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A critical review and some results of recently developed refined theories of fiber-reinforced laminated composites and sandwiches

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A critical review of literature pertinent to the subject matter of this paper was carried out under the following two broad headings: free vibration and transient dynamics. Each of these groups describes the various theoretical developments in fiber reinforced laminated composite and sandwich plates. The theoretical developments are further classified according to the refinement/accuracy of the theories developed, such as the classical theory, the first-order shear deformation theory, and the three-dimensional elasticity/higher-order shear deformation theories. The present literature review is limited to linear free vibration and transient dynamic analyses, and geometric nonlinear transient response of multilayer sandwich/fiber-reinforced composite plates. A comparative study of recently developed refined theories in conjunction with the C^0 isoparametric finite element formulation has been made and the conclusions were drawn based on the literature review and the refined theories results. In order to compare the present results with the available results and to provide an easy means for future comparisons by other investigators, the numerical results are presented in tabular form.

INTRODUCTION

Engineers have a wide scope in composite structural design because of the variety of constituent materials which can be employed and the numerous options in fiber orientations and the laminas' arrangements. Most structures, whether they are used on land, sea or in the air are subjected to dynamic loads during their operation. Thus, theories which can predict the complete behavior become necessary for better understanding of the complex failure mechanism and strength of multilayer composite structures.

For mathematical modeling purposes, the individual layer (lamina) is considered to be homogeneous and orthotropic while the laminate is heterogeneous through the thickness and generally anisotropic. The greater differences in elastic properties between fiber filaments and matrix materials lead to a high ratio of in-plane Young's modulus to transverse shear modulus for most of the composite laminates developed to date. This makes the classical lamination theory, which neglects the effect of out-of-plane strains, inadequate for the analysis of multilayer composite plates. Thus, in order to have a reliable analysis and safe designs, more accurate theories which include the effects of transverse shear deformation become necessary. As a result, a considerable amount of work in investigating the effects of transverse shear deformation and rotatory inertia has been conducted leading to the so-called first-order shear deformation theories. The first-order theories assume a constant shear rotation through the plate thickness and thus require the use of a shear correction coefficient whose accurate prediction for anisotropic laminates is cumbersome and problem dependent. It is clear that these theories do not include the effects of cross-sectional warping which is definitely essential for thick sandwich plates which are generally composed of a middle weak core sandwiched between stiff facings. Further, the effects of transverse normal stress/strain which are neglected in first-order theories should also be evaluated. These limitations of the first-order shear deformation theories forced the development of refined theories which include the consideration of realistic parabolic variation of transverse shear stresses through the plate thickness, warping of the transverse cross-section and consideration of the complete material constitutive Hooke's law.

A review of available literature is classified under two broad headings: free vibration and transient dynamics.

FREE VIBRATION

The oldest (approximate) plate theory which still enjoys considerable status is the so-called classical (or Germain–Lagrange–Cauchy) plate theory (CPT). In this theory, straight lines originally normal to the plate median surface are constrained to remain straight and normal during the process of deformation (Kirchhoff's assumption). This assumption is equivalent to neglecting transverse shear deformation in the plate. Because of this (artificial) constraint, free vibration frequencies calculated from classical plate theory are always higher than those obtained by more precise means, the deviation becoming larger with increasing mode numbers. To remove this as well as other deficiencies, and to retain the comparative analytical simplicity of a plate theory, an improved plate theory was advanced.[1] This theory accounts for the effects of shear deformation and rotatory inertia in addition to transverse inertia.

Sandwich plates

Sandwich plates have been the subject of many investigations; a large amount of literature has been devoted to the development of theories for conventional sandwich structures and to the study of their static and dynamic behaviors by analytical and numerical methods. A detailed historical review is given in books by Plantema[2] and Allen[3] and in two papers by Habip.[4,5] Originally, most authors dealt with sandwiches in which the facings were thin, stiff and heavy as compared with the core. Henceforth, this configuration will be referred to as classical. The pioneer worker Reissner[6] in 1947 suggested a simple and useful model to describe such plates. He only took into account the transverse shear stiffness of the core and the in-plane or membrane stiffness of the facings. Subsequently Reissner[7,8] studied finite deflection of such plates.

The main purpose of further investigations apparently was to include more and more 'physical mechanisms'. This was done in order to relax or even eliminate the restriction of dealing only with classical sandwiches. However the governing equations of motion of these more complicated 'strength-of-materials theories' have necessarily become more involved as well.

Yu[9] studied the propagation of plane harmonic waves in sandwich plates where no limitation was imposed upon the magnitude of the ratios between the thickness, material densities and elastic constants of the core and the facings. He applied Mindlin's bending theory of plates[1] to all layers of the sandwich and obtained extremely complicated equations of motion. And Yu[10–13] further published a series of papers on vibration of sandwich plates including viscous damping and large deflections. He accommodated in the theory the transverse shear deformation and rotary inertia effects, the importance of which was clearly demonstrated when the theory was used in dealing with conventional sandwiches. Chu[14] presented a set of approximate equations governing the extensional motion of a sandwich plate, together with the associated initial and boundary conditions. Frequency wavelength curves were obtained from these equations for an infinite plate in plane strain. A solution of the exact equations was obtained and approximate solutions compared with the exact solution.

Exact elasticity solutions for some particular sandwich plate bending problems were obtained by Pagano[15] while many other researchers, such as Monforton and Schmit[16] and Ahmed[17,18] adopted the versatile finite element method in analysing conventional sandwich structures. Monforton and Schmit[16] have presented an 80 degrees of freedom rectangular element for analysing sandwich plates. The element formulation uses a Hermitian polynomial to describe the membrane displacements of the upper and lower faces and the transverse displacement. The formulation was presented in terms of membrane, bending and coupling stiffnesses of the faces and transverse shear stiffness for the antiplane core. Chan & Cheung[19] used a finite strip method for static and dynamic analysis of a sandwich plate without considering common shear angle for the cores and have shown that the assumption of common shear rotation for all cores leads to erroneous results for certain cases. The generalized formulations applicable to multilayer sandwich plates have been reported by Khatua and Cheung[20] using non-conforming rectangular elements. The condition of common shear angle for all cores has not been assumed in the formulation.

Lee and Chang[21] derived analytical frequency equations for plane waves from the three-dimensional equations of elasticity. They computed dispersion curves for two very different sandwich plates and discussed the corresponding difference in the physical behavior. Their treatment could be very useful in testing numerically the accuracy of approximate theories. Ng and Lam[22] presented displacement-based finite element formulations using a parallelogrammic element having five degrees of freedom per node for static and dynamic analysis of skew sandwich plates. The formulations are valid for a sandwich plate made up of isotropic faces of equal thickness and an orthotropic core. Sayir and Koller[23] discussed the physical behavior of bending waves in sandwich plates in which the facings are thin, stiff and heavy as compared with the core. By means of asymptotic expansions of the basic equations of linear elasticity, it was shown that different 'physical mechanisms' predominate in different frequency ranges. Raville and Ueng[24] experimentally determined the natural frequencies of vibration of a sandwich plate.

Fiber reinforced composite plates

The available literature related to free vibration analysis of fiber reinforced composite plates is classified into three groups. The first group describes the classical (thin plate) theory which neglects the effects of shear strains, normal strain and normal stress in the transverse direction. This is commonly known as the classical lamination theory (CLT) and it is an extension of the classical plate theory[25–28] to laminated plates. The limitations of CLT have led to the development of the first-order shear deformation theory (FOST) which requires the use of shear correction coefficients and the literature related to this is described in the second group. The third group describes the refined theories which are either based on the three-dimensional (3D) approach or the two-dimensional (2D) approach with higher-order displacement models giving parabolic variation of transverse shear strains through the plate thickness and thus requiring no shear correction coefficients. Bert and Francis,[29] and Bert[30–33] have presented a detailed review of the literature related to the structural mechanics (statics and dynamics) aspects of composite beams, plates and shells. A review on finite element modeling of plates is given in Refs 34 and 35.

Classical lamination theory

The classical laminate theory ignores the three transverse strain components and the transverse normal stress components and models the laminate as a two-dimensional equivalent single layer. This simple theory can provide reasonably accurate prediction only for relatively thin plates. Various texts[36–44] have described this theory and its application to the analysis of laminated composite structures. Laminated plate theories based on the Kirchhoff hypotheses have been developed by Reissner and Stavsky,[45] Dong *et al.*[46] and Stavsky,[47] and these developments are summarized in Ref. 40. These works are based on a linear longitudinal displacement distribution across the entire laminate with shear deformation neglected.

For bending, buckling and free vibrational analyses of a clamped anisotropic plate a modified Fourier series method was used by Whitney.[48] Tsay and Reddy[49] presented a mixed finite element formulation based on Reissner's variational principle using a rectangular element for bending, stability and free vibration analyses of isotropic and orthotropic thin plates. It was shown by Laura *et al.*[50,51] that the polynomial coordinate functions used to obtain the fundamental frequency of transverse vibration of thin, elastic plates by making use of the Rayleigh–Ritz method yield better accuracy. Dong and Lopez[52] determined the natural frequencies and mode shapes of a clamped circular plate with rectilinear orthotropy by a modified application of the interior collocation method. A least-squares error fit was then used to generate the governing eigenvalue problem. This method was simple to implement and had an advantage over the Rayleigh–Ritz and Galerkin methods where energy or error integrals had to be evaluated analytically before numerical analysis. However, collocation does not provide an upper bound as in Rayleigh–Ritz. Iyengar & Umaretiya[53] made an attempt to obtain the free vibration response of hybrid, laminated rectangular and skew plates. The Galerkin technique was employed to obtain an approximate solution of the governing differential equations. Optimal designs of laminated plates have been given in several studies[54,55] of which only a few are referenced here, with respect to natural frequencies in which the effect of shear deformation was neglected. The neglect of shear deformation leads to designs that are only suboptimal. Indeed, the values of optimum fiber orientations and layer thicknesses depend on the side-to-thickness ratio and consequently FOST and CPT yield different optimum points.

First-order shear deformation theories

The classical plate theory which ignores the transverse shear and transverse normal deformations is inadequate for the analysis of moderately thick plates. Further, the shear deformation effects are more pronounced in fiber-reinforced composite laminates in comparison with the isotropic plates due to the high ratio of in-plane modulus to transverse shear modulus of fiber reinforced materials. These high ratios make classical lamination theory inadequate for the analysis of fiber-reinforced composite plates. An adequate theory must account for transverse shear effects. The development of FOST began with the work of Reissner[56] and Mindlin[1] for isotropic plates. The approach was extended to embrace laminated composite plates by Yang, Norris and Stavsky (YNS)[57] and Whitney and Pagano[58,59] for dynamic analysis. For bending, stability and vibration analysis of specially orthotropic or transversely isotropic plates, Ambartsumyan[60] has systematically presented the governing equations which incorporate the transverse shear effects. On similar lines, Vinson & Chou[61] published a book which includes composite plate as well as shell structures. Whitney[62] noted that the values of shear correction coefficients for orthotropic laminates depended on the details of the laminate construction. Dong and Nelson[63] studied the vibrations of a laminated plate composed of an arbitrary number of bonded elastic, orthotropic layers. The analysis was carried out within the framework of linear elasticity for plane-strain behavior.

Sun and Whitney[64] investigated the effect of heterogeneous shear deformation over the thickness of plate on the dynamic behavior of laminated plates. Three sets of governing equations were derived according to different assumptions on the local transverse shear deformation and the interface conditions. Fortier & Rossettos[65] analysed free vibration of rectangular plates of unsymmetric cross-ply construction while Sinha and Rath[66] considered both vibration and buckling for the same type of plates. Using a thick finite strip approach, Hinton[67] presented a note on the free vibration of laminated plates including transverse shear effects and rotary inertia. Bert and Chen[68] have given a closed form solution using a YNS theory for the free vibration of simply supported rectangular plates of antisymmetric angle-ply laminates. The effect of deleting rotary inertia and in-plane inertia, singly and in combination were also investigated. Craig and Dawe[69,70] studied the flexural vibration of rectangular laminated plates using FOST. Two numerical techniques were employed in the study, viz. the Rayleigh–Ritz method and the finite-strip method, and in both, the trial displacement functions make use of the normal modes of vibration of Timoshenko Beams. Chen & Ramkumar[71] formulated a theory for the analysis of clamped orthotropic plates by using a Lagrangian multiplier technique for the solution of the static and eigenvalue problem.

While considerable effort has been expanded in the finite element vibration analysis of isotropic plates, only limited investigations of laminated anisotropic plates can be found in the literature. Hinton *et al.*[72,73] outlined a particular lumping process to show that good accuracy can be obtained in a linear and non-linear dynamic problem using isoparametric parabolic elements. The procedure of lumping recommended in view of the infinite possibilities offered is to compute the diagonal terms of the consistent mass matrix and then scale these terms so as to preserve the total mass of the element. Reddy[74] used the YNS theory for free vibration of antisymmetric angle-ply laminated plates with a finite element formulation.

Refined theories

The first-order shear deformation theory which ignores the effects of cross-sectional warping leads to an unrealistic (constant) variation of the transverse shear stresses through the laminate thickness. Development of refined 2D theories, which incorporate higher-order modes of transverse cross-sectional deformation and account for 2D/3D state of stress/strain has been attempted in recent years. These theories depict a realistic parabolic variation of transverse shear stresses through the laminate thickness and do not require the use of assumed shear correction coefficients as in the case of first-order Reissner/Mindlin theory.

Srinivas and Rao[75,76] derived the governing equations for the bending, free vibration and buckling analyses of simply supported thick isotropic and orthotropic rectangular laminates using three-dimensional theory of elasticity. Solutions of the 3D elasticity theory for free vibration of multilayered composite plates were obtained by Noor[77] using a higher-order finite difference scheme. Such a scheme was shown to give highly accurate results for the response characteristics of the plate. Reissner[78] formulated a theory for flexural response of isotropic plates by assuming a cubic variation of in-plane displacements and parabolic variation of transverse displacement across the plate thickness. Lo *et al.*[79,80] presented a theory for homogeneous isotropic[79] and laminated composite[80] plates which is of the same order of approximation as that of Reissner,[78] but includes the terms contributing to the in-plane modes of deformation. Thus, the theory accounts for the parabolic variation of transverse shear stresses, transverse normal strain/stress and a cubic variation of the in-plane displacements across the plate thickness. The principle of stationary potential energy has been used to derive the governing differential equations.

Based on the assumed displacement field of Reissner,[78] both Levinson[81] and Murthy[82] used the equilibrium equations of the FOST which are variationally inconsistent for the higher-order displacement field used by them with those derived from the principle of virtual displacements. This fact was noted by Reddy[83] and he presented a consistent derivation of the associated equilibrium equations. The displacement model used by Murthy, Levinson and Reddy is the same and it contains the same number of dependent variables as in the FOST and the theory implicitly satisfies the free transverse shear conditions on top and bottom surfaces of the plate. Closed-form solutions were presented for simply supported symmetric cross-ply laminates. Murthy[84] realised the inability of earlier[81,83] higher-order theories to evaluate the transverse shear strains at points in the plate where displacements are constrained to be zero, such as those on fixed edges. To overcome this limitation, an additional partial shear deflection variable was introduced. Thus, based on four basic displacement variables (two partial transverse deflections and two in-plane displacements), the governing equations have been derived using a variational principle and are presented in the form of four simultaneous partial differential equations. The free transverse shear conditions on the bounding plane of the plate are not satisfied due to the introduction of partial shear deflection in the formulation.

Kant[85] adopted the segmentation method and derived the governing equations for linear elastic analysis of homogeneous isotropic plates. Later, Kant *et al.*[86] presented a displacement-based finite element formulation using the displacement model of Kant.[85] Pandya and Kant[87-90] investigated the behavior of anisotropic laminated composite plates based on various assumed displacement fields with simple isoparametric finite element formulations. These studies[78-80,82-90] were confined to static analysis.

Whitney and Sun[91] extended the theories of YNS[57] and Whitney and Pagano[59] to include the first symmetric thickness shear and thickness stretch modes by including higher-order terms in the displacement expansion about the mid-plane of the laminate in a manner similar to that of Mindlin & Medick[92] for homogeneous isotropic plates. Bhimaraddi and Stevens[93] have given some results for free vibration of orthotropic plates by using a higher-order theory with closed form solution. They considered a total of five unknowns, which are the middle surface displacement quantities. They maintain the higher-order (cubic) polynomial form for in-plane displacement expressions and at the same time the more realistic parabolic variation for transverse shear strains is achieved.

To determine the natural frequencies and buckling loads of orthotropic laminated plates, Reddy and Phan[94] and Putcha and Reddy[95] presented a closed form solution and mixed finite element formulation, respectively, with the assumed displacement field used earlier in Ref. 83. Owen and Li[96,97] presented a local finite element model based on an approximate theory for thick anisotropic laminated plates. The three-dimensional problem was reduced to a two-dimensional one by assuming piecewise linear variation of the in-plane displacements u and v, and a constant value of the lateral displacement w across the thickness. A substructuring technique was used in the bending, vibration and buckling analysis. A 3D eight-node hybrid stress finite element was developed for the free vibration analyses of laminated plates by Sun and Liou.[98] This hybrid stress model was based on the modified complementary energy principle and all three displacement components were assumed to vary linearly through the thickness of each lamina.

Recently, Mallikarjuna and Kant[99-102] emphasized on establishing the credibility of higher-order theories with different displacement models for free vibration analysis of anisotropic laminated composite and sandwich plates. A simple isoparametric C^0 finite element formulation was presented. The special mass matrix diagonalization scheme was adopted which conserves the total mass of the element and included the effects of mass inertia terms corresponding to all the degrees of freedom.

TRANSIENT DYNAMICS

The analyst usually has at his disposal a variety of mathematical models of varying analytical complexity and physical fidelity. If considerations are restricted to sufficiently small elastic deformations, the resulting theories can be considered to be more or less satisfactory approximations of 3D classical elasticity theory. The latter represents the undisputed standard of accuracy within the limitations of elastic action and small deformations. The static behavior of composite laminates has been reasonably well established analytically and experimentally. In the case of dynamic loading, however, the state of the art is in a developing stage.

The linear elastic transient response of isotropic plates has been investigated by several researchers. Reismann and his colleagues[103-105] analysed a simple supported, rectangular, isotropic plate subjected to a suddenly applied uniformly distributed load over a square area of the plate. Exact solution was obtained using (classical) 3D elasticity theory, and classical improved theories. The finite difference method, widely used in the solution of the equations of motion which govern the transient response of structures such as plates and shells, can become unstable unless the ratio of the time mesh to the space mesh satisfies a certain condition. The condition of stability of the finite difference equation for the transient response of a thin flat plate and moderately thick plate has been given by Leech,[106] and Tsui and Tong,[107] respectively. Rock and Hinton[108] presented transient finite element analysis of thick and thin isotropic plates. The element is based on the Reissner–Mindlin (R–M) thick plate theory for homogeneous, isotropic plates. Excellent agreement of the finite element solutions with the analytical solution of Reismann & Lee[103] was obtained. Hinton[109] adopted the FOST for circular plate bending problems by using axisymmetric parabolic isoparametric elements and an explicit time marching scheme with a special mass lumping procedure. A uniform reduced integration technique was used. These papers[103-109] dealt only with linear transient response of isotropic plates.

The solution of linear and nonlinear dynamic transient plate bending problems was considered by Hinton *et al.*[110] and three situations were examined: small deformation and large deformation elastic response, and small deformation elasto-plastic response employing the yield criteria of Von Mises and Tresca. An estimate of the critical time step length for the transient solution of R–M plates, given by Tsui and Tong[107] was used with minor modification in Refs 109 and 110. Shantaram *et al.*[111] employed the FEM in the prediction of the transient response of 2D and 3D solids exhibiting geometric (large deformation) and material (elasto-plastic) nonlinearities. Pica and Hinton[112,113] presented a unified approach for the static and transient dynamic linear and geometrically nonlinear analysis of R–M plates including initial imperfections. A finite element idealization was adopted and the quadratic Lagrangian elements were used together with selective integration. Akay[114] analysed large deflection transient response of isotropic plates using a four-node isoparametric mixed quadrilateral element. Dynamic Von Karman plate equations are modified to include the effect of transverse shear deformations as in Reissner plate theory. Finite element equations of motion are obtained via a mixed Galerkin approach with three moment and three displacement components as dependent variables. All of these studies were confined to homogeneous, isotropic plates.

Moon[115,116] investigated the response of infinite laminated plates subjected to transverse impact loads at the center of the plate. Five partial-differential equations of motion with a mathematical model based on the work of Mindlin and co-workers were obtained[115] for orthotropic symmetry in which the in-plane and flexural motion were described. The two-dimensional velocity and wave surfaces and the principal vibratory direction of particle motion for each wave normal were presented. The analysis[116] was based on the use of a Laplace transform on time and a 2D Fourier transform on the space variables. The solution permits the analytical inversion of the Laplace transform while a computational tool called the Fast Fourier Transform was used to numerically invert the Fourier transform solution.

Chow[117] employed the Laplace transform technique to study the dynamic response of orthotropic laminated plates. The dynamic equations were derived from the concepts of Timoshenko's beam theory to include the effects of transverse shear and rotary inertia. The influence of internal friction related to the damping on the response of the plate was also considered. Wang *et al.*[118] applied the method of characteristics to unsymmetrical orthotropic laminated plates. In a series of papers, Sun and his colleagues[119-122] used the classical method of separation of variables combined with the Mindlin–Goodman[123] procedure for treating time-dependent boundary conditions and/or dynamic external loadings. Yu[124] subsequently applied it to sandwich plates. However, these papers[115-124] were confined to plates under cylindrical bending.

For two different lamination schemes, under appropriate boundary conditions and sinusoidal distribution of the load, the exact form of the spatial variation of the solution was obtained by Reddy[125] and the problem was reduced to the solution of a system of ordinary differential equations in time. Reddy[126] has also presented the linear transient response of composite plates using finite elements. In both the papers,[125,126] the theory used was a generalization of the R–M thick plate theory for homogeneous, isotropic plates to arbitrarily laminated anisotropic plates and included shear deformation and rotary inertia effects.

A generalization of the Von Karman[127] nonlinear plate theory for isotropic plates to include the effects of transverse shear and rotary inertia in the theory of orthotropic plates is due to Medwadowski[128] and that for anisotropic plates is due to Ebcioglu.[129] Forced motions of laminated composite plates are investigated by Reddy[130] using a finite element that accounts for the transverse shear strains, rotary inertia, and large rotations (in the Von Karman sense). Chen & Sun[131] used the FEM for nonlinear transverse response of laminated composite plates under initial deformation and initial stress according to the R–M plate theory and Von Karman large deformation assumptions. Kant & Mallikarjuna[132] presented the linear transient response of composite-sandwich plates using a FOST with 4-, 8-, and 9-noded isoparametric quadrilateral elements. All of the above studies[103-132] were based on either the classical 3D elasticity theory or the classical (Kirchhoff) plate theory or the first-order shear deformation (Reissner–Mindlin) theory. Recently Mallikarjuna and Kant used the higher-order displacement models (see below eqns (1)–(5)) with simple C° finite element formulation for free vibration[99-102] and transient dynamic[99,133-143] analyses, and in obtaining solutions to general laminated fiber-reinforced composite and sandwich plate problems.

HIGHER-ORDER SHEAR DEFORMATION THEORIES

Refined theories[99] for free vibration and transient dynamic analysis of anisotropic composite and sandwich laminates have been developed, separately for symmetric and unsymmetric lamination schemes, using the following displacement fields based on the Taylor's series expansion:

Symmetric laminate

(a) Higher-order shear deformable theory (HOST5), 5 d.o.f./node

$$\begin{aligned} u(x,y,z,t) &= z\theta_x(x,y,t) + z^3\theta_x^*(x,y,t) \\ v(x,y,z,t) &= z\theta_y(x,y,t) + z^3\theta_y^*(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{1}$$

(b) Higher-order shear deformable theory (HOST6), 6 d.o.f./node

$$\begin{aligned} u(x,y,z,t) &= z\theta_x(x,y,t) + z^3\theta_x^*(x,y,t) \\ v(x,y,z,t) &= z\theta_y(x,y,t) + z^3\theta_y^*(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) + z^2 w_0^*(x,y,t) \end{aligned} \tag{2}$$

Unsymmetric laminate

(a) Higher-order shear deformable theory (HOST7), 7 d.o.f./node

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\,\theta_x(x,y,t) \\ &\quad + z^3\theta_x^*(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\,\theta_y(x,y,t) \\ &\quad + z^3\theta_y^*(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{3}$$

(b) Higher-order shear deformable theory (HOST9), 9 d.o.f./node

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\,\theta_x(x,y,t) \\ &\quad + z^2 u_0^*(x,y,t) + z^3\theta_x^*(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\,\theta_y(x,y,t) \\ &\quad + z^2 v_0^*(x,y,t) + z^3\theta_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{4}$$

(c) Higher-order shear deformable theory (HOST11), 11 d.o.f./node

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\,\theta_x(x,y,t) \\ &\quad + z^2 u_0^*(x,y,t) + z^3 \theta_x^*(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\,\theta_y(x,y,t) \\ &\quad + z^2 v_0^*(x,y,t) + z^3 \theta_y^*(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) + z\,\theta_z(x,y,t) \\ &\quad + z^2 w_0^*(x,y,t) \end{aligned} \tag{5}$$

where t is the time, u, v and w define the displacements of any generic point (x, y, z) in the plate space, u_0, v_0 and w_0 denote the displacements of a generic point (x, y) on the midplane, θ_x and θ_y are the rotations of normals to midplane about the y and x axes, respectively. The parameters u_0^*, v_0^*, w_0^*, θ_x^*, θ_y^* and θ_z are higher-order terms in the Taylor's series expansion and are also defined at the mid-surface. The development of the refined theories and the isoparametric C° finite element formulation can be seen in Ref. 99.

NUMERICAL RESULTS AND DISCUSSION

To demonstrate the versatility of the refined theories developed, various numerical examples drawn from the literature are described, evaluated and discussed. The finite element solution technique adopted here has a wide range of applicability for laminates with arbitrary geometry, loading and boundary conditions. The computer programs have been developed separately to predict the free vibration and transient response of symmetric and unsymmetric laminates. In addition to the refined theories, programs were developed for the first-order shear deformation theory with three degrees of freedom (FOST3) i.e. w, θ_x, θ_y for symmetric laminates and with five degrees of freedom (FOST5) i.e. u, v, w, θ_x, θ_y for unsymmetric laminates. It is well known that the shear correction coefficients depend on the lamination scheme and the lamina material properties. But due to lack of well accepted coefficients for finite plates, the transverse shear energy term in FOST is corrected using a multiplier 5/6 for all the materials except for the core of a sandwich plate where a coefficient of unity has been used. The results of the present refined theories have been compared with the present FOST wherever solutions by other methods are not available in the literature.

For the 9-noded Lagrangian quadrilateral isoparametric element used throughout here, the selective numerical integration scheme, based on the Gauss-quadrature rules, viz. 3×3 for membrane, flexure and coupling between membrane and flexure terms, and 2×2 for shear terms in the energy expression is employed in the evaluation of the element stiffness property. The element mass matrix is evaluated using a 3×3 Gauss quadrature rule. An explicit central difference technique and subspace iteration scheme for the solution of transient dynamics and free vibration, respectively, are employed with a special mass matrix diagonalization scheme applicable to quadrilateral isoparametric C° finite elements. A convergence study was carried out with a view to getting reasonably convergent reliable solutions with an optimum number of elements. A 2×2 mesh (4 elements) in a quarter plate and a 4×4 mesh (16 elements) in a full plate discretization were seen to give generally converged displacements, stresses and stress-resultants, and therefore unless otherwise specified, these discretizations were adopted in the present work. After having established an optimum space discretization, the associated critical time step was obtained for the transient dynamic analyses. A quarter plate is used for isotropic, 0° — orthotropic and cross-ply (0°/90°/...) laminates, while a full plate geometric model is used for angle-ply laminates. Further, a full plate discrete model is invariably used in a free vibration analysis for obtaining higher frequencies and modes. In transient response analyses, zero initial conditions on displacements and their time derivatives were assumed for all the cases. All the computations were carried out in single precision on a CDC CYBER 180/840 computer at the Indian Institute of Technology, Bombay, India. The boundary conditions corresponding to different types of edges most commonly occurring in practice, namely, the simple supported, the clamped support and symmetry conditions along the edge are listed in Table 1. The material characteristics of the individual layers are given in Table 2 for different sets of data.

Example 1. Simply supported square plates of multilayered symmetric cross-ply were analysed with the material properties typical of high fibrous composites (DATA-1) given in Table 2. The ratio of E_1/E_2 was varied between 3 and 40, and the number of layers between 3 and 9. Because of the existence of biaxial symmetry in the cross-ply laminates, only a quadrant of the laminate is

Table 1. Details of boundary conditions for laminated plates

Edge	Theory	Simply-supported		Clamped	Symmetry line[a]
		SS1	SS2		
$x=$ constant	FOST5	$v_0=w_0=\theta_y=0$	$u_0=w_0=\theta_y=0$	$u_0=v_0=w_0=0$ $\theta_x=\theta_y=0$	$u_0=\theta_x=0$
	HOST7	$v_0=w_0=0$ $\theta_y=\theta_y^*=0$	$u_0=w_0=0$ $\theta_y=\theta_y^*=0$	$u_0=v_0=w_0=0$ $\theta_x=\theta_y=\theta_x^*=\theta_y^*=0$	$u_0=0$ $\theta_x=\theta_x^*=0$
	HOST9	$v_0=v_0^*=w_0=0$ $\theta_y=\theta_y=0$	$u_0=u_0^*=w_0=0$ $\theta_y=\theta_y^*=0$	$u_0=v_0=u_0^*=v_0^*=0$ $\theta_x=\theta_y=\theta_x^*=\theta_y^*=0$ $w_0=0$	$u_0=u_0^*=0$ $\theta_x=\theta_x^*=0$
	HOST11	$v_0=v_0^*=0$ $w_0=w_0^*=0$ $\theta_y=\theta_y^*=\theta_z=0$	$u_0=u_0^*=0$ $w_0=w_0^*=0$ $\theta_y=\theta_y^*=\theta_z=0$	$u_0=v_0=u^*=v_0^*=0$ $\theta_x=\theta_y=\theta_x^*=\theta_y^*=0$ $w_0=w_0^*=\theta_z=0$	$u_0=u_0^*=0$ $\theta_x=\theta_x^*=0$
$y=$ constant	FOST5	$u_0=w_0=\theta_x=0$	$v_0=w_0=\theta_x=0$	$u_0=v_0=w_0=0$ $\theta_x=\theta_y=0$	$v_0=\theta_y=0$
	HOST7	$u_0=w_0=0$ $\theta_x=\theta_x^*=0$	$v_0=w_0=0$ $\theta_x=\theta_x^*=0$	$u_0=v_0=w_0=0$ $\theta_x=\theta_y=\theta_x^*=\theta_y^*=0$	$v_0=0$ $\theta_y=\theta_y^*=0$
	HOST9	$u_0=u_0^*=w_0=0$ $\theta_x=\theta_x^*=0$	$v_0=v_0^*=w_0=0$ $\theta_x=\theta_x^*=0$	$u_0=v_0=u_0^*=v_0^*=0$ $\theta_x=\theta_y=\theta_x^*=\theta_y^*=0$ $w_0=0$	$v_0=v_0^*=0$ $\theta_y=\theta_y^*=0$
	HOST11	$u_0=u_0^*=0$ $w_0=w_0^*=0$ $\theta_x=\theta_x^*=\theta_z=0$	$v_0=v_0^*=0$ $w_0=w_0^*=0$ $\theta_x=\theta_x^*=0$	$u_0=v_0=u_0^*=v_0^*=0$ $\theta_x=\theta_y=\theta_x^*=\theta_y^*=0$ $w_0=w_0^*=\theta_z=0$	$v_0=v_0^*=0$ $\theta_y=\theta_y^*=0$

[a]Boundary conditions along the symmetry lines are used only for quarter plate analyses. SS1 is used for both quarter and full plate analyses, but SS2 is used only for full plate analysis.

Table 2. Material properties

DATA	Material properties
1	Typical high-modulus graphite/epoxy (dimensionless properties) $E_1/E_2=$ Open; $E_2=E_3=1$; $G_{12}=G_{13}=0{\cdot}6\ E_2$; $G_{23}=0{\cdot}5\ E_2$; $\nu_{12}=\nu_{23}=\nu_{13}=0{\cdot}25$; $\rho=1{\cdot}0$
2	For face sheets, the assumed ply data based on Hercules AS1/3501-6 graphite/epoxy prepreg system $E_1=13{\cdot}08\times10^6$ N/cm^2; $E_2=E_3=1{\cdot}06\times10^6$ N/cm^2; $G_{12}=G_{13}=0{\cdot}6\times10^6$ N/cm^2; $G_{23}=0{\cdot}39\times10^6$ N/cm^2; $\nu_{12}=\nu_{13}=0{\cdot}28$; $\nu_{23}=0{\cdot}34$; $\rho=15{\cdot}8\times10^{-6}$ N-sec^2/cm^4 Core material is of U.S. Commercial aluminium honeycomb (1/4 inch cell size, 0·003 inch foil) $G_{23}=G_{yz}=1{\cdot}772\times10^4$ N/cm^2; $G_{13}=G_{xz}=5{\cdot}206\times10^4$ N/cm^2; $E_3=E_z=3{\cdot}013\times10^5$ N/cm^2; $\rho=0{\cdot}1009\times10^{-5}$ N-sec^2/cm^4
3	For face sheets (typical graphite/epoxy) $E_1=0{\cdot}12\times10^8$ N/cm^2; $E_2=E_3=0{\cdot}79\times10^6$ N/cm^2; $G_{12}=G_{23}=G_{13}=0{\cdot}55\times10^6$ N/cm^2; $\nu_{12}=\nu_{23}=\nu_{13}=0{\cdot}3$; $\rho=1{\cdot}58\times10^{-5}$ N-sec^2/cm^4 For core material (U.S. Commercial aluminium honeycomb, 1/4 inch cell size, 0·007 inch foil) $G_{23}=G_{yz}=0{\cdot}7034\times10^4$ N/cm^2; $G_{13}=G_{xz}=0{\cdot}1407\times10^5$ N/cm^2; $\rho=0{\cdot}3415\times10^{-6}$ N-sec^2/cm^4

modelled for finding fundamental frequencies. The effects of the number of layers and degree of orthotropy of the individual layers on the dimensionless fundamental frequency are presented in Table 3.

The results are compared with Noor's solution[77] of the 3D elasticity theory using higher-order finite difference schemes. Very good agreements are observed between the present refined theories and 3D elasticity theory. The results obtained using classical lamination theory,[97] hybrid stress finite element method[98] and a local finite element model based on a refined approximate theory[97] are also included for comparison. The CPT overestimates the fundamental frequencies, especially when the degree of anisotropy is greater. The fundamental frequencies increase with the increase in degree of orthotropy and also the increase in number of layers.

Example 2. To show the effects of transverse shear rigidities of stiff layers and length/thickness ratio on the natural frequencies, a seven-layer (0°/45°/90°/core/90°/45°/0°) square symmetric composite-sandwich plate is analysed with different boundary conditions: simply-supported (SS2) and clamped. The material properties (DATA-2) given in Table 2 are used.

The results obtained using the present refined theories and the present FOST are presented in Tables 4 and 5 for simply-supported and clamped boundary conditions, respectively. It is seen that for a moderately thick plate ($a/h = 10$) with the transverse shear modulii (G_{23} and G_{13}) of stiff layers included, the difference between the predictions of natural frequencies using theories HOST and FOST increases with increasing mode numbers. The FOST estimates higher frequencies. This is due to simplifying assumptions made in FOST. It is concluded that the effect of transverse shear modulii of stiff layers is more pronounced in thicker laminates (low a/h ratio) than for thin laminates (high a/h ratio). The effect of boundary condition can be seen from the Tables 4 and 5. The frequencies for clamped laminate are always higher than that of simply-supported laminate.

Example 3. Simply supported square orthotropic laminates having skew-symmetric laminations with respect to the middle plane are considered.

Table 3. Effect of number of layers and degree of orthotropy of individual layers on the fundamental frequency of simply-supported square multilayered symmetric composite plates (DATA-1 with varying E_1/E_2, 0°/90°/0°/.../0°, 2 × 2 mesh, quarter plate, $a/h = 5$, $\bar{\omega} = \omega(\rho h^2/E_2)^{1/2} \times 10$)

No. of layers	Source	E_1/E_2 = 3	10	20	30	40
3	Noor	2·6474	3·2841	3·8341	4·1089	4·3006
	HOST5	2·6260(−0·8)	3·2672(−0·5)	3·7801(−1·4)	4·0300(−1·9)	4·1998(−2·3)
	HOST6	2·6126(−1·3)	3·2528(−0·9)	3·7253(−2·5)	3·9884(−2·9)	4·1521(−3·4)
	FOST3	2·6124(−1·3)	3·2519(−0·9)	3·7221(−2·6)	3·9721(−3·3)	4·1501(−3·5)
	Owen & Li	2·6948(+1·8)	3·3917(+3·2)	3·8979(+1·9)	4·1941(+2·1)	4·3951(+2·2)
	Sun & Liou	2·6524(+0·2)	3·3364(+1·6)	3·8289(−0·1)	4·1142(+0·1)	4·3062(+0·1)
	CPT	2·9198(+10)	4·1264(+25)	5·4043(+41)	6·4336(+56)	7·3196(+70)
5	Noor	2·6587	3·4089	3·9792	4·3140	4·5374
	HOST5	2·6389(−0·7)	3·3766(−0·9)	3·9337(−1·1)	4·2622(−1·2)	4·4831(−1·1)
	HOST6	2·6255(−1·2)	3·3621(−1·3)	3·9192(−1·5)	4·2482(−1·5)	4·4695(−1·5)
	FOST3	2·6255(−1·2)	3·3622(−1·3)	3·9190(−1·5)	4·2456(−1·6)	4·4628(−1·6)
	Owen & Li	2·6988(+1·5)	3·4534(+1·3)	4·0297(+1·3)	4·3704(+1·3)	4·5992(+1·4)
	Sun & Liou	2·6608(0·08)	3·4103(0·04)	3·9803(0·03)	4·3149(0·02)	4·5380(0·01)
	CPT	2·9198(+9·8)	4·1264(+21)	5·4043(+36)	6·4336(+51)	7·3196(+61)
9	Noor	2·6640	3·4432	4·0547	4·4210	4·6679
	HOST5	2·6433(−0·7)	3·4184(−0·7)	4·0259(−0·7)	4·3904(−0·7)	4·6367(−0·6)
	HOST6	2·6298(−1·3)	3·4035(−1·1)	4·0107(−1·1)	4·3755(−1·0)	4·6222(−0·9)
	FOST3	2·6297(−1·3)	3·4035(−1·1)	4·0107(−1·1)	4·3756(−1·0)	4·6225(−0·9)
	Owen & Li	2·6971(+1·2)	3·4708(+0·8)	4·0746(+0·5)	4·4360(+0·5)	4·6803(+0·3)
	CPT	2·9198(+9·6)	4·1264(+20)	5·4043(+33)	6·4336(+45)	7·3196(+57)

Values in brackets give percentage errors with respect to 3D-elasticity solution.

Table 4. Effect of shear rigidity of stiff layers and length-to-thickness ratio on the natural frequencies ($\omega/2\pi$ cycles/sec) of seven-layer (0°/45°/90°/core/90°/45°/0°) simply-supported square symmetric composite-sandwich plates (DATA-2, 4 × 4 mesh, full plate, $a = b = 100$ cm, $h_0 = h_{45} = h_{90} = 0{\cdot}05\ h$, $h_{core} = 0{\cdot}7\ h$)

Modal No.	Considering G_{23} and G_{13} of stiff layers						Neglecting G_{23} and G_{13} of stiff layers					
	$a/h = 10$			$a/h = 100$			$a/h = 10$			$a/h = 100$		
	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3
1	473	473	593	70	70	70	333	334	356	69	69	69
2	775	774	1203	166	166	168	517	518	524	161	161	161
3	1004	1003	1331	194	194	196	616	617	707	188	188	189
4	1096	1097	1363	267	268	271	718	720	713	256	257	257
5	1173	1173	1719	344	345	358	729	731	802	322	323	321
6	1320	1321	2005	400	400	407	862	865	820	377	375	383
7	1376	1376	2172	408	409	421	870	872	827	382	383	385
8	1436	1476	2180	478	479	492	884	885	907	442	444	450

Table 5. Effect of shear rigidity of stiff layers and length-to-thickness ratio on the natural frequencies ($\omega/2\pi$ cycles/sec) of seven-layer (0°/45°/90°/core/90°/45°/0°) clamped square symmetric composite-sandwich plates (DATA-2, 4 × 4 mesh, full plate, $a=b=100$ cm, $h_0=h_{45}=h_{90}=0{\cdot}05\ h$, $h_{core}=0{\cdot}7\ h$)

Modal No.	Considering G_{23} and G_{13} of stiff layers						Neglecting G_{23} and G_{13} of stiff layers					
	$a/h=10$			$a/h=100$			$a/h=10$			$a/h=100$		
	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3
1	605	606	866	133	134	135	384	385	407	128	128	128
2	854	856	1399	250	251	259	545	547	546	232	233	232
3	1080	1081	1512	290	291	297	661	663	717	270	271	275
4	1173	1176	1877	373	375	388	740	741	728	342	344	346
5	1244	1245	2128	475	477	514	764	766	815	415	416	410
6	1377	1383	2180	550	552	584	875	876	820	483	485	484
7	1450	1455	2285	564	566	587	893	896	831	499	501	519
8	1549	1556	2408	644	647	683	940	938	916	559	560	558

The fiber orientations of the different laminae alternate between 0° and 90° with respect to the x-axis. The material characteristics of the individual layers are taken to be those of high fibrous composites (DATA-1) which are given in Table 2. Two parameters were varied, namely the degree of orthotropy of the individual layers (E_1/E_2), and the length-to-thickness ratio of the laminate. The ratio E_1/E_2 was varied between 3 and 40, and number of layers varied between 2 and 10.

In Table 6, the fundamental frequencies, obtained by the present theories, 3D elasticity theory[77] using higher-order finite difference schemes, a 3D eight-node hybrid stress finite element solution,[98] a local finite element model based on a refined approximate theory,[97] a mixed FEM based on a higher-order theory[95] and a classical thin plate theory,[97] are presented. It is found that for skew-symmetric laminates, as the number of layers increases from 2 to 4, the accuracy of the CPT sharply deteriorates. Further increase of the number of layers does not have a significant effect on the accuracy. On the other hand, for symmetrically laminated plates (see Example 1), the error decreases as the number of layers increases. The error in the CPT predictions is mainly attributed to the neglect of shear deformation. When the results of present theories are compared with the 3D elasticity solution,[77] the agreement is seen to be excellent. The error in the predictions of HOST7, HOST9 and HOST11 did not exceed 2·59%, 1·3% and 1·63% respectively, even for the case of a highly orthotropic thick laminate with $E_1/E_2=40$. The corresponding error estimate for the present FOST5 and CFS of a higher order theory[95] is seen to be 5·1% and 6·12% respectively, whereas for small degrees of orthotropy ($E_1/E_2=3$–10), error is almost negligible. From Table 6, it is concluded that the results reaffirm the fact that the effect of coupling between bending and stretching and orthotropy cannot be ignored even at low modulus ratio. The fundamental frequency increases with the increase in degree of orthotropy and/or increase in number of layers.

Example 4. An eight-layer (0°/45°/90°/core/90°/45°/30°/0°) unsymmetric square composite-sandwich plate is analysed for two different boundary conditions: simply-supported (SS2) and clamped. The elastic material properties (DATA-2) given in Table 2 are used. The natural frequencies obtained using the present refined theories (HOST) and the present FOST are presented in Tables 7 and 8 for simply-supported and clamped boundary conditions, respectively. A comparison of the effects of transverse shear rigidity of stiff layers and length/thickness ratio on the natural frequencies of unsymmetric laminates is made.

It is seen from Tables 7 and 8 that the effect of transverse shear modulii (G_{23} and G_{13}) of stiff layers is more pronounced in thicker laminates (low a/h ratio) than for thin laminates (high a/h ratio). For a moderately thick plate, the difference in the predictions of FOST5 with HOST9 and HOST11 is more than with HOST7. This discrepancy is due to simplifying assumptions made in FOST, whereas the present refined theories represent the realistic cross-sectional deformation. In HOST7 the higher-order in-plane degrees-of-freedom (u_0^*, v_0^*) are neglected, the effect of which is seen in Tables 7 and 8. The difference in the results between refined theories and FOST increases with increasing mode

Table 6. Effect of number of layers and degree of orthotropy of individual layers on the fundamental frequency of simply-supported (SS1) square mutlilayered skew-symmetric composite plates (DATA-1 with varying E_1/E_2, 0°/90°/.../90°, 2 × 2 mesh, quarter plate, $a/h = 5$, $\bar{\omega} = \omega(\rho h^2/E_2)^{1/2} \times 10$)

No. of layers	Source	E_1/E_2				
		3	10	20	30	40
	Noor	2·5031	2·7938	3·0698	3·2705	3·4250
	HOST11	2·4782(−0·99)	2·7764(−0·62)	3·0737(+0·12)	3·3003(+0·91)	3·4810(+1·63)
	HOST9	2·4909(−0·48)	2·7905(−0·11)	3·0702(+0·01)	3·2979(+0·83)	3·4698(+1·30)
	HOST7	2·4909(−0·48)	2·7981(+0·15)	3·1252(+1·80)	3·3414(+2·16)	3·5138(+2·59)
2	FOST5	2·4829(−0·80)	2·7751(−0·67)	3·0998(+0·98)	3·3771(+3·26)	3·5995(+5·10)
	Putcha & Reddy	2·4868(−0·65)	2·7955(+0·06)	3·1284(+1·91)	3·4020(+4·02)	3·6348(+6·12)
	Owen & Li	2·5601(+2·27)	2·8712(+2·77)	3·1558(+2·80)	3·3610(+2·76)	3·5185(+2·73)
	Sun & Liou	2·5148(+0·47)	2·8030(+0·33)	3·0768(+0·23)	3·2763(+0·18)	3·4301(+0·15)
	CPT	2·7082(+8·19)	3·0968(+10·8)	3·5422(+15·3)	3·9335(+20·2)	4·2884(+25·2)
	Noor	2·6182	3·2578	3·7622	4·0660	4·2719
	HOST11	2·5997(−0·70)	3·2486(−0·28)	3·7801(+0·47)	4·1041(+0·93)	4·3240(+1·21)
	HOST9	2·6037(−0·55)	3·2621(+0·13)	3·7835(+0·56)	4·0923(+0·64)	4·3069(+0·81)
	HOST7	2·6055(−0·48)	3·2870(+0·89)	3·8014(+1·04)	4·1247(+1·44)	4·3786(+2·49)
4	FOST5	2·6012(−0·65)	3·2889(+0·95)	3·8741(+2·97)	4·2462(+4·43)	4·5062(+5·48)
	Putcha & Reddy	2·6003(−0·68)	3·2782(+0·62)	3·8506(+2·35)	4·2139(+3·64)	4·4686(+4·60)
	Owen & Li	2·6691(+1·94)	3·3250(+2·06)	3·8454(+2·21)	4·1612(+2·34)	4·3763(+2·44)
	Sun & Liou	2·6219(+0·14)	3·2621(+0·13)	3·7675(+0·14)	4·0719(+0·14)	4·2780(+0·14)
	CPT	2·8676(+9·52)	3·8877(+19·3)	4·9907(+32·6)	5·8900(+44·8)	6·6690(+56·1)
	Noor	2·6440	3·3657	3·9359	4·2783	4·5091
	HOST11	2·6194(−0·93)	3·3423(−0·69)	3·9249(−0·27)	4·2766(−0·04)	4·5141(+0·11)
	HOST9	2·6243(−0·74)	3·3545(−0·33)	3·9373(+0·03)	4·2890(+0·25)	4·5262(+0·37)
	HOST7	2·6275(−0·62)	3·3712(+0·16)	3·9784(+1·07)	4·3526(+1·73)	4·6090(+2·21)
6	FOST5	2·6222(−0·82)	3·3664(+0·02)	3·9756(+1·00)	4·3512(+1·70)	4·6083(+2·19)
	Putcha & Reddy	2·6223(−0·82)	3·3621(−0·11)	3·9672(+0·79)	4·3419(+1·48)	4·6005(+2·02)
	Owen & Li	2·6839(+1·50)	3·4085(+1·27)	3·9758(+1·01)	4·3233(+1·05)	4·5558(+1·03)
	Sun & Liou	2·6458(+0·06)	3·3666(+0·02)	3·9359(+0·00)	4·2775(−0·02)	4·5077(−0·01)
	CPT	2·8966(+9·55)	4·0215(+19·5)	5·2234(+32·7)	6·1963(+44·8)	7·0359(+56·0)
	Noor	2·6583	3·4250	4·0337	4·4011	4·6498
	HOST11	2·6331(−0·94)	3·3989(−0·76)	4·0069(−0·66)	4·3780(−0·52)	4·6295(−0·43)
	HOST9	2·6385(−0·74)	3·4083(−0·48)	4·0221(−0·28)	4·3929(−0·18)	4·6441(−0·12)
	HOST7	2·6389(−0·72)	3·4142(−0·31)	4·0377(+0·09)	4·4178(+0·37)	4·6771(+0·58)
10	FOST5	2·6329(−0·96)	3·4043(−0·60)	4·0239(−0·24)	4·4003(−0·02)	4·6554(+0·12)
	Putcha & Reddy	2·6337(−0·92)	3·4050(−0·58)	4·0270(−0·16)	4·4079(+0·15)	4·6692(+0·41)
	Owen & Li	2·6916(+1·25)	3·4527(+0·80)	4·0526(+0·47)	4·4140(+0·29)	4·6590(+0·19)
	Sun & Liou	—	—	—	—	—
	CPT	2·9115(+9·52)	4·0888(+19·4)	5·3397(+32·4)	6·3489(+44·2)	7·2184(+55·2)

Values in brackets give percentage errors with respect to the 3D-elasticity solution.

numbers. The effect of boundary conditions on the frequencies can be seen in Tables 7 and 8. The frequencies of clamped laminates are higher than those of simply-supported laminates, which is obvious.

Example 5. An anisotropic laminated composite-sandwich plate (0°/30°/45°/60°/core/60°/45°/30°/0°) clamped on all the four sides is analysed for suddenly applied uniformly distributed pulse loading. The length-to-thickness ratio $a/h = 10$ (moderately thick plate) and $a/h = 50$ (reasonably thin plate) are considered. A full plate is discretized with 4 × 4 mesh. The elastic material properties (DATA-3) given in Table 2 are used. A comparison of the results obtained by the refined theories (HOST) with those of FOST results is made in Tables 9–12. The static results of FOST3 and HOST5 are also included in these tables. The main purpose of tabulating these results is to provide an easy means for future comparison by other investigators.

The variation of in-plane displacements (u and v) and center transverse deflection (w_0) with respect to time for $a/h = 10$ and 50 is shown in

Table 7. Effect of shear rigidity of stiff layers and length-to-thickness ratio on the natural frequencies ($\omega/2\pi$ cycles/sec) of eight-layer (0°/45°/90°/core/90°/45°/30°/0°) simply-supported (SS2) square unsymmetric composite-sandwich plates
(DATA-2, 4 × 4 mesh, full plate, $a = b = 100$ cm, thickness of each top stiff layer = 0·025 h, thickness of each bottom stiff layer = 0·08125 h, thickness of core = 0·6 h)

Modal No.	Considering G_{23} and G_{13} of stiff layers								Neglecting G_{23} and G_{13} of stiff layers							
	$a/h = 10$				$a/h = 100$				$a/h = 10$				$a/h = 100$			
	HOST11	HOST9	HOST7	FOST5	HOST11	HOST9	HOST7	FOST5	HOST11	HOST9	HOST7	FOST5	HOST11	HOST9	HOST7	FOST5
1	464	464	485	516	59	59	59	59	281	280	305	297	57	57	58	58
2	853	853	926	1013	127	127	127	127	431	430	452	430	120	120	123	123
3	943	934	1063	1154	154	154	154	154	530	528	580	579	142	141	150	150
4	956	941	1355	1501	211	210	210	211	582	581	619	582	192	191	202	201
5	1002	1000	1531	1773	264	263	265	265	603	602	673	656	236	235	246	243
6	1201	1188	1747	1993	321	320	321	322	628	624	731	673	279	278	299	297
7	1226	1224	1781	2042	326	325	326	327	638	636	737	678	282	280	309	309
8	1245	1246	1791	2173	387	386	387	389	665	659	780	744	327	325	359	357

Table 8. Effect of shear rigidity of stiff layers and length-to-thickness ratio on the natural frequencies ($\omega/2\pi$ cycles/sec) of eight-layer (0°/45°/90°/core/90°/45°/30°/0°) clamped square unsymmetric composite-sandwich plates
(DATA-2, 4 × 4 mesh, full plate, $a = b = 100$ cm, thickness of each top stiff layer = 0·025 h, thickness of each bottom stiff layer = 0·08125 h, thickness of core = 0·6 h)

Modal No.	Considering G_{23} and G_{13} of stiff layers								Neglecting G_{23} and G_{13} of stiff layers							
	$a/h = 10$				$a/h = 100$				$a/h = 10$				$a/h = 100$			
	HOST11	HOST9	HOST7	FOST5	HOST11	HOST9	HOST7	FOST5	HOST11	HOST9	HOST7	FOST5	HOST11	HOST9	HOST7	FOST5
1	641	639	686	754	103	102	102	102	321	319	341	332	94	93	98	98
2	995	988	1093	1244	192	191	192	192	456	455	470	446	168	167	177	176
3	997	994	1238	1382	231	230	231	231	580	578	607	586	194	193	216	216
4	1053	1043	1508	1706	295	293	295	296	597	596	628	595	245	244	269	268
5	1161	1160	1664	1961	374	371	375	378	621	620	691	666	302	300	320	314
6	1385	1396	1825	2150	440	437	440	444	641	638	735	674	346	344	380	374
7	1399	1410	1916	2173	459	456	459	462	673	671	737	680	375	345	411	411
8	1429	1432	1921	2222	525	522	526	531	678	673	792	750	396	394	445	432

Tables 9 and 10 respectively. The in-plane displacements in refined theories (HOST) are about 50% larger compared to those of FOST for moderately thick laminate ($a/h=10$), and for thin laminate ($a/h=50$) it is about 12% larger, whereas the transverse displacements in HOSTs are about 70% and 24% larger compared to that in FOST for $a/h=10$ and 50 respectively.

The variation of center normal stresses and corner in-plane shear stress with time for $a/h=10$ and 50 is given in Tables 11 and 12 respectively. The maximum central normal stress obtained by HOSTs is about 40% larger compared to that of FOST for $a/h=10$, and it is about 12% for $a/h=50$. The in-plane shear stress obtained by HOSTs is about 50% larger compared to that of

Table 9. Comparison of in-plane displacements ($u^{top}\times 10^4$ cm at $x=21{\cdot}875$ cm and $y=12{\cdot}5$ cm, $v^{top}\times 10^4$ cm at $x=12{\cdot}5$ cm and $y=21{\cdot}875$ cm) and center transverse displacements ($w_0\times 10^3$ cm) of a clamped, nine-layer (0°/30°/45°/60°/core/60°/45°/30°/0°) composite-sandwich plate under suddenly applied uniformly distributed pulse loading (DATA-3, 4×4 mesh, full plate, $a=b=25$ cm, $a/h=10$, $\Delta t=1{\cdot}0$ μsec, $q_0=1$ N/cm², $h_0=h_{30}=h_{45}=h_{60}=0{\cdot}0625$ cm, $h_{core}=2{\cdot}0$ cm)

Time (μsec)	In-plane displacement, $u\times 10^4$			In-plane displacement, $v\times 10^4$			Transverse displacement, $w_0\times 10^3$		
	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6
40	0·0626	0·0500	0·0484	0·1094	0·0937	0·0887	0·0934	0·1046	0·1037
80	0·2451	0·1587	0·1564	0·3996	0·3145	0·3110	0·4278	0·4158	0·4152
120	0·4502	0·3229	0·3209	0·6286	0·6352	0·6218	0·7970	0·9390	0·9380
160	0·4791	0·5090	0·5083	0·6850	0·9821	0·9676	0·8104	1·6410	1·6363
200	0·3336	0·7011	0·6951	0·5391	1·2376	1·2202	0·6022	2·3691	2·3606
240	0·1534	0·8372	0·8283	0·1975	1·3695	1·3456	0·2658	2·8936	2·8857
280	0·0069	0·9074	0·8940	0·0270	1·3616	1·3393	−0·0247	3·0243	3·0205
320	0·0038	0·8519	0·8452	0·0372	1·2766	1·2564	0·0253	2·7628	2·7508
360	0·1627	0·6916	0·6859	0·2416	1·1524	1·1370	0·2727	2·2270	2·2050
400	0·3761	0·4719	0·4624	0·5508	0·9483	0·9246	0·6479	1·5665	1·5515
440	0·4858	0·2716	0·2638	0·7117	0·6332	0·6119	0·8563	0·9074	0·9070
480	0·4119	0·1187	0·1144	0·5953	0·3184	0·3094	0·7058	0·3541	0·3526
520	0·2371	0·0335	0·0382	0·3608	0·0456	0·0390	0·4225	−0·0004	−0·0089
560	0·0555	0·0240	0·0263	0·1092	−0·0259	−0·0350	0·0874	−0·08867	−0·0924
600	−0·0138	0·0726	0·0686	−0·0328	0·0800	0·0944	−0·0511	0·1274	0·1323
640	0·0664	0·1771	0·1790	0·1589	0·3867	0·3799	0·1508	0·5915	0·6038
680	0·2989	0·3491	0·3502	0·4227	0·7269	0·7064	0·4852	1·1845	1·1847
Static	0·2319	0·4293	—	0·3470	0·7466	—	0·4006	1·3889	—

Table 10. Comparison of in-plane displacements ($u^{top}\times 10^4$ cm at $x=21{\cdot}875$ cm and $y=12{\cdot}5$ cm, $v^{top}\times 10^4$ cm at $x=12{\cdot}5$ cm and $y=21{\cdot}875$ cm) and center transverse displacements ($w_0\times 10^3$ cm) of a clamped, nine-layer (0°/30°/45°/60°/core/60°/45°/30°/0°) composite-sandwich plate under suddenly applied uniformly distributed pulse loading (DATA-3, 4×4 mesh, full plate, $a=b=25$ cm, $a/h=50$, $\Delta t=0{\cdot}5$ μsec, $q_0=1$ N/cm², $h_0=h_{30}=h_{45}=h_{60}=0{\cdot}0125$ cm, $h_{core}=0{\cdot}4$ cm)

Time (μsec)	In-plane displacement, $u\times 10^4$			In-plane displacement, $v\times 10^4$			Transverse displacement, $w_0\times 10^3$		
	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6
80	0·9266	0·7509	0·7469	1·1267	0·8759	0·8673	1·5314	2·1392	2·1379
160	2·7955	2·4368	2·4255	4·1145	3·2871	3·2612	9·2532	8·0655	8·0586
240	5·4656	4·6502	4·6309	6·4896	6·5753	6·5250	22·397	21·407	21·406
320	8·6644	7·7323	7·6993	8·3435	8·9394	8·8624	34·646	37·566	37·547
400	10·132	10·705	10·647	10·932	10·577	10·487	43·617	50·339	50·256
480	10·906	12·534	12·455	12·233	12·296	12·193	46·183	57·967	57·813
560	10·990	12·300	12·196	11·203	13·733	13·606	42·433	60·547	60·292
640	8·4234	11·184	11·077	9·3373	13·314	13·159	35·161	54·825	54·506
720	5·3966	9·9867	9·8870	6·2327	10·782	10·619	23·542	44·791	44·449
800	3·2790	7·4292	7·3164	3·1421	7·5245	7·3841	11·143	32·956	32·582
880	0·9129	4·0222	3·9323	1·6724	4·6662	4·5678	2·6100	20·364	20·023
960	0·1013	1·6976	1·6388	1·1519	2·4086	2·3473	−2·5596	8·2169	7·9374
1040	0·7382	0·4594	0·4251	1·1877	0·8293	0·7938	0·9209	−1·7155	−1·8653
1120	2·4923	−0·0726	−0·0498	2·6283	0·6960	0·6941	11·325	−2·4563	−2·3686
1200	5·6634	0·9318	0·9694	5·7087	1·9628	1·9856	21·065	3·5303	3·7405
Static	5·6275	6·2546	—	6·2417	7·0081	—	22·464	28·936	—

Table 11. Comparison of center normal stresses (σ_x^{top} and σ_y^{top} in N/cm^2) and in-plane shear stresses (τ_{xy}^{top} in N/cm^{top} of a clamped, nine-layer (0°/30°/45°/60°/core/60°/45°/30°/0°) composite-sandwich plate under suddenly applied uniformly distributed pulse loading
(DATA-3, 4×4 mesh, full plate, $a=b=25$ cm, $a/h=10$, $\Delta t=1{\cdot}0$ μsec, $q_0=1$ N/cm^2, $h_0=h_{30}=h_{45}=h_{60}=0{\cdot}0625$ cm, $h_{core}=2{\cdot}0$ cm)

Time (μsec)	Bending stress σ_x			Bending stress σ_y			Shear stress τ_{xy}		
	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6
40	1·9236	4·0605	5·3069	−0·0476	0·6385	1·6677	−0·5358	−0·8014	−0·7834
80	65·836	21·526	22·775	5·3596	2·2350	3·6950	−1·3958	−2·3640	−2·3611
120	148·55	54·841	57·173	17·108	4·6381	5·2782	−2·2142	−3·5661	−3·5336
160	140·81	130·07	133·36	16·103	11·339	12·686	−2·6347	−4·0055	−3·9217
200	104·78	207·64	207·59	8·8519	19·057	21·014	−1·7777	−4·1059	−4·0360
240	36·692	249·67	249·49	5·1846	27·238	28·899	−0·7928	−4·3655	−4·3489
280	−16·263	245·07	248·31	−2·1548	29·358	32·290	−0·3114	−4·9960	−4·9997
320	7·4574	212·49	218·32	−1·4023	26·353	28·508	−0·0866	−5·2157	−5·1711
360	20·940	177·86	180·34	3·1356	18·282	19·578	−0·9115	−4·5711	−4·4927
400	122·06	132·90	130·79	12·922	10·384	11·696	−2·0724	−3·4561	−3·4092
440	164·21	80·666	81·140	16·751	4·7036	5·1654	−2·4718	−2·5517	−2·5460
480	102·83	20·261	23·308	12·365	1·2816	1·5971	−2·2054	−1·8727	−1·8907
520	77·434	−19·594	−21·286	7·3379	−1·0825	0·1142	−1·2958	−1·3733	−1·3162
560	17·677	−27·577	−28·916	0·7743	−0·8900	−0·8138	−0·4365	−0·8591	−0·8063
600	−37·018	3·5468	4·4951	−2·3994	1·1277	1·3388	−0·0747	−0·8079	−0·7667
640	22·641	50·359	54·425	−0·0631	3·9069	5·9228	−0·4440	−1·6855	−1·6901
680	88·304	104·86	106·04	9·3167	8·7733	9·6446	−1·6735	−3·2167	−3·2045
Static	65·58	109·60	—	6·769	11·24	—	−1·286	−3·130	—

Table 12. Comparison of center normal stresses (σ_x^{top} and σ_y^{top} in N/cm^2) and in-plane shear stresses (τ_{xy}^{top} in N/cm^{top} of a clamped, nine-layer (0°/30°/45°/60°/core/60°/45°/30°/0°) composite-sandwich plate under suddenly applied uniformly distributed pulse loading
(DATA-3, 4×4 mesh, full plate, $a=b=25$ cm, $a/h=50$, $\Delta t=0{\cdot}5$ μsec, $q_0=1$ N/cm^2, $h_0=h_{30}=h_{45}=h_{60}=0{\cdot}0125$ cm, $h_{core}=0{\cdot}4$ cm)

Time (μsec)	Bending stress σ_x			Bending stress σ_y			Shear stress τ_{xy}		
	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6	FOST3	HOST5	HOST6
80	−156·06	−29·528	−29·026	−13·868	1·2555	2·0682	−9·7087	−8·0415	−7·9791
160	486·77	111·16	111·26	1·4424	−7·9831	−7·9845	−16·469	−19·005	−18·836
240	1796·5	1276·2	1276·9	112·07	49·284	49·585	−21·244	−25·492	−25·234
320	2663·6	2456·8	2460·2	249·34	181·59	184·76	−39·140	−34·807	−34·452
400	3786·8	3297·5	3295·8	297·86	298·48	300·85	−49·506	−48·820	−48·333
480	3816·3	3775·0	3769·0	276·69	351·66	355·00	−46·901	−58·642	−58·036
560	2991·7	4268·1	4256·0	257·97	333·42	335·21	−44·679	−60·287	−59·556
640	2827·3	3777·5	3758·1	215·91	257·96	258·65	−35·255	−54·727	−53·920
720	2080·4	2554·2	2531·1	147·65	210·76	212·38	−31·047	−44·600	−43·842
800	654·22	1851·3	1830·9	67·442	169·00	167·20	−18·812	−32·648	−32·039
800	51·075	1520·0	1503·2	−19·063	109·40	110·39	−1·7287	−22·319	−21·890
960	−54·437	619·88	599·06	−84·155	40·529	39·609	−4·2819	−15·680	−15·356
1040	−186·29	−511·43	−524·92	−28·272	−48·132	−50·059	−10·761	−9·8985	−9·6544
1120	1016·9	−511·55	−508·48	90·696	−66·862	−66·040	−11·740	−3·0369	−2·9355
1200	1390·8	−87·275	−70·608	119·60	−35·858	−36·030	−25·815	−1·9424	−2·0707
Static	1751·0	1807·0	—	128·6	139·7	—	−26·43	−31·46	—

FOST for $a/h=10$ and it is about 18% for $a/h=50$.

As expected, the maximum displacement for a constant force applied suddenly is twice the displacement caused by the same force applied statistically (slowly). Unlike in isotropic plates, the internal forces and stresses for dynamic load are about 2·2–2·5 times the forces and stresses obtained by static load. The refined theories, for a composite-sandwich laminate with high-strength facing layers and a soft core layer, have thus shown that the dynamic loads induce considerable warping of the transverse cross-section. The FOST predicts significantly lower values of deflection, period and stresses. As a/h ratio increases, the difference between the results of HOSTs and FOST decreases.

CONCLUSIONS

The conclusions were drawn based on the available literature and the results of the present refined theories. The general remarks from the current literature survey are as follows:

(i) Investigation of transient response of anisotropic composite and sandwich laminates under time-dependent boundary conditions and/or dynamic external loadings is scarce.

(ii) In most of the practical situations, composite laminate problems involved complicated geometry, boundary conditions and loadings, but closed-form analytical solutions exist for simple geometry, loading and boundary conditions. This has led to adoption of some approximate numerical solution techniques. Of such techniques available, it is seen that the finite element method is not only simple but straightforward for efficient programming and also versatile enough to cover all types of problems relevant to situations in practice.

(iii) The 3D analysis using a numerical technique is not cost effective as it needs large computer core storage and time in comparison with 2D analysis. Thus, it was recognized that efforts ought to be made to restrict to 2D analysis without much loss of accuracy of results in comparison with 3D analysis.

(iv) In most of the 2D theories developed to predict the transient dynamic analysis of laminated plates, the assumed cross-sectional deformations of a laminate appear to be reasonably unrealistic. Thus, there is a need to consider the accurate and realistic deformation of the transverse cross-section and stress variation across the thickness to predict the behavior of composites and sandwiches.

(v) Among finite element formulations, the hybrid-stress approach gave realistic and accurate predictions of transverse cross-sectional deformations and stress variation across the thickness but it involved a large number of unknown variables in comparison with the conventional displacement approach.

(vi) There is a need to consider the refined higher-order displacement models for free vibration and transient dynamics of anisotropic composite and sandwich laminates and adopt a simple C° isoparametric approach for the finite element formulation. Such an approach is generally attractive due to ease of software development and implementation in an existing general purpose program.

The present refined theories[99] and their applications to numerous composite and sandwich laminates have led to the following general conclusions:

(i) The evaluations using higher-order theories show considerable warping of the transverse cross-section for composite-sandwich laminates. This true behavior is not possible to model with a first-order shear deformation theory. For a sandwich type of laminate with high-strength facing layers and a soft core layer, it is shown that the dynamic loads induce considerable warping across the thickness.

(ii) The arbitrary shear correction coefficients for fiber reinforced composite and sandwich laminates used in the FOST are not justified. These coefficients depend on the lamina material properties, stacking sequence and are thus problem dependent. The present refined theories account for parabolic variation of the transverse shear stresses through the plate thickness and thus require no such shear correction coefficients.

(iii) The effect of including higher-order degrees of freedom (θ_z, w_0^*) in the transverse direction is not appreciable, but the inclusion of the higher-order in-plane degrees of freedom (u_0^*, v_0^*) does improve the response of unsymmetrical laminates. It is thus concluded that the HOST9 is the optimal one out of the three theories considered for the analysis of unsymmetric laminates, while for symmetric laminates, it turns out to be HOST5.

(iv) The FOST predicts significantly lower values of deflection and period for composite-sandwich laminates. The prevalent sandwich plate theories, which account only for bending rigidities of the facings and shear rigidities of the core material, predict lower values of frequencies and overestimates deflections. Considerable deviation in the results is noted if the shear rigidities for the facings is also accounted for in addition to the bending rigidities.

(v) The variety of case studies performed[99] aid in identifying the effect of time step, finite element mesh, aspect ratio, side to thickness ratio, transverse shear rigidity of facings and core layers, lamination scheme, number of layers through the thickness, material anisotropy on the dynamic response of the laminated composite and sandwich plates.

(vi) Finally, it is believed that the refined theories presented here are definitely an improvement over the FOST and are essential for the anisotropic composite and sandwich laminates in which the elastic properties vary drastically from layer to layer. Further their integration with the simple C° finite element formulation has enhanced the practical applicability of such a theoretical development.

An extension of the present work to reduce the higher-order degrees of freedom, and express them in terms of primary degrees of freedom u, v, w, θ_x, θ_y, θ_z for making it more compatible with the general purpose program (GPP) is currently under investigation at the Department of Civil Engineering, Indian Institute of Technology, Bombay, India.

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REFERENCES

1. Mindlin, R. D., Influence of rotary inertia and shear deformation on flexural motions of isotropic elastic plates. *ASME, J. Appl. Mech.*, **18** (1951) 31–8.
2. Plantema, F. J., *Sandwich Construction: The Bending and Buckling of Sandwich Beams, Plates and Shells.* John Wiley & Sons, Inc., New York, 1966.
3. Allen, H. G., *Analysis and Design of Structural Sandwich Panels.* Pergamon Press, London, 1969.
4. Habip, L. M., A Survey of modern developments in analysis of sandwich structures. *Appl. Mech. Rev.*, **18** (1965) 93–8.
5. Habip, L. M., A review of recent work on multilayered structures. *Int. J. Mech. Sci.*, **7** (1965) 389–93.
6. Reissner, E., On bending of elastic plates. *Q. Appl. Math.*, **5** (1947) 55–68.
7. Reissner, E., Finite deflections of sandwich plates. *J. Aerospace Sci.*, **15** (1948) 435–40.
8. Reissner, E., Finite deflections of sandwich plates. *J. Aerospace Sci.*, **17** (1950) 125.
9. Yu, Y. Y., A new theory of elastic sandwich plates — one-dimensional case. *ASME, J. Appl. Mech.*, **26** (1959) 415–21.
10. Yu, Y. Y., Flexural vibrations of elastic sandwich plates. *J. Aerospace Sci.*, **27** (1960) 272–82.
11. Yu, Y. Y., Simplified vibration analysis of elastic sandwich plates. *J. Aerospace Sci.*, **27** (1960) 894–900.
12. Yu, Y. Y., Damping of flexural vibrations of sandwich plates. *J. Aerospace Sci.*, **29** (1962) 790–803.
13. Yu, Y. Y., Nonlinear flexural vibrations of sandwich plates. *J. Acoust. Soc. Am.*, **34** (1962) 1176–83.
14. Chu, H. N., High-frequency extensional vibrations of sandwich plates. *J. Acoust. Soc. Am.*, **34** (1962) 1184–90.
15. Pagano, N. J., Exact solutions for rectangular composites and sandwich plates. *J. Compos. Mater.*, **4** (1970) 20–34.
16. Monforton, G. R. & Schmit, L. A., Finite element analysis of sandwich plates and cylindrical shells with laminated faces. *Proceedings of the Second Conference on Matrix Methods in Structural Mechanics*, Wright-Patterson Air Force Base, Ohio, 573–616 (1968).
17. Ahmed, K. M., Free vibration of curved sandwich beams by the method of finite elements. *J. Sound Vib.*, **18** (1971) 61–74.
18. Ahmed, K. M., Static and dynamic analysis of sandwich structures by the finite element method. *J. Sound Vib.*, **18** (1971) 75–91.
19. Chan, H. C. & Cheung, Y. K., Static and dynamic analysis of multilayered sandwich plates. *Int. J. Mech. Sci.*, **14** (1972) 399–406.
20. Khatua, T. P. & Cheung, Y. K., Bending and vibration of multilayer sandwich beams and plates. *Int. J. Numer. Meth. Engng.*, **6** (1973) 11–24.
21. Lee, P. C. Y. & Chang, N., Harmonic waves in elastic sandwich plates. *J. Elasticity*, **9** (1979) 51–79.
22. Ng, S. S. F. & Lam, D. K. Y., Dynamic and static analysis of skew sandwich plates. *J. Sound Vib.*, **99** (1985) 393–401.
23. Sayir, M. & Koller, M. G., Dynamic behaviour of sandwich plates. *J. Appl. Math. Phys.*, **37** (1986) 76–103.
24. Raville, M. E. & Ueng, C. E. S., Determination of natural frequencies of vibration of a sandwich plate. *Exp. Mech.*, **7** (1967) 490–93.
25. Timoshenko, S. P. & Krieger, S. W., *Theory of Plates and Shells.* McGraw-Hill, New York, 1959.
26. Szilard, R., *Theory and Analysis of Plates — Classical and Numerical Methods.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.
27. Gorman, D. J., *Free Vibration Analysis of Rectangular Plates.* Elsevier North Holland, Inc., New York, 1930.
28. Timoshenko, S., *Vibration Problems in Engineering.* Constable, London, (Second edition) 1937.
29. Bert, C. W. & Francis, P. H., Composite material mechanics: Structural Mechanics. *AIAA J.*, **12** (1974) 1173–86.
30. Bert, C. W., Dynamics of composite and sandwich panels — Parts I and II. *Shock Vib. Dig.*, **8** (1976) 15–24, 37–48.
31. Bert, C. W., Recent research in composite and sandwich plate dynamics. *Shock Vib. Dig.*, **11** (1979) 13–23.

32. Bert, C. W., Research on dynamics of composite and sandwich plates, 1979-1981. *Shock Vib. Dig.*, **14** (1982) 17-34.
33. Bert, C. W., A critical evaluation of new plate theories applied to laminated composites. *Compu. Struc.*, **2** (1984) 329-47.
34. Reddy, J. N., Finite-element modelling of layered, anisotropic composite plates and shells: A review of recent research. *Shock Vib. Dig.*, **13** (1981) 3-12.
35. Reddy, J. N., A review of the literature on finite-element modelling of laminated composite plates. *Shock Vib. Dig.*, **17** (1985) 3-8.
36. Lekhnitskii, S. G., *Theory of Elasticity of an Anisotropic Elastic Body.* Holden-Day, San Francisco, 1963.
37. Lekhnitskii, S. G., *Anisotropic Plates.* Gordon and Breach, New York, 1968.
38. Ashton, J. E., Halpin, J. C. & Petit, P. H., *Primer on Composite Materials: Analysis.* Technomic Publishing Co., Westport, Conn., 1969.
39. Calcote, L. R., *The Analysis of Laminated Composite Structures.* Van Nostrand Reinhold Co., New York, 1969.
40. Ashton, J. E. & Whitney, J. M., *Theory of Laminated Plates, Progress in Material Science Series*, Vol. IV. Technomic Publication, Stanford, 1970.
41. Jones, R. M., *Mechanics of Composite Materials.* McGraw-Hill Kogakusha Ltd., Tokyo, 1975.
42. Christensen, R. M., *Mechanics of Composite Materials.* John Wiley & Sons, New York, 1980.
43. Agarwal, B. D. & Broutman, L. J., *Analysis and Performance of Fibre Composites.* John Wiley & Sons, New York, 1980.
44. Holmes, M. & Just, D. J., *GRP in Structural Engineering*, Applied Science Publishers, London, 1983.
45. Reissner, E. & Stavsky, Y., Bending and stretching of certain types of heterogeneous aeolotropic elastic plates. *ASME, J. Appl. Mech.*, **28** (1961) 402-8.
46. Dong, S. B., Pister, K. & Taylor, R. L., On the theory of laminated anisotropic shells and plates. *J. Aerospace Sci.*, **29** (1962) 969-75.
47. Stavsky, Y., Bending and stretching of laminated aeolotropic plates. *ASCE, J. Engng Mech.*, **87** (1961) 31-56.
48. Whitney, J. M., Fourier analysis of clamped anisotropic plates. *ASME, J. Appl. Mech.*, **38** (1971) 530-2.
49. Tsay, C. S. & Reddy, J. N., Bending, stability and free vibration of thin orthotropic plates by simplified mixed finite elements. *J. Sound Vib.*, **59**, (1978) 307-11.
50. Laura, P. A. A., Luisoni, L. E. & Sarmiento, G. S., A method for the determination of the fundamental frequency of orthotropic plates of polygonal boundary shape. *J. Sound Vib.*, **4** (1980) 69.
51. Laura, P. A. A., Luisoni, L. E. & Sarmiento, G. S., Vibrations of fibre-reinforced plates of complicated shape. *Fibre Sci. Technol.*, **14** (1981) 165-70.
52. Dong, S. B. & Lopez, A. E., Natural vibrations of a clamped circular plate with rectilinear orthotropy by least-square collocation. *Int. J. Solids Struc.*, **21** (1985) 515-26.
53. Iyengar, N. G. R. & Umaretiya, J. R., Transverse vibrations of hybrid laminated plates. *J. Sound Vib.*, **104** (1986) 425-35.
54. Bert, C. W., Optimal design of a composite material plate to maximize its fundamental frequency. *J. Sound Vib.*, **50** (1977) 229-37.
55. Bert, C. W., Design of clamped composite material plates to maximize fundamental frequency. *ASME, J. Mech. Des.*, **100** (1978) 274-8.
56. Reissner, E., The effect of transverse shear deformation on the bending of elastic plates. *ASME, J. Appl. Mech.*, **12** (1945) A69-A77.
57. Yang, P. C., Norris, C. H. & Stavsky, Y., Elastic wave propagation in heterogeneous plates. *Int. J. Solids Struc.*, **2** (1966) 665-84.
58. Whitney, J. M., The effect of transverse shear deformation on the bending of laminated plates. *J. Compos. Mater.*, **3** (1969) 534-47.
59. Whitney, J. M. & Pagano, N. J., Shear deformation in heterogeneous anisotropic plates. *ASME, J. Appl. Mech.*, **37** (1970) 1031-6.
60. Ambartsumyan, S. A., *Theory of Anisotropic Plates.* Technomic Publishing Co., Westport, Conn., 1970.
61. Vinson, J. R. & Chou, T. W., *Composite Materials and their use in Structures.* Applied Science Publishers, London, 1975.
62. Whitney, J. M., Shear correction factors for orthotropic laminates under static loads. *ASME, J. Appl. Mech.*, **40** (1973) 302-4.
63. Dong, S. B. & Nelson, R. B., On natural vibrations and waves in laminated orthotropic plates. *ASME, J. Appl. Mech.*, **39** (1972) 739-45.
64. Sun, C. T. & Whitney, J. M., Theories for the dynamic response of laminated plates. *AIAA J.*, **11** (1973) 178-83.
65. Fortier, R. C. & Rossettos, J. N., On the vibration of shear deformable curved anisotropic composite plates. *ASME, J. Appl. Mech.*, **40** (1973) 299-301.
66. Sinha, P. K. & Rath, A. K., Vibration and buckling of cross-ply laminated circular cylindrical panels. *Aeron. Quart.*, **26** (1975) 211-18.
67. Hinton, E., A note on a thick finite strip method for the free vibration of laminated plates (short communication). *Earthq. Engng Struc. Dyna.*, **4** (1976) 511-14.
68. Bert, C. W. & Chen, T. L. C., Effect of shear deformation on vibration of antisymmetric angle-ply laminated rectangular plates. *Int. J. Solids Struc.*, **14** (1978) 465-73.
69. Craig, T. J. & Dawe, D. J., Flexural vibration of symmetrically laminated composite rectangular plates including transverse shear effects. *Int. J. Solids Struc.*, **22** (1986) 155-69.
70. Craig, T. J. & Dawe, D. J., Vibration of shear-deformable laminated plates structures by the finite strip method. *Compu. Struc.*, **27** (1987) 61-72.
71. Chen, P. C. & Ramkumar, R. L., Static and dynamic analysis of clamped orthotropic plates using Lagrangian multiplier technique. *AIAA J.*, **25** (1987) 316-23.
72. Hinton, E., Rock, T. & Zienkiewicz, O. C., A note on mass lumping and related processes in the finite element method. *Earthq. Engng Struc. Dyna.*, **4** (1976) 245-9.
73. Rock, T. A. & Hinton, E., A finite element method for the free vibration of plates allowing for transverse shear deformation. *Compu. Struc.*, **6** (1976) 37-44.
74. Reddy, J. N., Free vibration of antisymmetric, angle-ply laminated plates including transverse shear deformation by the finite element method. *J. Sound Vib.*, **4** (1979) 565-76.
75. Srinivas, S., Joga Rao, C. V. & Rao, A. K., An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates. *J. Sound Vib.*, **12** (1970) 187-99.
76. Srinivas, S. & Rao, A. K., Bending, vibration and buckling of simply supported thick orthotropic plates and laminates. *Int. J. Solids Struc.*, **6** (1970) 1463-81.
77. Noor, A. K., Free vibrations of multilayered composite plates. *AIAA J.*, **11** (1973) 1038-9.
78. Reissner, E., On transverse bending of plates, including the effects of transverse shear deformation. *Int. J. Solids Struc.*, **11** (1975) 569-73.

79. Lo, K. H., Christensen, R. M. & Wu, E. M., A high-order theory of plate deformation — Part 1: Homogeneous plates. *ASME, J. Appl. Mech.*, **44** (1979) 663-8.
80. Lo, K. H., Christensen, R. M. & Wu, E. M., A high-order theory of plate deformation — Part 2: Laminated plates. *ASME, J. Appl. Mech.*, **44** (1977) 669-76.
81. Levinson, M., An accurate, simple theory of the statistics and dynamics of elastic plates. *Mech. Res. Commun.*, **7** (1980) 343-50.
82. Murthy, M. V. V., An improved transverse shear deformation theory for laminated anisotropic plates. NASA Technical Paper 1903, 1981.
83. Reddy, J. N., A simple higher-order theory for laminated composite plates. *ASME, J. Appl. Mech.*, **51** (1984) 745-52.
84. Murthy, A. V. K., Flexural of composite plates. *Proceedings of the 19th Midwestern Mechanics Conference*, Columbus, Ohio, U.S.A., 1985.
85. Kant, T., Numerical analysis of thick plates. *Compu. Meth. Appl. Mech. Engng*, **31** (1982) 1-18.
86. Kant, T., Owen, D. R. J. & Zienkiewicz, O. C., A refined higher-order C° plate bending element. *Compu. Struc.*, **15** (1982) 177-83.
87. Pandya, B. N. & Kant, T., A consistent refined theory for flexure of a symmetric laminate. *Mech. Res. Commun.*, **14** (1987) 107-13.
88. Pandya, B. N. & Kant, T., A refined higher-order generally orthotropic C° plate bending element. *Compu. Struc.*, **28** (1988) 119-33.
89. Pandya, B. N. & Kant, T., Flexural analysis of laminated composites using refined higher-order C° plate bending elements. *Compu. Meth. Appl. Mech. Engng*, **66** (1988) 173-98.
90. Kant, T. & Pandya, B. N., A simple finite element formulation of a higher-order theory for unsymmetrically laminated composite plates. *Compos. Struc.*, **9** (1988) 215-46.
91. Whitney, J. M. & Sun, C. T., A higher-order theory for extensional motion of laminated composites. *J. Sound Vib.*, **30** (1973) 85-97.
92. Mindlin, R. D. & Medick, M. A., Extensional vibrations of elastic plates. *ASME, J. Appl. Mech.*, **26** (1959) 561-9.
93. Bhimaraddi, A. & Stevens, L. K., A higher order theory for free vibration of orthotropic, homogeneous, and laminated rectangular plates. *ASME, J. Appl. Mech.*, **51** (1984) 195-8.
94. Reddy, J. N. & Phan, N. D., Stability and vibration of isotropic and laminated plates according to a higher-order shear deformation theory. *J. Sound Vib.*, **98** (1985) 157-70.
95. Putcha, N. S. & Reddy, J. N., Stability and vibration analysis of laminated plates by using a mixed element based on a refined plate theory. *J. Sound Vib.*, **104** (1986) 285-300.
96. Owen, D. R. J. & Li, Z. H., A refined analysis of laminated plates by finite-element displacement methods — I. Fundamentals and static analysis. *Compu. Struc.*, **26** (1987) 907-14.
97. Owen, D. R. J. & Li, Z. H., A refined analysis of laminated plates by finite-element displacement methods — II. Vibration and stability. *Compu. Struc.*, **26** (1987) 915-23.
98. Sun, C. T. & Liou, W. J., A three-dimensional hybrid-stress finite element formulation for free vibrations of laminated composite plates. *J. Sound Vib.*, **119** (1987) 1-14.
99. Mallikarjuna, Refined theories with C finite elements for free vibration and transient dynamics of anisotropic composite-sandwich plates. PhD thesis, Indian Institute of Technology, Bombay, India, 1988.
100. Mallikarjuna & Kant, T., Free vibration of symmetrically laminated plates using a higher order theory with finite element technique. *Int. J. Numer. Meth. Engng*, **28** (1989) 1875-89.
101. Kant, T. & Mallikarjuna, A higher order theory for free vibration of unsymmetrically laminated composite and sandwich plates — finite element evaluations. *Compu. Struc.*, **32** (1989) 1125-32.
102. Kant, T. & Mallikarjuna, Vibrations of unsymmetrically laminated plates analyzed by using a higher order theory with a C° finite element formulation. *J. Sound Vib.*, **134** (1989) 1-16.
103. Reismann, H. & Lee, Y., Forced motions of rectangular plates. *Developments in Theoretical and Applied Mechanics*, **4** (1969) 3-18.
104. Reismann, H., Forced motion of elastic plates. *ASME, J. Appl. Mech.*, **35** (1968) 510-15.
105. Lee, Y. & Reismann, H., Dynamics of rectangular plates. *Int. J. Engng Sci.*, **7** (1969) 93-113.
106. Leech, J. W., Stability of finite difference equations for transient response of flat plates. *AIAA J.*, **3** (1965) 1772-3.
107. Tsui, T. Y. & Tong, P., Stability of transient solution of moderately thick plates by finite difference methods. *AIAA J.*, **9** (1971) 2062-3.
108. Rock, T. & Hinton, E., Free vibration and transient response of thick and thin plates using the finite element method. *Earthq. Engng Struc. Dyna.*, **3** (1974) 51-63.
109. Hinton, E., The dynamic transient analysis of axisymmetric circular plates by the finite element method. *J. Sound Vib.*, **46** (1976) 465-72.
110. Hinton, E., Owen, D. R. J. & Shantaram, D., Dynamic transient linear and nonlinear behaviour of thick and thin plates. *Proceedings of the Conference on Mathematics of Finite Elements and Applications*, Brunel University, April 1975.
111. Shantaram, D., Owen, D. R. J. & Zienkiewicz, O. C., Dynamic transient behaviour of two-and-three-dimensional structures including plasticity, large deformation effects and fluid interaction. *Earthq. Engng Struc. Dyna.*, **4** (1976) 561-78.
112. Pica, A. & Hinton, E., Transient and pseudo transient analysis of Mindlin plates. *Int. J. Numer. Meth. Engng*, **15** (1980) 189-208.
113. Pica, A. & Hinton, E., Further developments in transient and pseudo-transient analysis of Mindlin plates. *Int. J. Numer. Meth. Engng*, **17** (1981) 1749-61.
114. Akay, H. U., Dynamic large deflection analysis of plates using mixed finite elements. *Compu. Struc.*, **11** (1980) 1-11.
115. Moon, F. C., Wave surface due to impact on anisotropic plates. *J. Compos. Mater.*, **6** (1972) 62-79.
116. Moon, F. C., One dimensional transient waves in anisotropic plates. *ASME, J. Appl. Mech.*, **40** (1973) 485-90.
117. Chow, T. S., On the propagation of flexural waves in an orthotropic laminated plate and its response to an impulsive load. *J. Compos. Mater.*, **5** (1971) 306-19.
118. Wang, A. S. D., Chou, P. C. & Rose, J. L., Strongly coupled stress waves in heterogeneous plates. *AIAA J.*, **10** (1972) 1088-90.
119. Sun, C. T., Propagation of shock waves in anisotropic composite plates. *J. Compos. Mater.*, **7** (1973) 366-82.
120. Sun, C. T. & Whitney, J. M., Forced vibrations of laminated composite plates in cylindrical bending. *J. Acoust. Soc. Am.*, **55** (1974) 1003-8.
121. Sun, C. T. & Whitney, J. M., Dynamic response of

laminated composite plates. *AIAA J.*, **13** (1975) 1259–60.
122. Whitney, J. M. & Sun, C. T., Transient response of laminated composite plates subjected to transverse dynamic loading. *J. Acoust. Soc. Am.*, **61** (1977) 101–4.
123. Mindlin, R. D. & Goodman, L. F., Beam vibrations with time-dependent boundary conditions. *ASME, J. Appl. Mech.*, **17** (1950) 377–80.
124. Yu, Y. Y., Forced flexural vibrations of sandwich plates in plane strain. *ASME, J. Appl. Mech.*, **27** (1960) 535–40.
125. Reddy, J. N., On the solutions to forced motions of rectangular composite plates. *ASME, J. Appl. Mech.*, **49** (1982) 403–8.
126. Reddy, J. N., Dynamic (transient) analysis of layered anisotropic composite material plates. *Int. J. Numer. Meth. Engng*, **19** (1983) 237–55.
127. Von Karman, T., Festigkeitsprobleme in Maschinenbau. *Encyklopadie der Mathematischen Wissenschaften*, Teubner, Leipzig, 4, Art. 27, 350 (1907–1914).
128. Medwadowski, S. J., A refined theory of elastic, orthotropic plates. *ASME, J. Appl. Mech.*, **25** (1958) 437–43.
129. Ebcioglu, I. K., A large deflection theory of anisotropic plates. *Ingenieur-Archiv*, **33** (1964) 396–403.
130. Reddy, J. N., Geometrically nonlinear transient analysis of laminated composite plates. *AIAA J.*, **21** (1983) 621–9.
131. Chen, J. K. & Sun, C. T., Nonlinear transient response of initially stressed composite plates. *Compu. Struc.*, **21** (1985) 513–20.
132. Kant, T. & Mallikarjuna, Transient dynamics of composite plates using 4-, 8-, 9-noded isoparametric quadrilateral elements. *Int. J. Appl. Finite Elements & Compu. Aided Engng — Finite Element in Analysis & Design*, **6** (1989) 307–18.
133. Mallikarjuna & Kant, T., Finite element formulation of a higher order theory for dynamic response of laminated composite plates. *Int. J. Compu. Aided Engng & Software — Engng Compu.*, **6** (1989) 198–208.
134. Mallikarjuna & Kant, T., Finite element transient response of composite and sandwich plates with a refined theory. *ASME, J. Appl. Mech.*, **57** (1991) 1084–6.
135. Mallikarjuna & Kant, T., Analysis of anisotropic composite sandwich shells using a new displacement model with the superparametric element. *J. Struc. Engng*, **17** (1990) 91–100.
136. Mallikarjuna, An important note on symmetry line boundary conditions in fibre-reinforced laminated anisotropic composites. *Compu. Struc.*, **38** (1991) 669–71.
137. Kant, T. & Mallikarjuna, Impulse response of anisotropic composite plates with a higher order theory and finite element discretization. *Transactions of the 10th International Conference on Structural Mechanics in Reactor Technology*, Anaheim, California, U.S.A., 1989.
138. Mallikarjuna & Keshava Rao, M. N., Static and transient dynamic analysis of beam, plate and shell structures using a shear deformation theory with the superparametric element. Research Report (RD-22), Structural Engineering Research Centre, Madras, 1990.
139. Mallikarjuna & Kant, T., A general fibre-reinforced composite shell element based on a refined shear deformation theory. *Compu. Struct.*, **42** (1992) 381–8.
140. Mallikarjuna & Kant, T., Effect of cross-sectional warping of anisotropic sandwich laminates due to dynamic loads using a refined theory and C° finite elements. *Int. J. Numer. Meth. Engng*, **35** (1992) 2031–47.
141. Mallikarjuna & Kant, T., Dynamics of laminated composite plates with a higher-order theory and finite element discretization. *J. Sound Vib.*, **126** (1988) 463–75.
142. Kant, T. & Mallikarjuna, Nonlinear dynamics of laminated plates with a higher order theory and C° finite elements. *Int. J. Nonlinear Mech.*, **26** (1991) 335–44.
143. Mallikarjuna, Kant, T. & Fafard, M., Transient response of isotropic, orthotropic and anisotropic composite-sandwich shells with the superparametric element. *Int. J. Appl. Finite Elements of Compu. Aided Engng — Finite Element in Analysis of Design*, **12** (1992) 63–73.