Journal of Reinforced Plastics and Composites
http://jrp.sagepub.com

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Journal of Reinforced Plastics and Composites 1991; 10; 102
DOI: 10.1177/073168449101000107

Buckling Loads of Sandwich Columns with a Higher-Order Theory

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ABSTRACT: The application of a higher-order shear deformable theory, which assumes a realistic cross-sectional deformation pattern and eliminates the use of a shear correction coefficient, is presented to evaluate the buckling loads of a symmetric, three-layered, simply supported, sandwich column when subjected to compressive edge loads. A closed form solution of the equilibrium equations is obtained by employing Von Karman strain displacement relations in the variational statement of the total potential energy and neglecting any stretching of the middle line. The results, when compared with those obtained from Euler-Bernoulli and Timoshenko theories, show that the high shear deformability of the core significantly reduces the buckling load, highlighting the efficacy of a higher order theory.

INTRODUCTION

THE USE OF sandwich construction in aerospace and automobile engineering has resulted into a spurt in research activity in the field of mechanics of composite materials. Relatively thinner element sections having highly shear deformable core material and stiff faces demand the study of stability under axial loads. Classical Euler-Bernoulli beam theory completely disregards the transverse shear deformation whereas the Timoshenko theory assumes a uniform shear rotation of the normal to the middle line requiring the use of a shear correction coefficient in the estimation of the transverse shear strain energy. Kant (1982) has presented a higher order theory which assumes a more realistic cross-sectional deformation pattern and eliminates the use of a shear correction coefficient. Recently Kant and Gupta (1988) have presented static and vibration results by employing a finite element model for a higher-order shear deformable beam theory. An attempt is made in this note, to highlight the efficacy of a higher-order theory as applied to sandwich beams for estimating buckling loads.

DISPLACEMENT MODEL AND FORMULATION

A three-layered (each layer made up of homogeneous and isotropic material), simply supported, sandwich beam of unit width having core thickness c, total

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thickness h and span length l is considered here for the analysis. The parameters (x,y,z) represent the orthogonal Cartesian co-ordinate system in the axial, lateral and thickness directions, respectively. The displacement field is assumed to be of a form given as:

$$u(x,z) = u_o(x) + z\theta_x(x) + z^2 u_o^*(x) + z^3\theta_x^*(x)$$
$$w(x,z) = w_o(x) \tag{1}$$

where the parameters u and w define the displacements at any point (x,z) in the beam domain in the x and z directions, respectively. The parameters u_o, w_o, θ_x, u_o^* and θ_x^* are the appropriate one-dimensional terms in the Taylor series and are defined along the x axis at $z = 0$. The symbols u_o, w_o and θ_x denote the axial-x and transverse-z displacements of a point on the reference x axis and the rotation of the transverse normal to the reference x axis in the x-z plane, respectively. While these parameters are physical quantities, the parameters u_o^* and θ_x^* are the higher-order terms in the Taylor series expansion and their physical interpretation is difficult indeed, except that they represent higher-order axial and flexural deformation modes, respectively.

The von-Karman strain-displacement relations, expressed as,

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \; ; \; \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial z}\right) \; ;$$
$$\epsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 \tag{2}$$

reduce to a simpler form, as given in Equation (3), since in the present case, the strain in the z direction is zero.

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \; ; \; \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \; ; \; \epsilon_z = 0 \tag{3}$$

Assuming elastic behaviour of the isotropic layers, the stress–strain relationships are given as:

$$\sigma_x = E\epsilon_x \; ; \; \tau_{xz} = G\gamma_{xz} \tag{4}$$

where σ_x and τ_{xz} are the in-plane and shear stress components while E and G are Young's and shear moduli, respectively.

The variational statement of the total potential energy Π, is given as:

$$\delta\Pi = \delta U - \delta W_{ex} = \int_V (\sigma_x\delta\epsilon_x + \tau_{xz}\delta\gamma_{xz})dV - \int_y\int_z (\bar{\sigma}_x\delta u)dz\,dy = 0 \tag{5}$$

where U and W_{ex} represent the internal strain energy and work done by edge stresses ($\bar{\sigma}_x$) on edges x = constant, respectively; body forces and tractive forces on other edges are assumed to be zero and δ indicates the variation taken over the parameter attached to it. Equation (5), when integrated through the thickness with the help of Equations (1) and (3), results in five equilibrium equations:

$$\frac{dN_x}{dx} = 0 \quad ; \quad \frac{dN_x^*}{dx} - 2S_x = 0 \tag{6a}$$

$$\frac{dM_x}{dx} - Q_x = 0 \quad ; \quad \frac{dM_x^*}{dx} - 3Q_x = 0 \quad ; \quad \frac{dQ_x}{dx} + \bar{N}_x \frac{d^2w_o}{dx^2} = 0 \tag{6b}$$

with boundary conditions,

$$\begin{aligned} N_x &= \bar{N}_x &\quad \text{or} \quad u_o &= \bar{u}_o \\ N_x^* &= \bar{N}_x^* &\quad \text{or} \quad u_o &= \bar{u}_o^* \\ M_x &= \bar{M}_x &\quad \text{or} \quad \theta_x &= \bar{\theta}_x \\ M_x^* &= \bar{M}_x^* &\quad \text{or} \quad \theta_x^* &= \bar{\theta}_x^* \\ Q_x + N_x \frac{dw_o}{dx} &= \bar{Q}_x &\quad \text{or} \quad w_o &= \bar{w}_o \end{aligned} \tag{7}$$

where N_x, N_x^*, M_x, M_x^*, Q_x, Q_x^* and S_x are the stress resultants defined as:

$$[N_x, M_x, N_x^*, M_x^*] = \sum_{l=1}^{l=n} \int_{h_l}^{h_{l+1}} \sigma_x[1, z, z^2, z^3]dz \tag{8a}$$

$$[Q_x, S_x, Q_x^*] = \sum_{l=1}^{l=n} \int_{h_l}^{h_{l+1}} \tau_{xz}[1, z, z^2]dz \tag{8b}$$

and the quantities with a bar indicate the edge values. $\bar{N}_x$ is the compressive edge load which is applied at the edges, $x = 0$ and $x = 1$, of the sandwich beam.

Combining Equations (1), (3), (4), (8a) and (8b) and neglecting any stretching of the middle line due to transverse z displacement, the stress resultants for a symmetrically layered beam are expressed in their final form as:

$$N_x = E_1 \frac{du_o}{dx} + E_3 \frac{du_o^*}{dx} \tag{9a}$$

$$N_x^* = E_3 \frac{du_o}{dx} + E_5 \frac{du_o^*}{dx} \tag{9b}$$

$$M_x = E_3 \frac{d\theta_x}{dx} + E_5 \frac{d\theta_x^*}{dx} \tag{9c}$$

$$M_x^* = E_5 \frac{d\theta_x}{dx} + E_7 \frac{d\theta_x^*}{dx} \tag{9d}$$

$$Q_x = G_1 \left[\frac{dw_o}{dx} + \theta_x\right] + 3G_3\theta_x^* \tag{9e}$$

$$Q_x^* = G_3 \left[\frac{dw_o}{dx} + \theta_x\right] + 3G_5\theta_x^* \tag{9f}$$

$$S_x = 2G_3 u_o^* \tag{9g}$$

where

$$E_i = \sum_{l=1}^{l=n} E^l \frac{h_l^i - h_{l-1}^i}{i} \tag{10a}$$

$$G_i = \sum_{l=1}^{l=n} G^l \frac{h_l^i - h_{l-1}^i}{i} \quad ; \quad i = 1, 3, 5, 7 \tag{10b}$$

The terms E^l and G^l define Young's and shear moduli of the l^{th} layer, respectively, and n represents the number of layers. Equations (9a) to (9g), when substituted in the equilibrium Equations (6), result in Equations (6a) getting decoupled from Equation (6b).

A closed form solution for the equilibrium Equations (6b) is attempted here, for a simply supported, three layered, symmetric sandwich beam by assuming the following forms:

$$w_o = \sum_{m=1}^{\infty} w_{om} \sin \alpha x$$

$$\theta_x = \sum_{m=1}^{\infty} \theta_{xm} \cos \alpha x \tag{11}$$

$$\theta_x^* = \sum_{m=1}^{\infty} \theta_{xm}^* \cos \alpha x \quad ; \quad \alpha = m\pi/l$$

with parameters l and m representing span length of the beam and a harmonic integer taking values 1, 2, 3, . . . ∞, respectively, and w_{om}, θ_{xm}, θ^*_{xm} are the modal values of the displacement components.

Equations (11) satisfy the equilibrium Equations (6b) together with boundary conditions (7), only if

$$\begin{bmatrix} (G_1\alpha^2 - \overline{N}_x\alpha^2) & (G_1\alpha) & (3G_3\alpha) \\ (G_1\alpha) & (E_3\alpha^2 + G_1) & (E_5\alpha^2 + 3G_3) \\ (3G_3\alpha) & (E_5\alpha^2 + 3G_3) & (E_7\alpha^2 + 9G_5) \end{bmatrix} = 0 \tag{12}$$

or

$$\overline{N}_x = P_h = \frac{9E_3(G_1G_5 - G_3^2)\alpha^2 + G_1(E_3E_7 - E_5^2)\alpha^4}{9(G_1G_5 - G_3^2) + (E_7G_1 + 9E_3G_5 - 6E_5G_3)\alpha^2 + (E_3E_7 - E_5^2)\alpha^4} \tag{13}$$

where $\overline{N}_x$ denotes the buckling load and P_h represents the same according to the present higher-order theory.

The well-known classical Euler-Bernoulli theory [Timoshenko and Gere (1936)] gives the buckling load for an indentical beam, with core and faces rigid in shear, as:

$$P_e = E_3\alpha^2 \qquad (\text{for } m = 1) \tag{14}$$

while for the Timoshenko theory [Timoshenko and Gere (1936)] the expression for the same is given as:

$$P_t = \frac{P_e}{[1 + (P_e/kG_1)]} \qquad (\text{for } k = 5/6) \tag{15}$$

where k is the shear correction coefficient for a rectangular cross-section.

Plantema (1966) has given the buckling load expression, for thick as well as thin, three layered, symmetric sandwich beams assuming shear deformation of the core as independent of the face deformation (Euler buckling mode modified for shear deformable core), as:

$$P_p = \alpha^2B\{[s_o/(1 + s_o)] + c_f\} \tag{16}$$

where

$$\alpha = \pi/l \; ; \qquad B = E_ff(c + f)^2/2 \; ; \qquad s_o = s(1 + c_f)^2 \; ;$$

$$c_f = 1/\{3[1 + (c/f)]^2\} \; ; \qquad s = S/(\alpha^2B) \; ; \qquad S = (c + f)^2G_ck/c \tag{17}$$

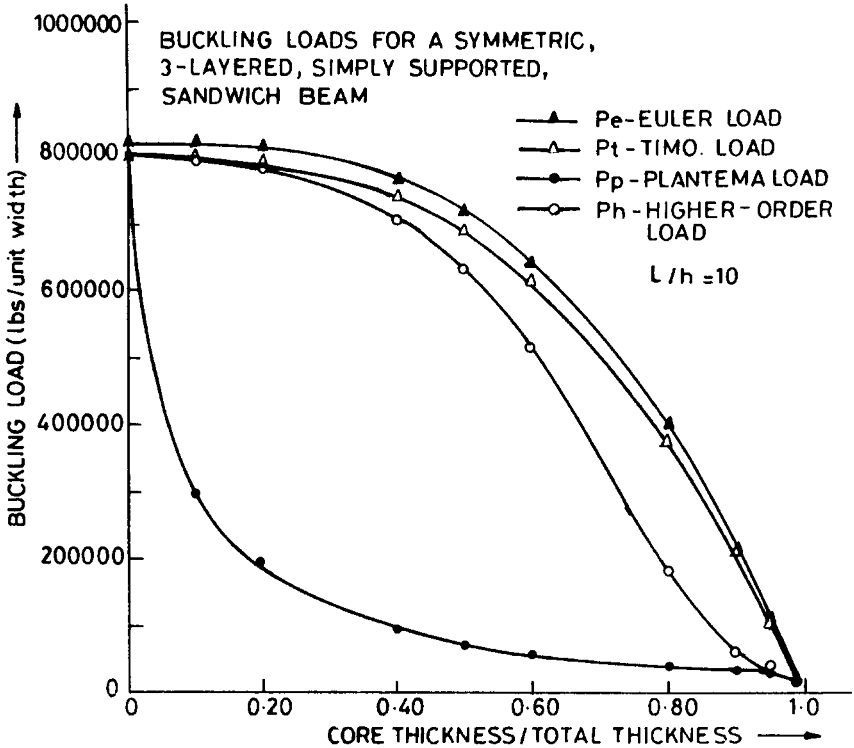

Figure 1. *Buckling load (P_e, P_t, P_p, P_h) vs. (core thickness/total thickness).*

In the above expressions, c and f are the core and face thicknesses, E_f and G_c denote Young's modulus of the face material and shear modulus of the core material, respectively.

NUMERICAL RESULTS

Numerical results are obtained for a symmetric, three layered, simply supported, sandwich beam with the following material properties:

Isotropic stiff layers:

$$E_f = 10^7 \text{ psi}$$
$$G_f = E_f/[2(1 + \nu)]$$
$$\nu = 0.30$$

Central core:

$$G_c = 5000 \text{ psi}$$
$$\nu = 0$$

The beam dimensions:

l = 100 in. (span length)
b = 1 in. (width)

The buckling loads obtained by the use of Equations (13), (14), (15) and (16) for different length to total thickness ratios (L/h) and varying core thickness to total thickness ratios (c/h), are given in Table 1 and the nature of variation of the buckling loads is presented in Figure 1. In Table 1, the symbols P_e, P_t, P_p and P_h represent buckling loads (lbs/unit width) for the sandwich beam, according to Euler-Bernoulli, Timoshenko, Plantema and higher-order theories, respectively.

CONCLUSION

The application of a higher-order shear deformable theory is presented to evaluate the buckling loads of a symmetric, three layered, simply supported, sandwich beam of unit width when subjected to compressive edge loads. The following conclusions were reached based on the closed-formed solution obtained.

The high shear deformability of the core (E_f/G_c ratio being significantly high) significantly reduces the buckling load, as compared to Timoshenko and Euler-Bernoulli theories. With the increasing core thickness to total thickness ratio the buckling load reduces progressively.

As the length to total thickness ratio increases, the buckling load approaches

Table 1. Buckling loads for sandwich beam (lbs./unit width).

c/h	L/h	P_e, Euler	P_t, Timoshenko	P_p, Plantema	P_h, Present
0.99	5	0.1954×10^6	0.1539×10^6	0.5842×10^5	0.7395×10^5
	10	0.2443×10^5	0.2288×10^5	0.1540×10^5	0.1732×10^5
	50	0.1954×10^3	0.1949×10^3	0.1909×10^3	0.1923×10^3
	100	0.2443×10^2	0.2441×10^2	0.2429×10^2	0.2433×10^2
0.90	5	0.1783×10^7	0.1399×10^7	0.8161×10^5	0.1628×10^6
	10	0.2229×10^6	0.2086×10^6	0.3544×10^5	0.6360×10^5
	50	0.1783×10^4	0.1778×10^4	0.1471×10^4	0.1620×10^4
	100	0.2229×10^3	0.2227×10^3	0.2116×10^3	0.2174×10^3
0.50	5	0.5757×10^7	0.4882×10^7	0.3046×10^6	0.3705×10^7
	10	0.7197×10^6	0.6888×10^6	0.7270×10^5	0.6315×10^6
	50	0.5757×10^4	0.5747×10^4	0.3786×10^4	0.5725×10^4
	100	0.7197×10^3	0.7193×10^3	0.6357×10^3	0.7187×10^3
0.00	5	0.6580×10^7	0.5967×10^7	0.6580×10^7	0.5967×10^7
	10	0.8225×10^6	0.8019×10^6	0.8225×10^6	0.8019×10^6
	50	0.6580×10^4	0.6573×10^4	0.6580×10^4	0.6573×10^4
	100	0.8225×10^3	0.8223×10^3	0.8225×10^3	0.8223×10^3

the value predicted by Euler's theory in the limit. This proves the soundness of and consistency in the theoretical formulations.

Plantema results should be relied only in cases where core thickness is considerable.

The results presented establish the usefulness of a higher-order theory where thick core, thick sandwiches are encountered.

ACKNOWLEDGEMENT

Partial support of this research by the Aeronautics Research and Development Board, Ministry of Defence, Government of India through its Grant No. Aero/RD-134/100/10/88-89/518 is gratefully acknowledged.

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