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Free vibration of composite and sandwich laminates with a higher-order facet shell element

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Abstract

A simple C^0 isoparametric finite element formulation based on a shear deformable model of higher-order theory using a higher-order facet shell element is presented for the free vibration analysis of isotropic, orthotropic and layered anisotropic composite and sandwich laminates. This theory incorporates a realistic non-linear variation of displacements through the shell thickness, and eliminates the use of shear correction coefficients. The validity and efficiency of the present formulation is established by obtaining solutions to a wide range of problems and comparing them with the available three-dimensional closed-form and finite element solutions. In addition, other plate and shell solutions of different kind and available in the literature are also compiled and tabulated for the sake of completeness. The parametric effects of degree of orthotropy, length-to-thickness ratio, plate aspect ratio, number of layers and fibre orientation upon the frequencies and mode shapes are discussed.

Keywords: Free vibration; Facet shell element; Shear deformation; Higher-order theory; Composite and sandwich laminates; Plates and shells

1. Introduction

Laminated composites are being used more extensively as structural components in aerospace, automobile, civil, marine and other related weight sensitive engineering applications requiring high strength to weight and stiffness to weight ratios. The mechanical behaviour of laminated composites is strongly dependent on the degree of orthotropy of the individual layers, the ratio of transverse shear modulus to the in-plane modulus and the stacking sequence of laminae. By appropriate orientation of the fibres in each lamina, desired strength and stiffness parameters can be achieved.

The simplifying assumptions, made in classical and first-order theories, are reflected by the high percentage error in the results of thick composite and sandwich plates with highly stiff facings. The effect of plate aspect ratio and transverse shear rigidities of stiff layers on fundamental frequencies are more pronounced in thicker plates than they are for thin plates. In contrast to the first-order shear deformation theories, higher-order shear deformation theories do not require a shear correction coefficient, owing to more realistic representation of the cross-sectional deformation. Because of these limitations, the need is obvious to use the refined theories, which include the consideration of realistic parabolic variation of transverse shear stress through the laminate thickness and warping of the transverse cross-section. Thus, the use of higher-order shear deformation theory is very important for the vibration analysis of laminated composite plates, especially for thick sandwich laminates.

Many analytical methods of analysis have been used to study the vibration of plates and shells. In closed form solutions, the analytical difficulties in solving the equations have until now been overcome only in some special cases, while the general case has not yet received a satisfactory treatment.

The finite element approach has proved to be a powerful and widely applicable method for the vibration analysis of complex problems for which analytical solutions are nearly impossible to find. A variety of new elements have been proposed based on different structural theories, interpolation functions and formulation procedures in order to achieve a more accurate prediction of the free vibration of plates and shells.

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A number of theories exist in the literature for the analysis of laminated composite structures. The classical lamination theory due to Kirchhoff is based on the assumption that the normals to the undeformed midplane remain straight and normal to the deformed midplane of laminate, and therefore the corresponding neglect of the transverse shear effects was the starting point in the development of general plate and shell theories. The assumptions of classical theories lead to an increase in the stiffness of the structure and hence displacements are under-predicted and their natural frequencies over-predicted.

The need to include transverse shear effects was first recognized by Reissner [1], followed by Mindlin [2], who included rotary inertia effects in the dynamic analysis of plates. The first-order shear deformation theory of Reissener and Mindlin was extended by Yang et al. [3] to laminated plates, followed by many variants of first-order theory. Reissner [4], Noor and Burton [5] and Reddy [6] have reviewed these developments. Noor [7] used the first-order shear deformation theory to analyse the free vibration of cross-ply laminated plates, while Bert and Chen [8] developed a closed-form solution for angle-ply laminated plates. Reddy [9] presented a finite element model based on Yang–Norris–Stavsky theory and its application to the free vibration of antisymmetric, angle-ply laminated plates. Sinha and Rath [10] presented a closed form solution of vibration and buckling of simply supported cross-ply laminated cylindrical panels based on the first-order shear deformation Donnell's shell theory. Carrera [11] analysed the vibration of cross-ply laminated cylindrical panels by the Navier method based on the first-order shear deformation Flugge's shell theory. Lim and Liew [12] implemented the first-order shear deformation theory for the prediction of vibratory characteristics of shallow conical shell panels, while Chakravorty et al. [13] implemented it for the finite element vibration analysis of doubly curved laminated composite shells.

In order to overcome the limitations of first-order shear deformation theories, higher-order shear deformation theories that involve higher-order terms in the Taylor's series expansions of the displacements in the thickness coordinate were developed. In these higher-order theories an additional dependent variable is introduced into the theory with each additional power of the thickness coordinate. Hildebrand et al. [14] were the first to introduce this approach to derive improved theories of plates and shells. Nelson and Lorch [15] presented higher-order displacement based shear deformation theory for the analysis of laminated plates.

Kant et al. [16] are the first to present finite element formulation of a higher-order flexure theory. This theory considers three-dimensional Hooke's law and incorporates the effect of transverse normal strain in addition to transverse shear deformations. Reddy [17] later proposed a higher-order shear deformation theory utilizing a displacement field with cubic variations with respect to the thickness direction. This higher-order shear deformation theory was applied by Reddy and Phan [18] for the determination of the natural frequencies of elastic plates, by Khdeir [19,20] for free vibration analysis of cross-ply laminated plates, and by Putcha and Reddy [21] to develop a mixed finite element consisting of 11 degrees of freedom. Kant and Mallikarjuna [22] streamlined the higher-order shear deformation theory by allowing the displacement in the thickness direction to be quadratic with respect to the thickness co-ordinate z, and developed a simple C^0 finite element formulation and presented solutions for the free vibration analysis of general laminated composite and sandwich plate problems.

Bhimaraddi [23] presented a higher-order theory for free vibration analysis of circular cylindrical shells. Liew and Lim [24] developed a higher-order shear deformation theory for vibration analysis of thick, doubly curved shallow shells. Matsunaga [25] presented a global higher-order theory for analyzing natural frequencies and buckling stress of cross-ply laminated composite plates.

Reddy and Khdeir [26] developed analytical and finite element solutions of the classical, first-order and third-order laminate theories to study the buckling and free vibration behaviour of cross-ply rectangular composite laminates under various boundary conditions. Wang and Lin [27] developed the finite strip method based on the higher-order plate theory for the free vibration of the laminated plates. Mizusawa and Kito [28–30] presented an application of the spline strip method based on the first-order and higher-order shear deformation Donnell's shell theories to analyse vibration of thick laminated cylindrical panels.

Srinivas et al. [31], Srinivas and Rao [32], Noor and Burton [33] and Chao and Chern [34] presented exact three-dimensional elasticity solutions for the free vibration of isotropic, orthotropic and anisotropic composite laminated plates, while Bhimaraddi [35] and Chern and Chao [36] provided solutions for curved panels.

Flat facet shell elements are popular and are integral parts of any general purpose finite element code. With the advent of high-speed computers it is also possible now a days to employ large number of elements, to approximate even a curved shell by flat facet elements. In facet shell elements, which have flat surface, the membrane and bending stiffness are superposed at the element level. The coupling between membrane and bending is realized at the assemblage level by transforming the local degrees of freedom to the global ones. Because of the simplicity of the formulation, the effectiveness of the computation and the flexibility in applications such as shells of regular and irregular shapes and folded plate structures, flat shell elements are extensively useful in engineering practice. Allman [37] used a triangular flat facet element for the dynamic analysis of General thin shells. Kant and Khare [38] presented a C^0 finite element formulation of a flat faceted element based on a higher-order displacement model for the static analysis of general, thin-to-thick, fibre reinforced composite laminated plates and shells. Batoz et al. [39] presented a quadrilateral discrete Kirchhoff flat shell element with 16 degrees of freedom for the linear analysis of plates and shells.

Mukherjee [40] presented a higher-order quadratic isoparametric element for the free vibration analysis of laminated composite plates. Shankara and Iyengar [41] developed a C^0 continuous finite element with five and seven degrees of freedom, while Maiti and Sinha [42] employed the higher-order shear deformation theory to develop a finite element formulation using an eight-noded isoparametric element for the free vibration analysis of laminated composite plates.

Chandrashekhara [43] presented free vibration characteristics of laminated composite shells using an isoparametric doubly curved quadrilateral shear flexible element. Beakou and Touratier [44] developed a four-noded C^1 rectangular finite element for the analysis of composite multilayered shallow shells. Gautham and Ganeshan [45] presented a two-noded finite element for the vibration analysis of thick orthotropic layered shells of revolution based on the discrete layer theory. Chakravorty et al. [46] developed a finite element formulation based on the first-order shear deformation theory for the free vibration analysis of point supported laminated composite thin, shallow cylindrical shells using the eight-noded curved quadrilateral isoparametric element. Aksu [47] formulated a curved isoparametric trapezoidal finite element for the free vibration analysis of shells of general shape. This shell element with eight nodes and 40 degrees of freedom is applicable for both thin and moderately thick shell analysis. Lee and Han [48] developed a nine-noded degenerated shell element for the free vibration analysis of plates and shells.

In the present work, a C^0 continuous shear deformable finite element formulation of the facet quadrilateral elements family based on a higher-order displacement model is presented, which does not require the use of a shear correction coefficient and includes the rotational degree of freedom. The element is used for the free vibration analysis of composite and sandwich laminates. Accuracy of the present formulation is established by obtaining solutions to a wide range of problems and comparing them with the available three-dimensional closed-form and finite element solutions for isotropic, orthotropic and layered anisotropic composite and sandwich plates and shells.

2. Theory and formulation

A facet element of a composite laminates consisting of laminas with isotropic/orthotropic material properties oriented arbitrarily in space is considered and shown in Figs. 1 and 2. The higher-order shear deformation theory considered for investigation in the present work is based on the assumption of the displacement field in the following form.

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z\theta_y(x,y,t) + z^2 u_0^*(x,y,t) \\
&\quad + z^3\theta_y^*(x,y,t), \\
v(x,y,z,t) &= v_0(x,y,t) - z\theta_x(x,y,t) + z^2 v_0^*(x,y,t) \\
&\quad - z^3\theta_x^*(x,y,t), \\
w(x,y,z,t) &= w_0(x,y,t),
\end{aligned}
\tag{1}
$$

where t is the time, u, v and w are the displacements of a general point (x,y,z) in an element of the laminate domain in the x, y and z directions respectively. The parameters u_0, v_0 are the inplane displacements and w_0 is the transverse displacement of a point (x, y) on the laminate middle plane. The functions, θ_x, θ_y are the

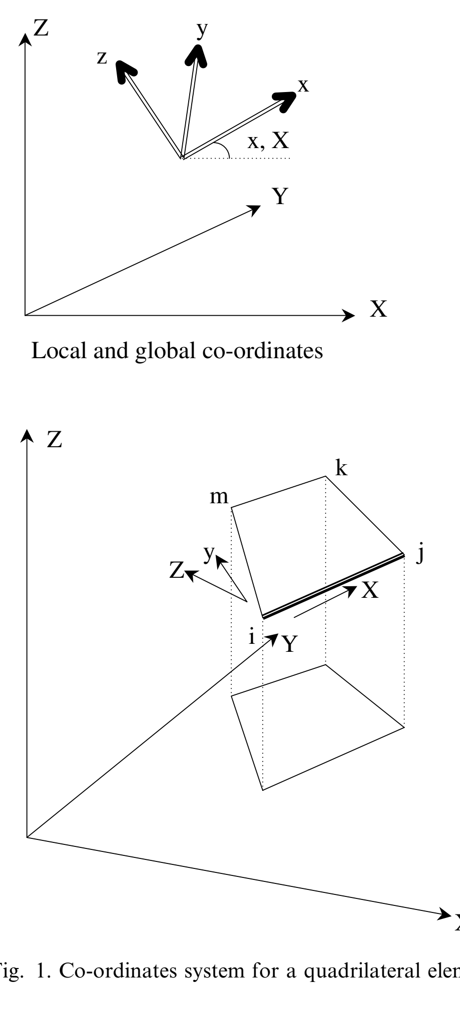

Fig. 1. Co-ordinates system for a quadrilateral element.

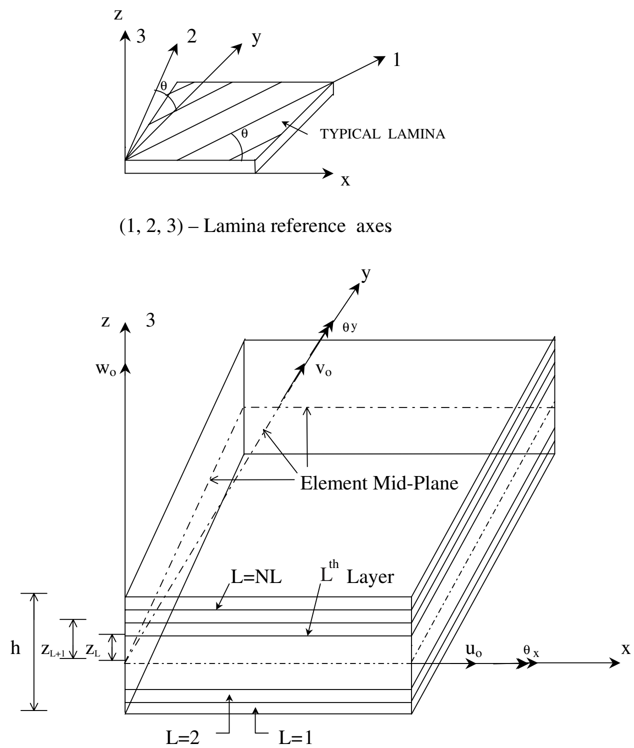

Fig. 2. Element laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

rotations of the normal to the laminate middle plane about x- and y-axes respectively. The parameters u_0^*, v_0^*, θ_x^*, θ_y^* are the higher-order terms in the Taylor's series expansion and they represent higher-order transverse cross sectional deformation modes. Though-thickness variation of various components of the displacement field is shown in Fig. 3.

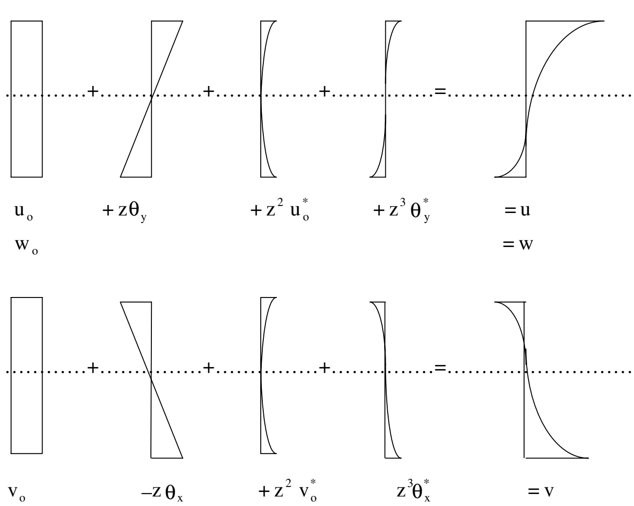

Fig. 3. Through-thickness variation of various components of the displacement field.

By substituting Eq. (1) into the general linear strain–displacement relations, the following relations are obtained.

$$
\begin{aligned}
\varepsilon_x &= \varepsilon_{x0} + z\chi_x + z^2\varepsilon_{x0}^* + z^3\chi_x^*,\\
\varepsilon_y &= \varepsilon_{y0} + z\chi_y + z^2\varepsilon_{y0}^* + z^3\chi_y^*,\\
\gamma_{xy} &= \varepsilon_{xy0} + z\chi_{xy} + z^2\varepsilon_{xy0}^* + z^3\chi_{xy}^*,\\
\gamma_{xz} &= \varphi_x + z\chi_{xz} + z^2\varphi_x^*,\\
\gamma_{yz} &= \varphi_y + z\chi_{yz} + z^2\varphi_y^*,
\end{aligned}
\tag{2a}
$$

where

$$
\begin{aligned}
&\varepsilon_{x0} = \frac{\partial u_0}{\partial x}, \quad \varepsilon_{y0} = \frac{\partial v_0}{\partial y}, \quad \varepsilon_{xy0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x},\\
&\varepsilon_{x0}^* = \frac{\partial u_0^*}{\partial x}, \quad \varepsilon_{y0}^* = \frac{\partial v_0^*}{\partial y}, \quad \varepsilon_{xy0}^* = \frac{\partial u_0^*}{\partial y} + \frac{\partial v_0^*}{\partial x},\\
&\chi_x = \frac{\partial \theta_y}{\partial x}, \quad \chi_y = -\frac{\partial \theta_x}{\partial y}, \quad \chi_{xy} = \frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x},\\
&\chi_x^* = \frac{\partial \theta_y^*}{\partial x}, \quad \chi_y^* = -\frac{\partial \theta_x^*}{\partial y}, \quad \chi_{xy}^* = \frac{\partial \theta_y^*}{\partial y} - \frac{\partial \theta_x^*}{\partial x},\\
&\varphi_x = \theta_y + \frac{\partial w_0}{\partial x}, \quad \varphi_y = -\theta_x + \frac{\partial w_0}{\partial y}, \quad \chi_{xz} = 2u_0^*,\\
&\varphi_x^* = 3\theta_y, \quad \varphi_y^* = -3\theta_x^*, \quad \chi_{yz} = 2v_0^*.
\end{aligned}
\tag{2b}
$$

The constitutive relations for a typical lamina L with reference to the fibre-matrix co-ordinate axes $(1,2,3)$ can be written as

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix}^L,
\tag{3a}
$$

where $(\sigma_1, \sigma_2, \tau_{12}, \tau_{13}, \tau_{23})$, $(\varepsilon_1, \varepsilon_2, \gamma_{12}, \gamma_{13}, \gamma_{23})$ are the stresses and the linear strain components respectively with reference to lamina co-ordinates in the element, C_{ij}'s of constitutive matrix $\mathbf{C}$ are the elastic constants of the Lth lamina and are related to engineering constants by the following relations:

$$
\begin{aligned}
&C_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}; \quad C_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}};\\
&C_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}; \quad C_{33} = G_{12}; \quad C_{44} = G_{13};\\
&C_{55} = G_{23}; \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}.
\end{aligned}
\tag{3b}
$$

The stress–strain relations for the Lth lamina in the element co-ordinates (x, y, z) are written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^L, \tag{3c}$$

or in short form $\sigma = \mathbf{Q}\varepsilon$, (3d)

where $\sigma = (\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz,})^{\mathrm{T}}$ and $\varepsilon = (\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz,}\gamma_{yz})^{\mathrm{T}}$ are the stress and strain vectors, respectively with respect to the element co-ordinates. Q_{ij}'s are the plane stress reduced stiffness coefficients, and are derived following the usual transformation rule of stress/strain between the lamina $(1, 2, 3)$ and the element (x, y, z) co-ordinate system as described in Cook [49].

Integrating the Eq. (3d) through the laminate thickness the stress–strain relations can be written in matrix form as

$$\int \boldsymbol{\sigma}\,\mathrm{d}z = \int (\mathbf{Q}\boldsymbol{\varepsilon})\,\mathrm{d}z, \tag{4a}$$

or $\bar{\boldsymbol{\sigma}} = \mathbf{D}\bar{\boldsymbol{\varepsilon}}$, (4b)

in which $\bar{\boldsymbol{\varepsilon}}$ is the mid-surface strain vector, $\bar{\boldsymbol{\sigma}}$ is the stress-resultant vector and $\mathbf{D}$ is the rigidity matrix composed of membrane ($\mathbf{D}_{\mathrm{m}}$), bending ($\mathbf{D}_{\mathrm{b}}$), coupling ($\mathbf{D}_{\mathrm{c}}$) and shear ($\mathbf{D}_{\mathrm{s}}$) rigidity matrices. Writing them in an equation form

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_{\mathrm{m}} & \mathbf{D}_{\mathrm{c}} & \mathbf{0} \\ \mathbf{D}_{\mathrm{c}}^{\mathrm{t}} & \mathbf{D}_{\mathrm{b}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathrm{m}} \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ \chi_0 \\ \phi_0 \end{Bmatrix}, \tag{5}$$

in which

$$\begin{aligned} \mathbf{N} &= \left(N_x, N_y, N_{xy}, N_x^*, N_y^*, N_{xy}^*\right)^{\mathrm{t}}, \\ \varepsilon_0 &= \left(\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{xy0}, \varepsilon_{x0}^*, \varepsilon_{y0}^*, \varepsilon_{xy0}^*\right)^{\mathrm{t}}, \\ \mathbf{M} &= \left(M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*\right)^{\mathrm{t}}, \\ \chi_0 &= \left(\chi_{x0}, \chi_{y0}, \chi_{xy0}, \chi_{x0}^*, \chi_{y0}^*, \chi_{xy0}^*\right)^{\mathrm{t}}, \\ \mathbf{Q} &= \left(Q_x, Q_y, Q_x^*, Q_y^*, S_x, S_y\right)^{\mathrm{t}}, \\ \phi_0 &= \left(\phi_{x0}, \phi_{y0}, \phi_{x0}^*, \phi_{y0}^*, \chi_{xz}, \chi_{yz}\right)^{\mathrm{t}}. \end{aligned} \tag{6}$$

The individual sub-matrices of the rigidity matrix $\mathbf{D}$ are defined in Appendix A.

3. Finite element formulation

For the present study, a nine-noded quadrilateral (Lagrangian family) two-dimensional C^0 continuous isoparametric element with nine degrees of freedom per node is used. The displacement vector $\mathbf{d}$ at any point on the mid-surface is given by

$$\mathbf{d} = \sum_{i=1}^{NN} \mathbf{N}_i(x, y)\mathbf{d}_i, \tag{7}$$

where $\mathbf{d}_i$ is the displacement vector corresponding to node i, N_i is the interpolating or shape function associated with node i, and NN is the total number of nodes per element (nine in this case).

Knowing the generalized displacement vector $\mathbf{d}$ at all points within the element, the generalized mid-surface strains at any point given by Eq. (2) can be expressed in terms of nodal displacements in matrix form as follows:

$$\bar{\boldsymbol{\varepsilon}} = \sum_{i=1}^{NN} \mathbf{B}_i\mathbf{d}_i, \tag{8}$$

where $\mathbf{B}_i$ is a differential operator matrix of shape functions [50].

The governing differential equations of motion can be derived using Hamilton's principle

$$\delta \int_{t_1}^{t_2} (\Pi - E)\,\mathrm{d}t = 0, \tag{9}$$

where t is the time, E is the total kinetic energy of the system and Π is the potential energy of the system, including both strain energy and potential of conservative external forces. For the ideal case in which the system has no damping and no external forcing function, the mathematical statement of Hamilton's principle can be written as

$$\int_{t_1}^{t_2} (\delta U - \delta E)\,\mathrm{d}t = 0, \tag{10}$$

where δU and δE are the first variation of the strain energy and the kinetic energy respectively. Using the standard finite element technique, the total domain is discretized into NE sub-domains or elements such that

$$\int_{t_1}^{t_2} \sum_{i=1}^{NE} (\delta U^{\mathrm{e}} - \delta E^{\mathrm{e}}) = 0. \tag{11}$$

The first variation of the strain energy δU^{e} and kinetic energy δE^{e} for an element can be written in matrix form as

$$\delta U^e = \delta \mathbf{d}_{\mathrm{e}}^{\mathrm{t}} \mathbf{K}^{\mathrm{e}} \mathbf{d}_{\mathrm{e}} \quad \text{and} \quad \delta E^{\mathrm{e}} = -\delta \mathbf{d}_e^{\mathrm{t}} \mathbf{M}^e \mathbf{d}_{\mathrm{e}}, \tag{12}$$

in which $\mathbf{K}^e$ is the stiffness matrix for an element 'e' which includes membrane, flexure and the transverse shear effects and $\mathbf{M}^e$ is the element mass matrix, which are given by

$$\mathbf{K}^{\mathrm{e}} = \int_A \mathbf{B}^{\mathrm{t}}\mathbf{D}\mathbf{B}\,\mathrm{d}A \quad \text{and} \quad \mathbf{M}^e = \int_A \mathbf{N}^{\mathrm{t}}\mathbf{m}\mathbf{N}\,\mathrm{d}A. \tag{13}$$

The element stiffness matrix and mass matrix can be obtained by using the standard relation

$$\mathbf{K}_{ij}^e = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}_i^{\mathrm{t}}\mathbf{D}\mathbf{B}_j|\mathbf{J}|\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{14}$$

and

$$\mathbf{M}_{ij}^{e} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{N}_i^{t}\mathbf{m}\mathbf{N}_j|\mathbf{J}|\,d\xi\,d\eta, \tag{15}$$

where $|\mathbf{J}|$ is the determinant of the standard Jacobian matrix, $\mathbf{D}$ is the rigidity matrix and $\mathbf{m}$ is the inertia matrix, which is defined in Appendix B.

Before assembly, the stiffness matrix and mass matrix are transformed to the global co-ordinate system (X, Y, Z) by the simple transformation rules as described in Zienkiewicz and Taylor [51]. The displacements of a node are transformed from the global to the local system by a matrix $\mathbf{L}_g$ giving

$$\mathbf{d}_i = \mathbf{L}_g\mathbf{d}_i^{g}, \tag{16}$$

in which matrix $\mathbf{L}_g$ is defined in Appendix C and $\mathbf{d}_i^g$ is the displacement vector in the global co-ordinate system corresponding to node i defined as

$$\mathbf{d}_i^{g} = (u_0^g, v_0^g, w_0^g, \theta_x^g, \theta_y^g, u_0^{*g}, v_0^{*g}, \theta_x^{*g}, \theta_y^{*g}, \theta_z^g), \tag{17}$$

superscript ‘g’ indicates the components in the global co-ordinate system. As the element is a flat shell element, which does not have rotational degree of freedom about the normal, initially at element level formulation only nine degrees of freedom are there. In the global co-ordinate system the rotation about the global z-direction, i.e. the rotational degree of freedom (Zienkiewicz and Taylor [51] and Cook [60]) is introduced with zero value to facilitate the desired transformation to global co-ordinate system where rotation about global z-direction exists. For the whole set of displacements on nodes of an element, Eq. (16) can therefore be expressed as

$$\mathbf{d}_e = \mathbf{T}_g\mathbf{d}_e^{g}. \tag{18}$$

By the rules of orthogonal transformation the stiffness matrix of an element in the global co-ordinate becomes

$$\mathbf{K}^{ge} = \mathbf{T}_g^{t}\mathbf{K}^{e}\mathbf{T}_g, \tag{19}$$

in both the above equations $\mathbf{T}_g$ is given by

$$\mathbf{T}_g = \begin{bmatrix} \mathbf{L}_g & 0 & 0 & \cdots \\ 0 & \mathbf{L}_g & 0 & \\ 0 & 0 & \mathbf{L}_g & \\ \vdots & & & \end{bmatrix}, \tag{20}$$

a diagonal matrix built of $\mathbf{L}_g$ matrices in a number equal to that of the nodes in the element. Knowing the element stiffness matrix and the element mass matrix in the common global co-ordinate system, they are assembled to represent a particular geometry with prescribed boundary conditions, in the global co-ordinate system.

The governing equation is obtained by substituting Eq. (12) in Eq. (11), resulting in:

$$\int_{t_1}^{t_2} \sum_{e=1}^{NE} \delta\mathbf{d}_e^{t}[\mathbf{K}^e\mathbf{d}_e + \mathbf{M}^e\ddot{\mathbf{d}}_e]dt = 0. \tag{21}$$

This relation is valid for every virtual displacement in an arbitrary time interval t_1 and t_2, we have

$$\sum_{e=1}^{NE}\mathbf{K}^e \mathrm{d}_e + \sum_{e=1}^{NE}\mathbf{M}^e\ddot{\mathbf{d}}_e = 0. \tag{22}$$

The global discrete equation for free vibration in matrix form can be written as

$$\mathbf{K}\mathbf{d} + \mathbf{M}\ddot{\mathbf{d}} = 0, \tag{23}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the global stiffness and mass matrices, respectively for the structure, $\mathbf{d}$ is the nodal displacement vector and $\ddot{\mathbf{d}}$ is the second derivative of the displacements of the structure with respect to time.

To find natural modes and frequencies, we assume that the field variables can be expressed as

$$\mathbf{d} = \bar{\mathbf{d}}e^{i\omega t}, \tag{24}$$

where $\bar{\mathbf{d}}$ is the vector of unknown amplitudes at time $t = 0$ at the nodes, and ω is the circular natural frequency of the system. Substituting Eq. (25) in Eq. (24), we get

$$(\mathbf{K} - \omega^2\mathbf{M})\bar{\mathbf{d}} = 0. \tag{25}$$

The above equation can be solved after imposing boundary conditions of the problem, by any standard eigenvalue problem solving technique.

The subspace iteration method is used here to obtain the numerical solution of the eigenvalue problems. Although all the eigenvalues and eigenvectors can be computed by this method for each deformation mode m and n, the dominant eigenvalues that correspond to the lower natural frequencies are of most concern.

4. Numerical results and discussion

A computer program has been developed, based on the foregoing theoretical formulation, for the free vibration analysis of laminated composite and sandwich plates and shells. An 8×8 mesh and a 16×16 mesh of nine-noded Lagrangian higher-order faceted shell elements have been used in the computations for plates and shells respectively. This scheme is arrived at on the basis of a convergence study in which the fundamental natural frequency converges monotonically from a higher value. The details of the convergence study are not presented for the sake of brevity. The full integration scheme (3×3) is used. A parallel computer code was also developed based on the Reissner–Mindlin’s first-order shear deformation theory (FOST) in order to compare its results with those of higher-order shear deformation theory (HOST). A shear correction factor of 5/6 is used with this theory.

To demonstrate the efficiency and versatility of the present formulation, a considerable number of examples, including isotropic, laminated composite and sandwich plates and shells with simply supported boundary conditions are investigated. The following simply supported boundary condition is used:

$$u_0 = w_0 = \theta_y = u_0^{\bullet} = \theta_y^{\bullet} = 0, \quad \text{at } x = 0, a,$$
$$v_0 = w_0 = \theta_x = v_0^{\bullet} = \theta_x^{\bullet} = 0, \quad \text{at } y = 0, b.$$

Results have been validated by comparing them with those of 3-D elasticity theory in addition to analytical and finite element analysis results available in the literature.

4.1. Thick isotropic plate

The non-dimensional natural frequencies of a thick isotropic square plate with a side-to-thickness ratio of 10 were first obtained to check the numerical accuracy of the present finite element formulation and results are listed in Table 1. The non-dimensional natural frequencies obtained by present FOST and HOST is found to be in excellent agreement with the exact solutions of the linear three-dimensional theory of elasticity. Classical plate theory over-predicts the natural frequencies.

4.2. Cross-ply $(0°/90°)_s$ laminated composite plate

Simply supported cross-ply laminates having antisymmetric laminations are considered here for free vibration analysis. The laminates are square in plan form with $a/h = 5$. The non-dimensional fundamental frequencies, for various degree of orthotropy of individual layers ($E_1/E_2 = 3$, 10, 20, 30, 40) are obtained by the present higher-order and first-order finite element formulations. The number of layers of antisymmetric cross-ply $(0/90)_s$ is varied from 2 to 10. The results are presented in Table 2 along with those obtained by 3-D elasticity results of Noor [7] and finite element results using a higher-order plate formulation given by Putcha and Reddy [21]. The corresponding classical lamination plate theory (CPT) results given by Putcha and Reddy [21] are also included. The orthotropic material properties in all the above laminates considered are $E_1/E_2 = open$, $E_2 = E_3$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$.

For all the laminate types considered, the results of the present higher order theory show excellent agreement with the 3-D elasticity solutions. The present results are much closer to elasticity solutions than those obtained by Putcha and Reddy [21] and present first-order theory. It is observed that the fundamental frequency increases with the increase in number of layers as well as increase of degree of orthotropy. The Classical lamination plate theory, as observed earlier, over-predicts the fundamental frequency and the effect of transverse shear deformation is seen to increase with increasing degree of material orthotropy.

4.3. Antisymmetric angle-ply (45°/−45°/45°/−45°) laminated composite plate

The effect of aspect ratio and side to thickness ratio on the non-dimensional fundamental frequency for a

Table 1
Non-dimensionalized natural frequencies $\bar{\omega} = \omega(\rho h^2/G)^{1/2}$ of a square simply supported plate, $\nu = 0.3$, $a/h = 10$

Mode		3-D elasticity theory [31]	Present FOST	Present HOST	Reddy and Phan [18]		
m	n				FSDPT	HSDPT	CPT
1	1	0.0932	0.09305 (−0.16)	0.09306 (−0.15)	0.0930 (−0.21)	0.0931 (−0.11)	0.0955 (2.47)
2	1	0.2226	0.22231 (−0.13)	0.22236 (−0.11)	0.2219 (−0.31)	0.2222 (−0.18)	0.2360 (6.02)
2	2	0.3421	0.34105 (−0.31)	0.34116 (−0.27)	0.3406 (−0.44)	0.3411 (−0.29)	0.3732 (9.09)
1	3	0.4171	0.41727 (0.04)	0.41759 (0.12)	0.4149 (−0.53)	0.4158 (−0.31)	0.4629 (10.98)
2	3	0.5239	0.52257 (−0.25)	0.52298 (−0.17)	0.5206 (−0.63)	0.5221 (−0.34)	0.5951 (13.59)
1	4	–	0.66030	0.66150	0.6520	0.6545	0.7668
3	3	0.6889	0.68621 (−0.39)	0.68697 (−0.28)	0.6834 (−0.80)	0.6862 (−0.39)	0.8090 (17.43)
2	4	0.7511	0.75198 (0.12)	0.75119 (0.01)	0.7446 (−0.86)	0.7481 (−0.40)	0.8926 (18.84)
3	4	–	0.89669	0.89845	0.8896	0.8949	1.0965
1	5	0.9268	0.93937 (1.36)	0.94267 (1.71)	0.9174 (−1.01)	0.9230 (−0.41)	1.1365 (22.63)
2	5	–	0.99479	0.99479	0.9984	1.0053	1.2549
4	4	1.0889	1.08618 (−0.25)	1.08900 (−0.00)	1.0764 (−1.15)	1.0847 (−0.38)	1.3716 (25.96)

Numbers in parentheses are the percentage error with respect to three-dimensional elasticity values of Srinivas et al. [31].

Table 2
Non-dimensionalized fundamental frequencies $\bar{\omega} = \omega(\rho h^2/E_2)^{1/2}$ of a simply supported cross-ply square laminated plate with $a/h = 5$

Lamination and number of layers	Source	E_1/E_2				
		3	10	20	30	40
$(0/90)_1$	Noor [7]	0.25031	0.27938	0.30698	0.32705	0.34250
	Present FOST	0.24837	0.27759	0.30826	0.33287	0.35335
		(−0.77)	(−0.64)	(−0.42)	(1.78)	(3.17)
	Present HOST	0.24869	0.27843	0.30810	0.33069	0.34870
		(−0.65)	(−0.34)	(0.36)	(1.11)	(1.81)
	Putcha and Reddy [21]	0.24868	0.27955	0.31284	0.34020	0.36348
		(−0.65)	(0.06)	(1.91)	(4.02)	(6.12)
	CPT	0.27082	0.30968	0.35422	0.39335	0.42884
		(8.19)	(10.84)	(15.39)	(20.27)	(25.21)
$(0/90)_2$	Noor [7]	0.26182	0.32578	0.37622	0.40660	0.42719
	Present FOST	0.26019	0.32900	0.38756	0.42481	0.45085
		(−0.62)	(0.99)	(3.01)	(4.48)	(5.54)
	Present HOST	0.25985	0.32514	0.37794	0.41023	0.43225
		(−0.75)	(−0.20)	(0.46)	(0.89)	(1.18)
	Putcha and Reddy [21]	0.26003	0.32782	0.38506	0.42139	0.44686
		(−0.68)	(0.63)	(2.35)	(3.64)	(4.60)
	CPT	0.28676	0.38877	0.49907	0.58900	0.66690
		(9.52)	(19.33)	(32.65)	(44.86)	(56.11)
$(0/90)_3$	Noor [7]	0.26440	0.33657	0.39359	0.42783	0.45091
	Present FOST	0.26222	0.33666	0.39764	0.43525	0.46100
		(−0.82)	(0.03)	(1.03)	(1.73)	(2.24)
	Present HOST	0.26204	0.33459	0.39260	0.42775	0.45158
		−0.89	(−0.59)	(−0.25)	(−0.02)	(0.15)
	Putcha and Reddy [21]	0.26223	0.33621	0.39672	0.43419	0.46005
		(−0.82)	(−0.11)	(0.79)	(1.49)	(2.03)
	CPT	0.28966	0.40215	0.52234	0.61963	0.70359
		(9.55)	(19.48)	(32.71)	(44.83)	(56.04)
$(0/90)_5$	Noor [7]	0.26583	0.34250	0.40337	0.44011	0.46498
	Present FOST	0.26337	0.34055	0.40257	0.44025	0.46579
		(−0.92)	(−0.57)	(−0.20)	(0.03)	(0.17)
	Present HOST	0.26334	0.33990	0.40106	0.43819	0.46345
		(−0.94)	(−0.76)	(−0.57)	(−0.44)	(−0.33)
	Putcha and Reddy [21]	0.26337	0.34050	0.40270	0.44079	0.46692
		(−0.92)	(−0.58)	(−0.17)	(0.15)	(0.42)
	CPT	0.29115	0.40888	0.53397	0.63489	0.72184
		(9.52)	(19.38)	(32.37)	(44.25)	(55.24)

Numbers in parentheses are the percentage error with respect to three-dimensional elasticity results of Noor [7].

45° antisymmetric laminated composite simply supported plate is presented in Table 3. The orthotropic material properties for the antisymmetric angle ply laminates considered are $E_1/E_2 = 40$, $G_{12} = 0.6E_2$, $G_{13} = G_{23} = 0.5E_2$, $\nu_{12} = 0.25$. Results of HOST are in good agreement with the closed form solution of Bert and Chen [8] and finite element solutions using higher-order shear deformation theory given by Shankara and Iyengar [41], but the results of the FOST and finite element results of Reddy [9] using laminated anisotropic plate theory of Yong et al. [3] are much higher.

4.4. Rectangular honeycomb sandwich plate

The natural frequencies of a 72 in. long by 48 in. wide simply supported sandwich plate are presented in Table 4. The plate has two identical aluminium faceplates, 0.016 in. thick, and an aluminium honeycomb core, 0.25 in. thick. The physical properties for the faceplates are

$$E = 1.00 \times 10^7\ \mathrm{lb/in^2},\quad G = 3.76 \times 10^6\ \mathrm{lb/in^2},$$
$$\nu = 0.33,\quad \rho = 2.59 \times 10^{-4}\ \mathrm{lb\,s^2/in^4}$$

and for the core

$$G_{yz} = 7500\ \mathrm{lb/in^2},\quad G_{xz} = 19\,500\ \mathrm{lb/in^2},$$
$$\rho = 1.14 \times 10^{-5}\ \mathrm{lb\,s^2/in^4}.$$

To be consistent with the simple representation of core behaviour of Ref. [52], it is assumed that core in-plane direct and shear stiffnesses are zero. The results of present theories are in good agreement with those of

Table 3
Non-dimensionalized fundamental frequencies $\bar{\omega} = \omega a^2(\rho/E_2h^2)^{1/2}$ of a simply supported antisymmetric angle-ply laminate [45°/−45°/45°/−45°]

a/h	Source	Aspect ratio (a/b)						
		0.2	0.6	0.8	1.0	1.2	1.6	2.0
10	Reddy [9]	8.724	12.965	15.712	18.609	21.567	27.736	34.247
	Bert and Chen [8]	8.664	12.82	15.54	18.46	21.51	27.95	34.87
	Shankara [41]	8.5557	12.5588	15.1802	17.9735	20.8797	26.9916	33.5534
	Present FOST	8.9601	13.0028	15.6867	18.5627	21.5347	27.8101	34.5742
	Present HOST	8.7898	12.3692	14.7843	17.4136	20.1811	26.1788	32.7950
20	Reddy [9]	9.475	14.896	18.557	22.584	26.857	36.249	46.789
	Bert and Chen [8]	9.300	14.45	17.97	21.87	26.12	35.56	46.26
	Shankara [41]	9.3011	14.3856	17.8458	21.6808	25.8363	35.0421	45.4096
	Present FOST	9.7382	14.9123	18.4897	22.4810	26.7672	36.2992	47.1127
	Present HOST	9.6614	14.5341	17.8993	21.6564	25.6981	34.7466	45.1144
30	Reddy [9]	9.667	15.385	19.304	23.676	28.381	38.940	51.132
	Bert and Chen [8]	9.436	14.84	18.56	22.74	27.35	37.82	49.98
	Shankara [41]	9.4880	14.8427	18.5390	22.6911	27.2555	37.5907	49.5474
	Present FOST	9.9205	15.3890	19.2213	23.5546	28.2727	38.9952	51.4993
	Present HOST	9.8792	15.1665	18.8622	23.0340	27.5692	37.8867	49.9614
40	Reddy [9]	9.759	15.853	19.604	24.118	29.003	40.071	53.012
	Bert and Chen [8]	9.485	14.98	18.78	23.08	27.83	38.72	51.52
	Shankara [41]	9.5724	15.0248	18.8134	23.0940	27.8286	38.6523	51.3324
	Present FOST	9.9769	15.5495	19.4762	23.9405	28.8290	40.0441	53.2926
	Present HOST	9.9514	15.4074	19.2434	23.5965	28.3534	39.2591	52.1559
50	Reddy [9]	9.816	15.689	19.759	24.343	29.321	40.653	53.989
	Bert and Chen [8]	9.507	15.04	18.89	23.24	28.06	39.17	52.29
	Shankara [41]	9.6216	15.1177	18.9510	23.2956	28.1168	39.1932	52.2539
	Present FOST	10.0105	15.6350	19.6094	24.1400	29.1150	40.5835	54.2223
	Present HOST	9.9932	15.5375	19.4485	23.8997	28.7779	40.0099	53.3680

Table 4
Natural frequencies (Hz) of a simply supported rectangular honeycomb sandwich plate

Mode	Present FOST	Present HOST	Raville [52]	Zhou and Li [53]	Yuan and Dawe [54]	Bardell et al. [55]
1	23.5994	23.4841	23	23.29	23.41	23.05
2	45.4641	44.9546	44	44.47	44.64	43.91
3	73.7553	72.4881	71	71.15	71.50	71.06
4	82.6571	80.5639	80	78.78	79.26	78.37
5	95.2236	93.4366	91	91.57	92.19	90.85
6	131.7518	128.2182	126	125.10	125.94	123.82

Table 5
Non-dimensionalized fundamental frequencies $\bar{\omega} = \omega a^2/h(\rho/E_2)_f^{1/2}$ of a simply supported antisymmetric (0°/90°/core/0°/90°) sandwich plate with $a/b = 1$ and $t_c/t_f = 10$

a/h	Present FOST	Present HOST	Kant and Swaminathan [56]		Reddy [17][a]	Whiteny and Pagano [57][a]
			Model-1	Model-2		
02	4.5076	1.1938	1.1941	1.1734	1.6252	5.2017
04	9.0160	2.1334	2.1036	2.0913	3.1013	9.0312
10	13.9939	4.9522	4.8594	4.8519	7.0473	13.8694
20	15.6078	8.7303	8.5955	8.5838	11.2664	15.5295
30	15.9816	11.2250	11.0981	11.0788	13.6640	15.9155
40	16.1232	12.7874	12.6821	12.6555	14.4390	16.0577
50	16.1921	13.7764	13.6899	13.6577	15.0323	16.1264
60	16.2364	14.4226	14.3497	14.3133	15.3868	16.1612
70	16.2664	14.8582	14.7977	14.7583	15.6134	16.1845
80	16.2760	15.1539	15.1119	15.0702	15.7660	16.1991
90	16.2984	15.3793	15.3380	15.2946	15.8724	16.2077
100	16.3093	15.5421	15.5093	15.4647	15.9522	16.2175

[a] Results of these theories are as given by Kant and Swaminathan [56].

analytical results of Raville and Veng [52], spline finite point method results of Zhou and Li [53], spline finite strip method result of Yuan and Dawe [54] and finite element method results of Bardell et al. [55].

4.5. Antisymmetric (0°/90°/core/0°/90°) sandwich plate

The variation of fundamental frequency with respect to the side-to-thickness ratio (a/h) of a five-layer square sandwich plate with antisymmetric cross-ply face-sheets is given in Table 5. The thickness of the core to thickness of the flange (t_c/t_f) ratio is taken equal to 10. The physical properties for the face sheets are

$$E_1 = 19\times10^6\ \text{lb/in}^2,\quad E_2 = 1.5\times10^6\ \text{lb/in}^2,\quad E_2 = E_3,$$
$$G_{12} = G_{23} = 1\times10^6\ \text{lb/in}^2,\quad G_{13} = 0.90\times10^6\ \text{lb/in}^2,$$
$$\nu_{12} = \nu_{13} = 0.22,\quad \nu_{23} = 0.49,\quad \rho = 0.057\ \text{lb/in}^3$$

and for the core

$$E_1 = E_2 = E_3 = 1000\ \text{lb/in}^2,$$
$$G_{12} = G_{23} = G_{13} = 500\ \text{lb/in}^2,$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0,\quad \rho = 0.003403\ \text{lb/in}^3.$$

The fundamental frequencies obtained by the present HOST are in good agreement with the analytical solutions given by the Kant and Swaminathan [56] using two

Table 6
Natural frequencies ω_{mn} (rad/s) of isotropic cylindrical and spherical shell panels

Panels	Mode		3D Elasticity [36]	Present FOST	Present HOST	Shen and Wan [58]	Geannakakes and Wang [59]
	m	n					
Cylindrical	1	1	0.28016	0.28046 (0.11)	0.28055 (0.14)	0.28285 (0.96)	0.28220 (0.73)
	2	1	0.29214	0.29197 (−0.06)	0.29234 (0.07)	0.30285 (3.66)	0.31593 (8.14)
	1	2	0.49846	0.49969 (0.25)	0.49991 (0.29)	0.50551 (1.41)	0.49745 (−0.20)
	2	2	0.50971	0.50706 (−0.52)	0.50790 (−0.35)	0.52489 (2.98)	0.51843 (1.71)
	3	1	0.54862	0.55309 (0.81)	0.55358 (0.90)	0.57185 (4.23)	0.58479 (6.59)
	1	3	0.72361	0.72570 (0.29)	0.72608 (0.34)	0.73930 (2.17)	0.71976 (−0.53)
	3	2	0.72931	0.72638 (−0.40)	0.72773 (−0.22)	0.75758 (3.88)	0.75347 (3.31)
	2	3	0.79998	0.79629 (−0.46)	0.79752 (−0.31)	0.82441 (3.05)	0.80075 (0.10)
	4	1	0.92680	0.94051 (1.48)	0.94121 (1.55)	0.96993 (4.65)	0.97587 (5.29)
	3	3	1.01875	1.01109 (−0.75)	1.01323 (−0.54)	1.05842 (3.89)	1.05203 (3.27)
Spherical	1	1	0.52543	0.50211 (−4.44)	0.50223 (−4.41)	0.52835 (0.55)	0.53146 (1.14)
	2	1	0.58420	0.56247 (−3.72)	0.56276 (−3.66)	0.59151 (1.25)	0.59114 (1.19)
	1	2	0.58487	0.56248 (−3.83)	0.56277 (−3.78)	0.59253 (1.31)	0.59641 (1.97)
	2	2	0.67676	0.65706 (−2.91)	0.65788 (−2.79)	0.69040 (2.01)	0.68980 (1.93)
	3	1	0.75219	0.73915 (−1.73)	0.73966 (−1.66)	0.77070 (2.46)	0.76283 (1.41)
	1	3	0.75220	0.74035 (−1.57)	0.74081 (−1.51)	0.77307 (2.77)	0.77390 (2.88)
	3	2	0.87811	0.86359 (−1.65)	0.86493 (−1.50)	0.90372 (2.92)	0.89397 (1.81)
	2	3	0.87804	0.86360 (−1.64)	0.86494 (−1.49)	0.90744 (3.35)	0.89940 (2.43)
	4	1	1.06018	1.06310 (0.27)	1.06383 (0.34)	1.10283 (4.02)	1.09537 (3.32)
	1	4	1.06234	1.06310 (0.07)	1.06384 (0.14)	1.15263 (8.50)	1.13240 (6.59)

Numbers in parentheses are the percentage error with respect to three-dimensional elasticity results of Chern and Chao [36].

higher-order displacement models having 12 and 9 degrees of freedom respectively and using Reddy's [17] higher-order theory for laminated composite plates. Present FOST and the Whitney–Pagano's [57] first-order theory predict much higher frequencies, particularly for the lower a/h ratios.

4.6. Isotropic cylindrical and spherical panels

Natural frequencies of the first ten modes of the isotropic cylindrical and spherical panels of square planform are given in Table 6. The following physical and geometric properties are considered.

Curved lengths of shell panels $(a, b) = 1.0118$, Thickness of shell panels $(h) = 0.0191$, Radius $(R) = 1.91$, $E = 1$, $\nu = 0.3$, $\rho = 1$.

The natural frequencies of cylindrical and spherical panels obtained by the present FOST and HOST are compared with those of the 3-D elasticity results of Chern and Chao [36], B-spline function results of Shen and Wan [58] and the B_3-spline finite strips results of Geannakakes and Wang [59]. The natural frequencies for all modes obtained by present FOST and HOST are found to be in good agreement with the exact solutions of the three-dimensional elasticity theory.

4.7. Cross-ply (0°/90°) orthotropic cylindrical and spherical panels

The non-dimensional fundamental frequencies for shallow cross-ply orthotropic cylindrical and spherical shells obtained by the present theories has been given in Tables 7, 8 along with the frequencies given by the three-dimensional elasticity theory of Bhimaraddi [35], parabolic shear deformation theory (PSD), constant shear deformation theory (CSD) and thin shell theory (TST). The orthotropic material properties correspond to:

$$E_1 = 25E_2, \quad E_2 = E_3, \quad G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2, \quad \nu_{12} = 0.25, \quad \nu_{13} = 0.03, \quad \nu_{23} = 0.40.$$

The non-dimensional fundamental frequencies obtained by present FOST and HOST is found to be in good agreement with the exact solutions of the three-dimensional theory of elasticity except in the case of R/a ratio as 1 in orthotropic cross-ply spherical shell. The differences in the results of spherical shells are due to the inclined boundary conditions considered in the three-dimensional elasticity solutions, while boundary conditions considered in present finite element solutions of FOST and HOST are in global directions. In cylindrical shells a transformation is applied for inclined boundary conditions thereby getting excellent agree-

Table 7
Non-dimensionalized fundamental frequencies $\bar{\omega} = \omega a(\rho/E_2)^{1/2}$ of orthotropic cross-ply cylindrical shell.

R/a	h/a	Present FOST	Present HOST	Bhimaraddi [35]			
				3-D	PSD	CSD	TST
1	0.05	0.78567	0.78627	0.78683	0.79993	0.79798	0.80580
	0.10	1.04062	1.04357	1.04085	1.09189	1.07475	1.14313
	0.15	1.29578	1.30351	1.29099	1.38174	1.33274	1.54124
2	0.05	0.56568	0.56684	0.57252	0.58000	0.57733	0.58723
	0.10	0.91367	0.91801	0.93627	0.95664	0.93653	1.01398
	0.15	1.22101	1.23107	1.25377	1.28933	1.23527	1.45781
3	0.05	0.51084	0.51224	0.52073	0.52516	0.52222	0.53294
	0.10	0.88528	0.88993	0.91442	0.92642	0.90563	0.98505
	0.15	1.20295	1.21344	1.24500	1.20563	1.21316	1.43751
4	0.05	0.48994	0.49141	0.50110	0.50415	0.50109	0.51217
	0.10	0.87452	0.87942	0.90613	0.91506	0.89403	0.97408
	0.15	1.19547	1.20629	1.24090	1.25977	1.20454	1.42910
5	0.05	0.47978	0.48128	0.49167	0.49402	0.49091	0.50216
	0.10	0.86919	0.87402	0.90200	0.90953	0.88840	0.96870
	0.15	1.19147	1.20216	1.23849	1.25551	1.20020	1.42464
10	0.05	0.46566	0.46722	0.47859	0.47997	0.47677	0.48827
	0.10	0.86115	0.86603	0.89564	0.90150	0.88026	0.96074
	0.15	1.18452	1.19525	1.23374	1.24875	1.19342	1.41709
20	0.05	0.46190	0.46348	0.47509	0.47625	0.47304	0.48459
	0.10	0.85852	0.86340	0.89341	0.89904	0.87779	0.95819
	0.15	1.18166	1.19238	1.23140	1.24626	1.19100	1.41400
∞	0.05	0.47374	0.47448	0.47365	0.47483	0.47161	0.48317
	0.10	0.89001	0.89493	0.89179	0.89761	0.87640	0.95661
	0.15	1.22349	1.23625	1.22905	1.24437	1.18923	1.41139

Table 8
Non-dimensionalized fundamental frequencies $\bar{\omega} = \omega a(\rho/E_2)^{1/2}$ of orthotropic cross-ply spherical shell

R/a	h/a	Present FOST	Present HOST	Bhimaraddi [35]			
				3-D	PSD	CSD	TST
1	0.05	1.20901	1.20915	1.29835	1.32595	1.32483	1.33000
	0.10	1.42129	1.42323	1.39974	1.49075	1.48008	1.52391
	0.15	1.64689	1.65369	1.51936	1.68141	1.64797	1.78940
2	0.05	0.79140	0.79172	0.79577	0.81059	0.80870	0.81618
	0.10	1.08551	1.08886	1.05528	1.09708	1.08054	1.14507
	0.15	1.36643	1.37633	1.31111	1.38083	1.33375	1.52705
3	0.05	0.64333	0.64379	0.64044	0.64949	0.64713	0.65602
	0.10	0.98501	0.98900	0.96917	0.99330	0.97455	1.04657
	0.15	1.28989	1.30102	1.26650	1.30815	1.25698	1.46512
4	0.05	0.57653	0.57708	0.57419	0.58038	0.57775	0.58749
	0.10	0.94350	0.94778	0.93637	0.95306	0.93332	1.00862
	0.15	1.25914	1.27077	1.25032	1.28092	1.22810	1.44211
5	0.05	0.54091	0.54151	0.54039	0.54500	0.54219	0.55247
	0.10	0.92202	0.92644	0.92065	0.93361	0.91338	0.99034
	0.15	1.24310	1.25495	1.24272	1.26797	1.21434	1.43120
10	0.05	0.48706	0.48776	0.49127	0.49341	0.49031	0.50149
	0.10	0.88986	0.89445	0.89912	0.90679	0.88584	0.96519
	0.15	1.21845	1.23053	1.23249	1.25034	1.19559	1.41639
20	0.05	0.46995	0.47070	0.47812	0.47955	0.47636	0.48782
	0.10	0.87685	0.88140	0.89363	0.89992	0.87877	0.95876
	0.15	1.20639	1.21832	1.22992	1.24586	1.19083	1.41264
∞	0.05	0.47374	0.47448	0.47365	0.47483	0.47161	0.48317
	0.10	0.89001	0.89493	0.89179	0.89761	0.87640	0.95661
	0.15	1.22349	1.23625	1.22905	1.24437	1.18923	1.41139

ment in three-dimensional and finite element solutions. It is observed that PSD and TST consistently over-predict the frequency as compared to the three-dimensional theory of elasticity.

5. Conclusions

A simple C^0 isoparametric finite element formulation based on the higher-order shear deformation theory using a higher-order facet shell element is presented for the free vibration analysis of fibre reinforced composite and sandwich laminates. The accuracy of the present formulation is evaluated by obtaining the solutions to a wide range of problems and comparing them with the available three-dimensional closed-form and finite element solutions alongwith the solutions obtained using first-order shear deformation theory. The parametric effects of degree of orthotropy, length-to-thickness ratio, plate aspect ratio, number of layers and fibre orientation upon the frequencies and mode shapes are discussed. The results show that the difference in predictions of FOST and HOST are small for composite laminates. However, for sandwich panels, in comparison to HOST, FOST over-predicts the natural frequency by a significant margin and the margin increases as the thickness of laminate increases. It is shown that the element has fairly good accuracy and is promising for further engineering applications.

Appendix A. Rigidity matrices

Assuming, $H_i = (z_{L+1}^i - z_L^i)/i$, where i takes an integer value from 1–7, the elements of the submatrices of the rigidity matrix can be readily obtained in the following forms:

$$\mathbf{D}_\mathrm{m} = \sum_{L=1}^{NL} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{13}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_3 \\ & Q_{22}H_1 & Q_{23}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_3 \\ & & Q_{33}H_1 & Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_3 \\ & & & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_5 \\ & \text{Symm.} & & & Q_{22}H_5 & Q_{23}H_5 \\ & & & & & Q_{33}H_5 \end{bmatrix},$$

$$\mathbf{D}_\mathrm{s} = \sum_{L=1}^{NL} \begin{bmatrix} Q_{44}H_1 & Q_{45}H_1 & Q_{44}H_3 & Q_{45}H_3 & Q_{44}H_2 & Q_{45}H_2 \\ & Q_{55}H_1 & Q_{45}H_3 & Q_{55}H_3 & Q_{45}H_2 & Q_{55}H_2 \\ & & Q_{44}H_5 & Q_{45}H_5 & Q_{44}H_4 & Q_{45}H_4 \\ & & & Q_{55}H_5 & Q_{45}H_4 & Q_{55}H_4 \\ & \text{Symm.} & & & Q_{44}H_3 & Q_{45}H_3 \\ & & & & & Q_{55}H_3 \end{bmatrix}.$$

The elements of the $\mathbf{D}_\mathrm{c}$ and $\mathbf{D}_\mathrm{b}$ matrices are obtained by replacing (H_1, H_3 and H_5) by (H_2, H_4 and H_6) and (H_3, H_5 and H_7) respectively in the $\mathbf{D}_\mathrm{m}$ matrix.

Appendix B. Inertia matrix

The inertia matrix **m** for the present higher order theory is given by

$$\mathbf{m} = \begin{bmatrix} I_1 & 0 & 0 & 0 & I_2 & I_3 & 0 & 0 & I_4 \\ 0 & I_1 & 0 & -I_2 & 0 & 0 & I_3 & -I_4 & 0 \\ 0 & 0 & I_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & I_3 & 0 & 0 & -I_4 & I_5 & 0 \\ I_2 & 0 & 0 & 0 & I_3 & I_4 & 0 & 0 & I_5 \\ I_3 & 0 & 0 & 0 & I_4 & I_5 & 0 & 0 & I_6 \\ 0 & I_3 & 0 & -I_4 & 0 & 0 & I_5 & -I_6 & 0 \\ 0 & -I_4 & 0 & I_5 & 0 & 0 & -I_6 & I_7 & 0 \\ I_4 & 0 & 0 & 0 & I_5 & I_6 & 0 & 0 & I_7 \end{bmatrix}.$$

The parameters I_1, I_2 and (I_5, I_7) are linear inertia, rotary inertia and higher-order inertia terms respectively. The parameters, I_2, I_4 and I_6 are the coupling inertia terms. They are defined as follows:

$$(I_1, I_2, I_3, I_4, I_5, I_6, I_7) = \sum_{L=1}^{NL} \int_{Z_L}^{Z_{L+1}} (1, z, z^2, z^3, z^4, z^5, z^6)\rho_L \, dz,$$

where ρ_L is the material density of the Lth layer.

Appendix C. Matrix L_g

Matrix $\mathbf{L}_g$ for the transformation of nodal displacement may be defined using a matrix λ, a 3×3 matrix of direction cosines of the angles formed between the two sets of axes i.e.,

$$\lambda = \begin{bmatrix} \lambda_{xX} & \lambda_{xY} & \lambda_{xZ} \\ \lambda_{yX} & \lambda_{yY} & \lambda_{yZ} \\ \lambda_{zX} & \lambda_{zY} & \lambda_{zZ} \end{bmatrix},$$

in which $\lambda_{xX} =$ cosine of angle between x (local) and X (global) axes etc.

The matrix $\mathbf{L}_g$ is then given by,

$$\mathbf{L}_g = \begin{bmatrix} \lambda_{xX} & \lambda_{xY} & \lambda_{xZ} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_{yX} & \lambda_{yY} & \lambda_{yZ} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_{zX} & \lambda_{zY} & \lambda_{zZ} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{xX} & \lambda_{xY} & 0 & 0 & 0 & 0 & \lambda_{xZ} \\ 0 & 0 & 0 & \lambda_{yX} & \lambda_{yY} & 0 & 0 & 0 & 0 & \lambda_{yZ} \\ 0 & 0 & 0 & 0 & 0 & \lambda_{xX} & \lambda_{xY} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{yX} & \lambda_{yY} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{xX} & \lambda_{xY} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{yX} & \lambda_{yY} & 0 \\ 0 & 0 & 0 & \lambda_{zX} & \lambda_{zY} & 0 & 0 & 0 & 0 & \lambda_{zZ} \end{bmatrix}.$$

Element stiffness sub-matrix for each node has the size of 10×10 with 10th row and column as zero, but when the elements are coplanar, provision is made in the computer programme to modify the element stiffness matrix, introducing the drilling degree of freedom concept [60].

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