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Large Amplitude Vibration of Polymer Composite Stiffened Laminates by the Finite Element Method

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ABSTRACT: Large amplitude free flexural vibration analyses of composite stiffened plates have been carried out using a C^0 nine noded Lagrangian element. The element, capable of incorporating curved boundaries, is based on the first order shear deformation theory. The large deformation effect of the stiffened plated structures has been taken care of by the dynamic version of von Karman's field equations. The resulting nonlinear equations have been solved by the direct iteration technique using linear mode shapes as the starting vectors. The stiffeners have been modeled with the same shape functions as that of plate element enforcing compatibility without introduction of additional unknowns. A large number of problems of isotropic and composite bare and stiffened plates have been analyzed to validate the formulation and some new results have been put forward for future reference.

INTRODUCTION

MOST OF THE aerospace and aircraft structures require that they be light weight without sacrificing any strength. Stiffened structural configurations are the perfect answer for this necessity. The newly developed light weight, stiff and strong composite materials are ideal for this type of structure where material economy and strength are needed. The plated structures are the most common in aerospace and civil engineering disciplines. These structures are often stiffened by ribs to achieve proper strength with comparatively less amount of materials. These plated structures are often subjected to dynamic loading in their service life. Due to their thin walled configuration these structures often vibrate with large geometrical deformation in the form of large amplitude. At the large amplitude level the

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strain-displacement relationships become nonlinear which makes the entire analysis nonlinear.

Considering the practical applicability of the analysis, large amplitude vibration problem has attracted many research workers. Sathyamoorthy [1] presented a comprehensive study of large amplitude vibration analysis of plates using finite element method. Large amplitude vibration analyses of orthotropic plates have been studied by Ambartsumyan [2] and Hassert and Nowinsky [3]. The nonlinear frequencies of orthotropic plates and symmetric laminates have been evaluated by Wu and Vinson [4,5] using Berger's [6] approach. Whitney and Leissa [7] included the bending-stretching coupling in non-linear plate analysis. Large amplitude flexural vibration of cross-ply laminates is studied by Chandra [8] and Chandra and Raju [9] using two term perturbation technique for unsymmetric laminates. Rao et al. [10] have presented finite element large amplitude analysis of isotropic plates/stiffened plates. A finite element solution for large amplitude flexural vibration analysis of laminated plates using a first order shear deformation theory has been put forward by Reddy and Chao [11]. Kant and Kommineni [12] have presented a higher order finite element study of large amplitude vibration of composite and sandwich plates where parabolic variation of shear and cubical variation of in-plane deformations has been considered in the analysis. Ganapathi et al. [13] have presented large amplitude vibration of composite bare plates using field consistent isoparametric elements.

To the best of the authors' knowledge, there is no published work available on the large amplitude free flexural vibration analysis of stiffened composite laminates based on first order shear deformation theory. Considering the importance of this frequently occurring stiffened structural configuration, a first order shear deformation theory including the non-linear effects in the sense of von Karman is presented here for large amplitude free vibration analysis of cross and angle ply stiffened laminates to fill the gap.

PROPOSED ANALYSIS

The laminated plate element used here throughout the study is a C^0 nine noded Lagrangian isoparametric element with five degrees of freedom (u, v, w, θ_x and θ_y) per node. The formulation of the element and the stiffener has been given in detail with the formulation of mass matrix in Reference [18]. So the stiffness and mass matrix formulation has not been elaborated here for brevity. Only the geometrically nonlinear analysis procedure has been described here in detail.

The formulation of large amplitude vibration of composite laminates includes geometrically nonlinear treatment of structural system with in-plane strains coupled with moderately large rotations.

The membrane strains, curvatures and shear strains including the nonlinear effects can be expressed as follows:

$$\varepsilon_x = u_x + \frac{1}{2} w_x^2$$

$$\varepsilon_y = v_y + \frac{1}{2} w_y^2$$

$$\gamma_{xy} = u_y + v_x + w_x w_y$$

$$\chi_x = \theta_{x,x} \tag{1}$$

$$\chi_y = \theta_{y,y}$$

$$\chi_{xy} = \theta_{x,y} + \theta_{y,x}$$

$$\gamma_{xz} = \theta_x - w_x$$

$$\gamma_{yz} = \theta_y - w_y$$

The structural behavior of the element is modeled on a shear flexible lamination theory. If N represents the membrane stress resultants (N_{xx}, N_{yy}, N_{xy}) and M, the bending stress resultants (M_{xx}, M_{yy}, M_{xy}), we may relate these to membrane strains and curvature through the constitutive relations as

$$N = A\,\varepsilon + B\,\chi \text{ and } M = B\,\varepsilon + D\,\chi \tag{2}$$

where A_{ij}, D_{ij} and B_{ij} (i,j = 1,2,6) are extensional, bending and bending-extensional stiffness coefficients of the composite laminate. Similarly, the transverse shear forces Q representing the quantities (Q_{xz} and Q_{yz}) are related to the transverse shear strains through the constitutive relations as

$$Q = E\,\gamma \tag{3}$$

where E_{ij} (i,j = 4,5) are the transverse shear stiffness coefficients of the laminate.

The formulation has been done in the total Lagrangian coordinate system which is given in detail in Reference [19]. So the final expressions are directly written here using standard notations. The governing equation of the nonlinear free flexural vibration is given as follows:

$$[K]\{\delta\} + [M]\{\ddot{\delta}\} = \{0\} \tag{4}$$

where $[K]$ and $[M]$ are the secant stiffness matrix (nonlinear) and mass matrix of the structure, respectively, and $\{\delta\}$ is the nodal displacements of the structure.

The secant stiffness matrix can be written as follows:

$$[K]=\int_A [B_0]^T[D][B_0]dA+\frac{1}{2}\int_A [B_0]^T[D][B_L]dA+\frac{1}{2}\int_A [B_L]^T[D][B_0]dA$$

$$+\frac{1}{3}\int_A [B_L]^T[D][B_L]dA+\frac{1}{2}\int_A [G]^T[S_0][G]dA+\frac{1}{3}\int_A [G]^T[S_L][G]dA \quad (5)$$

where $[D]$ is the elasticity matrix, $[S_0]$ and $[S_L]$ are the initial stress matrices corresponding to linear and nonlinear strain terms, respectively, and $[B_0]$ and $[B_L]$ are the linear and nonlinear strain matrices, respectively.

The nonlinear strain matrix can again be written as

$$[B_L]=[A][G] \quad (6)$$

where $[A]$ is dependent on the displacements but $[G]$ is independent of displacements. Now, from Equation (1), we can write separating linear and nonlinear strain as

$$\varepsilon=\varepsilon_0+\varepsilon_L \quad (7)$$

With standard finite element notations,

$$\varepsilon_0=\sum_{r=1}^{NN}[B_0]_r\{\delta\}_r \quad (8)$$

where r varies from 1 to number of nodes (NN).

Similarly, the nonlinear strain, considering only in-plane action, can be written as

$$\varepsilon_L=\begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{Bmatrix}=\frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial x}\right) & 0 \\ 0 & \left(\frac{\partial w}{\partial y}\right) \\ \left(\frac{\partial w}{\partial y}\right) & \left(\frac{\partial w}{\partial x}\right) \end{bmatrix}\begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} \quad (9)$$

or

$$\varepsilon_L = \frac{1}{2}[A]\{\theta\} \tag{10}$$

Here, matrix $[A]$ is displacement dependent as mentioned in Equation (6). Matrix $[A]$ and $\{\theta\}$ can be expressed as follows:

$$[A] = \begin{bmatrix} \sum_{r=1}^{NN} \frac{\partial N_r}{\partial x} w_r & 0 \\ 0 & \sum_{r=1}^{NN} \frac{\partial N_r}{\partial y} w_r \\ \sum_{r=1}^{NN} \frac{\partial N_r}{\partial y} w_r & \sum_{r=1}^{NN} \frac{\partial N_r}{\partial x} w_r \end{bmatrix} \tag{11}$$

and

$$\{\theta\} = \sum_{r=1}^{NN} [G]_r \{\delta\}_r \tag{12}$$

The initial stress matrices for both linear and nonlinear strain terms can be found directly from the relations as

$$[S_0] = [D]\{\varepsilon_0\} \tag{13}$$

and

$$[S_L] = [D]\{\varepsilon_L\} \tag{14}$$

NUMERICAL RESULTS

Some problems related to bare isotropic and composite plates and isotropic stiffened plates as available in the open literature have been studied to check the present formulation implemented in a FORTRAN 77 code developed in the Civil Engineering Department, IIT, Bombay. Finally, some new results have been put forward for future research.

In order to test the accuracy and efficiency of developed algorithm, and to investigate effects of transverse shear deformations, the following material property sets were used in obtaining the numerical results.

Material set 1:

$$E_1 = 40.0E_2, G_{12} = G_{31} = 0.6E_2, G_{23} = 0.5E_2, E_2 = 1.0, \rho = 1.0$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

Material set 2:

$$E_1 = 25.0E_2, G_{12} = G_{31} = 0.2E_2, G_{23} = 0.5E_2, E_2 = 1.0, \rho = 1.0$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

Material set 3:

$$E_1 = 40.0E_2, G_{12} = G_{31} = G_{23} = 0.5E_2, E_2 = 1.0, \rho = 1.0$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

RESULTS—ISOTROPIC BARE PLATES

Simply Supported Square Plate

This problem has been solved by research workers as a basic check of the formulation (References [10,13 and 17]) and can be considered as a bench mark problem. Here, the same problem has been solved to validate the computer program. The results (shown in Table 1) are found to be in good agreement with the other numerical solutions.

Simply Supported Rectangular Plate (*a*/*b* = 2)

This problem has also been solved here and the results checked with the other

Table 1. ***Frequency ratios (w_{NL}/w_L) for simply supported isotropic square plate.***

w/h	Reference [17]	Reference [13]	Reference [10]	Present
0.2	1.02599	1.02504	1.0261	1.0263
0.4	1.10027	1.10021	1.1009	1.1012
0.6	1.21402	1.20803	1.2162	1.2165
0.8	1.35735	1.35074	1.3624	1.3629
1.0	1.52192	1.51347	1.5314	1.5325

Table 2. Frequency ratios (w_{NL}/w_L) for simply supported isotropic rectangular plate.

w/h	Reference [14]	Reference [10]	Present
0.2	1.0238	1.0324	1.0323
0.4	1.0918	1.1252	1.1251
0.6	1.1957	1.2682	1.2680
0.8	1.3264	1.4403	1.4500
1.0	1.4758	1.6613	1.6614

published data (References [10,14]). Results (shown in Table 2) are found to be sufficiently close to the other solutions.

Square Clamped Plate

This problem has been solved by some other investigators (References [10,14]) to check the effect of clamped boundary conditions. So, this problem is solved here and the results are compared (shown in Table 3) with the other published results. The comparison shows good agreement.

Square Plate with Four Stiffeners

A square plate having four concentric stiffeners as shown in the Figure 1 has been analyzed here with the given formulation. The stiffeners are identical and placed along one direction of the plate. The investigation has been carried out for both simply supported and clamped edges of the plate. The nonlinear to linear frequency ratios (w_{NL}/w_L) obtained by the proposed formulation have been presented with that of Prathap and Varadan [15] in Tables 4 and 5 for different amplitude ratios. The results have agreed reasonably well.

Cross-Stiffened Plate

A simply supported square plate stiffened along both the directions by central

Table 3. Frequency ratios (w_{NL}/w_L) for clamped square isotropic plate.

w/h	Reference [14]	Reference [10]	Present
0.2	1.0070	1.0095	1.0096
0.4	1.0276	1.0375	1.0382
0.6	1.0608	1.0825	1.0840
0.8	1.1047	1.1424	1.1448
1.0	1.1578	1.2149	1.2183

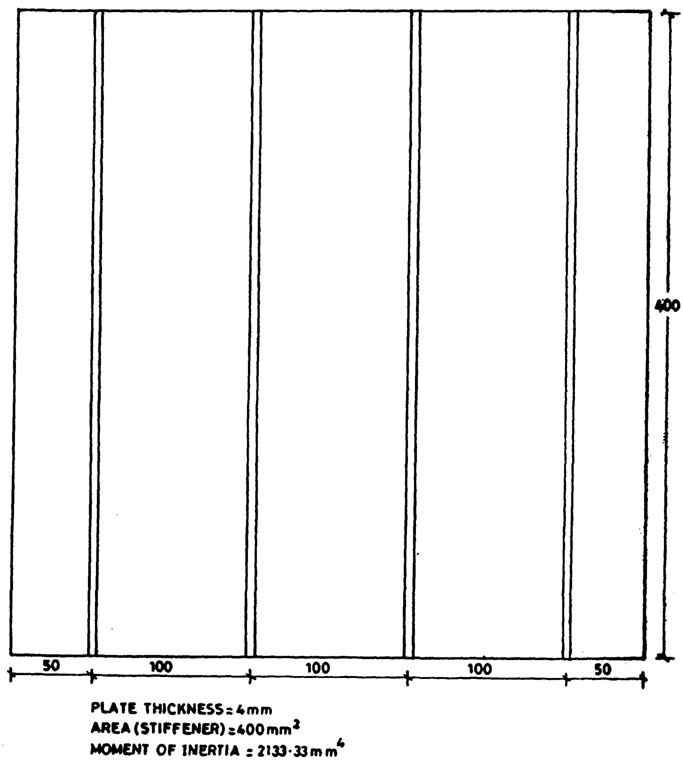

Figure 1. *Plate configuration and material properties of the square plate with four stiffeners.*

Table 4. ***Frequency ratios (w_{NL}/w_L) for simply supported square isotropic plate with four stiffeners.***

w/h	Reference [15]	Reference [10]	Present
0.2	1.0183	1.0146	1.0149
0.4	1.0713	1.0573	1.0576
0.6	1.1540	1.1250	1.1251
0.8	1.2610	1.2140	1.2139
1.0	1.3860	1.3200	1.3205

Table 5. ***Frequency ratios (w_{NL}/w_L) for clamped square isotropic plate with four stiffeners.***

w/h	Reference [15]	Reference [10]	Present
0.2	1.0030	1.0044	1.0046
0.4	1.0119	1.0173	1.0181
0.6	1.0266	1.0386	1.0391
0.8	1.0469	1.0676	1.0680
1.0	1.0724	1.1039	1.1042

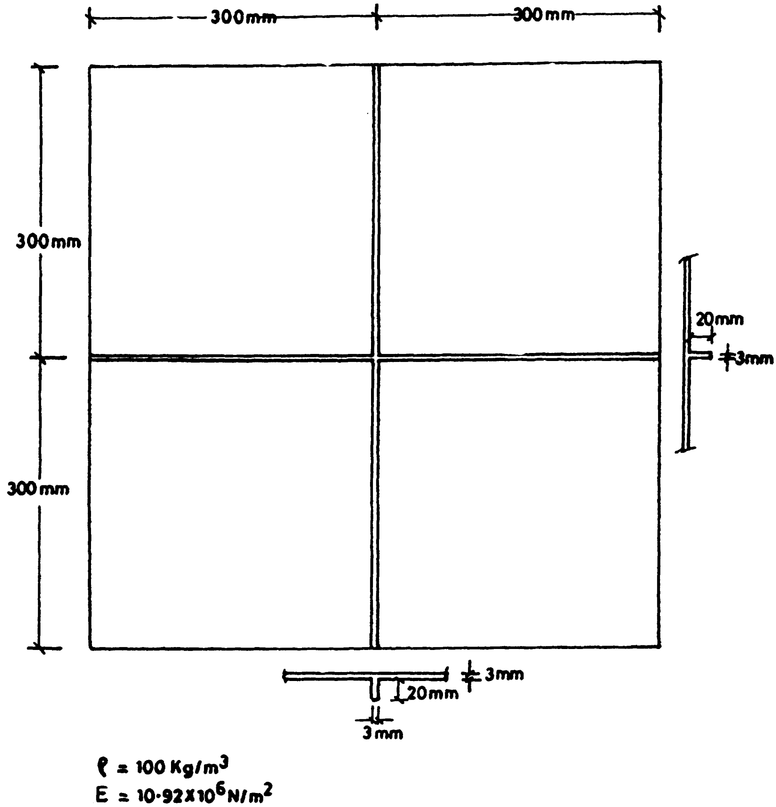

Figure 2. Plate and stiffener configuration of cross-stiffened square plate.

eccentric stiffeners as shown in Figure 2 has been analyzed here. This problem is solved previously in References [10] and [16]. The results obtained by the proposed approach has been shown in the Table 6 with the other available results. The results obtained by this method are found to be less than Reference [10] but closer to Reference [16].

Table 6. ***Frequency ratios (w_{NL}/w_L) for simply supported square isotropic cross-stiffened plate.***

w/h	Reference [16]	Reference [10]	Present
0.2	1.0013	1.0087	1.0071
0.4	1.0122	1.0269	1.0229
0.6	1.0318	1.0540	1.0452
0.8	1.0592	1.0889	1.0708
1.0	1.0938	1.1310	1.0996

RESULTS—COMPOSITE PLATES

Symmetric Laminate

A simply supported cross-ply (0°/90°/90°/0°) with in-plane motions at the boundary restricted with $a/h = 10$ and 1000 and the material set 1 and 2 is considered. The present results for the ratio of nonlinear to linear frequency against the amplitude ratio (w_0/h) are compared with Ganapathi et al. [13] and Kant et al. [12] and are presented in Table 7. The table shows that the results of higher order shear deformation theory (HOST) of Kant and Kommineni [12] is somewhat lower than that of first order shear deformation theory by Ganapathy et al. [13] and the present one. This may be due to the better representation of in-plane deformation in higher order shear deformation theory by Kant and Kommineni [12] which is important in nonlinear analysis and also the real parabolic distribution of shear stress which is important for moderately thick laminated plates.

Table 7. Frequency ratios (w_{NL}/w_L) of nonlinear vibration of a simply supported cross-ply (0°/90°/90°/0°) laminate.

a/h	w_0/h	Material Set	Kant et al. [12]	Ganapathy et al. [13]	Present
	0.2		1.02843	1.04125	1.04165
	0.4		1.14575	1.15093	1.15809
1000	0.6	1	1.29166	1.31825	1.33133
	0.8		1.48372	1.51495	1.53466
	1.0		1.70091	1.73828	1.73824
	0.2		1.02808	1.04108	1.04160
	0.4		1.13436	1.15029	1.15508
1000	0.6	2	1.28324	1.31653	1.32903
	0.8		1.47890	1.51394	1.52936
	1.0		1.68399	1.73650	1.73322
	0.2		1.04843	1.06453	1.06600
	0.4		1.21575	1.22915	1.24285
10	0.6	1	1.42617	1.44215	1.45123
	0.8		1.63372	1.66125	1.68725
	1.0		1.82091	1.85671	1.86809
	0.2		1.04082	1.06016	1.05822
	0.4		1.20057	1.21973	1.22015
10	0.6	2	1.40401	1.43125	1.44534
	0.8		1.60105	1.65078	1.68062
	1.0		1.81449	1.85126	1.86625

PARAMETRIC STUDY

Composite Stiffened Plates

As there are no published results available for the large amplitude free flexural vibration of composite stiffened plates, some new results have been generated in the form of parametric study and are given in a graphical manner. For all the cases, the material properties are taken as material set 3. For all the problems tackled here, the plate is taken as simply supported and of square configuration with in-plane motion restricted. For all the following cases, the laminae orientations for the stiffeners are considered as running along the length of it (0°/0° for x-directional stiffener and 90°/ 90° for y-directional stiffener).

CASE 1

For this case, the a/h ratio is taken as 1000. The plate thickness is taken as 0.1 unit which is same as width of the stiffener. The depth of the stiffener is taken as five times its width. Different laminae orientations considered for the plate are given as follows:

(a) Symmetric cross-ply: 0°/90°/90°/0°
(b) Symmetric-angle-ply: 45°/−45°/−45°/45°
(c) Anti-symmetric cross-ply: 0°/90°/0°/90°
(d) Anti-symmetric angle-ply: 45°/−45°/45°/−45°
(e) Unsymmetric cross-ply: 0°/90°

The results are plotted in Figure 3.

CASE 2

For this case, all the other information is exactly the same as the previous problem except the ratio of depth to width for both the stiffeners is taken as 10 (double that of the previous case). This study reveals that the heavier stiffeners have got a stabilizing effect over the entire structural system with reducing amplitude ratio which is helpful for stiffened configurations vibrating in nonlinear domain. The ply orientations are exactly the same as that of the previous case (five cases). Figure 4 shows the variation of nonlinear to linear frequency ratio with amplitude ratio.

CASE 3

This study contains some results of large amplitude vibration of moderately

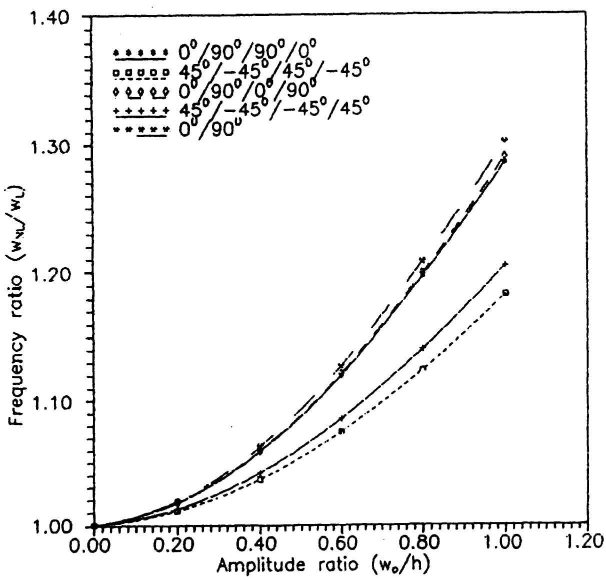

Figure 3. *Variation of frequency ratio (w_{NL}/w_L) a square cross-stiffened composite plate with amplitude ratio (w_0/h) for five different ply orientations (Case 1).*

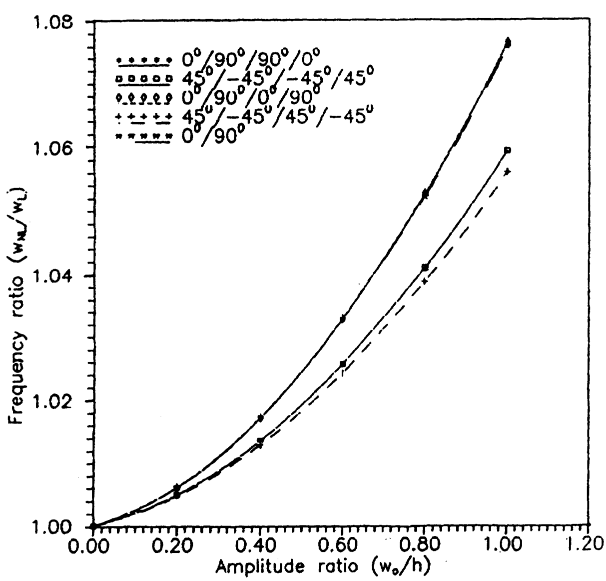

Figure 4. *Variation of frequency ratio (w_{NL}/w_L) of a square cross-stiffened composite plate with amplitude ratio (w_0/h) for five different ply orientations (Case 2).*

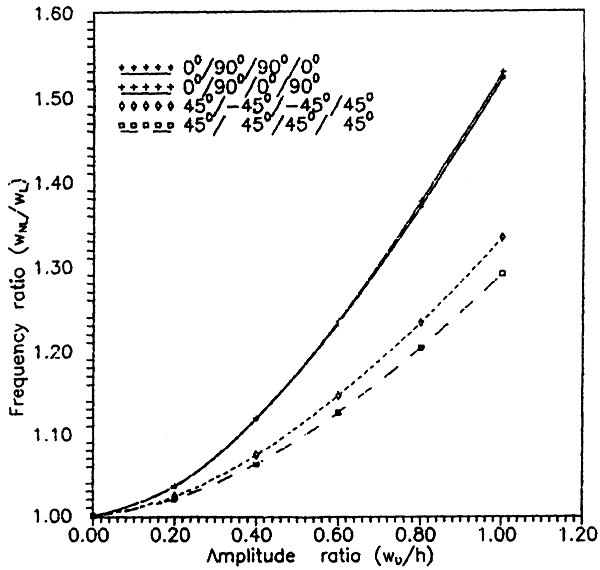

Figure 5. *Variation of frequency ratio (w_{NL}/w_L) of a square cross-stiffened composite plate with amplitude ratio (w_0/h) for five different ply orientations (Case 3).*

thick cross-stiffened plates. Here, *a*/*h* ratio is taken as 20. Thickness of the plate is taken as 0.5. The width of the stiffener is half that of its plate thickness and the depth of the stiffener is twice that of its width. Four types of ply orientations are considered here

(a) Symmetric cross-ply: 0°/90°/90°/0°
(b) Symmetric-angle-ply: 45°/−45°/−45°/45°
(c) Anti-symmetric cross-ply: 0°/90°/0°/90°
(d) Anti-symmetric angle-ply: 45°/−45°/45°/−45°

Figure 5 shows the frequency ratio variation with amplitude ratio.

From the figures, it can be observed that the cross-ply laminates (symmetric and anti-symmetric) always create more hardening with higher frequency ratio than the angle-ply laminates. The unsymmetric two layer (0°/90°) cross-ply laminates produce the hardest structural configuration. In between angle-ply laminates, symmetric configurations produce more hardening effects than anti-symmetric ply orientations.

CONCLUSIONS

The geometrically nonlinear free flexural vibration analysis of composite stiffened laminates has been presented in this paper. A first order shear deformation theory is used throughout the work in conjunction with a C^0 continuous finite element. The results of various problems of isotropic and composite plates as available in the literature have been compared and finally, a parametric study has been conducted for various laminae orientations, aspect ratios and stiffener eccentricities. As far as the authors' knowledge goes, the results for the large amplitude vibration of composite stiffened laminates are given for the first time in the published literature. Results indicate that the difference of frequency ratio increases for higher order shear theory and first order shear theory from high aspect ratio to low aspect ratio.

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