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On the angle ply higher order beam vibrations

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Abstract Vibrations of angle ply laminated beams are studied using the higher order theory and isoparametric 1d finite element formulations through proper constitution of elasticity matrix. Subsequent to the validation of the formulation, deep sandwich and composite beams are critically analyzed for various boundary conditions. Frequencies classified based on their spectrum are presented along with those of first order theories for comparison.

Keywords Angle ply · Laminates · Sandwich · Composite · Higher order theory · Beams · Free vibrations

1 Introduction

Studies on composite/ sandwich beam vibrations invariably focus only on cross ply laminates, except for few rare exceptions. Chandrashekhara et al [1,2] studied symmetric angle ply laminates through analytical solutions. Teh and Huang [3] employed finite element approach to study an angle ply graphite/ epoxy cantilever beam.

Though many works on cross ply beam vibrations are available in the open literature, only representative samples are cited here, while detailed discussions can be seen in ref. [6]. Ahmed [4] evaluated the vibration characteristics of sandwich beams using finite elements with six degrees of freedom per node. Abramovich and Livshits [5] studied the vibrations of unsymmetric but cross ply beams. Marur and Kant [6,7] studied the cross ply beam vibrations using higher order theories. Yildirim et al. [8] compared the classical and first order beam theories for symmetric cross-ply laminated beam vibrations. Matsunaga [9] studied the vibration of cross-ply beams using higher order theories.

The aim of this paper is to present a formulation to study symmetric and unsymmetric, deep and thin, angle ply beam vibrations through a higher order model with isoparametric one dimensional elements. Suitable conclusions are drawn from studying beams with various boundary conditions.

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2 Theoretical Formulation

The higher order displacement model, based on Taylor's series expansion [10], can be expressed, for a beam, as follows:

$$u = u_0 + z\theta_x + z^2u_0^* + z^3\theta_x^* \quad (1)$$

$$w = w_0 + z\theta_z + z^2w_0^* \quad (2)$$

where u_0 and w_0 are axial and transverse displacements in x–z plane, $\theta_x$is the face rotation about y-axis and $u_0^*, \theta_x^*, \theta_z, w_0^*$ are the higher order terms arising out of Taylor's series expansion and defined at the neutral axis.

The Lagrangian function, in the absence of external and damping forces can be given as,

$$\mathrm{L} = \mathrm{T} - \mathrm{U} \quad (3)$$

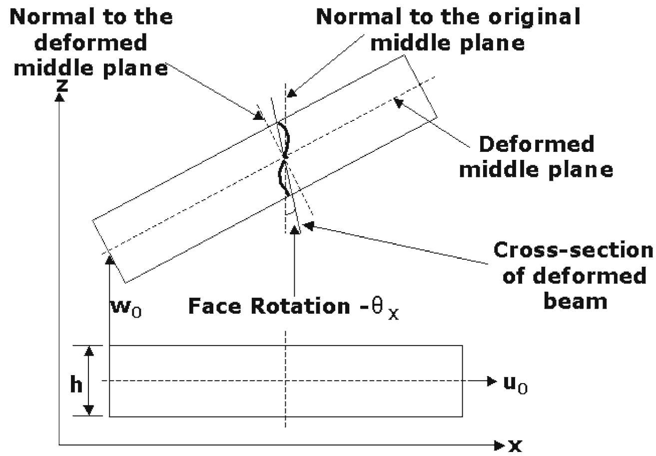

Fig. 1 Beam geometry with displacement components

where T is the kinetic energy and U is the internal strain energy. The same can be expressed as,

$$L = \frac{1}{2}\int \dot{u}^t \rho \dot{u} \mathrm{d}v - \frac{1}{2}\int \varepsilon^t \sigma \mathrm{d}v \tag{4}$$

where,

$$u = [u \;\; w]^t, \dot{u} = [\dot{u} \;\; \dot{w}]^t, \quad \varepsilon = [\varepsilon_x \;\; \varepsilon_z \;\; \gamma_{xz}]^t, \\ \sigma = [\sigma_x \;\; \sigma_z \;\; \tau_{xz}]^t \tag{4a}$$

The field variables can be expressed in terms of nodal degrees of freedom as,

$$u = Z_d d \tag{5}$$

where

$$d = [u_0 \;\; w_0 \;\; \theta_x \;\; u_0^* \;\; \theta_x^* \;\; \theta_z \;\; w_0^*]^t \tag{5a}$$

$$Z_d = \left[\begin{array}{ccccccc} 1 & 0 & z & z^2 & z^3 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & z & z^2 \end{array}\right] \tag{5b}$$

Now, the axial, transverse normal and shear strains for a beam can be expressed as,

$$\varepsilon_x = Z_a^t \varepsilon_a + Z_b^t \varepsilon_b \tag{6}$$

$$\varepsilon_z = Z_t^t \varepsilon_t \tag{7}$$

$$\gamma_{xz} = Z_s^t \gamma_s \tag{8}$$

where,

$$\varepsilon_a = [\varepsilon_{x0} \;\; \varepsilon_{x0}^*]^t = [u_{0,x} \;\; u_{0,x}^*]^t \tag{8a}$$

$$\varepsilon_b = [\kappa_x \;\; \kappa_x^*]^t = [\theta_{x,x} \;\; \theta_{x,x}^*]^t \tag{8b}$$

$$\varepsilon_t = [\varepsilon_{z0} \;\; \kappa_z]^t = [\theta_z \;\; 2w_0^*]^t \tag{8c}$$

$$\gamma_s = [\phi \;\; \phi^* \;\; \chi_{xz}]^t \\ = [(w_{0,x} + \theta_x)(w_{0,x}^* + 3\theta_x^*)(\theta_{z,x} + 2u_0^*)]^t \tag{8d}$$

$$Z_a = \left[\, 1 \; z^2 \,\right]^t \tag{8e}$$

$$Z_b = \left[\, z \; z^3 \,\right]^t \tag{8f}$$

$$Z_t = [1 \; z]^t \tag{8g}$$

$$Z_s = [1 \; z^2 \; z]^t \tag{8h}$$

The strains of eqn. (6–7) can be rewritten in a combined matrix form as,

$$\varepsilon_{xz} = [\varepsilon_x \; \varepsilon_z]^t = \overline{Z}\overline{\varepsilon} \tag{9}$$

where,

$$\bar{Z} = \left[\begin{array}{ccc|ccc} 1 & z^2 & 0 & z & z^3 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & z \end{array}\right] \tag{9a}$$

$$\overline{\varepsilon} = [\varepsilon_{x0} \;\; \varepsilon_{x0}^* \;\; \varepsilon_{z0} \;\; \kappa_x \;\; \kappa_x^* \;\; \kappa_z]^t \tag{9b}$$

The stress–strain relationship of an orthotropic lamina in a 3[d] state of stress can be expressed as [11],

$$\sigma^\circ = Q\varepsilon^\circ \tag{10}$$

where

$$\sigma^\circ = [\sigma_x \; \sigma_y \; \sigma_z \; \tau_{xy} \; \tau_{yz} \; \tau_{xz}]^t \tag{10a}$$

$$\varepsilon^\circ = [\varepsilon_x \; \varepsilon_y \; \varepsilon_z \; \gamma_{xy} \; \gamma_{yz} \; \gamma_{xz}]^t \tag{10b}$$

and Q is given in Appendix-1 [(40)–(53)].

By setting $\sigma_y, \tau_{xy}, \tau_{yz}$ equal to zero in eq. (10) and deriving the remaining stress components from the same eq. [12], one gets the stress–strain relationship as,

$$\sigma = C\varepsilon \tag{11}$$

where

$$\sigma = [\sigma_x \; \sigma_z \; \tau_{xz}]^t \tag{11a}$$

$$C = \left[\begin{array}{cc|c} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ \hline 0 & 0 & C_{33} \end{array}\right] \tag{11b}$$

and the expressions for various C matrix elements are given in Appendix-2 [(54)–(59)].

The internal strain energy can be evaluated using eqs. (8), (9) and (11) as,

$$U = \frac{1}{2}\int \varepsilon^t \sigma \, \mathrm{d}v = \frac{1}{2}\int \overline{\varepsilon}^t \overline{D}\overline{\varepsilon} \mathrm{d}x + \frac{1}{2}\int \gamma_s^t D_s \gamma_s \mathrm{d}x \quad (12)$$

where,

$$\bar{D} = b\int \bar{Z}^t \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \bar{Z} dz \quad (12a)$$

$$D_s = b\int Z_s C_{33} Z_s^t \mathrm{d}z \quad (12b)$$

and the expansions for $\overline{D}$ and D_s matrices are given in Appendix-3 [(60), (61)].

The kinetic energy can be expressed using eq. (5) as,

$$T = \frac{1}{2}\int (\dot{d}^t \overline{m}\dot{d}) \mathrm{d}x \quad (13)$$

where,

$$\overline{m} = b\sum_{L=1}^{NL} \int \left(z_d^t \rho_L z_d \right) \mathrm{d}z \quad (14)$$

where ρ_L is the mass density of a layer and $\overline{m}$ is given in Appendix 3 (62).

Now, the Lagrangian function can be re-stated with eqs. (12) and (13) as,

$$L = \frac{1}{2}\int (\dot{d}^t \overline{m}\dot{d}) \mathrm{d}x - \left(\frac{1}{2}\int (\overline{\varepsilon}^t \overline{D}\overline{\varepsilon}) \mathrm{d}x + \frac{1}{2}\int \left(\gamma_s^t D_s \gamma_s \right) \mathrm{d}x \right) \quad (15)$$

3 Finite element modeling

The displacements within an element can be expressed in terms of its nodal displacements in isoparametric formulations as,

$$d = N a_e \quad (16)$$

where a_e is a vector containing nodal displacement vectors of an element with n nodes and can be expressed as,

$$a_e = [d_1^t \; d_2^t, \ldots, d_n^t]^t \quad (17)$$

Similarly, the strains with in an element can be written through eqs. (5a), (9b) and (8d) as,

$$\overline{\varepsilon} = \overline{B} a_e \quad (18a)$$

$$\gamma_s = B_s a_e \quad (18b)$$

where, for a given node i, the strain displacement matrix can be computed as,

$$\bar{B} = \begin{bmatrix} N_{,x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N & 0 \\ 0 & 0 & N_{,x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{,x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}_i \quad (19)$$

$$B_s = \begin{bmatrix} 0 & N_{,x} & N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3N & 0 & N_{,x} \\ 0 & 0 & 0 & 2N & 0 & N_{,x} & 0 \end{bmatrix}_i \quad (20)$$

By substituting eqs. (16) and (18) in eq. (15), one gets,

$$L = \frac{1}{2}\dot{a}_e^t \int N^t \overline{m} N \mathrm{d}x \dot{a}_e - \frac{1}{2} a_e^t \int \left(\overline{B}^t \overline{D}\,\overline{B} + B_s^t D_s B_s \right) \mathrm{d}x a_e \quad (21)$$

Applying Hamilton's principle on L, we get the governing equation of motion as,

$$M\ddot{d} + Kd = 0 \quad (22)$$

where

$$M = \int N^t \overline{m} N \mathrm{d}x \quad (22a)$$

and

$$K = \int (\overline{B}^t \overline{D}\,\overline{B} + B_s^t D_s B_s) \mathrm{d}x \quad (22b)$$

This equation of motion can be solved by expressing the displacement vector as,

$$d = \overline{d} e^{i\omega t} = \overline{d}(\cos \omega t + i \sin \omega t) \quad (23)$$

where $\overline{d}$ is the modal vector and ω is the natural frequency. Substituting eq. (23) into eq. (22), one gets,

$$(K - \omega^2 M)\overline{d} = 0 \quad (24)$$

By solving eq. (24), using standard Eigen value solvers [13], after applying suitable boundary conditions, the natural frequencies and corresponding mode shapes are directly obtained.

4 Numerical experiments

Numerical experiments have been carried out to study the natural frequencies of angle ply beams. First, validation runs with the available results in the literature and subsequently, vibration analysis of deep beams have

Table 1 Data for numerical experiments

No	Details	Ref
Data 1 Material Data		[1,2]
	$E_1 = 21 \times 10^6$ psi, $E_2 = E_3 = 1.4\times10^6$ psi	
	$G_{12} = G_{13} = 0.6 \times 10^6$ psi, $G_{23} = 0.5 \times 10^6$ psi	
	$\rho = 0.13\times10^{-3}$ lb s^2 /in^4	
	$\nu = 0.3$, $S/t = 15$, $t = 1$ in	
	FNP: $\omega S^2\sqrt{\frac{\rho}{E_1 t^2}}$	
	BC: CF	
Data 2		[3]
	$E_{11} = 1.8726 \times 10^7$ psi, $E_{22} = E_{33} = 1.3638 \times 10^6$ psi	
	$G_{12} = 6.242 \times 10^5$ psi, $G_{23} = 3.686 \times 10^5$ psi, $G_{13} = 7.479 \times 10^5$ psi	
	$\rho = 1.427\times 10^{-4}$ lb s^2 /in^4	
	$\nu = 0.3$, $b = 1.2$ in, $t = 0.125$ in, $S = 7.5$ in	
	FNP: $\omega S^2\sqrt{\frac{12\rho}{E_1 t^2}}$	
	BC: CF	
Data 3		[5]
	$E_1 = 1.45 \times 10^{11}$ N/m^2, $E_2 = E_3 = 9.6 \times 10^9$ N/m^2	
	$G_{12} = G_{13} = 4.1 \times 10^9$ N/m^2, $G_{23} = 3.4 \times 10^9$ N/m^2	
	$\rho = 1.0$ N s^2 /m^4	
	$\nu = 0.3$, $b = 1\ m, t = 1\ m, S = 10$ m	
	FNP: As in data 1	
Data 4		[14,15]
	$S = 30$ in, $b = 1$ in, $S/t = 5$	
	Face: graphite/ epoxy	
	t_f (top, bot) = 0.6 in	
	$E_x = 0.1742\times10^8$ psi	
	$E_y = E_z = 0.1147\times10^7$ psi	
	$G_{xy} = G_{yz} = G_{xz} = 0.7983 \times 10^6$ psi	
	$\rho = 0.1433\times10^{-3}$ lb.s^2 /in^4	
	$\nu = 0.3$	
	Core: aluminium honeycomb (0.25 in cell size, 0.007 in foil)	
	$t_c = 4.8$ in	
	$E_x = E_y = E_z = G_{xy} = 0$	
	$G_{yz} = 0.1021 \times 10^5$ psi	
	$G_{xz} = 0.2042 \times 10^5$ psi	
	$\rho = 0.3098\times10^{-5}$ lb.s^2 /in^4	
	$t_c/t_f = 8$	
	Lamination scheme: 0/30/45/60/core/60/45/30/0	
	FNP: $\omega S^2\sqrt{\frac{12\rho_f}{E_{fx} t^2}}$	
Data 5		[14,15]
	Lamination scheme: 0/15/core/0/60	
	Rest are same as data 4	
Data 6		[16]
	$E_x = 0.762 \times 10^8$ psi	
	$E_y = E_z = 0.3048 \times 10^7$ psi	
	$G_{xy} = G_{yz} = G_{xz} = 0.1524 \times 10^7$ psi	
	$\rho = 0.72567\times10^{-4}$ lb.s^2/in^4	
	$\nu = 0.25$, $S = 30$ in, $b = 1$ in, $S/t = 5$	
	Lamination scheme: 30/-30/30	
	FNP: as in data 2	
Data 7		[16]
	Lamination scheme: 0/45/-45/90	
	Rest are same as data 6	

Nomenclature:

S—Beam Length,
t—thickness of cross section,
S/t—Aspect ratio,
FNP—Frequency normalizing parameter,

FOST — First order shear theory, Shear correction factor — 1.2

Boundary Conditions (1 – arrested; 0 – free)

Simply supported (SS): $u_0 = u_0^* = w_0 = \theta_z = w_0^* = 1$, at $x = 0$ and $x = S$;

Clamped-clamped (CC): $u_0 = u_0^* = \theta_x = \theta_x^* = w_0 = \theta_z = w_0^* = 1$, at $x = 0$ and $x = S$;

Clamped-free (CF): $u_0 = u_0^* = \theta_x = \theta_x^* = w_0 = \theta_z = w_0^* = 1$, at $x = 0$

$u_0 = u_0^* = \theta_x = \theta_x^* = w_0 = \theta_z = w_0^* = 0$, at $x = S$

been taken up with the present model. All the details about various examples that have been solved are given in Table 1.

4.1 Validation experiments

Normalized natural frequencies of two cantilevers with cross ply lamination are taken up for comparison, as shown in Table 2 and 3. Subsequently, frequencies of a cross ply beam for all boundary conditions had been compared, as in Table 4. Through the close correlation observed between the present model and the earlier works, accuracy and adequacy of the higher order model is established. In order to benchmark the present higher order finite element formulation, results of a simply supported sandwich beam are compared with those from analytical solutions [18] and a close correlation had been observed between them, as in Table 5. It can be seen that the higher order frequencies are almost half of those from first order shear theory (FOST), based on Timoshenko's beam theory [17], as the higher order beam becomes more flexible with increased degrees of freedom.

4.2 Higher order experiments

Next, the normalized natural frequencies of symmetric and unsymmetric composite and sandwich beams with an aspect ratio of five are studied (Data 4–7, Table 6–9) for various boundary conditions. The frequencies obtained through the higher order model are classified and compared according to their modes of vibration – axial, flexural and shear.

It can be seen for symmetric sandwich beams that higher order predictions are less by 25–50% of those of FOST, for all modes of vibrations and for all boundary conditions. Similarly, higher order predictions are lower for unsymmetric configuration.

In the case of composites the order of difference between the two is marginal, while the higher order frequencies are relatively higher. A similar pattern observed even for cross ply beams had been discussed in great detail in ref. [6,7].

Table 2 Normalized natural frequencies of angle ply cantilever (Data 1)

Present	Ref. [2]	Present	Ref. [2]	Present	Ref. [2]
45/-45/-45/45		30/50/50/30		0/90/90/0	
0.2848	0.3143	0.3485	0.3818	0.9214	0.9231
1.7445	1.9177	2.1169	2.3069	4.8919	4.8884
4.7206	5.1597	5.6634	6.1268	11.4758	11.4331
8.8415	9.5963	10.4645	11.2242	18.8162	18.6890
13.8763	14.9197	16.1992	17.2044	26.4660	26.2033

Table 3 Normalized natural frequencies of composite cantilever (Data 2)

Lamina angle =15°		Lamina angle =75°		Mode
Present	Ref. [3]	Present	Ref. [3]	
2.1113	2.2300	0.9470	0.9600	Bending
13.1787	13.2300	5.9250	5.9900	Bending
36.6660	35.1100	16.5459	16.7200	HO shear
71.1954	63.8700	32.2984	32.6300	HO shear
116.3320	104.8700	53.1277	53.6500	HO shear

5 Conclusions

A higher order model with transverse shear and normal strain components is formulated, with 1d elements, for studying the free vibrations of cross ply beams. Through the constitutive relationship, adapted from the 3d stress–strain relationship of an orthotropic lamina, even angle-ply laminates can be analyzed using beam formulations. The frequency responses of higher order model with sandwich and composite material are presented for various end conditions.

Appendix 1

The stress–strain relationship at a point of an orthotropic lamina in a 3d state of stress/ strain can be expressed, along the lamina axes, as [11],

$$\sigma' = D\varepsilon' \tag{25}$$

where

$$\sigma' = \left[\sigma_1\ \sigma_2\ \sigma_3\ \tau_{12}\ \tau_{23}\ \tau_{13}\right] \tag{26}$$

$$\varepsilon' = \left[\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \gamma_{12}\ \gamma_{23}\ \gamma_{13}\right] \tag{27}$$

$$D = \frac{1}{\Delta}\left[\begin{array}{ccc|ccc} E_1(1 - \nu_{23}\nu_{32}) & E_1(\nu_{21}+\nu_{31}\nu_{23}) & E_1(\nu_{31}+\nu_{21}\nu_{32}) & 0 & 0 & 0 \\ E_2(\nu_{12}+\nu_{13}\nu_{32}) & E_2(1 - \nu_{13}\nu_{31}) & E_2(\nu_{32}+\nu_{12}\nu_{31}) & 0 & 0 & 0 \\ E_3(\nu_{13}+\nu_{12}\nu_{23}) & E_3(\nu_{23}+\nu_{21}\nu_{13}) & E_3(1 - \nu_{12}\nu_{21}) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \Delta G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta G_{13} \end{array}\right] \tag{28}$$

Table 4 Comparison of normalized natural frequencies symmetric cross ply beam (data 3)

Lamination scheme: 0/90/90/0					
SS		CF		CC	
Present	Ref. [5]	Present	Ref. [5]	Present	Ref. [5]
2.3094	2.3194	0.8846	0.8819	3.7090	3.7570
6.9808	7.0029	4.1214	4.0259	7.8271	7.8700
12.0500	12.0370	9.0231	9.1085	12.5878	12.5730
17.1358	17.0150	11.4422 (a)	12.193 (a)	17.5105	17.3730
22.2158	21.9070	14.0717	14.0800	22.5217	22.2000
22.7246 (a)	23.337 (a)	19.2073	19.0660	22.7246 (a)	23.337 (a)
27.3044	26.7360	24.3004	23.9380	27.5612	27.2540

a Axial frequencies; rest are flexural frequencies

Table 5 Normalized natural frequencies of symmetric sandwich beam: data-4 (0/90/core/90/0)

Mode	Analytical—First order theory Ref. [18]	Analytical—higher order theory Ref. [18]	Present
1	2.2290	1.2930	1.3184
2	5.5150	2.7440	2.8351
3	8.6990	4.0730	4.3621

$$\Delta = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31}) \quad (29)$$

The relation between engineering and tensor strain vectors, along lamina and laminate axes, can be given as

$$\varepsilon' = R\varepsilon'_{ts} \quad (30)$$

$$\varepsilon^\circ = R\varepsilon^\circ_{ts} \quad (31)$$

where

$$R = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&2&0&0 \\ 0&0&0&0&2&0 \\ 0&0&0&0&0&2 \end{bmatrix} \quad (32)$$

If the angle between lamina and laminate axes can be defined as α, then the lamina to laminate axis transformation is given by,

$$T = \begin{bmatrix} c^2 & s^2 & 0 & 2sc & 0 & 0 \\ s^2 & c^2 & 0 & -2sc & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -sc & sc & 0 & (c^2 - s^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{bmatrix} \quad (33)$$

where,

$$c = \cos\,\alpha$$
$$s = \sin\,\alpha \quad (34)$$

and the stress and strain along the lamina and laminate axes can be equated as,

$$\sigma' = T\sigma^\circ \quad (35)$$

$$\varepsilon'_{ts} = T\varepsilon^\circ_{ts} \quad (36)$$

By making use of eqs. (30–36), one can get the laminate stress–strain relationship as,

$$\sigma^\circ = Q\varepsilon^\circ \quad (37)$$

where

$$Q = T^{-1}D(T^{-1})^t \quad (38)$$

$$(T^{-1})^t = RTR^{-1} \quad (39)$$

$$Q = \left[\begin{array}{cccc|cc} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & 0 & 0 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q_{55} & Q_{56} \\ 0 & 0 & 0 & 0 & Q_{65} & Q_{66} \end{array}\right] \quad (40)$$

$$Q_{11} = D_{11}c^4 + 2(D_{12} + 2D_{44})s^2c^2 + D_{22}s^4 \quad (41)$$

$$Q_{12} = D_{12}(s^4 + c^4) + (D_{11} + D_{22} - 4D_{44})s^2c^2 \quad (42)$$

$$Q_{13} = D_{31}c^2 + D_{32}s^2 \quad (43)$$

$$Q_{14} = (D_{11} - D_{12} - 2D_{44})sc^3 + (D_{12} - D_{22} + 2D_{44})s^3c \quad (44)$$

$$Q_{22} = D_{11}s^4 + 2(D_{12} + 2D_{44})s^2c^2 + D_{22}c^4 \quad (45)$$

$$Q_{23} = D_{13}s^2 + D_{23}c^2 \quad (46)$$

Table 6 Normalized natural frequencies of symmetric sandwich beam: data 4 (0/30/45/60/core/60/45/30/0)

SS				CC				CF			
Mode	Present	Mode	FOST	Mode	Present	Mode	FOST	Mode	Present	Mode	FOST
Axial frequencies											
7	30.3061	4	30.2873	6	30.3061	4	30.2873	4	15.2445	3	15.1437
12	47.5957	9	60.5747	11	47.5957	8	60.5747	11	43.2504	7	45.4310
13	47.7246	14	90.8621	12	47.7246	13	90.8621	12	47.2367	12	75.7184
Bending frequencies											
1	4.3382	1	6.8358	1	5.0333	1	9.5010	1	2.0060	1	2.7615
2	9.6090	2	18.2552	2	10.1534	2	19.0400	2	6.4712	2	10.7782
3	14.8964	3	29.7059	3	15.6885	3	30.0317	3	12.2122	4	22.4397
5	20.3614	6	40.9185	4	21.3861	5	40.8797	5	17.4739	5	32.9357
6	26.0888	8	51.9714	5	27.3707	7	51.9699	6	22.3598	6	42.1634
8	32.1321	10	62.9263	7	33.6420	9	63.0122	7	24.2600	8	47.1246
10	38.5257	11	73.8198	9	40.2871	10	73.5372	8	29.6224	9	57.1684
11	45.2900	13	84.6743	10	47.2724	11	77.7307	9	35.7281	10	63.0247
Shear frequencies											
4	16.8080	5	38.9651	8	35.4922	6	51.7405	19	49.7140	29	186.1219
9	35.4732	7	51.7345	24	64.3743	30	202.2575	29	71.9804	–	–

Table 7 Normalized natural frequencies of unsymmetric sandwich beam: data 5 (0/15/core/0/60)

SS				CC				CF			
Mode	Present	Mode	FOST	Mode	Present	Mode	FOST	Mode	Present	Mode	FOST
Axial frequencies											
8	39.7987	5	40.5783	7	39.6264	4	40.5393	5	20.4255	3	20.5222
13	49.4508	11	81.2725	11	49.5099	10	79.1865	11	48.5641	9	60.2157
14	49.5021	17	121.9963	12	49.5480	15	116.7815	14	49.3968	14	97.7863
Bending frequencies											
1	4.3291	1	7.7911	1	4.9133	1	9.9044	1	2.0746	1	3.2458
2	9.2779	2	19.2881	2	10.0351	2	19.7060	2	6.5352	2	11.6732
3	14.5932	3	30.5832	3	15.6723	3	30.7632	3	12.0345	4	23.6884
5	20.2806	6	41.7125	4	21.7457	5	41.7066	4	17.4763	5	34.1538
6	26.6401	7	52.5736	5	28.3183	6	52.5840	6	23.5463	6	43.5029
7	33.2033	9	63.4432	6	35.2981	8	63.5710	8	30.6368	7	48.2988
9	40.6008	10	74.2651	8	42.5322	9	74.2154	9	37.2450	8	58.0871
11	48.1055	12	85.0710	13	49.7788	11	85.0774	10	44.2879	10	68.2767
Shear frequencies											
4	16.8080	4	38.9651	9	46.7261	7	58.1498	7	27.2323	–	–
10	45.4617	8	57.7101	–	–	–	–	–	–	–	–

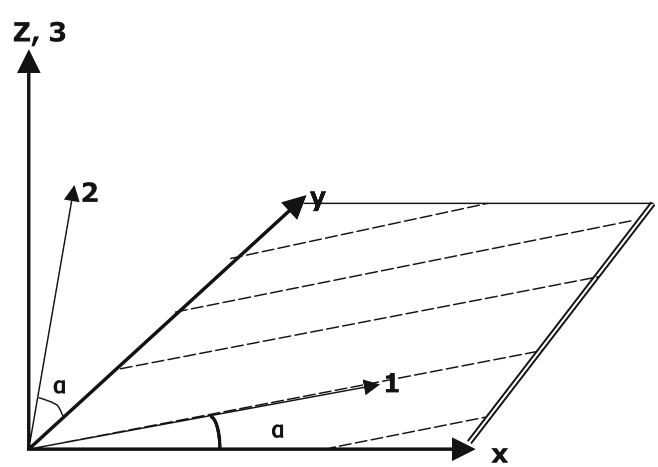

Fig. 2 x,y,z: Laminate axes; 1,2,3: Lamina axes

Table 8 Normalized natural frequencies of symmetric composite beam: data 6 (30/-30/30)

SS				CC				CF			
Mode	Present	Mode	FOST	Mode	Present	Mode	FOST	Mode	Present	Mode	FOST
				Axial frequencies							
3	16.1191	3	15.0004	3	16.1191	3	15.0004	3	8.0583	3	7.5002
7	32.1945	6	30.0008	6	32.1945	6	30.0008	6	24.1546	6	22.5006
12	48.0335	11	45.0012	11	48.0335	10	45.0012	10	40.1413	9	37.5010
				Bending frequencies							
1	2.6689	1	2.5060	1	4.8412	1	4.6188	1	0.9952	1	0.9303
2	8.8087	2	8.3846	2	10.6640	2	10.2374	2	5.0731	2	4.8103
4	16.1638	4	15.5326	4	17.4843	4	16.8007	4	11.6600	4	11.1348
5	23.8443	5	23.0304	5	24.7137	5	23.7566	5	18.8536	5	18.0928
6	31.5937	7	30.5742	7	32.2033	7	30.9523	7	26.3348	7	25.3417
9	39.3514	8	38.0743	8	39.6067	8	38.1162	8	33.5599	8	32.4110
11	47.1108	12	45.5119	10	47.2364	11	45.3599	9	39.7071	10	38.8487
14	54.8778	14	52.8905	13	55.1283	13	52.8120	11	42.2937	11	41.6110
				Shear frequencies							
8	38.4870	9	38.7296	9	42.3508	9	42.1534	25	69.0136	–	–
10	42.1553	10	42.0484	12	51.3563	12	50.4164	–	–	–	–

Table 9 Normalized natural frequencies of unsymmetric composite beam: data 7 (0/45/-45/90)

SS				CC				CF			
Mode	Present	Mode	FOST	Mode	Present	Mode	FOST	Mode	Present	Mode	FOST
				Axial frequencies							
4	18.0088	4	18.1889	4	22.0953	4	22.1685	4	13.9425	4	13.9178
10	43.4638	7	34.5643	9	43.4952	7	34.6038	9	39.0452	9	39.8291
12	51.3923	10	44.8213	11	52.7946	9	44.7163	12	48.1955	15	60.5323
				Bending frequencies							
1	4.2547	1	4.2584	1	5.2649	1	5.1979	1	1.1531	1	1.1498
2	9.0551	2	8.9636	2	11.1468	2	10.9428	2	5.4589	2	5.4099
3	17.3559	3	17.0973	3	18.0035	3	17.6292	3	11.9861	3	11.8596
5	25.3704	5	25.0181	5	25.5142	5	25.0345	5	19.7135	5	19.4755
6	31.3065	6	31.0095	6	31.6337	6	31.2696	6	25.7884	6	25.5902
7	34.1663	9	38.9361	7	34.3409	8	38.9346	7	30.1299	7	30.2844
9	39.6193	11	46.4678	8	39.6749	10	46.4697	8	33.8083	8	33.5761
11	47.3698	13	53.8794	10	47.4981	11	53.8377	10	41.1018	10	41.0383
				Shear frequencies							
8	38.4870	8	38.7296	12	54.4615	13	57.0349	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–	–

$$Q_{24} = (D_{11} - D_{12} - 2D_{44})s^3c + (D_{12} - D_{22} + 2D_{44})sc^3 \quad (47)$$

$$Q_{33} = D_{33} \quad (48)$$

$$Q_{34} = (D_{13} - D_{23})sc \quad (49)$$

$$Q_{44} = (D_{11} - 2D_{12} + D_{22} - 2D_{44})s^2c^2 + D_{44}(c^4 + s^4) \quad (50)$$

$$Q_{55} = D_{55}c^2 + D_{66}s^2 \quad (51)$$

$$Q_{56} = (D_{66} - D_{55})sc \quad (52)$$

$$Q_{66} = D_{55}s^2 + D_{66}c^2 \quad (53)$$

Appendix 2

$$\Phi = \left(Q_{22}Q_{44} - Q_{24}^2\right) \quad (54)$$

$$C_{11} = Q_{11} + \frac{Q_{12}}{\Phi}(Q_{14}Q_{24} - Q_{12}Q_{44}) + \frac{Q_{14}}{\Phi}(Q_{12}Q_{24} - Q_{14}Q_{22}) \quad (55)$$

$$C_{12} = Q_{13} + \frac{Q_{12}}{\Phi}(Q_{24}Q_{34} - Q_{23}Q_{44}) + \frac{Q_{14}}{\Phi}(Q_{23}Q_{24} - Q_{22}Q_{34}) \quad (56)$$

$$C_{21} = Q_{13} + \frac{Q_{23}}{\Phi}(Q_{14}Q_{24} - Q_{12}Q_{44}) + \frac{Q_{34}}{\Phi}(Q_{12}Q_{24} - Q_{14}Q_{22}) \quad (57)$$

$$C_{22} = Q_{33} + \frac{Q_{23}}{\Phi}(Q_{24}Q_{34} - Q_{23}Q_{44}) + \frac{Q_{34}}{\Phi}(Q_{23}Q_{24} - Q_{22}Q_{34}) \quad (58)$$

$$C_{33} = Q_{66} - \frac{Q_{56}^2}{Q_{55}} \quad (59)$$

Appendix 3

$$\bar{D} = b\sum_{L=1}^{NL}\left[\begin{array}{ccc|ccc} C_{11}H_1 & C_{11}H_3 & C_{12}H_1 & C_{11}H_2 & C_{11}H_4 & C_{12}H_2 \\ & C_{11}H_5 & C_{12}H_3 & C_{11}H_4 & C_{11}H_6 & C_{12}H_4 \\ & & C_{22}H_1 & C_{12}H_2 & C_{12}H_4 & C_{22}H_2 \\ \hline & & & C_{11}H_3 & C_{11}H_5 & C_{12}H_3 \\ & \textbf{\textit{sym}} & & & C_{11}H_7 & C_{12}H_5 \\ & & & & & C_{22}H_3 \end{array}\right] \quad (60)$$

$$D_s = b\sum_{L=1}^{NL} C_{33}\begin{bmatrix} H_1 & H_3 & H_2 \\ H_3 & H_5 & H_4 \\ H_2 & H_4 & H_3 \end{bmatrix} \quad (61)$$

$$\bar{m} = b\sum_{L=1}^{NL} \rho_L \begin{bmatrix} H_1 & 0 & H_2 & H_3 & H_4 & 0 & 0 \\ & H_1 & 0 & 0 & 0 & H_2 & H_3 \\ & & H_3 & H_4 & H_5 & 0 & 0 \\ & & & H_5 & H_6 & 0 & 0 \\ & \textbf{\textit{sym}} & & & H_7 & 0 & 0 \\ & & & & & H_3 & H_4 \\ & & & & & & H_5 \end{bmatrix} \quad (62)$$

In eqs. (60, 61, 62), for a given layer L,

$$H_k = \frac{1}{k}(h_L^k - h_{L-1}^k) \quad (63)$$

where

NL = Total Number of Layers of a cross section

k = constant varying from 1 to 7

h_L = Distance from the neutral axis to the top of a layer, L

h_{L-1}= Distance from the neutral axis to the top of layer $L-1$ or bottom of layer L

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