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A New Partial Finite Element Model for Statics of Sandwich Plates

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ABSTRACT: A new partial discretization formulation with four-noded bi-linear finite elements (FEs) has been developed in this study for flexural analysis of sandwich plates. Partial discretization results in solution of a two-point boundary value problem (BVP) governed by a system of coupled first-order ordinary differential equations (ODEs). Mixed degrees of freedom, displacements (u, v, w), and transverse stresses $(\tau_{xz}, \tau_{yz}, \sigma_z)$ are the dependent variables and thus continuity of transverse stresses and displacements are implicitly enforced in the present formulation. Numerical investigations on symmetric and unsymmetric sandwich plates are performed and presented, involving both validation and solution of new problems.

KEY WORDS: sandwich, partial finite element, boundary value problem, numerical integration algorithm.

INTRODUCTION

SANDWICHES, BASICALLY SPECIAL forms of fiber reinforced polymer composite (FRPC) material composed of two thin, strong stiff layers (face sheets) and thick flexible core, are very useful for weight sensitive engineering applications. The main function of face sheets is to resist bending, bonded to a relatively thicker and less dense layer (core) which is provided to resist shear stress. The face sheets are usually prepared from unidirectional FRPC and the core is a thick layer of a low density material made up of foam polymer or honeycomb material. Owing to large difference in the face sheets and core material stiffnesses, an accurate analysis of the sandwich laminates is difficult to achieve.

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An accurate estimation of interlaminar stresses plays a key role in predicating the delamination and matrix cracking of sandwich panels. The main requirements in the analysis of sandwich panels are that the transverse normal and the transverse shear strains as well as their variations through the thickness must be considered. In addition to these, elasticity solutions of layered components [1,2] indicate that the interlaminar continuity of transverse normal and shear fields as well as displacement fields through the thickness of a laminate is an essential requirement for their accurate analysis.

The simplifying assumptions made in the classical laminate plate theory (CLPT), first-order shear deformation theory (FOST), and higher-order shear deformation theory (HOST) are reflected by the deficiencies in the results of sandwich plates. All these simplified theories are termed as equivalent single layer (ESL) theories. Moreover, the transverse interlaminar stresses are most commonly estimated using a post-processing technique by integration of 3D equilibrium equations of elasticity along the laminate thickness in ESL theories and there are serious limitations. The estimates are not only inaccurate but the methods are unreliable and the whole methodology lacks robustness. Furthermore, continuity of transverse stresses could not be maintained at the laminae interfaces and thus a need is obvious to develop layerwise theories to take into account the continuity requirements.

Various displacement based layerwise theories have been proposed by Reddy [3], Soldatos [4], Wu and Kuo [5], Wu and Hsu [6], and others. These displacement based layerwise theories have been reported to provide satisfactory results for global values (deflections, flexural stresses) for thin and thick laminates. However, only continuity of displacement field through the thickness of a laminate could be satisfied in the displacement based layerwise models and continuity of transverse stresses at the laminae interfaces could not be enforced. To overcome this, a layerwise model with displacements and transverse stresses as primary variables, is proposed by various researchers. A group of researchers including Spilker [7], Wu and Lin [8], Shin and Chen [9], Ramtekkar et al. [10,11] have worked on development of layerwise mixed FE models. Such models satisfy the continuity requirements of displacements and transverse stresses at the laminae interfaces.

The various simple analytical/FE models developed in the past are based on the assumed variations over the global/element domain in all considered directions in space. An attempt is made here to present a general method starting from the fundamentals, that is, the exact three-dimensional (3D) partial differential equation (PDE) system of sandwich plate. A novel mixed partial FE model is developed for static analyses of sandwich plates based on the solution of a two-point BVP governed by a system of first-order ordinary differential equations (ODEs),

$$\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{y}(z) = \mathbf{A}(z)\boldsymbol{y}(z) + \boldsymbol{p}(z) \tag{1}$$

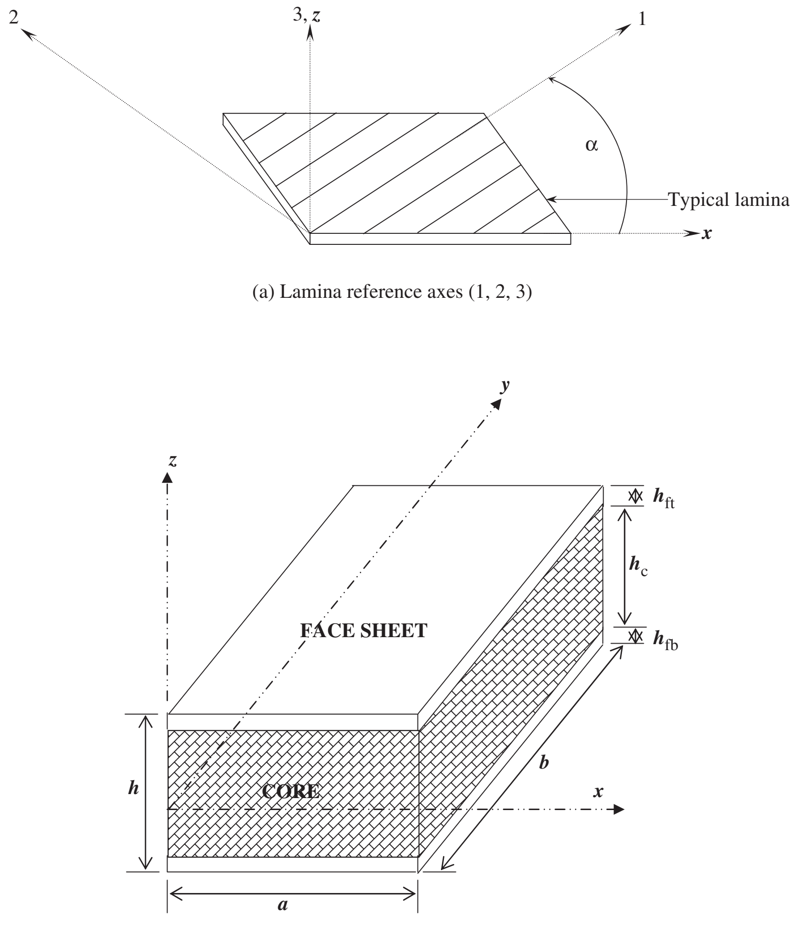
in the interval $-h/2 \leq z \leq h/2$ with any half of the dependent variables prescribed at the edges $z = \pm h/2$. The solution vector $\boldsymbol{y}(z)$ consists of a set of primary variables whose number equals the order of PDE system times the number of discrete FE mesh nodes. Availability of efficient, accurate and above all proven robust ODE numerical integrators for initial value problems (IVPs) helps in obtaining the set of primary variables at all nodal points through the thickness. Ingenuity lies here in transforming the boundary value problem (BVP) into a set of IVPs [12]. Once the fundamental set is known, the auxiliary set of dependent variables over the entire nodal set is simply computed by substitution of the values of the fundamental set of variables on the right-hand side of algebraic expressions node-by-node. The accuracy of the model is verified against the literature values for laminated sandwich plates. New results are presented for the clamped supported sandwich plates using the standard material properties available in the literature.

PARTIAL FINITE ELEMENT FORMULATION

An anisotropic sandwich laminate consisting of isotropic/orthotropic face sheets and flexible core with a plan dimension $a \times b$ and total thickness h $(h_{\mathrm{fb}} + h_{\mathrm{c}} + h_{\mathrm{ft}})$ is considered (Figure 1). Here h_{fb} and h_{ft} are the thicknesses of face sheets at the bottom and top of a laminate, respectively and h_{c} is the thickness of core material. The top surface of the laminate is loaded with transversely distributed load.

The constitutive relations for a typical ith lamina with reference to the principal material coordinate axes $(1, 2, 3)$ and with the consideration of each lamina to be in a 3D state of stress are written as:

$$\begin{aligned}
(\varepsilon_1)^i &= \left(\frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{31}}{E_3}\sigma_3\right)^i \\
(\varepsilon_2)^i &= \left(-\frac{\nu_{12}}{E_1}\sigma_1 + \frac{1}{E_2}\sigma_2 - \frac{\nu_{32}}{E_3}\sigma_3\right)^i \\
(\varepsilon_3)^i &= \left(-\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 + \frac{1}{E_3}\sigma_3\right)^i \\
(\gamma_{12})^i &= \left(\frac{\tau_{12}}{G_{12}}\right)^i; \quad (\gamma_{13})^i = \left(\frac{\tau_{13}}{G_{13}}\right)^i; \quad \text{and} \quad (\gamma_{23})^i = \left(\frac{\tau_{23}}{G_{23}}\right)^i.
\end{aligned} \tag{2}$$



(b) Sandwich plate reference axes (x, y, z)

***Figure 1.** Sandwich plate geometry with positive set of lamina/plate reference axes and fiber orientation.*

These can also be written as:

$$\boldsymbol{\sigma}^{\iota} = \mathbf{C}\boldsymbol{\varepsilon}^{\iota} \tag{3}$$

where $\{\sigma\}^i = [\sigma_1\, \sigma_2\, \sigma_3\, \tau_{12}\, \tau_{13}\, \tau_{23}]^{iT}$ and $\{\varepsilon\}^i = [\varepsilon_1\, \varepsilon_2\, \varepsilon_3\, \gamma_{12}\, \gamma_{13}\, \gamma_{23}]^{iT}$ are the stress and strain vectors with reference to the lamina coordinates 1, 2, and 3. C_{mn}s (m, $n = 1, \ldots, 6$) are elasticity constants of the ith lamina defined in Appendix I.

The stress–strain relations for the *i*th lamina in reference coordinates (x, y, z) can be written as:

$$\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\varepsilon}. \tag{4}$$

Here, $\{\sigma\} = [\sigma_x\, \sigma_y\, \sigma_z\, \tau_{xy}\, \tau_{xz}\, \tau_{yz}]^{\mathrm{T}}$ and $\{\varepsilon\} = [\varepsilon_x\, \varepsilon_y\, \varepsilon_z\, \gamma_{xy}\, \gamma_{xz}\, \gamma_{yz}]^{\mathrm{T}}$ are the stress and strain vectors with respect to laminate axes and

$$\mathbf{Q} = \left[\begin{array}{c|c} Q_{ij} & 0 \\ \hline 0 & Q_{lm} \end{array}\right] \quad \begin{array}{c} i, j = 1, 2, 3, 4 \\ l, m = 5, 6 \end{array}$$

are the transformed 3D elasticity constants for the *i*th lamina with respect to the laminate reference axes (Figure 1) defined in Appendix I.

From the linear theory of elasticity, the general 3D linear strain–displacement relations are:

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \varepsilon_z = \frac{\partial w}{\partial z} \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}. \end{aligned} \tag{5}$$

Further, the 3D differential equations of equilibrium are:

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + B_x = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + B_y = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z = 0. \end{aligned} \tag{6}$$

Here, B_x, B_y, and B_z are components of body force per unit volume in the *x*-, *y*-, and *z*-directions, respectively.

Equations (4)–(6) have a total of fifteen unknowns, six stresses $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$, six strains $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$, and three displacements (u, v, w) in fifteen equations. It is to be noted that transverse stresses and the displacements (Figure 2) are to be continuous at the laminae interfaces for the accurate analysis [1,2]. These conditions are naturally enforced in the present formulation. Through a simple

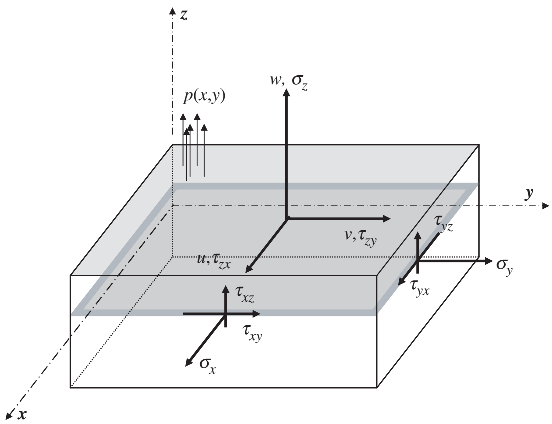


Figure 2. *3D domain subjected to transverse loading.*

algebraic manipulation, PDEs in terms of only six particular dependent variables $u, v, w, \tau_{xz}, \tau_{yz}$, and σ_z are obtained as:

$$
\begin{aligned}
\frac{\partial u}{\partial z} &= \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})}\left[-Q_{65}\tau_{yz} + Q_{66}\tau_{xz}\right] - \frac{\partial w}{\partial x} \\
\frac{\partial v}{\partial z} &= \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})}\left[Q_{55}\tau_{yz} - Q_{56}\tau_{xz}\right] - \frac{\partial w}{\partial y} \\
\frac{\partial w}{\partial z} &= \frac{1}{Q_{33}}\left[\sigma_z - Q_{31}\frac{\partial u}{\partial x} - Q_{34}\frac{\partial u}{\partial y} - Q_{32}\frac{\partial v}{\partial y} - Q_{34}\frac{\partial v}{\partial x}\right] \\
\frac{\partial \tau_{xz}}{\partial z} &= \left(-Q_{11} + \frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\partial^2 u}{\partial x^2} + \left(-Q_{41} - Q_{14} + \frac{Q_{13}Q_{34}}{Q_{33}} + \frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial^2 u}{\partial x \partial y} \\
&\quad + \left(-Q_{44} + \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 u}{\partial y^2} \\
&\quad + \left(-Q_{14} + \frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\partial^2 v}{\partial x^2} + \left(-Q_{12} - Q_{44} + \frac{Q_{13}Q_{32}}{Q_{33}} + \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 v}{\partial x \partial y} \\
&\quad + \left(-Q_{42} + \frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial^2 v}{\partial y^2} - \left(\frac{Q_{13}}{Q_{33}}\right)\frac{\partial \sigma_z}{\partial x} - \left(\frac{Q_{43}}{Q_{33}}\right)\frac{\partial \sigma_z}{\partial y} - B_x \\
\frac{\partial \tau_{yz}}{\partial z} &= \left(-Q_{41} + \frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial^2 u}{\partial x^2} + \left(-Q_{21} - Q_{44} + \frac{Q_{23}Q_{31}}{Q_{33}} + \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 u}{\partial x \partial y} \\
&\quad + \left(-Q_{24} + \frac{Q_{23}Q_{34}}{Q_{33}}\right)\frac{\partial^2 u}{\partial y^2} + \left(-Q_{44} + \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 v}{\partial x^2} \\
&\quad + \left(-Q_{24} - Q_{42} + \frac{Q_{23}Q_{34}}{Q_{33}} + \frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial^2 v}{\partial x \partial y} + \left(-Q_{22} + \frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\partial^2 v}{\partial y^2} \\
&\quad - \left(\frac{Q_{43}}{Q_{33}}\right)\frac{\partial \sigma_z}{\partial x} - \left(\frac{Q_{23}}{Q_{33}}\right)\frac{\partial \sigma_z}{\partial y} - B_y \\
\frac{\partial \sigma_z}{\partial z} &= -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - B_z.
\end{aligned}
\tag{7}
$$

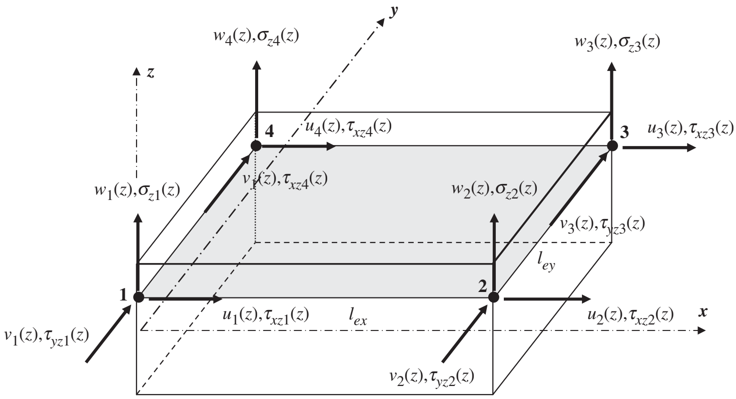


Figure 3. *Bi-linear plate element with dependent variables.*

This set of dependent variables is called a 'primary set' which is naturally defined at a plane $z = $ a constant, and the secondary variables σ_x, σ_y, and τ_{xy} can simply be expressed as a function of the primary set of variables as:

$$
\begin{aligned}
\sigma_x &= \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\partial u}{\partial x} + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&\quad + \left(Q_{12} - \frac{Q_{13}Q_{32}}{Q_{33}}\right)\frac{\partial v}{\partial y} + \frac{Q_{13}}{Q_{33}}\sigma_z \\
\sigma_y &= \left(Q_{21} - \frac{Q_{23}Q_{31}}{Q_{33}}\right)\frac{\partial u}{\partial x} + \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&\quad + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\partial v}{\partial y} + \frac{Q_{23}}{Q_{33}}\sigma_z. \\
\tau_{xy} &= \left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial u}{\partial x} + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&\quad + \left(Q_{22} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial v}{\partial y} + \frac{Q_{43}}{Q_{33}}\sigma_z.
\end{aligned}
\tag{8}
$$

The primary set of variables u, v, w, τ_{xz}, τ_{yz}, and σ_z is a function of independent coordinates x, y, and z. It is proposed here to carryout FE discretization in the x- and y-directions only, such that the discrete dependent vector $\boldsymbol{y}(z)$ will be a function of independent coordinate z and a system of coupled discrete first-order ODEs connecting all FE nodes results. This new formulation is described here with reference to a four-noded bi-linear elements in x- and y-directions with a mixed set of primary variables as nodal degrees of freedom (Figure 3).

The approximate variation of the displacement field over the element domain in the x–y plane can be written as:

$$u \simeq \hat{u}(x, y, z) = \sum_{i=1}^{4} u_i(z) N_i(x, y)$$
$$v \simeq \hat{v}(x, y, z) = \sum_{i=1}^{4} v_i(z) N_i(x, y) \tag{9}$$
$$w \simeq \hat{w}(x, y, z) = \sum_{i=1}^{4} w_i(z) N_i(x, y).$$

Further, from the basic 3D elasticity relations, it can be shown that,

$$\tau_{xz} \simeq \hat{\tau}_{xz}(x, y, z) = \sum_{i=1}^{4} \tau_{xz} i(z) N_i(x, y)$$
$$\tau_{yz} \simeq \hat{\tau}_{yz}(x, y, z) = \sum_{i=1}^{4} \tau_{yz} i(z) N_i(x, y) \tag{10}$$
$$\sigma_z \simeq \hat{\sigma}_z(x, y, z) = \sum_{i=1}^{4} \sigma_z i(z) N_i(x, y)$$

where,

$$N_1(x, y) = 1 - \frac{x}{l_{ex}} - \frac{y}{l_{ey}} - \frac{xy}{l_{ex} l_{ey}} \quad N_2(x, y) = \frac{x}{l_{ex}} - \frac{xy}{l_{ex} l_{ey}}$$
$$N_3(x, y) = \frac{xy}{l_{ex} l_{ey}} \quad N_4(x, y) = \frac{y}{l_{ey}} - \frac{xy}{l_{ex} l_{ey}}$$

and l_{ex}, l_{ey} are the length and width of element in the x- and y- directions, respectively.

Substituting Equations (9) and (10) into Equation (7), the domain residuals are obtained as:

$$\frac{\partial \hat{u}(x, y, z)}{\partial z} + \frac{1}{(Q_{55} Q_{66} - Q_{56} Q_{65})} \left[Q_{65} \hat{\tau}_{yz}(x, y, z) - Q_{66} \hat{\tau}_{xz}(x, y, z) \right] + \frac{\partial \hat{w}(x, y, z)}{\partial x} = R_{1D}(x, y) \tag{11}$$

$$\frac{\partial \hat{v}(x,y,z)}{\partial z} + \frac{1}{(Q_{55}Q_{66} - Q_{56}Q_{65})}\left[-Q_{55}\hat{\tau}_{yz}(x,y,z) + Q_{56}\hat{\tau}_{xz}(x,y,z)\right] + \frac{\partial \hat{w}(x,y,z)}{\partial y} = R_{2D}(x,y) \tag{12}$$

$$\frac{\partial \hat{w}(x,y,z)}{\partial z} - \frac{1}{Q_{33}}\left[\begin{array}{l} \hat{\sigma}_z(x,y,z) - Q_{31}\dfrac{\partial \hat{u}(x,y,z)}{\partial x} - Q_{34}\dfrac{\partial \hat{u}(x,y,z)}{\partial y} \\ -Q_{32}\dfrac{\partial \hat{v}(x,y,z)}{\partial y} - Q_{34}\dfrac{\partial \hat{v}(x,y,z)}{\partial x} \end{array}\right] = R_{3D}(x,y) \tag{13}$$

$$\begin{aligned} &\frac{\partial \hat{\tau}_{xz}(x,y,z)}{\partial z} + \left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x^2} + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial y^2} \\ &\quad + \left(Q_{41} + Q_{14} - \frac{Q_{43}Q_{31}}{Q_{33}} - \frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x \partial y} \\ &\quad + \left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial x^2} \\ &\quad + \left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial y^2} + \left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) \\ &\quad \times \frac{\partial^2 \hat{v}(x,y,z)}{\partial x \partial y} \\ &\quad + \frac{Q_{13}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial x} + \frac{Q_{43}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial y} + \hat{B}_x(x,y,z) = R_{4D}(x,y) \end{aligned} \tag{14}$$

$$\begin{aligned} &\frac{\partial \hat{\tau}_{yz}(x,y,z)}{\partial z} + \left(Q_{41} - \frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x^2} + \left(Q_{24} - \frac{Q_{23}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial y^2} \\ &\quad + \left(Q_{21} + Q_{44} - \frac{Q_{23}Q_{31}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x \partial y} \\ &\quad + \left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial x^2} \\ &\quad + \left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial y^2} + \left(Q_{24} + Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}} - \frac{Q_{23}Q_{34}}{Q_{33}}\right) \\ &\quad \times \frac{\partial^2 \hat{v}(x,y,z)}{\partial x \partial y} \\ &\quad + \frac{Q_{43}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial x} + \frac{Q_{23}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial y} + \hat{B}_y(x,y,z) = R_{5D}(x,y) \end{aligned} \tag{15}$$

$$\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial z} + \frac{\partial \hat{\tau}_{xz}(x,y,z)}{\partial x} + \frac{\partial \hat{\tau}_{yz}(x,y,z)}{\partial y} + \hat{B}_z(x,y,z) = R_{6D}(x,y). \tag{16}$$

Further, with the help of Equations (11)–(16) the strong Bubnov–Galerkin weighted residual statements [13] can be written as:

$$\iint_A N_i(x,y)\left(\frac{\partial \hat{u}(x,y,z)}{\partial z}+\frac{1}{(Q_{55}Q_{66}-Q_{56}Q_{65})}\left[Q_{65}\hat{\tau}_{yz}(x,y,z)-Q_{66}\hat{\tau}_{xz}(x,y,z)\right]+\frac{\partial \hat{w}(x,y,z)}{\partial x}\right)\mathrm{d}A=0 \tag{17}$$

$$\iint_A N_i(x,y)\left(\frac{\partial \hat{v}(x,y,z)}{\partial z}+\frac{1}{(Q_{55}Q_{66}-Q_{56}Q_{65})}\left[-Q_{55}\hat{\tau}_{yz}(x,y,z)+Q_{56}\hat{\tau}_{xz}(x,y,z)\right]+\frac{\partial \hat{w}(x,y,z)}{\partial y}\right)\mathrm{d}A=0 \tag{18}$$

$$\iint_A N_i(x,y)\left\{\frac{\partial \hat{w}(x,y,z)}{\partial z}-\frac{1}{Q_{33}}\left[\begin{array}{l}\hat{\sigma}_z(x,y,z)-Q_{31}\frac{\partial \hat{u}(x,y,z)}{\partial x}-Q_{34}\frac{\partial \hat{u}(x,y,z)}{\partial y}\\ -Q_{32}\frac{\partial \hat{v}(x,y,z)}{\partial y}-Q_{34}\frac{\partial \hat{v}(x,y,z)}{\partial x}\end{array}\right]\right\}\times\mathrm{d}A=0 \tag{19}$$

$$\iint_A N_i(x,y)\left\{\begin{array}{l}\frac{\partial \hat{\tau}_{xz}(x,y,z)}{\partial z}+\left(Q_{11}-\frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x^2}+\left(Q_{44}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial y^2}\\ +\left(Q_{41}+Q_{14}-\frac{Q_{43}Q_{31}}{Q_{33}}-\frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x\partial y}+\left(Q_{14}-\frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial x^2}\\ +\left(Q_{42}-\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial y^2}+\left(Q_{12}+Q_{44}-\frac{Q_{13}Q_{32}}{Q_{33}}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial x\partial y}\\ +\frac{Q_{13}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial x}+\frac{Q_{43}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial y}+\hat{B}_x(x,y,z)\end{array}\right\}\mathrm{d}A=0 \tag{20}$$

$$\iint_A N_i(x,y)\left\{\begin{array}{l}\frac{\partial \hat{\tau}_{yz}(x,y,z)}{\partial z}+\left(Q_{41}-\frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x^2}+\left(Q_{24}-\frac{Q_{23}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial y^2}\\ +\left(Q_{21}+Q_{44}-\frac{Q_{23}Q_{31}}{Q_{33}}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{u}(x,y,z)}{\partial x\partial y}+\left(Q_{44}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial x^2}\\ +\left(Q_{22}-\frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial y^2}+\left(Q_{24}+Q_{42}-\frac{Q_{43}Q_{32}}{Q_{33}}-\frac{Q_{23}Q_{34}}{Q_{33}}\right)\frac{\partial^2 \hat{v}(x,y,z)}{\partial x\partial y}\\ +\frac{Q_{43}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial x}+\frac{Q_{23}}{Q_{33}}\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial y}+\hat{B}_y(x,y,z)\end{array}\right\}\mathrm{d}A=0 \tag{21}$$

$$\iint_A N_i(x,y)\left(\frac{\partial \hat{\sigma}_z(x,y,z)}{\partial z}+\frac{\partial \hat{\tau}_{xz}(x,y,z)}{\partial x}+\frac{\partial \hat{\tau}_{yz}(x,y,z)}{\partial y}+\hat{B}_z(x,y,z)\right)\mathrm{d}A=0. \tag{22}$$

Equations (20) and (21) which contain second-order derivatives of $\hat{u}$ and $\hat{v}$, are replaced by their weak forms as:

$$
\begin{aligned}
&\iint_A N_i(x,y)\left[\frac{\partial\hat{\tau}_{xz}(x,y,z)}{\partial z}+\frac{Q_{13}}{Q_{33}}\frac{\partial\hat{\sigma}_z(x,y,z)}{\partial x}+\frac{Q_{43}}{Q_{33}}\frac{\partial\hat{\sigma}_z(x,y,z)}{\partial y}\right]\mathrm{d}A\\
&-\iint_A \frac{\mathrm{d}N_i(x,y)}{\mathrm{d}y}\left[\left(Q_{44}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}y}+\left(Q_{42}-\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}y}\right]\mathrm{d}A\\
&-\iint_A \frac{\mathrm{d}N_i(x,y)}{\mathrm{d}x}\left[\begin{array}{l}\left(Q_{11}-\frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}x}+\left(Q_{41}+Q_{14}-\frac{Q_{43}Q_{31}}{Q_{33}}-\frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}y}\\ +\left(Q_{14}-\frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}x}+\left(Q_{12}+Q_{44}-\frac{Q_{13}Q_{32}}{Q_{33}}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}y}\end{array}\right]\mathrm{d}A\\
&+\oint_s N_i(x,y)\left[\left(Q_{11}-\frac{Q_{13}Q_{31}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}x}n_x+\left(Q_{41}+Q_{14}-\frac{Q_{43}Q_{31}}{Q_{33}}-\frac{Q_{13}Q_{34}}{Q_{33}}\right)\right.\\
&\left.\times\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}y}n_y\right]\mathrm{d}s\\
&+\oint_s N_i(x,y)\left[\left(Q_{44}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}y}n_y+\left(Q_{12}+Q_{44}-\frac{Q_{13}Q_{32}}{Q_{33}}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}y}n_x\right]\mathrm{d}s\\
&+\oint_s N_i(x,y)\left[\left(Q_{14}-\frac{Q_{13}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}x}n_x+\left(Q_{42}-\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}y}n_y\right]\mathrm{d}s\\
&+\iint_A N_i(x,y)\hat{B}_x(x,y,z)\mathrm{d}A=0
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\iint_A N_i(x,y)\left[\frac{\partial\hat{\tau}_{yz}(x,y,z)}{\partial z}+\frac{Q_{23}}{Q_{33}}\frac{\partial\hat{\sigma}_z(x,y,z)}{\partial y}+\frac{Q_{43}}{Q_{33}}\frac{\partial\hat{\sigma}_z(x,y,z)}{\partial x}\right]\mathrm{d}A\\
&-\iint_A \frac{\mathrm{d}N_i(x,y)}{\mathrm{d}x}\left[\left(Q_{41}-\frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}x}+\left(Q_{44}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}x}\right]\mathrm{d}A\\
&-\iint_A \frac{\mathrm{d}N_i(x,y)}{\mathrm{d}y}\left[\left(Q_{24}-\frac{Q_{34}Q_{23}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}y}+\left(Q_{21}+Q_{44}-\frac{Q_{23}Q_{31}}{Q_{33}}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}x}\right.\\
&\left.+\left(Q_{22}-\frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}y}+\left(Q_{24}+Q_{42}-\frac{Q_{23}Q_{34}}{Q_{33}}-\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}x}\right]\mathrm{d}A\\
&+\oint_s N_i(x,y)\left[\left(Q_{41}-\frac{Q_{43}Q_{31}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}x}n_x+\left(Q_{21}+Q_{44}-\frac{Q_{23}Q_{31}}{Q_{33}}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}x}n_y\right]\mathrm{d}s\\
&+\oint_s N_i(x,y)\left[\left(Q_{24}-\frac{Q_{34}Q_{23}}{Q_{33}}\right)\frac{\mathrm{d}\hat{u}(x,y,z)}{\mathrm{d}y}n_y+\left(Q_{24}+Q_{42}-\frac{Q_{23}Q_{34}}{Q_{33}}-\frac{Q_{43}Q_{32}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}x}n_y\right]\mathrm{d}s\\
&+\oint_s N_i(x,y)\left[\left(Q_{44}-\frac{Q_{43}Q_{34}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}x}n_x+\left(Q_{22}-\frac{Q_{23}Q_{32}}{Q_{33}}\right)\frac{\mathrm{d}\hat{v}(x,y,z)}{\mathrm{d}y}n_y\right]\mathrm{d}s\\
&+\iint_A N_i(x,y)\hat{B}_y(x,y,z)\mathrm{d}A=0.
\end{aligned}
\tag{24}
$$

On substitution of discrete relations from Equations (9) and (10), the twenty-four coupled first-order ODEs are obtained and can be written as:

$$\begin{bmatrix} [\mathbf{A}_{01}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{03}^e] & [\mathbf{A}_{02}^e] \\ [\mathbf{A}_{02}^e] & [\mathbf{A}_{01}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{03}^e] \\ [\mathbf{A}_{03}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{01}^e] & [\mathbf{A}_{02}^e] \\ [\mathbf{A}_{02}^e] & [\mathbf{A}_{03}^e] & [\mathbf{A}_{02}^e] & [\mathbf{A}_{01}^e] \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}z} \begin{Bmatrix} \boldsymbol{y}_1^e(z) \\ \boldsymbol{y}_2^e(z) \\ \boldsymbol{y}_3^e(z) \\ \boldsymbol{y}_4^e(z) \end{Bmatrix}$$

$$= \begin{bmatrix} [\mathbf{B}_{01}^e] & [\mathbf{B}_{02}^e] & [\mathbf{B}_{03}^e] & [\mathbf{B}_{04}^e] \\ [\mathbf{B}_{05}^e] & [\mathbf{B}_{06}^e] & [\mathbf{B}_{07}^e] & [\mathbf{B}_{08}^e] \\ [\mathbf{B}_{09}^e] & [\mathbf{B}_{10}^e] & [\mathbf{B}_{11}^e] & [\mathbf{B}_{12}^e] \\ [\mathbf{B}_{13}^e] & [\mathbf{B}_{14}^e] & [\mathbf{B}_{15}^e] & [\mathbf{B}_{16}^e] \end{bmatrix} \begin{Bmatrix} \boldsymbol{y}_1^e(z) \\ \boldsymbol{y}_2^e(z) \\ \boldsymbol{y}_3^e(z) \\ \boldsymbol{y}_4^e(z) \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{p}_1^e \\ \boldsymbol{p}_2^e \\ \boldsymbol{p}_3^e \\ \boldsymbol{p}_4^e \end{Bmatrix} \tag{25}$$

in which vectors $\boldsymbol{y}_i^e(z)$ and $\boldsymbol{p}_i^e(x, y, z)$ are:

$$\boldsymbol{y}_i^e(z) = \left[u_i^e(z), v_i^e(z), w_i^e(z), \tau_{xzi}^e(z), \tau_{yzi}^e(z), \sigma_{zi}^e(z)\right]^t$$

$$\boldsymbol{p}_i^e(x, y, z) = \left[0, 0, 0, p_{i4}^e, p_{i5}^e, p_{i6}^e\right]^t$$

where, $i = 1$–4. The coefficients of individual submatrices and vectors are defined in Appendix II.

Equation (25) can be written in a compact form as:

$$\mathbf{C}^e(x, y)\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{y}^e(z) = \mathbf{D}^e(x, y, z)\boldsymbol{y}^e(z) + \boldsymbol{p}^e(x, y, z). \tag{26}$$

When the total plan area is discretized with n_x finite elements in the x-direction and n_y finite elements in the y-direction (Figure 4), then the semi-discrete system of equations for the entire domain turns out to be:

$$\sum_{k=1}^{n}\mathbf{C}^e(x, y)\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{y}^e(z) = \sum_{k=1}^{n}\mathbf{D}^e(x, y, z)\boldsymbol{y}^e(z) + \sum_{k=1}^{n}\boldsymbol{p}^e(x, y, z)$$

or

$$\mathbf{C}(x, y)\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{y}(z) = \mathbf{D}(x, y, z)\boldsymbol{y}(z) + \boldsymbol{p}(x, y, z). \tag{27}$$

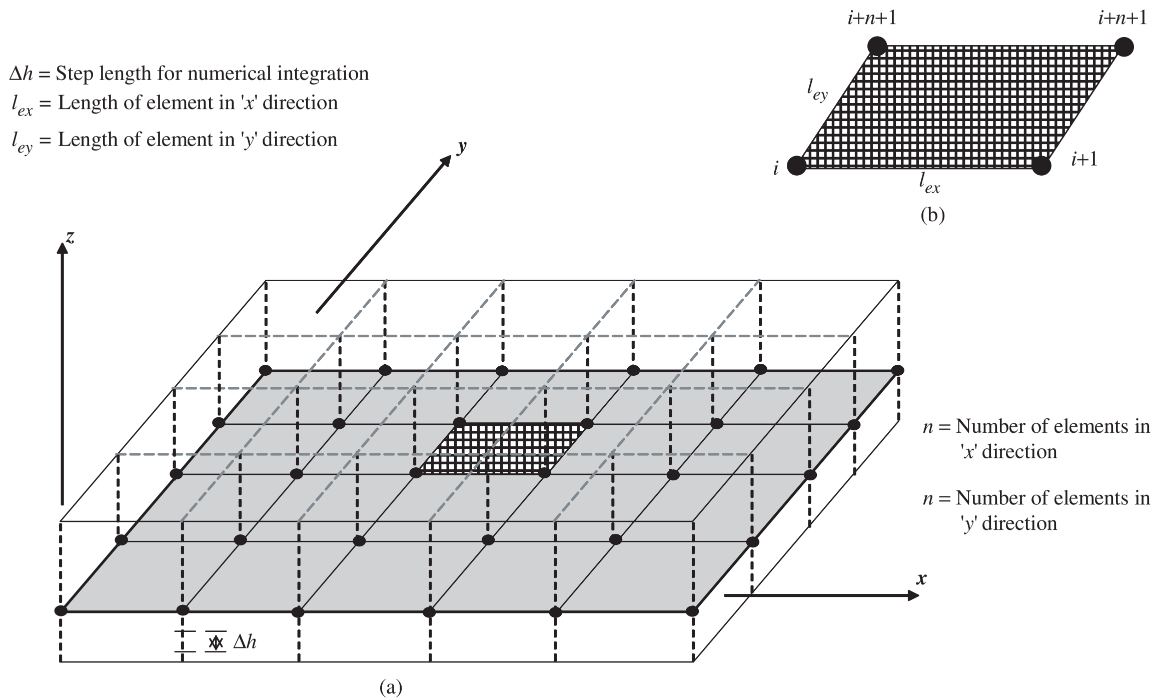


Figure 4. *Bi-linear plate elements (concept of partial discritizarion): (a) plate discretization and (b) typical bi-linear element.*

Multiplication of Equation (27) by $[\mathbf{C}(x,y)]^{-1}$ on both sides results in,

$$\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{y}(z) = \mathbf{K}(x,y,z)\boldsymbol{y}(z) + \boldsymbol{f}(x,y,z) \tag{28}$$

where, $\mathbf{K}(x,y,z) = [\mathbf{C}(x,y)]^{-1}\mathbf{D}(x,y,z)$ and $\boldsymbol{f}(x,y,z) = [\mathbf{C}(x,y)]^{-1}\boldsymbol{p}(x,y,z)$.

Equation (28) defines the governing equations of a two-point BVP in ODEs in the domain $-h/2 < z < h/2$. $\boldsymbol{y}(z)$ is an m-dimensional (m = no. of nodes $\times$ 6) vector of dependent variables, $\mathbf{K}(x,y,z)$ is an $m \times m$ coefficient matrix (which is a function of element geometry along in-plane directions, x and y material properties variation all in x-, y-, and z- directions) and $\boldsymbol{f}(x,y,z)$ is an m-dimensional vector of non-homogeneous (loading) terms. Any $m/2$ elements of $\boldsymbol{y}(z)$ are prescribed at the two ends, $z = -h/2$ and $h/2$ as boundary conditions. It is clearly seen that mixed and/or non-homogeneous boundary conditions are easily admitted in this formulation.

The basic approach to the numerical integration of the BVP defined by Equation (28) is to transform the given BVP into a set of IVPs – one particular (nonhomogeneous) and $m/2$ complimentary (homogeneous). Clearly the reason behind this is the availability of a number of successful and well-tested algorithms for numerical solution of IVPs in ODEs. The solution of the original BVP defined by Equation (28) is obtained by forming a linear combination of one nonhomogeneous and $m/2$ homogeneous solutions to satisfy the boundary conditions at $z = h/2$. This gives rise to a system of $m/2$ linear algebraic equations, the solution of which determines the unknown $m/2$ components of the vector of initial values $\boldsymbol{y}(z)$. Then a final numerical integration of Equation (28) with completely known initial vector of dependent variables $\boldsymbol{y}(z)$ produces the desired results. It is intended here to extend the applicability of this procedure, which is documented by Kant and Ramesh [12] and applied in its original form by Kant and Setlur [14], Ramesh et al. [15], Kant [16,17], and Kant and Hinton [18] for a class of two dimensions (2D) BVPs of plates and shells.

NUMERICAL INVESTIGATIONS

A computer code is developed in FORTRAN 90 by incorporating the foregoing partial FE formulation, for the static analyses of laminated composite and sandwich plates. A 10×10 full mesh of the four-noded bi-linear plate elements has been used in the computation. This scheme is arrived at on the basis of a convergence study in which mid-plane transverse displacement and transverse shear stress converge monotonically. The details of convergence studies are not presented here for the sake of brevity.

Table 1. Boundary conditions (BCs).

Description	Edge	BCs on displacement field	BCs on stress field
Nodes along simple support	$x=0$ and a	$v=w=0$	$\sigma_x=0$ (true)
	$y=0$ and b	$u=w=0$	$\sigma_y=0$ (true)
Nodes along clamped support	$x=0$ and a	$u=v=w=0$	–
	$y=0$ and b	$u=v=w=0$	–
For all nodes	$z=h/2$	–	$\tau_{xz}=\tau_{yz}=0$ and $\sigma_z=p_0(x,y)$
	$z=-h/2$	–	$\tau_{xz}=\tau_{yz}=\sigma_z=0$

'–' indicates no BCs.

To demonstrate the efficiency and versatility of the present formulation, symmetric and unsymmetric sandwich plates with simple (diaphragm) support end conditions on all four edges are investigated first. Results have been validated by comparing them with those of the 3D elasticity [2] and 2D/3D FE solutions [5,11,19–21] available in the literature. Further, new solutions are presented for clamped support conditions on all four edges for future reference. The boundary conditions used in the present work have been tabulated in Table 1 and the material properties are presented in Table 2.

Non-dimensionalized displacements and stresses reported in all tables and figures are defined by:

$$s=\frac{a}{h};\quad \bar{u}=\frac{E_2 u}{h p_0 s^3};\quad \bar{w}=\frac{100 E_2 h^3 w}{p_0 a^4};\quad \bar{\sigma}_z=\frac{\sigma_z}{p_0}$$
$$\left(\bar{\sigma}_x;\bar{\sigma}_y;\bar{\tau}_{xy}\right)=\frac{1}{p_0 s^2}\left(\sigma_x;\sigma_y;\tau_{xy}\right);\quad \left(\bar{\tau}_{xz};\bar{\tau}_{yz}\right)=\frac{1}{p_0 s}\left(\tau_{xz};\tau_{yz}\right) \tag{29}$$

for proper comparison of results in which bar over the variable defines its normalized value. Here E_2 is the modulus of elasticity of face sheet material presented in Table 2, along the lamina coordinate axis 2 (Figure 1(a)). Further, the intensity of bi-directional sinusoidal loading considered in the examples has been expressed as:

$$p(x,y)=p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{30}$$

where p_0 is the peak intensity of disturbed load.

Illustrative numerical examples considered in the present work are discussed next.

Table 2. Material properties.

Set	Source	Property		
I	Pagano [2]	Face sheet		
		$E_1 = 172.4$ GPa	$\nu_{12} = 0.25$	$G_{12} = 3.45$ GPa
		$E_2 = 6.89$ GPa	$\nu_{13} = 0.25$	$G_{13} = 3.45$ GPa
		$E_3 = 6.89$ GPa	$\nu_{23} = 0.25$	$G_{23} = 1.378$ GPa
		Core sheet		
		$E_1 = 0.276$ GPa	$\nu_{12} = 0.25$	$G_{12} = 0.1104$ GPa
		$E_2 = 0.276$ GPa	$\nu_{31} = 0.25$	$G_{13} = 0.414$ GPa
		$E_3 = 3.450$ GPa	$\nu_{32} = 0.25$	$G_{23} = 0.414$ GPa
II	Rao and Meyer- Piening [19]	Face sheet		
		$E_1 = 172.4$ GPa	$\nu_{12} = 0.25$	$G_{12} = 3.45$ GPa
		$E_2 = 6.89$ GPa	$\nu_{13} = 0.25$	$G_{13} = 3.45$ GPa
		$E_3 = 6.89$ GPa	$\nu_{23} = 0.25$	$G_{23} = 1.378$ GPa
		Core sheet		
		$E_1 = 0.100$ GPa	$\nu_{12} = 0.25$	$G_{12} = 0.040$ GPa
		$E_2 = 0.100$ GPa	$\nu_{31} = 0.25$	$G_{13} = 0.040$ GPa
		$E_3 = 0.100$ GPa	$\nu_{32} = 0.25$	$G_{23} = 0.040$ GPa

Example 1

A symmetric, square, three-layered (0°/core/0°) sandwich plate, supported on simple supports (Table 1) on all the four edges and subjected to bi-directional sinusoidal transverse load (Equation (30)) on its top surface has been considered here for validation. The material properties of face sheets and core are presented in Table 2(I). The thickness of each face sheet is $h/10$. The numerical results of normalized in-plane normal stresses ($\bar{\sigma}_x, \bar{\sigma}_y$), transverse shear stresses ($\bar{\tau}_{xz}, \bar{\tau}_{yz}$), and transverse displacement ($\bar{w}$) for aspect ratios, $s = 4$, 10, 20, and 50 are tabulated in Table 3 and compared with the 3D elasticity solution given by Pagano [2] as well as 2D/3D FE solutions presented by various authors [5,11,20,21]. Moreover, through thickness variations of in-plane normal stress ($\bar{\sigma}_x$), transverse normal stress ($\bar{\sigma}_z$), transverse shear stress ($\bar{\tau}_{xz}$), and in-plane displacement ($\bar{u}$) for an aspect ratio of 4 have been shown in Figure 5. The results obtained through the present investigation have been found to be in good agreement with the 3D elasticity solution.

Table 3. ***Maximum stresses ($\bar{\sigma}_x$, $\bar{\sigma}_{yz}$, $\bar{\tau}_{xz}$, $\bar{\tau}_{yz}$, and $\bar{\tau}_{xy}$) and mid-plane transverse displacement ($\bar{w}$) of symmetric (0°/core/0°) square sandwich plates under bi-directional transverse sinusoidal load.***

s	Source	$\bar{\sigma}_x\left(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2}\right)$		$\bar{\sigma}_y\left(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2}\right)$		$\bar{\tau}_{xy}\left(0,0,\pm\frac{h}{2}\right)$		$\bar{\tau}_{xz}\left(0,\frac{b}{2},0\right)$	$\bar{\tau}_{yz}\left(\frac{a}{2},0,0\right)$	$\bar{w}\left(\frac{a}{2},\frac{b}{2},0\right)$
4	Partial FEM	1.5220	−1.4786	0.2530	−0.2464	−0.1262	0.1296	0.2376	0.1083	7.5962
	[a]Elasticity analysis	1.5560	−1.5120	0.2590	−0.2530	−0.1440	–	0.2390	0.1070	–
	[b]FEM-HOST	1.5230	–	0.2410	–	0.1420	–	0.2750	0.1137	7.1600
	[c]Mixed FEM	1.5480	–	0.2410	–	0.1340	–	0.2490	–	
	[d]Mixed FEM	1.5700	−1.5240	0.2600	−0.2550	–	–	0.2370	0.1040	
10	Partial FEM	1.1220	−1.1205	0.1080	−0.1080	−0.0630	0.0636	0.2982	0.0517	2.1860
	[a]Elasticity analysis	1.1530	−1.1520	0.1100	−0.1100	−0.071	0.0720	0.3000	0.0530	–
	[b]FEM-HOST	1.1660	–	0.1050	–	−0.0690	–	0.3400	0.0564	2.0870
	[c]Mixed FEM	1.2100	–	0.1110	–	−0.0710	–	0.3240	–	–
	[d]Mixed FEM	1.1590	−1.1580	0.1110	−0.1100	−0.0710	0.0720	0.3030	0.0550	–
20	Partial FEM	±1.0868		±0.0690		±0.0470		0.3122	0.0344	1.2162
	[a]Elasticity analysis	±1.1100		±0.0700		±0.0510		0.3170	0.0360	–
	[c]Mixed FEM	1.1730		0.0720		0.0520		0.3530	–	–
	[d]Mixed FEM	±1.1150		±0.0700		±0.0510		0.3170	0.0360	–
50	Partial FEM	±1.0700		±0.0560		±0.0421		0.3200	0.0290	0.8220

[a]Pagano [2], [b]Pandya and Kant [20], [c]Wu and Kuo [5], and [d]Ramtekkar et al. [11].
'–' indicates results are not available.

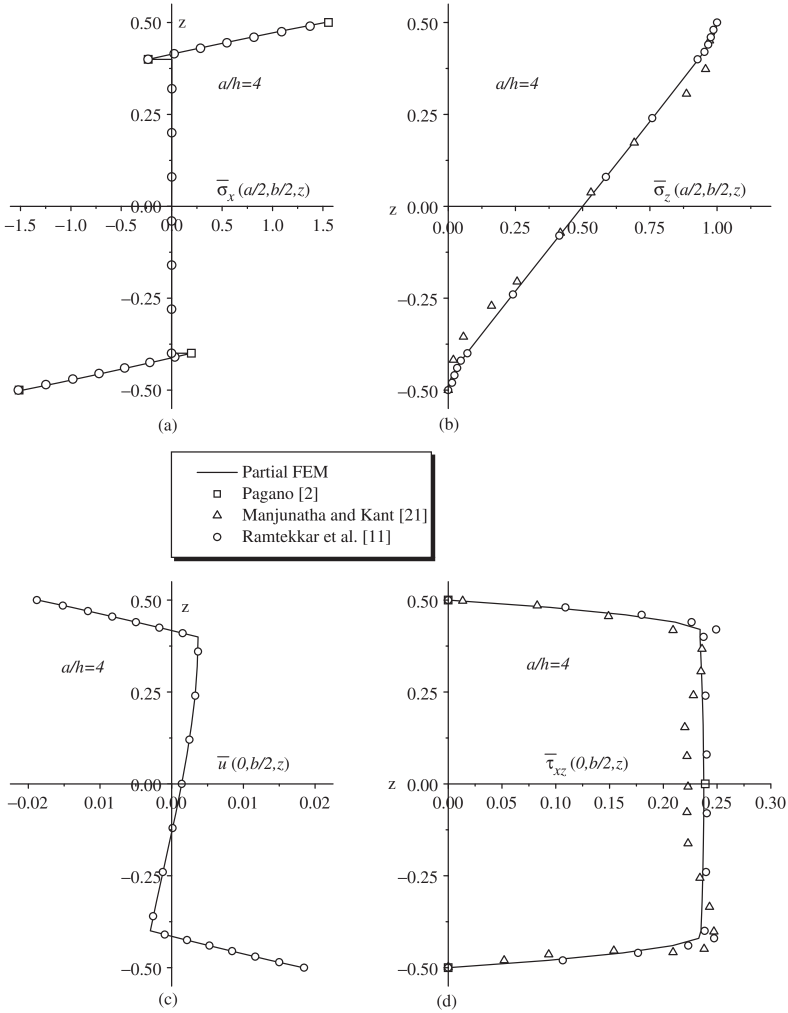


Figure 5. *Variation of normalized (a) in-plane normal stress $\bar{\sigma}_x$, (b) transverse normal stress $\bar{\sigma}_z$, (c) in-plane displacement $\bar{u}$, and (d) transverse shear stress $\bar{\tau}_{xz}$ through thickness of a simply supported, symmetric sandwich plate (0°/90°/0°) subjected to bidirectional sinusoidal load.*

Example 2

A square, unsymmetric sandwich plate consisting of angle-ply face sheets and flexible core (−45°/core/45°) with simple support end conditions on all four edges (Table 1) and bi-directional sinusoidal load (Equation (30)) on its top surface has been considered next for numerical investigation.

Table 4. Maximum stresses and transverse displacement ($\bar{\sigma}_x$, $\bar{\tau}_{xy}$, $\bar{\tau}_{xz}$, and $\bar{w}$) of unsymmetric (−45°/core/45°) square sandwich plates under transverse loading simply supported on all four edges.

s	Source	$\bar{\sigma}_x\left(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2}\right)$		$\bar{\tau}_{xy}\left(0,0,\pm\frac{h}{2}\right)$		$\bar{\tau}_{xz}\left(0,\frac{b}{2},0\right)$		$\bar{w}\left(\frac{a}{2},\frac{b}{2},0\right)$
4	Partial FEM	2.7746	−2.3294	−4.1390	0.7818	0.1166	0.6770 (0.44)	41.6800
10	Partial FEM	0.9622	−0.9529	−0.8256	0.7678	0.1684	0.2097 (0.44)	10.7694
20	Partial FEM	0.6201	−0.6205	−0.5613	0.5100		0.1933	5.0595
50	Partial FEM	0.5075	−0.5077	−0.4145	0.3941		0.2084	3.3118

Number within '()' indicates the position in the thickness dimension where the stress is maximum.

The thickness of each face sheets is $h/10$. The material properties are presented in Table 2(II). The ratio of shear modulus of the face sheet material with that of the core material is as high as 86.20 in the present example. Normalized in-plane normal stress ($\bar{\sigma}_x$), transverse shear stress ($\bar{\tau}_{xz}$), and transverse displacement ($\bar{w}$) for aspect ratios, 4, 10, 20, and 50 are tabulated in Table 4. Through thickness variations of normalized in-plane normal stress ($\bar{\sigma}_x$), in-plane shear stress ($\bar{\tau}_{xy}$), transverse shear stress ($\bar{\tau}_{yz}$), and transverse displacement ($\bar{w}$) have been shown graphically in Figure 6 for an aspect ratio of 4. The graphical results presented by Rao and Meyer-Piening [19] and Ramtekkar et al. [11] have been used for proper comparison. It is observed that though there is good agreement in the through thickness variation of numerical results for all quantities in the core region of the plate, there are small discrepancies in the face sheet regions (top as well as bottom face sheets).

Example 3

Clamped (Table 1) (a) symmetric (0°/core/0°) and (b) unsymmetric (−45°/core/45°) sandwich plates, subjected to bi-directional sinusoidal load (Equation (30)) on the top surface are considered here to show the ability of the present formulation to handle problems with general boundary conditions and high stress gradients. All geometrical and materials details are kept the same as in Examples 1 and 2. The numerical results of normalized in-plane normal stress ($\bar{\sigma}_x$), transverse shear stress ($\bar{\tau}_{xz}$), and transverse displacement ($\bar{w}$) for different aspect ratios are documented in Table 5. Through thickness variations of normalized in-plane stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{xz}$) for an aspect ratio of 4 are presented in Figures 7 and 8 for 0°/core/0° symmetric sandwich plate and −45°/core/45° unsymmetric sandwich plate, respectively. These results should serve as benchmark solutions for future reference.

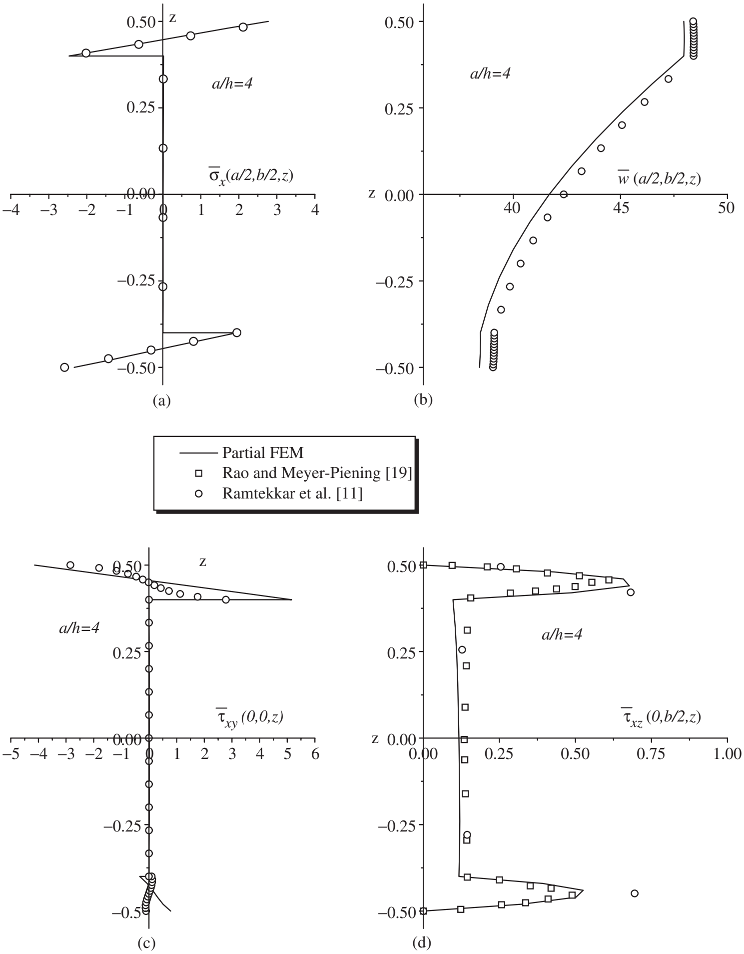


Figure 6. *Variation of normalized (a) in-plane normal stress $\bar{\sigma}_x$, (b) transverse displacement $\bar{w}$, (c) in-plane shear stress $\bar{\tau}_{xy}$, and (d) transverse shear stress $\bar{\tau}_{xz}$ through thickness of a simply supported, unsymmetric sandwich plate (−45°/90°/45°) subjected to bidirectional sinusoidal load.*

CONCLUSIONS

A novel, mixed partial FE formulation resulting in a solution of two-point BVP governed by a system of coupled first-order ODEs through thickness of plate is presented for the static analyses of sandwich plates for the first time. The present formulation is seen to be unique in its approach. The accuracy of the present model is evaluated by obtaining the solutions and comparing them with the available 3D solutions. Further, numerical investigation has also

Table 5. Normalized in-plane normal stress ($\bar{\sigma}_x$), transverse shear stresses ($\bar{\tau}_{xz}$), and transverse displacement ($\bar{w}$) of symmetric and unsymmetric clamped supported square sandwich plates subjected to bi-directional sinusoidal load.

0°/core/0° unsymmetric sandwich plate

s	Source	$\bar{\sigma}_x\left(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2}\right)$		$\bar{\tau}_{xz}\left(0,\frac{b}{2},\pm\frac{h}{2}\right)$		$\bar{w}\left(\frac{a}{2},\frac{b}{2},0\right)$
4	Partial FEM	0.6590	−0.6305	0.9433	0.9239	5.5864
10	Partial FEM	0.3954	−0.3932	1.0912	1.0941	1.2993
20	Partial FEM	0.3737	−0.3732	1.2667	1.2678	0.4935
50	Partial FEM	±0.3656		1.3408	1.3409	0.2142

−45°/core/45° unsymmetric sandwich plate

s	Source	$\bar{\sigma}_x\left(\frac{a}{2},\frac{b}{2};\pm\frac{h}{2}\right)$		$\bar{\tau}_{xz}\left(0,\frac{b}{2},\pm\frac{h}{2}\right)$		$\bar{w}\left(\frac{a}{2},\frac{b}{2},0\right)$
4	Partial FEM	1.9791	−1.6403	2.3597	1.7668	32.3627
10	Partial FEM	0.6346	−0.6468	0.8536	0.8444	8.0303
20	Partial FEM	0.3920	−0.3917	0.6687	0.6710	2.8666
50	Partial FEM	±0.2915		0.6686	0.6692	1.3117

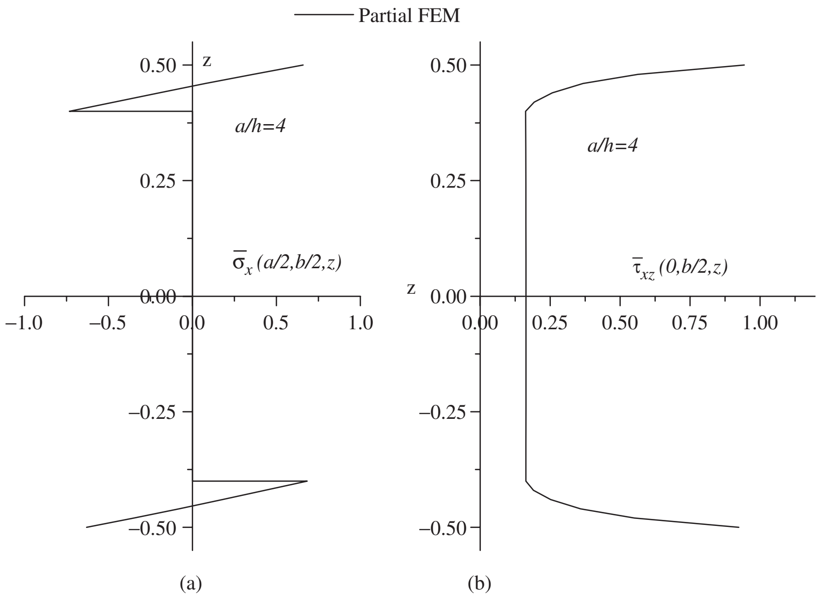


Figure 7. *Variation of normalized (a) in-plane normal stress $\bar{\sigma}_x$ and (b) transverse shear stress $\bar{\tau}_{xz}$ through thickness of a clamped supported, symmetric sandwich plate (0°/90°/0°) subjected to bidirectional sinusoidal load.*

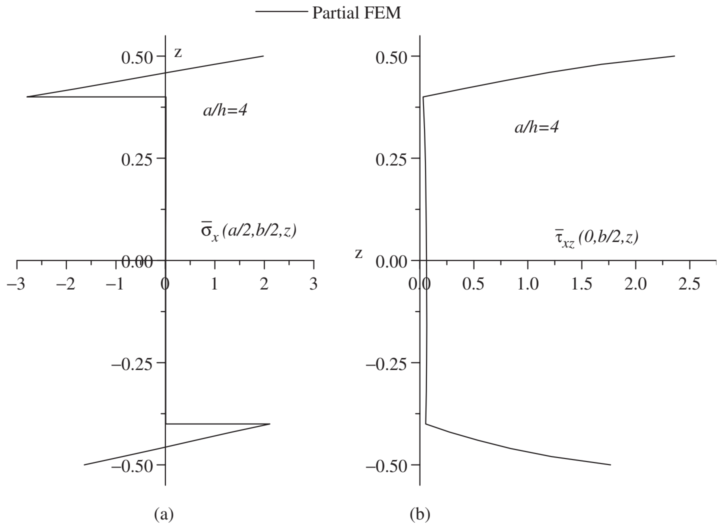


Figure 8. *Variation of normalized (a) in-plane normal stress $\bar{\sigma}_x$ and (b) transverse shear stress $\bar{\tau}_{xz}$ through thickness of a clamped supported, unsymmetric sandwich plate (−45°/90°/45°) subjected to bidirectional sinusoidal load.*

been carried out to show the generality of the formulation to handle different boundary conditions. The main advantage lies in the proposed model that both displacements and interlaminar stresses are evaluated simultaneously at a finite node with the same degree of accuracy through the numerical integration process. Post-processing module, which is required in other ESL models for the calculation of transverse stresses from in-plane stresses is altogether eliminated.

It is to be emphasized here, very clearly, that the semi-discrete form of Equation (27) is unique as long as all but one independent coordinates (z here) are discretized in a BVP. From this viewpoint also the proposed methodology can be considered as a novel and standard (modular) one.

APPENDIX I

Coefficients of [C] Matrix

$$C_{11} = \frac{E_1(1-\nu_{23}\nu_{32})}{\Delta}; \quad C_{12} = \frac{E_1(\nu_{21}+\nu_{31}\nu_{23})}{\Delta}; \quad C_{13} = \frac{E_1(\nu_{31}+\nu_{21}\nu_{32})}{\Delta};$$
$$C_{22} = \frac{E_2(1-\nu_{13}\nu_{31})}{\Delta}; \quad C_{23} = \frac{E_2(\nu_{32}+\nu_{12}\nu_{31})}{\Delta}; \quad C_{33} = \frac{E_3(1-\nu_{12}\nu_{21})}{\Delta};$$
$$C_{44} = G_{12}; \quad C_{55} = G_{13}; \quad C_{66} = G_{23}$$

where

$$\Delta = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31})$$

$$\text{and} \quad \frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}.$$

Coefficients of [Q] Matrix

$$Q_{11} = C_{11}c^4 + 2(C_{12} + 2C_{44})c^2s^2 + C_{22}s^4$$

$$Q_{12} = C_{12}(c^4 + s^4) + (C_{11} + C_{22} - 4C_{44})c^2s^2$$

$$Q_{13} = C_{13}c^2 + C_{23}s^2$$

$$Q_{14} = (C_{11} - C_{12} - 2C_{44})c^3s + (C_{12} - C_{22} + 2C_{44})cs^3$$

$$Q_{22} = C_{22}c^4 + 2(C_{12} + 2C_{44})c^2s^2 + C_{11}s^4$$

$$Q_{23} = C_{23}c^2 + C_{13}s^2$$

$$Q_{24} = (C_{12} - C_{22} + 2C_{44})c^3s + (C_{11} - C_{12} - 2C_{44})cs^3$$

$$Q_{33} = C_{33}$$

$$Q_{34} = (C_{31} - C_{32})cs$$

$$Q_{44} = (C_{11} - 2C_{12} + C_{22} - 2C_{44})c^2s^2 + C_{44}(c^4 + s^4)$$

$$Q_{55} = C_{55}c^2 + C_{66}s^2$$

$$Q_{56} = (C_{55} - C_{66})cs$$

$$Q_{66} = C_{55}s^2 + C_{66}c^2$$

where

$$c = \cos\alpha$$
$$s = \sin\alpha$$

and α is the angle between the fiber axis 1 and reference axis x (Figure 1(a)).

APPENDIX II

Coefficients of Vector, $p_i^e(x, y, z)$

$$p_{i4}^e = -\left[\begin{array}{l} \left(Q_{11} - \dfrac{Q_{13}Q_{31}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}u(x,y,z)}{\mathrm{d}x}n_x\mathrm{d}s + \left(Q_{44} - \dfrac{Q_{43}Q_{34}}{Q_{33}}\right) \\ \times \oint_s N_i(x,y)\dfrac{\mathrm{d}u(x,y,z)}{\mathrm{d}y}n_y\mathrm{d}s \\ +\left(Q_{14} - \dfrac{Q_{13}Q_{34}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}v(x,y,z)}{\mathrm{d}x}n_x\mathrm{d}s + \left(Q_{42} - \dfrac{Q_{43}Q_{32}}{Q_{33}}\right) \\ \times \oint_s N_i(x,y)\dfrac{\mathrm{d}v(x,y,z)}{\mathrm{d}y}n_y\mathrm{d}s \\ +\left(Q_{14} + Q_{41} - \dfrac{Q_{13}Q_{34}}{Q_{33}} - \dfrac{Q_{43}Q_{31}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}u(x,y,z)}{\mathrm{d}y}n_x\mathrm{d}s \\ +\left(Q_{12} + Q_{44} - \dfrac{Q_{13}Q_{32}}{Q_{33}} - \dfrac{Q_{43}Q_{34}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}v(x,y,z)}{\mathrm{d}y}n_x\mathrm{d}s \\ + \iint_A N_i(x,y)B_x(x,y,z)\mathrm{d}A \end{array}\right]$$

$$p_{i5}^e = -\left[\begin{array}{l} \left(Q_{41} - \dfrac{Q_{43}Q_{31}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}u(x,y,z)}{\mathrm{d}x}n_x\mathrm{d}s + \left(Q_{24} - \dfrac{Q_{34}Q_{23}}{Q_{33}}\right) \\ \times \oint_s N_i(x,y)\dfrac{\mathrm{d}u(x,y,z)}{\mathrm{d}y}n_y\mathrm{d}s \\ +\left(Q_{44} - \dfrac{Q_{43}Q_{34}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}v(x,y,z)}{\mathrm{d}x}n_x\mathrm{d}s + \left(Q_{22} - \dfrac{Q_{23}Q_{32}}{Q_{33}}\right) \\ \times \oint_s N_i(x,y)\dfrac{\mathrm{d}v(x,y,z)}{\mathrm{d}y}n_y\mathrm{d}s \\ +\left(Q_{12} + Q_{44} - \dfrac{Q_{23}Q_{31}}{Q_{33}} - \dfrac{Q_{43}Q_{34}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}u(x,y,z)}{\mathrm{d}x}n_y\mathrm{d}s \\ +\left(Q_{24} + Q_{42} - \dfrac{Q_{23}Q_{34}}{Q_{33}} - \dfrac{Q_{43}Q_{32}}{Q_{33}}\right)\oint_s N_i(x,y)\dfrac{\mathrm{d}v(x,y,z)}{\mathrm{d}x}n_y\mathrm{d}s \\ + \iint_A N_i(x,y)B_y(x,y,z)\mathrm{d}A \end{array}\right]$$

$$p_{i6}^e = -\iint_A N_i(x,y)B_z(x,y,z)\mathrm{d}A$$

where, n_x, n_y, and n_z are the direction cosine of the outward normal.

Coefficients of Diagonal Submatrices $[\mathbf{A}_s^e]$ (for $s = 1$–3)

$$\mathbf{A}_{01}^e = 2\mathbf{A}_{02}^e = 4\mathbf{A}_{03}^e$$
$$\mathbf{A}_{03}^e = a_{01}^e\mathbf{I}$$

where, $a_{01}^e = (l_{ex} l_{ey}/36)$ and **I** is 6×6 identity matrix.

Coefficients of Submatrices $[\mathbf{B}_j^e]$ (for $j = 1$–16)

$$\mathbf{B}_{01}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & 2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ k_{06}^e & k_{07}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{09}^e & k_{10}^e & 0 & 0 & 0 & k_{06}^e \\ k_{10}^e & k_{11}^e & 0 & 0 & 0 & k_{07}^e \\ 0 & 0 & 0 & 2k_{01}^e & 2k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{02}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{12}^e & k_{13}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{14}^e & k_{15}^e & 0 & 0 & 0 & k_{12}^e \\ k_{16}^e & k_{17}^e & 0 & 0 & 0 & k_{13}^e \\ 0 & 0 & 0 & -2k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{03}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ k_{18}^e & k_{19}^e & 0 & 0 & 0 & k_{08}^e \\ k_{20}^e & k_{21}^e & 0 & 0 & 0 & k_{18}^e \\ k_{21}^e & k_{22}^e & 0 & 0 & 0 & k_{19}^e \\ 0 & 0 & 0 & -k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{04}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{23}^e & k_{24}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{25}^e & k_{26}^e & 0 & 0 & 0 & k_{23}^e \\ k_{27}^e & k_{28}^e & 0 & 0 & 0 & k_{24}^e \\ 0 & 0 & 0 & k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{05}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{29}^e & k_{30}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{31}^e & k_{32}^e & 0 & 0 & 0 & k_{29}^e \\ k_{33}^e & k_{34}^e & 0 & 0 & 0 & k_{30}^e \\ 0 & 0 & 0 & 2k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{06}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & 2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ k_{35}^e & k_{36}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{37}^e & k_{38}^e & 0 & 0 & 0 & k_{35}^e \\ k_{38}^e & k_{39}^e & 0 & 0 & 0 & k_{36}^e \\ 0 & 0 & 0 & -2k_{01}^e & 2k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{07}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ k_{40}^e & k_{41}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{42}^e & k_{43}^e & 0 & 0 & 0 & k_{40}^e \\ k_{44}^e & k_{45}^e & 0 & 0 & 0 & k_{41}^e \\ 0 & 0 & 0 & -k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{08}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ k_{46}^e & k_{47}^e & 0 & 0 & 0 & k_{08}^e \\ k_{48}^e & k_{49}^e & 0 & 0 & 0 & k_{46}^e \\ k_{49}^e & k_{50}^e & 0 & 0 & 0 & k_{47}^e \\ 0 & 0 & 0 & k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{09}^e = \begin{bmatrix} 0 & 0 & k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ -k_{18}^e & -k_{19}^e & 0 & 0 & 0 & k_{08}^e \\ k_{20}^e & k_{21}^e & 0 & 0 & 0 & -k_{18}^e \\ k_{21}^e & k_{22}^e & 0 & 0 & 0 & -k_{19}^e \\ 0 & 0 & 0 & k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{10}^{e} = \begin{bmatrix} 0 & 0 & -k_{01}^{e} & 2k_{03}^{e} & -2k_{04}^{e} & 0 \\ 0 & 0 & 2k_{02}^{e} & -2k_{04}^{e} & 2k_{05}^{e} & 0 \\ -k_{23}^{e} & -k_{24}^{e} & 0 & 0 & 0 & 2k_{08}^{e} \\ k_{25}^{e} & k_{26}^{e} & 0 & 0 & 0 & -k_{23}^{e} \\ k_{27}^{e} & k_{28}^{e} & 0 & 0 & 0 & -k_{24}^{e} \\ 0 & 0 & 0 & -k_{01}^{e} & 2k_{02}^{e} & 0 \end{bmatrix}$$

$$\mathbf{B}_{11}^{e} = \begin{bmatrix} 0 & 0 & -2k_{01}^{e} & 4k_{03}^{e} & -4k_{04}^{e} & 0 \\ 0 & 0 & -2k_{02}^{e} & -4k_{04}^{e} & 4k_{05}^{e} & 0 \\ -k_{06}^{e} & -k_{07}^{e} & 0 & 0 & 0 & 4k_{08}^{e} \\ k_{09}^{e} & k_{10}^{e} & 0 & 0 & 0 & -k_{06}^{e} \\ k_{10}^{e} & k_{11}^{e} & 0 & 0 & 0 & -k_{07}^{e} \\ 0 & 0 & 0 & -2k_{01}^{e} & -2k_{02}^{e} & 0 \end{bmatrix}$$

$$\mathbf{B}_{12}^{e} = \begin{bmatrix} 0 & 0 & 2k_{01}^{e} & 2k_{03}^{e} & -2k_{04}^{e} & 0 \\ 0 & 0 & -k_{02}^{e} & -2k_{04}^{e} & 2k_{05}^{e} & 0 \\ -k_{12}^{e} & -k_{13}^{e} & 0 & 0 & 0 & 2k_{08}^{e} \\ k_{14}^{e} & k_{15}^{e} & 0 & 0 & 0 & -k_{12}^{e} \\ k_{16}^{e} & k_{17}^{e} & 0 & 0 & 0 & -k_{13}^{e} \\ 0 & 0 & 0 & 2k_{01}^{e} & -k_{02}^{e} & 0 \end{bmatrix}$$

$$\mathbf{B}_{13}^{e} = \begin{bmatrix} 0 & 0 & k_{01}^{e} & 2k_{03}^{e} & -2k_{04}^{e} & 0 \\ 0 & 0 & 2k_{02}^{e} & -2k_{04}^{e} & 2k_{05}^{e} & 0 \\ -k_{40}^{e} & -k_{41}^{e} & 0 & 0 & 0 & 2k_{08}^{e} \\ k_{42}^{e} & k_{43}^{e} & 0 & 0 & 0 & -k_{40}^{e} \\ k_{44}^{e} & k_{45}^{e} & 0 & 0 & 0 & -k_{41}^{e} \\ 0 & 0 & 0 & k_{01}^{e} & 2k_{02}^{e} & 0 \end{bmatrix}$$

$$\mathbf{B}_{14}^e = \begin{bmatrix} 0 & 0 & -k_{01}^e & k_{03}^e & -k_{04}^e & 0 \\ 0 & 0 & k_{02}^e & -k_{04}^e & k_{05}^e & 0 \\ -k_{46}^e & -k_{47}^e & 0 & 0 & 0 & k_{08}^e \\ k_{48}^e & k_{49}^e & 0 & 0 & 0 & -k_{46}^e \\ k_{49}^e & k_{50}^e & 0 & 0 & 0 & -k_{47}^e \\ 0 & 0 & 0 & -k_{01}^e & k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{15}^e = \begin{bmatrix} 0 & 0 & -2k_{01}^e & 2k_{03}^e & -2k_{04}^e & 0 \\ 0 & 0 & -k_{02}^e & -2k_{04}^e & 2k_{05}^e & 0 \\ -k_{29}^e & -k_{30}^e & 0 & 0 & 0 & 2k_{08}^e \\ k_{31}^e & k_{32}^e & 0 & 0 & 0 & -k_{29}^e \\ k_{33}^e & k_{34}^e & 0 & 0 & 0 & -k_{30}^e \\ 0 & 0 & 0 & -2k_{01}^e & -k_{02}^e & 0 \end{bmatrix}$$

$$\mathbf{B}_{16}^e = \begin{bmatrix} 0 & 0 & 2k_{01}^e & 4k_{03}^e & -4k_{04}^e & 0 \\ 0 & 0 & -2k_{02}^e & -4k_{04}^e & 4k_{05}^e & 0 \\ -k_{35}^e & -k_{36}^e & 0 & 0 & 0 & 4k_{08}^e \\ k_{37}^e & k_{38}^e & 0 & 0 & 0 & -k_{35}^e \\ k_{38}^e & k_{39}^e & 0 & 0 & 0 & -k_{36}^e \\ 0 & 0 & 0 & 2k_{01}^e & -2k_{02}^e & 0 \end{bmatrix}$$

where,

$$k_{01}^e = \frac{l_{ey}}{12}$$

$$k_{02}^e = \frac{l_{ex}}{12}$$

$$k_{03}^e = \frac{l_{ex}l_{ey}}{36}\left(\frac{Q_{66}}{Q_{55}Q_{66} - Q_{56}Q_{65}}\right)$$

$$k_{04}^e = \frac{l_{ex}l_{ey}}{36}\left(\frac{Q_{65}}{Q_{55}Q_{66} - Q_{56}Q_{65}}\right)$$

$$k_{05}^e = \frac{l_{ex}l_{ey}}{36}\left(\frac{Q_{55}}{Q_{55}Q_{66} - Q_{56}Q_{65}}\right)$$

$$k_{06}^e = \frac{l_{ey}}{6}\frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{07}^e = \frac{l_{ex}}{6}\frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{08}^e = \frac{l_{ex}l_{ey}}{36}\left(\frac{1}{Q_{33}}\right)$$

$$k_{09}^e = \frac{l_{ey}}{3l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) + \frac{l_{ex}}{3l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{10}^e = \frac{l_{ey}}{3l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{l_{ex}}{3l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{11}^e = \frac{l_{ey}}{3l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right) + \frac{l_{ex}}{3l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{12}^e = -\frac{l_{ey}}{6}\frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{13}^e = \frac{l_{ex}}{12}\frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{14}^e = -\frac{l_{ey}}{3l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{15}^e = -\frac{l_{ey}}{3l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{16}^{e} = -\frac{l_{ey}}{3l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$
$$+ \frac{l_{ex}}{6l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{17}^{e} = -\frac{l_{ey}}{3l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$
$$+ \frac{l_{ex}}{6l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{18}^{e} = -\frac{l_{ey}}{12}\frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{19}^{e} = -\frac{l_{ex}}{12}\frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{20}^{e} = -\frac{l_{ey}}{6l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right)$$
$$- \frac{l_{ex}}{6l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{21}^{e} = -\frac{l_{ey}}{6l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$
$$- \frac{l_{ex}}{6l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{22}^{e} = -\frac{l_{ey}}{6l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$
$$- \frac{l_{ex}}{6l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{23}^{e} = \frac{l_{ey}}{12}\frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{24}^{e} = -\frac{l_{ex}}{6}\frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{25}^{e} = \frac{l_{ey}}{6l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{26}^{e} = \frac{l_{ey}}{6l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{27}^{e} = \frac{l_{ey}}{6l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{28}^{e} = \frac{l_{ey}}{6l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{29}^{e} = \frac{l_{ey}}{6}\frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{30}^{e} = \frac{l_{ex}}{12}\frac{Q_{32}}{Q_{33}} + \frac{l_{ey}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{31}^{e} = -\frac{l_{ey}}{3l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{32}^{e} = -\frac{l_{ey}}{3l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{33}^{e} = -\frac{l_{ey}}{3l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{34}^{e} = -\frac{l_{ey}}{3l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{35}^e = -\frac{l_{ey}}{6}\frac{Q_{31}}{Q_{33}} + \frac{l_{ex}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{36}^e = \frac{l_{ex}}{6}\frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{37}^e = \frac{l_{ey}}{3l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) + \frac{l_{ex}}{3l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{38}^e = \frac{l_{ey}}{3l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{l_{ex}}{3l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{39}^e = \frac{l_{ey}}{3l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right) + \frac{l_{ex}}{6l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{40}^e = -\frac{l_{ey}}{12}\frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{6}\frac{Q_{34}}{Q_{33}}$$

$$k_{41}^e = -\frac{l_{ex}}{6}\frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{42}^e = \frac{l_{ey}}{6l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{43}^e = \frac{l_{ey}}{6l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{44}^e = \frac{l_{ey}}{6l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{45}^e = \frac{l_{ey}}{6l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right) - \frac{l_{ex}}{3l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right)$$

$$k_{46}^{e} = \frac{l_{ey}}{12}\frac{Q_{31}}{Q_{33}} - \frac{l_{ex}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{47}^{e} = \frac{l_{ex}}{12}\frac{Q_{32}}{Q_{33}} - \frac{l_{ey}}{12}\frac{Q_{34}}{Q_{33}}$$

$$k_{48}^{e} = -\frac{l_{ey}}{6l_{ex}}\left(Q_{11} - \frac{Q_{13}Q_{31}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{14} + Q_{41} - \frac{Q_{13}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{31}}{Q_{33}}\right) - \frac{l_{ex}}{6l_{ey}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right)$$

$$k_{49}^{e} = -\frac{l_{ey}}{6l_{ex}}\left(Q_{14} - \frac{Q_{13}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{12} + Q_{44} - \frac{Q_{13}Q_{32}}{Q_{33}} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) - \frac{l_{ex}}{6l_{ey}}\left(Q_{42} - \frac{Q_{43}Q_{32}}{Q_{33}}\right)$$

$$k_{50}^{e} = -\frac{l_{ey}}{6l_{ex}}\left(Q_{44} - \frac{Q_{43}Q_{34}}{Q_{33}}\right) + \frac{1}{4}\left(Q_{24} + Q_{42} - \frac{Q_{23}Q_{34}}{Q_{33}} - \frac{Q_{43}Q_{32}}{Q_{33}}\right) - \frac{l_{ex}}{6l_{ey}}\left(Q_{22} - \frac{Q_{23}Q_{32}}{Q_{33}}\right).$$

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