

TRANSIENT/PSEUDO-TRANSIENT FINITE ELEMENT SMALL/LARGE DEFORMATION ANALYSIS OF TWO-DIMENSIONAL PROBLEMS

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(Received 7 July 1989)

Abstract—A unified approach is presented for the static and transient linear and geometrically non-linear analyses of two-dimensional problems (plane stress/strain and axisymmetric). A finite element idealization with four-, eight- and nine-noded isoparametric quadrilateral elements is used for space discretization. An explicit central difference time-marching scheme is employed for time integration of the resulting discrete ordinary differential equations. Results of several numerical examples are presented and compared with the available data. A comparative study of the performance of various elements with different damping factors is also presented.

INTRODUCTION

The importance of investigating the dynamic behaviour of structures exhibiting geometric non-linearity is increasingly being recognised. At the same time, static, linear and non-linear analyses are inevitable parts of a thorough analysis. In this paper a unified approach for static and dynamic, linear and geometric-non-linear analyses of two-dimensional structures, viz. plane stress/strain and axisymmetric problems, is presented.

The origin of the approach presented herein dates back to 1965, when the method of dynamic relaxation was introduced by Day [1]. Since then many research workers have contributed to the development of this method by applying it to a variety of problems [2-4]. The method is essentially a step-by-step integration of critically damped vibration using inertia and viscous damping to ensure the attainment of a steady state solution. Even though the use of finite difference in space compares well with the finite element method, the difficulties encountered with complicated geometries makes its use unattractive. Thus, maintaining the generality of finite element method and the wide spectrum of software available thereon, it is necessary to develop the relaxation procedure with respect to space discretization by finite elements (pseudo-dynamic analysis). Only Mindlin plates have so far been analysed by this approach [5, 6].

BASIC THEORY

The static problem represented by the discrete equation

$$\mathbf{P} = \mathbf{K}\mathbf{u} = \mathbf{f} \quad (1a)$$

can be solved in a variety of ways which generally require the direct or factorized solution of simultaneous equations. An alternate solution procedure includes the transformation of eqn (1a) into a dynamic equation

$$\mathbf{P} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}(t) \quad (1b)$$

by inclusion of fictitious mass and/or damping matrices and carrying out the dynamic analysis until the steady state is reached. In the above equation, \mathbf{M} is the mass matrix; \mathbf{C} is the damping matrix; \mathbf{P} is the vector of internal resisting forces; $\mathbf{f}(t)$ is the vector of applied forces; \mathbf{u} is the vector of nodal displacements; and a dot denotes differentiation with respect to time.

The various terms are expressed in the standard finite element terminology [7-9] as follows:

$$\mathbf{M} = \int_{\text{vol}} \mathbf{N}^T \rho \mathbf{N} \, d(\text{vol}) \quad (2)$$

$$\mathbf{C} = \int_{\text{vol}} \mathbf{N}^T c \mathbf{N} \, d(\text{vol}) \quad (3)$$

$$\mathbf{f}^n = \int_{\text{vol}} \mathbf{N}^T \mathbf{b}^n \, d(\text{vol}) + \int_s \mathbf{N}^T \boldsymbol{\tau}^n \, ds \quad (4)$$

$$\mathbf{P}^n = \int_{\text{vol}} \mathbf{B}^T \boldsymbol{\sigma}^n \, d(\text{vol}). \quad (5)$$

The parameters ρ , c , b and τ are the mass density, velocity dependent damping coefficient, body forces per unit volume and boundary traction per unit area, respectively. \mathbf{B} and $\boldsymbol{\sigma}$ are the strain-displacement matrix and the stress vector, respectively.

In the explicit time-marching scheme used here the velocities and accelerations are approximated using central difference formulae as

$$\dot{\mathbf{u}}^n = (\mathbf{u}^{n+1} - \mathbf{u}^{n-1}) / (2 \Delta t) \tag{6}$$

$$\ddot{\mathbf{u}}^n = (\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) / \Delta t^2, \tag{7}$$

where $n - 1$, n and $n + 1$ denote three successive time stations. Using the above approximation, eqn (1) can be written as

$$\mathbf{M}(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) / \Delta t^2 + \mathbf{C}(\mathbf{u}^{n+1} - \mathbf{u}^{n-1}) / (2 \Delta t) + \mathbf{P}^n - \mathbf{f}^n = 0. \tag{8}$$

It becomes clear that value of \mathbf{u}^{n+1} can be determined from two previous displacements, \mathbf{u}^n and \mathbf{u}^{n-1} , by rewriting eqn (8) as

$$\mathbf{u}^{n+1} = \left(\mathbf{M} + \mathbf{C} \frac{\Delta t}{2} \right)^{-1} \times \left[\Delta t^2 (\mathbf{f}^n - \mathbf{P}^n) + 2\mathbf{M}\mathbf{u}^n - \left(\mathbf{M} - \mathbf{C} \frac{\Delta t}{2} \right) \mathbf{u}^{n-1} \right]. \tag{9}$$

If the mass matrix \mathbf{M} and the damping matrix \mathbf{C} are diagonalized, the above set of equations uncouple to give new displacement values directly without requiring matrix factorization or sophisticated solution techniques.

For the initial calculation the values of $\mathbf{u}_{-\Delta t}$ and \mathbf{u}_0 are required in order to obtain the displacement $\mathbf{u}_{\Delta t}$. A starting algorithm is therefore necessary to obtain $\mathbf{u}_{-\Delta t}$ from the initial values of $\dot{\mathbf{u}}_0$ and \mathbf{u}_0 . The central difference approximation gives

$$\mathbf{u}_{-\Delta t} = -2 \Delta t \dot{\mathbf{u}}_0 + \mathbf{u}_{\Delta t}. \tag{10}$$

On substituting eqn (10) into eqn (9) we obtain

$$\mathbf{u}_{\Delta t} = \mathbf{u}_1 = \frac{\Delta t^2}{2} \mathbf{M}^{-1} (-\mathbf{P}_0 + \mathbf{f}_0) + \mathbf{u}_0 + \mathbf{B} \Delta t \dot{\mathbf{u}}_0, \tag{11}$$

in which

$$\mathbf{B} = \mathbf{I} - \mathbf{C}\mathbf{M}^{-1} \frac{\Delta t}{2}. \tag{12}$$

Mass lumping

The inertia force vector is found for given shape functions by the mass matrix \mathbf{M} given by eqn (2). This matrix can be diagonalized in various ways to enable its use in explicit schemes. In this study a special mass lumping scheme is adopted which lumps the total mass in proportion to the diagonal terms of the original consistent mass matrix. The procedure involves calculation of diagonal elements m_{ii}^e of the

elemental mass matrix and total mass M^e by the following formulae:

$$m_{ii}^e = \int_{\text{vol}} N_i \rho N_i d(\text{vol}) \tag{13}$$

$$M^e = \int_{\text{vol}} \rho d(\text{vol}). \tag{14}$$

From these equations the lumped diagonal term m_{ii} is determined by

$$m_{ii} = \frac{m_{ii}^e}{\sum_i m_{ii}^e} M^e. \tag{15}$$

In the absence of any other information on damping, Rayleigh damping,

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}, \tag{16}$$

is adopted here with $\beta = 0$.

Critical time step

The critical time step length for linear problems is limited by the highest frequency ω_{max} of the finite element mesh, such that

$$\Delta t \leq 2 / \omega_{\text{max}}. \tag{17}$$

When Δt does not satisfy the above condition, a spurious increase in the computed displacements takes place and this leads to numerical instability. However, it is noted that the highest eigenvalue of the system must always be less than the highest eigenvalue of an individual element. Thus eigenvalue analysis of the system can be avoided by a conservative estimate of the elemental eigenvalue which leads to the following empirical formula for obtaining the critical time step length:

$$\Delta t \leq r l [\rho (1 + \nu) (1 - 2\nu) / (E (1 - \nu))]^{-1/2}, \tag{18}$$

where l is the minimum distance between adjacent nodes of the finite element mesh and r is a coefficient dependent on the problem and the type of element employed. The parameters E , ν and ρ are the usual material properties. In two-dimensional problems, the value of r is found to vary between 0.2 and 0.5.

PSEUDO-DYNAMIC ANALYSIS

In the pseudo-transient methods the static problem represented by eqn (1a) is transformed to a first-order or second-order transient problem. In this study we use a second-order transient problem, represented by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t). \tag{19}$$

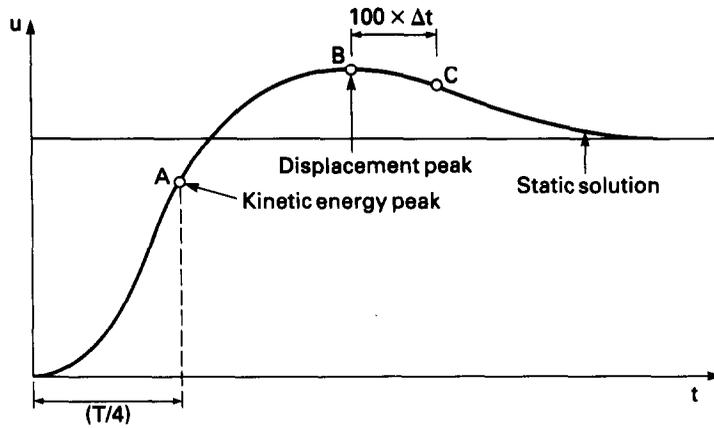


Fig. 1. Displacement-time response.

The solution of eqn (19) involves a step-by-step integration of critically damped vibration using viscous damping to ensure rapid attainment of the steady-state solution, by a suitable choice of damping matrix *C*. In this way the oscillations associated with the dynamic problem rapidly converge to the static solution. From available studies the use of the fictitious mass matrix, *M*, is found to be less effective than that of the fictitious damping matrix, *C* [5, 6]. A special lumped mass matrix adopted for dynamic analysis is used here along with the fictitious damping matrix.

Damping factor

As already discussed, the damping matrix *C* can be written as

$$C = \alpha M, \tag{20}$$

where α is the damping factor given as an input parameter in transient analysis. However, in the pseudo-transient analysis, α is automatically computed by the program as a critical damping factor given by

$$\alpha_c = 2\omega, \tag{21}$$

where ω is the dominant frequency of the system. Although the only way of evaluating ω exactly is by eigenvalue analysis, this is generally avoided. In fact, it would be rather expensive and contrary to the main philosophy of the pseudo-transient method, in which the aims are easy implementation and small computer core storage. Several methods are used for estimating ω : considering the motion of individual degree-of-freedom; the sum of squares of velocities; and the variation of the total kinetic energy during some 'non-productive' iteration. In the present work, the last alternative was found to be most suitable as it gives the frequency of the structure as a whole rather than that of an individual degree-of-freedom. The time employed by the structure to reach the maxi-

imum kinetic energy is estimated through the variation of total kinetic energy and is assumed as a quarter of the time period *T*, from which

$$\omega = 2\pi/T. \tag{22}$$

The dominant frequency, ω , is used to find the critical damping factor given by eqn (21). The procedure discussed above is described by a typical displacement response, shown in Fig. 1. The dynamic analysis of a structure without damping is carried out from the time of application of step load until it achieves its kinetic energy peak, represented by point A. The critical damping factor is obtained and is applied in further analysis.

Convergence

After the pseudo-transient analysis starts the convergence check should be applied to detect convergence. In the present study it is found to be more efficient to carry out a convergence check only after first displacement is determined. To avoid incorrect convergence at this displacement peak (represented by point B) the check is started at 100 step (represented by point C) onwards. This also avoids unnecessary computations. Two types of convergence checks are allowed: (i) convergence with respect to displacement, and (ii) convergence with respect to residual forces. The error percentages in two checks are given by

$$D = \frac{\sum_i |u_i^n - u_i^{n-1}|}{\sum_i |u_i^n|} \times 100 \tag{23}$$

and

$$R = \frac{\sum_i |f_i - p_i^n|}{\sum_i |f_i|} \times 100, \tag{24}$$

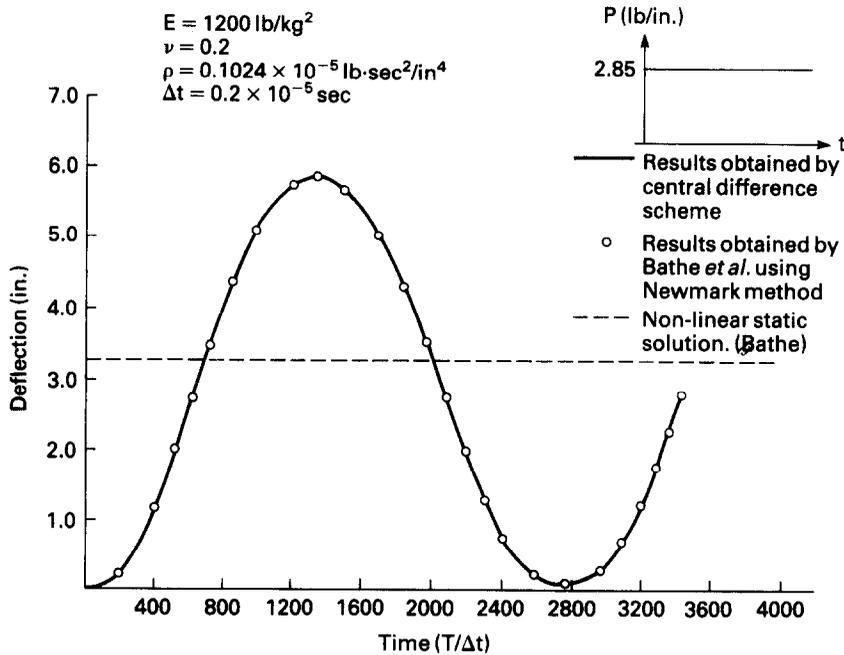


Fig. 2. Large deformation dynamic response of a cantilever beam under uniform load.

respectively. The convergence is assumed when the chosen parameter, R or D , becomes less than a given tolerance, typically 0.01.

COMPUTATION ALGORITHM

The essential steps of the computer program are as follows.

- (1) Read the input data—geometrical properties, material properties and time stepping parameters.
- (2) Form diagonal mass and damping matrix and calculate load vector for each element to form force array f .
- (3) Apply boundary conditions.
- (4) Evaluate residual force vector $f^n - P^n$.
- (5) Compute u^{n+1} from u^n and u^{n-1} .
- (6) Evaluate new internal force vector P^{n+1} from u^{n+1} .
- (7) Output required values.
- (8) Introduce iteration control. This is not activated during transient analysis and control is directly transferred back to Step 4. In the pseudo transient analysis the following operations are carried out.
 - The critical damping factor is determined by detecting the kinetic energy peak and is applied thereafter.
 - The first displacement peak is determined and a convergence check is started at 100 step onwards. On detection of convergence control execution stops, otherwise the control goes back to Step 4.

APPLICATION

The theory developed in the previous sections is now applied to a number of problems which illustrate the versatility of the unified approach. The applications also examine the suitability of various elements and the choice of parameters.

Example 1

A cantilever beam (two-dimensional plane stress) under a uniformly distributed step load, as shown in Figs 2–5, is analysed first. This problem is also attempted by Bathe *et al.* [10]. Both four and eight-noded elements are tried for the space discretizations. The full dynamic analysis with eight-noded elements, as shown in Fig. 2 agrees with [10]. Figure 3 shows the pseudo-transient analysis with the same eight-noded elements. With a damping factor of $\alpha = 0.8\alpha_{cr}$, the solution is seen to converge to the true solution of 3.65 in. in 2270 time steps. A large deformation analysis was carried out with damping factors $\alpha = \alpha_{cr}$ and $\alpha = 0.8\alpha_{cr}$. The results are shown in Fig. 4. The solution is seen to converge to a value of 3.25 in. in 2139 time steps and 3.34 in. in 2136 time steps with $\alpha = 0.8\alpha_{cr}$ and $\alpha = \alpha_{cr}$, respectively.

The same problem is also attempted with two layers of four-noded bilinear elements. Here the permissible time step length is obtained as 0.4×10^{-5} sec ($r = 0.415$), as against a step-size of 0.25×10^{-5} sec ($r = 0.259$) for the eight-noded elements. However, it is seen in Fig. 5 that the solution has converged to an incorrect solution of 2.26 in. as against 3.25 in. The possible reason for this type of behaviour is not yet apparent.

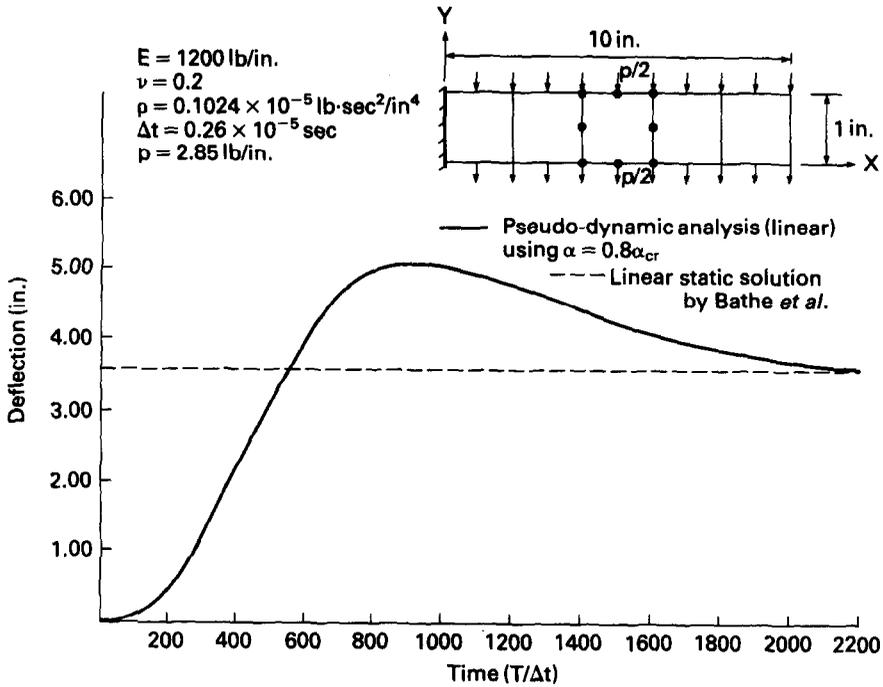


Fig. 3. Linear pseudo-dynamic analysis.

Example 2

A spherical cap (two-dimensional axisymmetric) subjected to an external step pressure of 600 lb/in² is considered. This example is also taken from Bathe *et al.* [10]. The critical time step is found to be 0.4×10^{-6} sec [$r = 0.468$ in eqn (18)] for an eight-noded element. The pseudo-transient analysis gives a converged crown displacement of 0.325 in., as shown

in Fig. 6. This is in agreement with the available results [10].

CONCLUSIONS

The numerical tests presented in the previous section show a generally good agreement of the present formulation with those from other sources in both transient and pseudo-transient analysis. Thus this

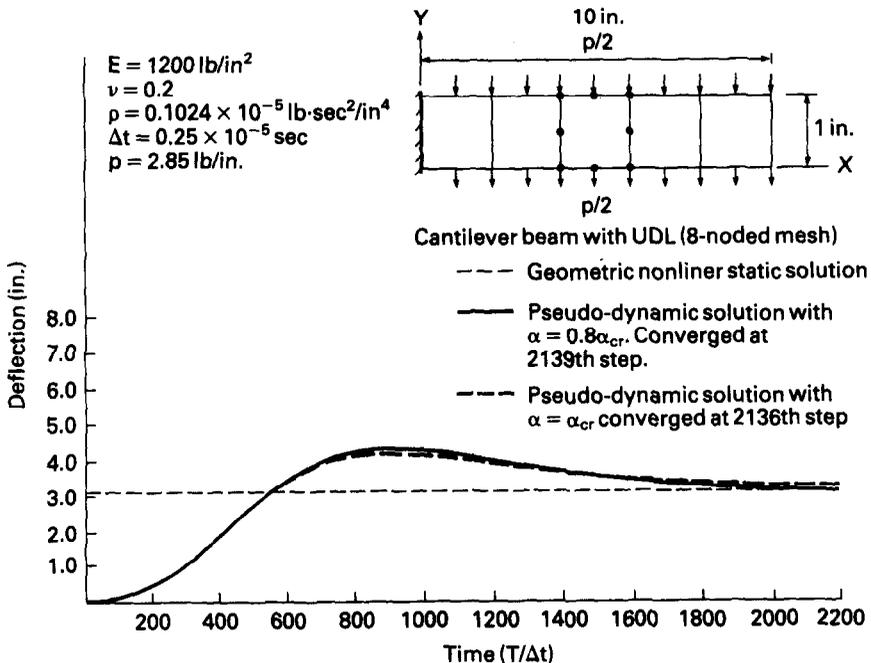


Fig. 4. Non-linear pseudo-dynamic analysis.

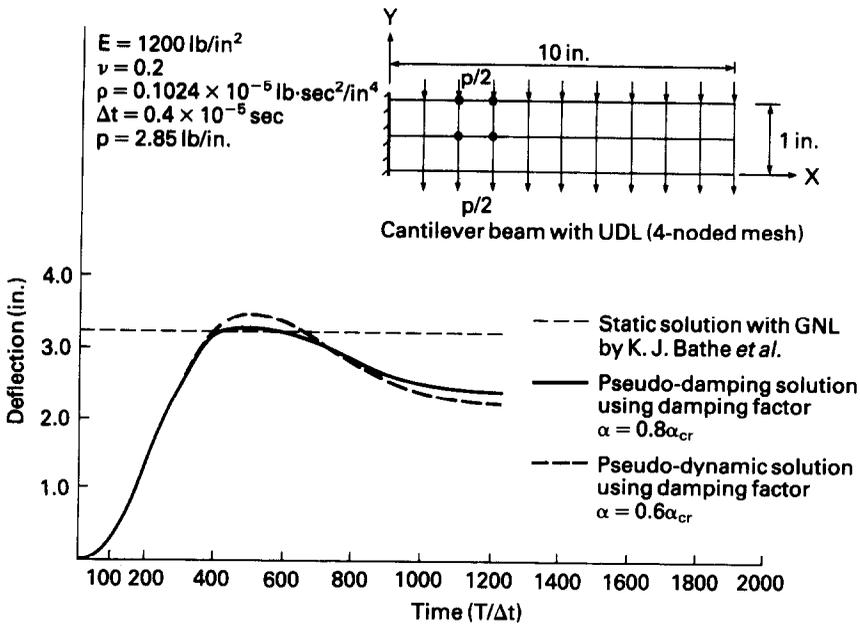


Fig. 5. Non-linear pseudo-dynamic analysis.

unified approach is capable of solving both static and dynamic, linear and non-linear two-dimensional problems. It is important to note that the program developed requires very small memory as compared to other programs which are for statics only. Another merit of the present unified approach is its easy implementation. It may be mentioned that the only computational effort involved is in the computation of the internal force vector P , in which any non-linearity can easily be introduced.

The test problems solved lead to the following important observations.

- (1) From the displacement response of a cantilever beam in pseudo-transient analysis (Fig. 4) it is clear that full critical damping should be avoided. The solution converged to a correct value of 3.25 in. when $\alpha = 0.8\alpha_{cr}$, while it converged to a higher value of 3.36 in. with $\alpha = \alpha_{cr}$. This shows that when full critical damping is used the vibrations die out so slowly that the

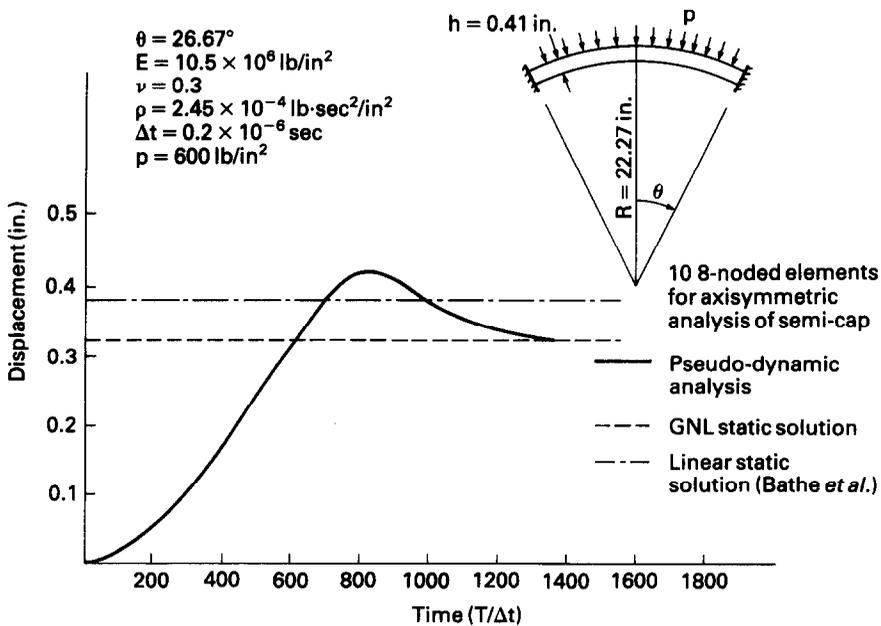


Fig. 6. Non-linear pseudo-dynamic analysis of a spherical cap under external pressure.

convergence criteria is satisfied at displacements far from the steady state value.

- (2) Finite element discretizations using four-noded mesh do not give correct results in spite of using a double layer. Furthermore, the results give a lower-bound value on displacements, i.e. it shows a stiff behaviour. When a higher value of time step is used (Δt increased from 2.5×10^{-6} sec), convergence is achieved in half the time steps (1190), thus increasing the economy of the solution; however the accuracy is unacceptable.
- (3) The present pseudo-transient analysis is found to be slower than the static-nonlinear analysis. However, it must be pointed out that this approach enables solution of very large linear and non-linear finite element problems on smaller computers. In the present work it is found that 9600 degrees-of-freedom can be solved using a generally available 16-bit microcomputer with 128 kbytes of core memory.

REFERENCES

1. A. S. Day, An introduction to dynamic relaxation. *Engr, Lond.* **219**, 218-221 (1965).
2. K. R. Rushton, Dynamic relaxation solution of elastic plate problems. *J. Strain Anal.* **3**, 24-32 (1968).
3. K. R. Rushton, Dynamic relaxation solution for large deflection of plates with specified boundary stresses. *J. Strain Anal.* **4**, 75-80 (1969).
4. J. S. Brew and D. M. Brotton, Nonlinear structural analysis by dynamic relaxation. *Int. J. Numer. Meth. Engng* **3**, 463-483 (1971).
5. A. Pica and E. Hinton, Transient and pseudo-transient analysis of Mindlin plates. *Int. J. Numer. Meth. Engng* **15**, 189-208 (1980).
6. A. Pica and E. Hinton, Further developments in transient and pseudo-transient analysis of Mindlin plates. *Int. J. Numer. Meth. Engng* **17**, 1749-1761 (1981).
7. O. C. Zienkiewicz, *The Finite Element Method*, 3rd Edn. McGraw-Hill, London (1977).
8. D. R. J. Owen and E. Hinton, *Finite Elements in Plasticity: Theory and Practice*. Pineridge Press, Swansea (1980).
9. K. J. Bathe, *Finite Element Procedures in Engineering Analysis*. Prentice-Hall, Englewood Cliffs, NJ (1982).
10. K. J. Bathe, H. Ozdemir and E. L. Wilson, Static and dynamic geometric and material nonlinear analysis. *SESM Report No. 74-4*, Department of Civil Engineering, University of California, Berkeley, CA (1974).