displacement model in the following form:

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y) \\
v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y) \\
w(x,y,y) &= w_0(x,y)
\end{aligned} \tag{1}
$$

in which u_0, v_0 and w_0 are the in-plane and transverse displacements of a point (x,y) on the mid-plane respectively and θ_x, θ_y are the rotations of normals to mid-plane about y and x axes, respectively. The parameters u_0^*, v_0^*, θ_x^*, θ_y^* are the corresponding higher-order deformation terms in the Taylor series expansion and are also defined at the mid-plane. The displacement model of eqn (1) differs from that given by Murthy[7] and Reddy[8] in the sense that the zero transverse shear stress conditions on the top and bottom surfaces of the plate are not enforced. This displacement model also differs from that adopted by Lo *et al.*[6] in the sense that the transverse displacement is assumed constant through the plate thickness. We attempt here, for the first time, with the displacement field given by eqn (1), the development of a theory and a simple C^0 isoparametric finite element formulation.

The strain expressions derived from the displacement model of eqn (1) are as follows:

$$
\begin{aligned}
\varepsilon_x &= \varepsilon_{x0} + z\kappa_x + z^2\varepsilon^*_{x0} + z^3\kappa^*_x \\
\varepsilon_y &= \varepsilon_{y0} + z\kappa_y + z^2\varepsilon^*_{y0} + z^3\kappa^*_y \\
\varepsilon_z &= 0 \\
\gamma_{xy} &= \varepsilon_{xy0} + z\kappa_{xy} + z^2\varepsilon^*_{xy0} + z^3\kappa^*_{xy} \\
\gamma_{yz} &= \Phi_y + z\varepsilon_{yz0} + z^2\Phi^*_y \\
\gamma_{xz} &= \Phi_x + z\varepsilon_{xz0} + z^2\Phi^*_x
\end{aligned}
\tag{2}
$$

where

$$
\varepsilon_{x0} = \frac{\partial u_0}{\partial x}, \qquad \varepsilon_{y0} = \frac{\partial v_0}{\partial y}, \qquad \varepsilon_{xy0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
$$

$$
\kappa_x = \frac{\partial \theta_x}{\partial x}, \qquad \kappa_y = \frac{\partial \theta_y}{\partial y}, \qquad \kappa_{xy} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}
$$

$$
\kappa^*_x = \frac{\partial \theta^*_x}{\partial x}, \qquad \kappa^*_y = \frac{\partial \theta^*_y}{\partial y}, \qquad \kappa^*_{xy} = \frac{\partial \theta^*_x}{\partial y} + \frac{\partial \theta^*_y}{\partial x}
$$

$$
\varepsilon^*_{x0} = \frac{\partial u^*_0}{\partial x}, \qquad \varepsilon^*_{y0} = \frac{\partial v^*_0}{\partial y}, \qquad \varepsilon^*_{xy0} = \frac{\partial u^*_0}{\partial y} + \frac{\partial v^*_0}{\partial x}
$$

$$
\Phi_y = \theta_y + \frac{\partial w_0}{\partial y}, \qquad \Phi_x = \theta_x + \frac{\partial w_0}{\partial x}
$$

$$
\varepsilon_{yz0} = 2v^*_0, \quad \varepsilon_{xz0} = 2u^*_0, \quad \Phi^*_y = 3\theta^*_y, \quad \Phi^*_x = 3\theta^*_x \tag{3}
$$

The stress–strain relations for a typical lamina L with reference to the lamina co-ordinate axes (1-2-3) are given by

$$
\begin{aligned}
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^L &= \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}^L \\
\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix}^L &= \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix}^L \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L
\end{aligned}
\tag{4}
$$

in which (σ_1, σ_2, τ_{12}, τ_{23}, τ_{13}) are the stress and (ε_1, ε_2, γ_{12}, γ_{23}, γ_{13}) are the linear strain components referred to the lamina co-ordinate axes (1-2-3) as shown in Fig. 1. The C_{ij}'s are the plane stress reduced elastic constants of the

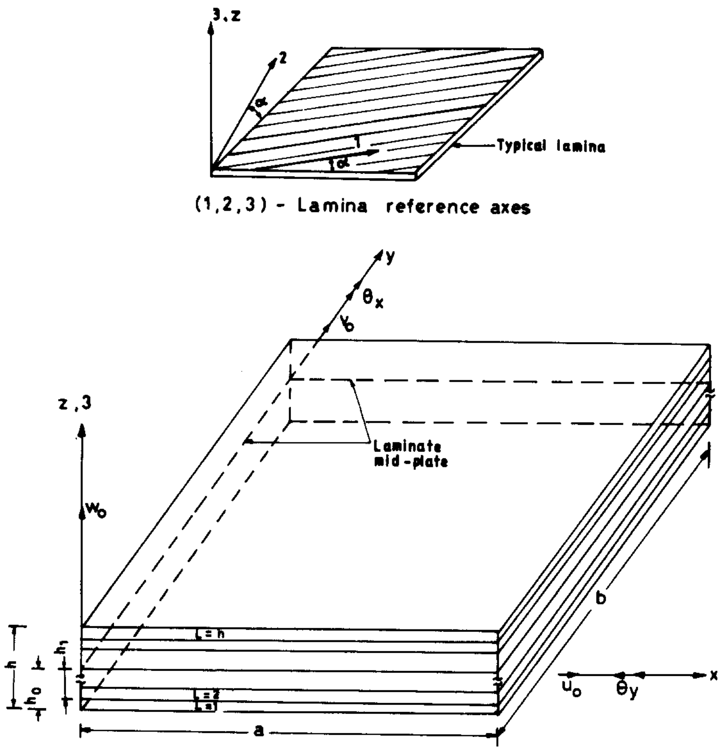

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

Lth lamina and the following relations hold between these and the engineering elastic constants.

$$\begin{gathered} C_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad C_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \qquad C_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\ C_{33} = G_{12}, \qquad C_{44} = G_{23}, \qquad C_{55} = G_{13} \end{gathered} \tag{5}$$

Following the usual transformation[18] rule of stresses/strains between the lamina and laminate coordinate systems, the stress–strain relations for the Lth lamina in the laminate coordinates (x-y-z) are written as

$$\begin{gathered} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^L \\ \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix}^L \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \end{gathered} \tag{6}$$

in which, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz})^t$ and $\boldsymbol{\varepsilon} = (\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})^t$ are the stress and linear strain vectors with respect to the laminate axes and Q_{ij}'s are the plane stress reduced elastic constants in the plate (laminate) axes of the Lth lamina. The superscript t denotes the transpose of a matrix.

The total potential energy π of the plate of mid-surface area A and volume V, loaded with an equivalent load vector $\mathbf{P}$ corresponding to the nine-degrees-of-freedom of a point on the mid-plane can be represented as

$$\pi = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^t \boldsymbol{\sigma}\, \mathrm{d}V - \int_A \mathbf{d}^t \mathbf{P}\, \mathrm{d}A \tag{7}$$

where

$$\mathbf{d} = (u_0, v_0, w_0, \theta_x, \theta_y, u_0^*, v_0^*, \theta_x^*, \theta_y^*)^t \tag{8}$$

The expressions for the strain components given by eqn (2) are substituted in the energy expression eqn (7). The functional given by eqn (7) is then minimized while carrying out explicit integration through the plate thickness. This leads to the following eighteen stress-resultants for the n-layered laminate:

$$\begin{Bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_{xy} & N_{xy}^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} [1 \mid z^2]\, \mathrm{d}z$$

$$\begin{Bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_{xy} & M_{xy}^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} [z \mid z^3]\, \mathrm{d}z \tag{9}$$

$$\begin{Bmatrix} Q_x & S_x & Q_x^* \\ Q_y & S_y & Q_y^* \end{Bmatrix} = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [1 \mid z \mid z^2]\, \mathrm{d}z$$

Upon integration, these expressions are rewritten in a matrix form which defines the stress-resultant/strain relations of the laminate and is given by,

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{N}^* \\ \hline \mathbf{M} \\ \mathbf{M}^* \\ \hline \mathbf{Q} \\ \mathbf{Q}^* \end{Bmatrix} = \left[\begin{array}{c|c|c} A & B & 0 \\ \hline B^t & D_b & 0 \\ \hline 0 & 0 & D_s \end{array}\right] \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\varepsilon}_0^* \\ \hline \mu \\ \mu^* \\ \hline \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^* \end{Bmatrix}$$

or

$$\bar{\sigma} = D\bar{\varepsilon} \tag{10}$$

in which

$$\begin{aligned}
\mathbf{N} &= (N_x, N_y, N_{xy})^t, & \mathbf{N}^* &= (N_x^*, N_y^*, N_{xy}^*)^t \\
\mathbf{M} &= (M_x, M_y, M_{xy})^t, & \mathbf{M}^* &= (M_x^*, M_y^*, M_{xy}^*)^t \\
\mathbf{Q} &= (Q_x, Q_y)^t, & \mathbf{Q}^* &= (S_x, S_y, Q_x^*, Q_y^*)^t \\
\boldsymbol{\varepsilon}_0 &= (\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{xy0})^t, & \boldsymbol{\varepsilon}_0^* &= (\varepsilon_{x0}^*, \varepsilon_{y0}^*, \varepsilon_{xy0}^*)^t \\
\boldsymbol{\kappa} &= (\kappa_x, \kappa_y, \kappa_{xy})^t, & \boldsymbol{\kappa}^* &= (\kappa_x^*, \kappa_y^*, \kappa_{xy}^*)^t \\
\boldsymbol{\Phi} &= (\Phi_x, \Phi_y)^t, & \boldsymbol{\Phi}^* &= (\varepsilon_{xz0}, \varepsilon_{yz0}, \Phi_x^*, \Phi_y^*)^t
\end{aligned}$$

$$A = \sum_{L=1}^{n} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{13}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_3 \\ & Q_{22}H_1 & Q_{23}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_3 \\ & & Q_{33}H_1 & Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_3 \\ & & & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 \\ & & & & Q_{22}H_5 & Q_{23}H_5 \\ \text{Symmetric} & & & & & Q_{33}H_5 \end{bmatrix}_{L\text{th layer}}$$

$$D_s = \sum_{L=1}^{n} \begin{bmatrix} Q_{55}H_1 & Q_{45}H_1 & Q_{55}H_2 & Q_{45}H_2 & Q_{55}H_3 & Q_{45}H_3 \\ & Q_{44}H_1 & Q_{45}H_2 & Q_{44}H_2 & Q_{45}H_3 & Q_{44}H_3 \\ & & Q_{55}H_3 & Q_{45}H_3 & Q_{55}H_4 & Q_{45}H_4 \\ & & & Q_{44}H_3 & Q_{45}H_4 & Q_{44}H_4 \\ & & & & Q_{55}H_5 & Q_{45}H_5 \\ \text{Symmetric} & & & & & Q_{44}H_5 \end{bmatrix}_{L\text{th layer}}$$

The elements of the B matrix can be obtained by replacing H_1 by H_2, H_3 by H_4 and H_5 by H_6 in the A matrix. Similarly, the elements of the D_b matrix can be obtained by replacing H_1 by H_3, H_3 by H_5 and H_5 by H_7 in A matrix, where

$$H_i = \frac{1}{i}(h_L^i - h_{L-1}^i) \qquad i = 1, 2, 3, \ldots, 7$$

The transverse shear stresses obtained by means of the stress–strain relations given by eqns (6) cannot satisfy the continuity condition at the interfaces of any two layers for the randomly laminated composite plate. For this reason, the interlaminar shear and normal stresses $(\tau_{xz}^L, \tau_{yz}^L, \sigma_z^L)$ between layers L and $(L + 1)$ at $z = h_L$ are obtained by integrating the three equilibrium equations of the theory of elasticity for each layer over the lamina thickness and summing over layers 1 through n as follows:

$$\tau_{xz}^{L}|_{z=h} = -\sum_{i=1}^{L} \int_{h_{i-1}}^{h_i} \left(\frac{\partial \sigma_x^i}{\partial x} + \frac{\partial \tau_{xy}^i}{\partial y} \right) \mathrm{d}z$$

$$\tau_{yz}^{L}|_{z=h} = -\sum_{i=1}^{L} \int_{h_{i-1}}^{h_i} \left(\frac{\partial \sigma_y^i}{\partial y} + \frac{\partial \tau_{xy}^i}{\partial x} \right) \mathrm{d}z \tag{11}$$

$$\sigma_{z}^{L}|_{z=h} = -\sum_{i=1}^{L} \int_{h_{i-1}}^{h_i} \left(\frac{\partial \tau_{xz}^i}{\partial x} + \frac{\partial \tau_{yz}^i}{\partial y} \right) \mathrm{d}z$$

3 FINITE ELEMENT FORMULATION

In the well-established finite element method, the total solution domain is discretized into 'ME' elements (sub-domains) such that

$$\pi(\mathbf{d}) = \sum_{e=1}^{\mathrm{ME}} \pi^{e}(\mathbf{d}) \tag{12}$$

where π and π^e are the potential energies of the total solution domain and the sub-domain respectively. The potential energy for an element 'e' can be expressed in terms of the internal strain energy, U^e, and the external work done, W^e, such that

$$\pi^{e}(\mathbf{d}) = U^{e} - W^{e} \tag{13}$$

in which $\mathbf{d}$ is the vector of nodal degrees-of-freedom of an element and is already defined by eqn (8). Adopting the same shape function 'N' to define all the components of the generalized displacement vector, $\mathbf{d}$, we can write

$$\mathbf{d} = \sum_{i=1}^{NE} \mathrm{N}_i \mathbf{d}_i \tag{14}$$

in which, NE is the number of nodes in the element. Now, referring to the expressions in eqn (3) the extensional strains (ε_0, ε_0^*), the bending curvatures

$(\boldsymbol{\kappa}, \boldsymbol{\kappa}^*)$ and the transverse shear strains $(\boldsymbol{\Phi}, \boldsymbol{\Phi}^*)$ can be written in terms of the nodal displacements $\mathbf{d}$ using the matrix notations as follows:

$$
\begin{aligned}
\begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\varepsilon}_0^* \end{Bmatrix} &= L_{\mathrm{E}}\mathbf{d} \\
\begin{Bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\kappa}^* \end{Bmatrix} &= L_{\mathrm{B}}\mathbf{d} \\
\begin{Bmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^* \end{Bmatrix} &= L_{\mathrm{S}}\mathbf{d}
\end{aligned}
\tag{15}
$$

in which the subscripts E, B and S refer to extension, bending and shear respectively and the matrices L_{E}, L_{B} and L_{S} attain the following form

$$
L_{\mathrm{E}} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0
\end{bmatrix}
$$

$$
L_{\mathrm{B}} = \begin{bmatrix}
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
$$

$$L_S = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad (16)$$

Knowing the generalized displacement vector, **d**, at all points within the element, the generalized strain vectors at any point are determined with the aid of eqns (14) and (16) as follows:

$$\begin{aligned} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\varepsilon}_0^* \end{Bmatrix} &= L_E \mathbf{d} = L_E \sum_{i=1}^{NE} N_i \mathbf{d}_i = \sum_{i=1}^{NE} B_{iE} \mathbf{d}_i = B_E \mathbf{a} \\ \begin{Bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\kappa}^* \end{Bmatrix} &= L_B \mathbf{d} = L_B \sum_{i=1}^{NE} N_i \mathbf{d}_i = \sum_{i=1}^{NE} B_{iB} \mathbf{d}_i = B_B \mathbf{a} \\ \begin{Bmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Phi}^* \end{Bmatrix} &= L_S \mathbf{d} = L_S \sum_{i=1}^{NE} N_i \mathbf{d}_i = \sum_{i=1}^{NE} B_{iS} \mathbf{d}_i = B_S \mathbf{a} \end{aligned} \quad (17)$$

in which

$$B_{iE} = L_E N_i, \qquad B_E = \sum_{i=1}^{NE} B_{iE}$$

$$B_{iB} = L_B N_i, \qquad B_B = \sum_{i=1}^{NE} B_{iB}$$

$$B_{iS} = L_S N_i, \qquad B_S = \sum_{i=1}^{NE} B_{iS}$$

and

$$\mathbf{a} = (\mathbf{d}_1^t, \mathbf{d}_2^t, \ldots, \mathbf{d}_{NE}^t) \quad (18)$$

Combining the expressions in eqn (18), the B matrix for the ith node can be written as

$$B_i = \begin{bmatrix} B_{iE} \\ B_{iB} \\ B_{iS} \end{bmatrix} \tag{19}$$

The internal strain energy of an element is determined by integrating the products of in-plane, moment and shear stress resultants with the extensional, bending and shear strains, respectively, over the area of an element. This is expressed as

$$U^e = \frac{1}{2} \int_A \left[(\varepsilon_0^t, \varepsilon_0^{*t}) \begin{Bmatrix} N \\ N^* \end{Bmatrix} + (\kappa^t, \kappa^{*t}) \begin{Bmatrix} M \\ M^* \end{Bmatrix} + (\Phi^t, \Phi^{*t}) \begin{Bmatrix} Q \\ Q^* \end{Bmatrix} \right] \mathrm{d}A \tag{20}$$

Replacing stress-resultants by the product of rigidity matrix and strains in the strain energy expression in eqn (20), we get

$$U^e = \frac{1}{2} \int_A \left[(\varepsilon_0^t, \varepsilon_0^{*t}) A \begin{Bmatrix} \varepsilon_0 \\ \varepsilon_0^* \end{Bmatrix} + (\varepsilon_0^t, \varepsilon_0^{*t}) B \begin{Bmatrix} \kappa \\ \kappa^* \end{Bmatrix} + (\kappa^t, \kappa^{*t}) B^t \begin{Bmatrix} \varepsilon_0 \\ \varepsilon_0^* \end{Bmatrix} \right.$$
$$\left. + (\kappa^t, \kappa^{*t}) D_B \begin{Bmatrix} \kappa \\ \kappa^{*t} \end{Bmatrix} + (\Phi^t, \Phi^{*t}) D_s \begin{Bmatrix} \Phi \\ \Phi^* \end{Bmatrix} \right] \mathrm{d}A \tag{21}$$

The internal strain energy expression in terms of the nodal displacements is derived by substituting relations in eqn (17) into eqn (21). The result is

$$U^e = \frac{1}{2} \int_A (\mathbf{a}^t B_E^t A B_E \mathbf{a} + \mathbf{a}^t B_B^t B B_E \mathbf{a} + \mathbf{a}^t B_E^t B B_B \mathbf{a}$$
$$+ \mathbf{a}^t B_B D_B B_B \mathbf{a} + \mathbf{a}^t B_S^t D_S B_S \mathbf{a}) \, \mathrm{d}A$$

or

$$U^e = \tfrac{1}{2} \mathbf{a}^t K^e \mathbf{a} \tag{22}$$

in which K^e is the element stiffness matrix and is expressed as

$$K^e = \int_A (B_E^t A B_E + B_B^t B B_E + B_E^t B B_B + B_B^t D_B B_B + B_S^t D_S B_S) \, \mathrm{d}A \tag{23}$$

The computation of the element stiffness matrix from eqn (23) is economised by explicit multiplication of the B_i, D and B_j matrices instead of carrying out the full matrix multiplication of the triple product. In addition, because of the symmetry of the stiffness matrix, only the blocks K_{ij} lying on one side of the main diagonal are formed. The integral is evaluated numerically using the Gauss quadrature rule,

$$K_{ij}^{e} = \int_{-1}^{1}\int_{-1}^{1} B_i^{t} D B_j |\mathrm{J}| \, d\varepsilon \, d\eta$$

$$K_{ij}^{e} = \sum_{a=1}^{g}\sum_{b=1}^{g} W_a W_b |\mathrm{J}| B_i^{t} D B_j \tag{24}$$

in which W_a and W_b are weighting coefficients, g is the number of numerical quadrature points in each of the two directions (x, y) and $|\mathrm{J}|$ is the determinant of the standard jacobian matrix. The subscripts i and j vary from one to the number of nodes per element. The matrices B_i and D are given by eqns (19) and (10) respectively and B_j is obtained by replacing i by j.

For the flexural analysis, the total external work done by the applied external loads for an element, e, is given by

$$W^{e} = \mathbf{a}^{t}\mathbf{F}_{c} + \mathbf{a}^{t}\int_{A} (N_i^{t} q + N_i^{t} P_{mn}) \, dA \tag{25}$$

in which suffix, i, varies from one to the number of nodes per element. $\mathbf{F}_c$ is the vector of concentrated nodal loads corresponding to nodal degrees-of-freedom. q and P_{mn} are the uniform and sinusoidal distributed load intensities acting over an element e in the z direction.

The integral of eqn (25) is evaluated numerically using the Gauss quadrature rule as follows

$$\mathbf{P}_i = \sum_{a=1}^{g}\sum_{b=1}^{g} W_a W_b |\mathrm{J}| N_i^{t} \{0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\}^{t} \left(q + P_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \right) \tag{26}$$

in which a and b are the plate dimensions, x and y are the Gauss point coordinates and m and n are the usual harmonic numbers.

4 NUMERICAL EXAMPLES

The validity of the theory, the finite element formulation, and its implementation in the computer program is established by comparison of numerical results for examples available in the literature. In examples 1 and 2, the individual laminae are taken to be of equal thickness whereas for the sandwich plate of example 3, the thickness of each face sheet is one-tenth of the total thickness of the plate. For all three examples considered, the plate is discretized with four nine-noded quadrilateral elements in a quarter plate. The numerical values of stress-resultants and stresses are at the nearest Gauss points for the finite element solutions. The superscripts c and e used in the various tables represent the values of stresses obtained from constitutive and equilibrium relations respectively. The material properties used for each lamina of the laminated composite or sandwich plate are as follows:

Material I

$$\frac{E_1}{E_2}=40, \quad \frac{G_{12}}{E_2}=0{\cdot}6, \quad \frac{G_{23}}{E_2}=0{\cdot}5, \quad E_2=E_3=10^6$$
$$G_{13}=G_{12} \quad \text{and} \quad \nu_{12}=\nu_{23}=\nu_{13}=0{\cdot}25 \tag{27}$$

Material II

$$\frac{E_1}{E_2}=25, \quad \frac{G_{12}}{E_2}=0{\cdot}5, \quad \frac{G_{23}}{E_2}=0{\cdot}2, \quad E_2=E_3=10^6$$
$$G_{13}=G_{12} \quad \text{and} \quad \nu_{12}=\nu_{23}=\nu_{13}=0{\cdot}25 \tag{28}$$

Material III

Material properties for each face sheet are given by eqn (28) with the fibres parallel to x-axis and the core material is transversely isotropic with respect to z and is characterized by the following properties:

$$\begin{aligned} &E_x=E_y=0{\cdot}4\times10^5, \qquad E_z=0{\cdot}5\times10^6 \\ &G_{xz}=G_{yz}=0{\cdot}6\times10^5, \quad G_{xy}=0{\cdot}16\times10^5 \\ &\nu_{xz}=\nu_{yz}=\nu_{xy}=0{\cdot}25 \end{aligned} \tag{29}$$

The deflection, internal stress-resultants, and stresses are presented here in non-dimensional form using the following multipliers:

$$m_1=\frac{10E_2h^3}{qa^4}, \quad m_2=\frac{10}{qa^2}, \quad m_3=\frac{10}{qa} \quad m_4=\frac{h^2}{qa^2}, \quad m_5=\frac{h}{qa} \tag{30}$$

The three examples selected from the literature are described below:

4.1 Example 1

A simply-supported square cross-ply (0°/90°) plate under uniform transverse load is considered for comparisons of maximum deflection and stress-resultants. The set of material properties used is given by eqn (27) and the results are presented in Table 1. Further, the behaviour of the same plate under sinusoidal load and a set of material properties given by eqn (28) is examined. The results for maximum stresses are compared with the three-dimensional elasticity solutions in Table 2.

TABLE 1

Maximum Deflection and Stress Resultants for a Simply-supported Unsymmetric Cross-ply (0°/90°) Square Plate under Uniform Transverse Load (Material I)

Source	a/h	$w_0 \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$M_x \times m_2$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$M_{xy} \times m_2$ $(0, 0)$	$N_x \times m_3$ $\left(\frac{3a}{8}, \frac{a}{4}\right)$	$N_{xy} \times m_3$ $\left(\frac{a}{8}, 0\right)$	$Q_x \times m_3$ $\left(0, \frac{a}{2}\right)$
Present		0·192 79	0·638 2	−0·155 9	0·085 9	0·054 8	2·897
Kant and Pandya[15]	5	0·190 72	0·638 7	−0·150 3	0·093 1	0·053 8	2·896
Turvey[17]		0·178 07	0·639 39	—	—	—	—
Present		0·141 90	0·649 5	−0·142 6	0·120 6	0·093 5	2·935
Kant and Pandya[15]	10	0·141 50	0·649 6	−0·141 3	0·127 8	0·095 3	2·935
Turvey[17]		0·138 02	0·643 91	—	—	—	—
Present		0·125 98	0·653 3	−0·136 1	0·202 2	0·325 5	2·951
Kant and Pandya[15]	40	0·125 95	0·653 3	−0·136 0	0·208 3	0·326 7	2·951
Turvey[17]		0·125 54	0·646 58	—	—	—	—

TABLE 2
Maximum Deflection and Stresses for a Simply-supported Unsymmetric Cross-ply (0°/90°) Square Plate under Sinusoidal Transverse Load (Material II)

Source	a/h	z/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\sigma_y \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$\tau_{xy} \times m_4$ $(0, 0)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0{\cdot}2h\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$\tau^e_{yz} \times m_5$ $\left(\frac{a}{2}, 0, -0{\cdot}2h\right)$	$\tau^c_{yz} \times m_5$ $\left(\frac{a}{2}, 0, 0\right)$	$w_0 \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$
Present		0·5	0·8056	0·0969	−0·0597	0·2843	0·2840	0·2745	0·2840	0·2055
		−0·5	−0·0969	−0·8056	0·0597					
Kant and Pandya[15]	4	0·5	0·8000	0·1038	−0·0579	0·2868	0·2845	0·2865	0·2845	0·2020
		−0·5	−0·1037	−0·8000	0·0579					
Pagano[16]		0·5	0·7807	0·0955	−0·0591	0·3127	—	0·3188	—	—
		−0·5	−0·1098	−0·8417	0·0588					
Present		0·5	0·7390	0·0871	−0·0540	0·2950	0·2887	0·2905	0·2887	0·1224
		−0·5	−0·0871	−0·7390	0·0540					
Kant and Pandya[15]	10	0·5	0·7367	0·0884	−0·05370	0·2957	0·2888	0·2956	0·2888	0·1220
		−0·5	−0·0884	−0·7367	0·05370					
Pagano[16]		0·5	0·7300	—	−0·0538	0·3310	—	—	—	—
		−0·5	−0·0890	—	0·0536					

TABLE 3

Maximum Deflection and Stress Resultants for a Simply-Supported Unsymmetric Angle-ply (15°/−15°) Square Plate under Uniform Transverse Load (Material I)

Source	a/h	$w_0 \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$M_x \times m_2$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$M_y \times m_2$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$-M_{xy} \times m_2$ (0, 0)	$N_x \times m_3$ (0, 0)	$N_y \times m_3$ (0, 0)	$N_{xy} \times m_3$ $\left(\frac{a}{2}, \frac{a}{2}\right)$	$Q_x \times m_3$ $\left(0, \frac{a}{2}\right)$	$Q_y \times m_3$ $\left(\frac{a}{2}, 0\right)$
Present		0·154 03	1·083 0	0·156 5	0·177 7	1·726	1·736	1·678	4·209	1·440
Kant & Pandya[15]	5	0·151 92	1·076 0	0·164 4	0·170 0	1·767	1·753	1·675	4·191	1·463
Turvey[17]		0·140 86	1·043 3	0·146 2	—	—	—	—	—	—
Present		0·091 87	1·140 0	0·128 1	0·158 8	4·014	4·030	3·840	4·302	1·281
Kant & Pandya[15]	10	0·091 42	1·138 0	0·130 0	0·156 9	4·109	4·116	3·839	4·297	1·287
Turvey[17]		0·088 36	1·115 1	0·125 3	—	—	—	—	—	—
Present		0·071 54	1·157 0	0·118 5	0·152 5	16·790	17·130	16·020	4·327	1·216
Kant & Pandya[15]	40	0·071 50	1·157 0	0·118 5	0·152 2	17·190	17·540	16·020	4·326	1·217
Turvey[17]		0·071 27	1·138 6	0·117 9	—	—	—	—	—	—

TABLE 4

Maximum Deflection and Stresses for a Simply-supported Square Sandwich Plate Subjected to Sinusoidal Transverse Load (Material III)

Source	a/h	$\sigma_x \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}, \frac{h}{2}\right)$	$\sigma_y \times m_4$ $\left(\frac{a}{2}, \frac{a}{2}, \frac{h}{2}\right)$	$\tau_{xy} \times m_4$ $\left(0, 0, \frac{h}{2}\right)$	$\tau^e_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$\tau^c_{xz} \times m_5$ $\left(0, \frac{a}{2}, 0\right)$	$\tau^e_{yz} \times m_5$ $\left(\frac{a}{2}, 0, 0\right)$	$\tau^c_{yz} \times m_5$ $\left(\frac{a}{2}, 0, 0\right)$	$w_0 \times m_1$ $\left(\frac{a}{2}, \frac{a}{2}, 0\right)$
Present		1·523	0·241 4	−0·141 9	0·220 0	0·275 0	0·088 98	0·113 7	0·716 0
Kant and Pandya[15]	4	1·533	0·267 1	−0·138 9	0·221 9	0·272 0	0·090 2	0·113 9	0·706 1
Pagano[16]		1·556	0·259 5	−0·143 7	0·239 0	—	0·107 2	—	—
Present		1·166	0·105 2	−0·069 2	0·268 5	0·340 0	0·044 62	0·056 42	0·208 7
Kant and Pandya[15]	10	1·168	0·111 1	−0·068 9	0·267 6	0·339 3	0·044 36	0·056 42	0·208 2
Pagano[16]		1·153	0·110 4	−0·070 7	0·300 0	—	0·052 70	—	—
Present		1·026	0·049 7	−0·044 0	0·288 0	0·362 7	0·027 04	0·033 22	0·089 1
Kant and Pandya[15]	100	1·110	0·056 5	−0·044 0	0·287 8	0·362 7	0·026 76	0·032 99	0·089 2
Pagano[16]		1·098	0·055 0	−0·043 7	0·324 0	—	0·029 7	—	—
CPT	—	1·097	0·054 3	−0·043 3	0·324 0	—	0·029 5	—	—

4.2 Example 2

A simply-supported square angle-ply $(15°/-15°)$ plate under uniform transverse load is considered here. The numerical results are presented in Table 3, considering the full plate.

4.3 Example 3

A simply-supported square sandwich plate under sinusoidal transverse load is considered for comparisons of stresses. The set of material properties used is given by eqn (29) and the results are presented in Table 4.

5 CONCLUSIONS

A simple C^0 isoparametric formulation of an assumed higher-order displacement model is presented. The present shear deformable theory does not require the usual shear correction factors generally associated with the Mindlin–Reissner type of theory. Comparisons of numerical results using two different displacement fields, with the 3D-elasticity and available closed-form solutions show that the use of the complete generalised Hooke's law which includes the effects of transverse normal stress/strain minimizes the errors. In general, the agreement of both the finite element solutions is excellent for thin-to-thick laminated composite and sandwich plates when compared with 3D-elasticity/closed-form solutions. With the present displacement model, it is not possible to satisfy the zero transverse shear stress conditions on the bounding plane of the plate. Further, the continuity conditions on the interfaces for the interlaminar stresses are also not met in the realm of any two-dimensional plate theory for laminates. For these reasons, the computer program developed makes use of three-dimensional equilibrium equations to predict the interlaminar stresses realistically. The in-plane lamina stresses are evaluated as usual from the plate constitutive relations. The difference in the results of transverse shear stresses obtained using equilibrium equations and plate constitutive relations is found to be a maximum for the sandwich plate rather than the laminated plates.

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