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Accurate numerical modeling for functionally graded (FG) cylinders of finite length subjected to thermo mechanical load

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A simplified and accurate analytical cum numerical model is presented here to investigate the behavior of FG cylinders of finite length subjected to thermo mechanical load. A diaphragm supported FG cylinder under symmetric thermal and mechanical load which is considered as a two dimensional (2D) plane strain problem of thermoelasticity in (r, z) direction. The boundary conditions are satisfied exactly in axial direction (z) by taking an analytical expression in terms of Fourier series expansion. Fundamental (basic) dependent variables are chosen in the radial coordinate of the cylinder. First order simultaneous ordinary differential equations are obtained as mathematical model which are integrated through an effective numerical integration technique by first transforming the BVP into a set of initial value problems (IVPs). For FG cylinders, the material properties have power law dependence in the radial coordinate. Effect of non homogeneity parameters on the stresses and displacements of FG cylinder are studied. The numerical results obtained are also first validated with existing literature for their accuracy. Stresses and displacements in axial and radial directions in cylinders having various l/r_i and r_o/r_i ratios parameter are presented for future reference.

KEYWORDS: Numerical integration; functionally graded materials; boundary value problem; thick cylinder.

The demand for improved structural efficiency in space structures and nuclear reactors has resulted in the development of a new class of materials, called functionally graded materials (FGMs). FGMs have become one of the major research topics in the mechanics of materials community during the past fifteen years. The concept of FGMs was proposed in 1984 by materials' scientists in the Sendai (Japan) area as a means of preparing thermal barrier materials[1]. Continuous changes in the composition, microstructure, porosity, etc. of these materials result in gradients in properties such as mechanical strength and thermal conductivity. Thus, FGMs are heterogeneous materials, characterized by spatially variable microstructures, and thus spatially variable macroscopic properties are introduced to enhance material or structural performance. Particularly, material properties can be designed to vary continuously along structural geometry to prevent delamination and stress concentration in traditional multilayered structures. The basic concept is to mix ceramic and metal such that the material properties continuously vary from one constituent material to the other. The spatially variable material properties make FGMs challenging to analyze. Before these material devices are used in engineering design, it is very important that these are analyzed very accurately. For such a reason, present study focuses the analysis of FG cylinders using the exact approach. The uniqueness of this approach is: it first requires algebraic manipulation of basic elasticity equations like equilibrium, strain displacement and constitute equations. After this manipulation, this becomes the two point boundary value problem which governs the behavior of finite length cylinder which is plane strain two dimensional problem in r, z plane and gives four first order simultaneous partial differential equations. This can be explained by the following equation[2,3].

$$y'(r) = A(r)y(r) + p(r) \qquad (1)$$

In the domain, $r_1 \leq r \leq r_2$, where, $y(r)$ is an n-dimensional vector of dependent variables; dependent variables in the present case can be described as $\underline{y} = (u, w, \sigma_r, \tau_{rz})^t$. Choice of dependent variables is an important task. The variables which naturally appear on r = constant are chosen as dependent variables; such variables are called intrinsic variables. Remaining variables are described as auxiliary dependent variables which are dependent on intrinsic dependent variables. $A(r)$ is a coefficient matrix of partial differential equations. $p(r)$ is an n-dimensional vector of non homogeneous (loading) terms. For boundary conditions, any $n/2$ elements of $y(r)$ are specified at the two termini edges; mixed type of boundary conditions can be specified in this type of formulation. Recently, Desai and Kant[4] have obtained accurate stresses in laminated finite length cylinders subjected to thermo elastic load using similar numerical model. Research results obtained thus far have demonstrated that FGMs have great potential for improving material/structural performance in many engineering applications precisely because of their spatially graded heterogeneous microstructure. Some of the recent literature relevant in this study is described as follows. Horgan and Chan[5] investigated the effects of material inhomogeneity in fundamental boundary-value problem of linear inhomogeneous isotropic pressurized hollow cylinder. The results are illustrated using a specific radially inhomogeneous material model for which explicit exact solutions are obtained. Chen et.al[6] considered the axisymmetric thermoelastic problem of a uniformly heated, functionally graded isotropic hollow cylinder and proposed an analytical form of solution. Ye et.al[7] studied the one-dimensional axisymmetric thermoelastic problem of a functionally graded transversely isotropic cylindrical shell and presented useful discussion and numerical results. Exact and explicit solution is derived. Tutuncu and Ozturk[8] obtained closed-form solutions for stresses and displacements in functionally graded cylindrical and spherical vessels subjected to internal pressure alone using the infinitesimal theory of elasticity. Jabbari et al[9] developed a general analysis procedure for tackling one-dimensional steady-state thermal stress problem of a hollow thick cylinder made of FGMs. Eslami et al[10] presented a general solution for the one-dimensional steady-state thermal and mechanical stresses in a hollow thick sphere made of functionally graded material. It is seen from the literature that accurate benchmark solutions using exact elasticity theory are rare for finite length cylinders under thermo-mechanical loadings. In this paper, governing elasticity equations of a simply (diaphragm) supported FG cylinder are used to predict its behaviour under longitudinally sinusoidal thermal and mechanical loads assuming that all material constants have a power-law dependence on the radial coordinate. By assuming a global analytical solution in the longitudinal direction satisfying the two end boundary conditions exactly. The equations are reformulated to enable application of an efficient and accurate numerical integration technique for the solution of the BVP of a cylinder in the radial coordinate. To enable application of numerical integration, BVP of a cylinder is converted into a set of IVPs. The basic approach to convert a BVP into a set of IVPs is also explained in the following sections. Finally, a comparison of the resulting stresses with the elasticity plane strain solution of infinitely long cylinder is carried out for ratios of the inner radius to outer radius of 1.5 and 1.05 for thermal loading and $r_i/r_o = 1/10$ for pressure loading; ratio of length to inner radius varying from 2 and 100. Results are validated with those given by Horgan and Chan[5].

In addition, one dimensional elasticity equations of an infinitely long axisymmetric cylinder are utilized to reformulate the mathematical model suitable for numerical integration. These equations are summarized in the Appendix. This has been done with a view to check and compares the results of the present formulation of finite length cylinder under uniform internal/external thermal and mechanical loads, when the length of the cylinder tends to infinity.

Problem Statement and Formulation

Basic governing equations of an axisymmetric cylinder in cylindrical coordinates are (Fig.1a).

Equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$
$$\frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0 \qquad (1a)$$

Strain displacement relations

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$
$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \tag{1b}$$

Strains in terms of stresses are

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_r + \sigma_\theta) + \alpha T$$
$$\varepsilon_r = \frac{\sigma_r}{E} - \frac{\nu}{E}(\sigma_z + \sigma_\theta) + \alpha T \tag{1c}$$
$$\varepsilon_\theta = \frac{\sigma_\theta}{E} - \frac{\nu}{E}(\sigma_z + \sigma_r) + \alpha T$$
$$\tau_{rz} = G_{rz}\gamma_{rz}$$

Stress-strains-temperature relations

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_r + \nu(\varepsilon_\theta + \varepsilon_z)\right] - \frac{\alpha T E}{1-2\nu}$$
$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_\theta + \nu(\varepsilon_r + \varepsilon_z)\right] - \frac{\alpha T E}{1-2\nu} \tag{1d}$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_r + \varepsilon_\theta)\right] - \frac{\alpha T E}{1-2\nu}$$
$$\tau_{rz} = G_{rz}\gamma_{rz}$$

It is now considered that all material constants have a power-law dependence on the radial coordinate, i.e.,

$E(r) = E^o\left(\frac{r}{r_i}\right)^n, \alpha(r) = \alpha^o\left(\frac{r}{r_i}\right)^n$ n = non dimensional arbitrary constant/ nonhomoginity parameter, E_o = constant parameter which has the same dimension as $E(r)$, r_i = inner radius. $E(r)$ and are function dependent on position. Spatial variation of Poisson's ratio is of much less practical significance than Young's modulus, hence, Poisson's ratio is assumed to be constant. This assumption, commonly made in the literature on FGMs, leads to considerable mathematical simplification. It can be easily proved that when the material is isotropic and if $n = 0$ for the homogeneous case, without taking thermal effect, results are same as given by Timoshenko and Goodier[11] for plane strain elasticity solution for Lame cylinder.

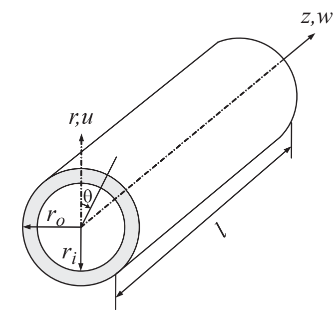

Fig. 1a Coordinate system and geometry of cylinder

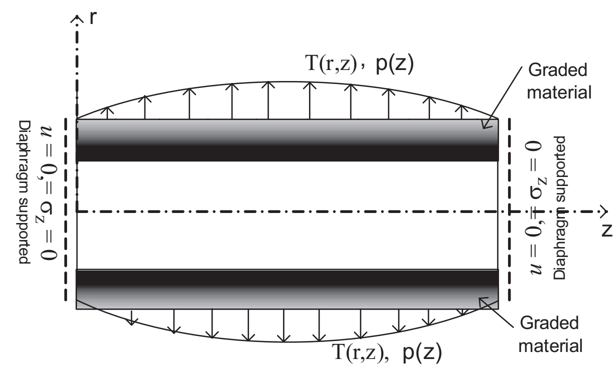

Fig. 1b Finite FG cylinder under sinusoidal external thermal and pressure loading

Stresses in terms of displacement components can be cast as follows:

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial r} + \nu\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right)\right] - \frac{\alpha T E}{1-2\nu}$$
$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{u}{r} + \nu\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right)\right] - \frac{\alpha T E}{1-2\nu} \tag{1e}$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right] - \frac{\alpha T E}{1-2\nu}$$
$$\tau_{rz} = G_{rz}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)$$

Boundary conditions in the longitudinal and radial directions are,

at $z = 0, l \quad u = 0; \sigma_z = 0;$ at $r = r_i$,
$\sigma_r = \tau_{rz} = 0$; at $r = r_o$, $\sigma_r = -p(z)$, $\tau_{rz} = 0$ (2)

where load can be represented in terms of Fourier series in general form as follows,

$$p(z) = \sum_{i=1,3,5..}^{N} p_i \sin\frac{i\pi z}{l} \tag{3a}$$

in which p_i is the Fourier load coefficient which can be determined by using the orthogonality conditions and for sinusoidal loading,

$$p(z) = p_0 \sin\frac{\pi z}{l} \tag{3b}$$

Radial direction r is chosen to be a preferred independent coordinate. Four fundamental dependent variables, viz., displacements, u and w and corresponding stresses σ_r, and τ_{rz} that occur naturally on a tangent plane r = constant, are chosen in the radial direction. Circumferential stress σ_θ and axial stress σ_z are treated here as auxiliary variables[4] since these are found to be dependent on the chosen fundamental variables. A set of four first order partial differential equations in independent coordinate r which involves only fundamental variables is obtained through algebraic manipulation of Eqs. (1a)-(1c). These are,

$$\frac{\partial u}{\partial r} = \frac{1}{(1-\nu)}\sigma_r - \frac{\nu}{1-\nu}\frac{u}{r} - \frac{\nu}{(1-\nu)}\frac{\partial w}{\partial z} + \frac{\alpha ET}{\lambda(1-\nu)(1-2\nu)} \tag{4a}$$

$$\frac{\partial w}{\partial r} = \frac{1}{G}\tau_{rz} - \frac{\partial u}{\partial z} \tag{4b}$$

$$\frac{\partial \sigma_r}{\partial r} = -\frac{\partial \tau_{rz}}{\partial z} - \frac{\sigma_r}{r} + \lambda\frac{u}{r^2}\frac{(1-2\nu)}{(1-\upsilon)} + \frac{\sigma_r}{r}\frac{\upsilon}{1-\upsilon} + \frac{\lambda}{r}\frac{\partial w}{\partial z}\frac{\upsilon(1-2\upsilon)}{(1-\upsilon)} - \frac{\alpha ET}{(1-\upsilon)} \tag{4c}$$

$$\frac{\partial \tau_{rz}}{\partial r} = -\frac{1}{r}\tau_{rz} - \frac{\lambda(1-2\upsilon)}{(1-\upsilon)}\frac{\partial^2 w}{\partial z^2} - \frac{\nu}{1-\nu}\frac{\partial \sigma_r}{\partial z} - \frac{\nu\lambda(1-2\nu)}{1-\nu}\frac{\partial}{\partial z}\left(\frac{u}{r}\right) - \alpha E\frac{(2\upsilon-1)}{(1-\upsilon)(1-2\upsilon)}\frac{\partial}{\partial z}\{T\} \tag{4d}$$

$$\text{where } \lambda = \frac{E}{(1+\nu)(1-2\nu)} \textit{ and } G = \frac{E}{2(1+\nu)}$$

and the auxiliary variables,

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\frac{u}{r} + \nu\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right)\right) - \frac{\alpha TE}{1-2\nu} \tag{5a}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\frac{\partial w}{\partial z} + \nu\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right) - \frac{\alpha TE}{1-2\nu} \tag{5b}$$

A longitudinally sinusoidal variation of temperature is assumed as follows,

$$T(r,z) = T_m \sin\frac{\pi z}{l} \tag{6}$$

Variations of the four fundamental dependent variables which completely satisfy the boundary conditions of simple (diaphragm) supports at z = 0, l can then be assumed as,

$$\begin{aligned} u(r,z) &= U(r)\sin\frac{\pi z}{l} \quad \sigma_r(r,z) = \sigma(r)\sin\frac{\pi z}{l} \\ w(r,z) &= W(r)\cos\frac{\pi z}{l} \quad \tau_{rz}(r,z) = \tau(r)\cos\frac{\pi z}{l} \end{aligned} \tag{7}$$

Substitution of Eq. (7) in Eqs. (4a)-(4d) and simplification resulting from orthogonality conditions of trigonometric functions leads to the following four simultaneous ordinary differential equations involving only fundamental variables. These are,

$$\frac{du}{dr} = \frac{1}{(1-\nu)\lambda}\sigma_r - \frac{\nu}{(1-\nu)}\frac{U}{r} + \frac{\nu}{(1-\nu)}\left(\frac{\pi}{l}\right)w + \frac{\alpha ET}{\lambda(1-\nu)(1-2\nu)} \tag{8a}$$

$$\frac{dw}{dr} = \frac{1}{G}\tau_{rz} - \left(\frac{\pi}{l}\right)U \tag{8b}$$

$$\frac{d\sigma_r}{dr} = \left(\frac{\pi}{l}\right)\tau_{rz} + \frac{\sigma_r}{r}\left[\frac{(2\upsilon-1)}{(1-\upsilon)}\right] + \lambda\frac{(1-2\upsilon)}{(1-\upsilon)}\frac{U}{r^2} - \lambda\left(\frac{\pi}{l}\right)\frac{\upsilon(1-2\upsilon)}{(1-\upsilon)}\frac{W}{r} + \frac{1}{r}\frac{\alpha ET}{(1-2\nu)}\left[\frac{2\upsilon-1}{1-\upsilon}\right] \tag{8c}$$

$$\frac{d\tau_{rz}}{dr} = -\frac{1}{r}\tau_{rz} + \left(\frac{\pi}{l}\right)^2\frac{\lambda(1-2\upsilon)}{(1-\upsilon)}W - \left(\frac{\nu}{1-\nu}\right)\left(\frac{\pi}{l}\right)\sigma_r - \frac{\nu\lambda(1-2\nu)}{1-\nu}\left(\frac{\pi}{l}\right)\frac{U}{r} - \left(\frac{\pi}{l}\right)\frac{\nu\alpha ET}{(1-\nu)(1-2\nu)} \tag{8d}$$

and the auxiliary variables,

$$\sigma_\theta = \lambda\left[(1-\nu)\frac{U}{r} + \nu\left(\begin{array}{l}\frac{\sigma}{\lambda(1-\nu)} - \frac{\nu}{(1-\nu)}\frac{U}{r} + \frac{\pi}{l} \\ \frac{\nu}{(1-\nu)}W + \frac{\alpha ET}{\lambda(1-\nu)(1-2\nu)} \\ -W\frac{\pi}{l}\end{array}\right)\right]$$

$$\sin\frac{\pi z}{l} \text{ - } \alpha T\frac{E}{(1-2\nu)}\sin\frac{\pi z}{l}$$

$$\sigma_z = \lambda\left[-(1-\nu)W\frac{\pi}{l} + \nu\left(\begin{array}{l}\frac{\sigma}{\lambda(1-\nu)} - \frac{\nu}{1-\nu}\frac{U}{r} + \\ \frac{\pi}{l}\frac{\nu}{(1-\nu)}W + \\ \frac{\alpha ET}{\lambda(1-\nu)(1-2\nu)} + \frac{U}{r}\end{array}\right)\right]$$

$$\sin\frac{\pi z}{l} \text{ - } \alpha T\sin\frac{\pi z}{l}$$

$$\text{where } \lambda = \frac{E}{(1+\nu)(1-2\nu)} \qquad \text{(9a-b)}$$

SOLUTION TECHNIQUE

The above system of first order simultaneous ordinary differential equations Eqs. (8a)-(8d) together with the appropriate boundary conditions at the inner and outer edges of the cylinder (Eq. (11)), forms a two-point BVP. However, a BVP in ODEs cannot be numerically integrated as only a half of the dependent variables (two) are known at the initial edge and numerical integration of an ODE is intrinsically an IVP. It becomes necessary to transform the problem into a set of IVPs. The initial values of the remaining two fundamental variables must be selected so that the complete solution satisfies the two specified conditions at the terminal boundary[12]. This technique has been successfully applied to solutions of plate[13-17]. However, problem of cylindrical coordinates is not covered in that literature using elasticity theory. The nth (N = 4 here) order BVP is transformed into a set of (N/2+1) IVPs. ODEs are integrated from initial edge to final edge using the initial values specified in Table 1. The N/2+1 solutions given in the Table 1 may be thought of as (i) one non-homogeneous integration which includes all the non-homogeneous terms (e.g., loading) and the known N/2 quantities at starting edge, with the unknown N/2 quantities at the starting edge set equal to zero, (ii) N/2 homogeneous integrations which are carried out by setting the known quantities at the starting edge as zero and choosing the N/2 unknown quantities at starting edge as unit values in succession and deleting the non-homogeneous terms from the ODEs. The solutions at the terminal boundary corresponding to the initial values are given in the right side columns in Table 1. A linear combination of the (N/2+1) solutions must satisfy the boundary conditions at the terminal edge, i.e.,

$$\begin{Bmatrix} Y_{3,0} \\ Y_{4,0} \end{Bmatrix} + \begin{pmatrix} Y_{3,1} & Y_{3,2} \\ Y_{4,1} & Y_{4,2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{Bmatrix} \overline{Y_3} \\ \overline{Y_4} \end{Bmatrix}$$

or

$$Y_{i,0} + Y_{i,j}X_j = \overline{Y_i} \qquad (10)$$

or

$$X_j = \left[Y_{i,j}\right]^{-1}\left(\overline{Y_i} - Y_{i,0}\right)$$

TABLE 1

INITIAL AND INTEGRATED VALUES

Integration number	Initial boundary				Terminal boundary				Load term
	u	w	σ_r	τ_{rz}	u	w	σ_r	τ_{rz}	
0	0	0	0(specified)	0(specified)	$Y_{1,0}$	$Y_{2,0}$	$Y_{3,0}$	$Y_{4,0}$	include
1	1	0	0	0	$Y_{1,1}$	$Y_{2,1}$	$Y_{3,1}$	$Y_{4,1}$	delete
2	0	1	0	0	$Y_{1,2}$	$Y_{2,2}$	$Y_{3,2}$	$Y_{4,2}$	delete
Final integration	X_1	X_2	0(specified)	0(specified)	Correct value	Correct value	Correct value	Correct value	include

where i indicates the n/2 variables consistent with the specified boundary values at terminal edge, j refers to solution number and ranges from 1 to n/2, $\overline{Y_i}$ is a vector of specified dependent variables at the terminal boundary and X_j is a vector of unknown dependent variables at the starting edge. Finally, a non-homogeneous integration with all the dependent variables known at the starting edge is carried out to get the desired results. Fourth order Runge-Kutta algorithm with modifications suggested by Gill[18] is used for the numerical integration of the IVPs. A computer code in FORTRAN 77 is written. A complete flowchart for numerical integration is given in Fig. 21.

RESULTS AND DISCUSSIONS

Example 1: FGM cylinder under pressure loading

A hollow cylinder is analyzed for external pressure loading by taking different ri/ro and l/ri ratios to cover both thick/thin and short/long cylinders. Three sets of numerical results are presented in Tables 2-3 and in Figs. 2-8, i. e., results from the present finite length cylinder formulation, computations on the analytical formulae available for infinitely long cylinder under plane strain condition[5] and numerically integrated values of the boundary value problem of the plane strain situation (1D), the first and the last are obtained by the specially developed numerical integration technique described in this paper.

Material properties and pressure loading are taken as follows.

$$\nu = 0,\ E_0 = 2 \times 10^8 \ \text{KN/m}^2,\ p_o = 1000 \ \text{KN/m}^2 \qquad (11)$$

Figure. 2-8 show the variation of radial and hoop stresses for r_i/r_o = 1/10 for various in-homogeneity parameters n = 0, 1/20, 1/5, 1/2 under external pressure loading. Radial and hoop stresses are compared with analytical solutions given by Horgan and Chan[5] for infinitely long cylinder shown in Figs. 2-5. It is clearly seen from Table 2-3, for higher l/r_i ratios, results are much closed to Horgan and Chan[5]. Eq. (13) represents the hoop and radial stress given by Horgan and Chan[5] and is used in the present work for comparison. When the cylinder is subjected to a sinusoidal pressure load, the results within the limited central length zone only are compared with the plane strain one dimensional solutions. Figs. 6-8 show radial and hoop stresses for different values of inhomogeneity parameters n. The effect of inhomogeneity parameter on stresses is clearly seen these figures. As seen from Figs. 6-7, for l/r_i ratios 20 and 1000, the value of radial hoop stress decreases for higher value of inhomogenity parameters. Thus, by selecting a proper value of this parameter n, it is possible to tailor the stresses as per the design requirements by engineers. Radial and hoop quantities are maximum at $z = l/2$ whereas axial quantities are maximum at $z = 0, l$.

TABLE 2A

COMPARISON OF NON-DIMENSIONAL RADIAL STRESS $\overline{\sigma_r}$ (z=l/2) THROUGH THICKNESS FOR DIAPHRAGM SUPPORTED ELASTIC CYLINDER UNDER PRESSURE LOADING FOR r_i/r_o = 1/10 AND n = 0 WITH ELASTICITY PLANE STRAIN SOLUTION FOR INFINITELY LONG CYLINDER AND FINITE CYLINDER FROM THE PRESENT WORK.

	Present - Finite length cylinder (2D) $\overline{\sigma_r}$ $(z = l/2)$			Analytical[9] and Present (1D)
$\bar{r}$	l/r_i =5	l/r_i =20	l/r_i =1000	
0.1	0.0000	0.0000	0.0000	0.0000
0.19	0.0506	0.6668	0.7268	0.7303
0.28	0.085	0.8134	0.8796	0.8813
0.37	0.1299	0.8742	0.9354	0.9363
0.46	0.1934	0.9101	0.9619	0.9624
0.55	0.283	0.9365	0.9764	0.9767
0.64	0.4057	0.9579	0.9852	0.9854
0.73	0.5651	0.9757	0.991	0.9911
0.82	0.7536	0.9896	0.995	0.9951
0.91	0.9337	0.9982	0.9979	0.9979
1	1.0000	1.0000	1.0000	1.0000

TABLE 2B

COMPARISON OF NON-DIMENSIONAL HOOP STRESS $\overline{\sigma_\theta}$ (z=l/2) THROUGH THICKNESS FOR DIAPHRAGM SUPPORTED ELASTIC CYLINDER UNDER PRESSURE LOADING FOR r_I/r_O = 1/10 AND n = 0 WITH ELASTICITY PLANE STRAIN SOLUTION FOR INFINITELY LONG CYLINDER AND FINITE CYLINDER FROM THE PRESENT WORK.

	Present- Finite length cylinder (2D) $\overline{\sigma_\theta}\ (z = l/2)$			Analytical[5]and Present (1D)
$\bar{r}$	l/r_i =5	l/r_i =20	l/r_i =1000	
0.1	0.1062	1.8278	2.0015	2.0202
0.19	0.0703	1.1814	1.2932	1.2899
0.28	0.0693	1.0441	1.1408	1.1389
0.37	0.0783	0.9963	1.085	1.0839
0.46	0.0942	0.9761	1.0586	1.0578
0.55	0.1174	0.9676	1.0441	1.0435
0.64	0.1489	0.9647	1.0352	1.0348
0.73	0.19	0.965	1.0295	1.0291
0.82	0.2414	0.967	1.0255	1.0251
0.91	0.3013	0.9697	1.0226	1.0223
1	0.3628	0.9724	1.0205	1.0202

TABLE 3A

COMPARISON OF NON-DIMENSIONAL RADIAL STRESS $\overline{\sigma_r}$ (z=l/2) THROUGH THICKNESS FOR DIAPHRAGM SUPPORTED ELASTIC CYLINDER UNDER PRESSURE LOADING FOR r_I/r_O = 1/10 AND n = 1/5 WITH ELASTICITY PLANE STRAIN SOLUTION FOR INFINITELY LONG CYLINDER AND FINITE CYLINDER FROM THE PRESENT WORK.

	Present - Finite length cylinder (2D) $\overline{\sigma_r}\ (z = l/2)$			Analytical[5] and Present (1D)
$\bar{r}$	l/r_i =5	l/r_i=20	l/r_i =1000	
0.1	0.0000	0.000	0.000	0.0000
0.19	0.0416	0.5609	0.6117	0.6148
0.28	0.0731	0.7120	0.7705	0.7720
0.37	0.1156	0.7876	0.8433	0.8442
0.46	0.1771	0.8388	0.887	0.8875
0.55	0.2652	0.8795	0.9173	0.9176
0.64	0.3876	0.9143	0.9403	0.9405
0.73	0.5488	0.9444	0.9589	0.9591
0.82	0.7416	0.9697	0.9746	0.9746
0.91	0.9285	0.9889	0.9881	0.9881
1	1.0000	1.0000	1.0000	1.0000

TABLE 3B

COMPARISON OF NON-DIMENSIONAL HOOP STRESS $\overline{\sigma_\theta}(z=l/2)$ THROUGH THICKNESS FOR DIAPHRAGM SUPPORTED ELASTIC CYLINDER UNDER PRESSURE LOADING FOR r_I/r_O = 1/10 AND n = 1/5 WITH ELASTICITY PLANE STRAIN SOLUTION FOR INFINITELY LONG CYLINDER AND FINITE CYLINDER FROM THE PRESENT WORK.

	Present- Finite length cylinder (2D) $\overline{\sigma_\theta}\ (z=l/2)$			Analytical[5] and Present (1D)
$\bar{r}$	l/r_i =5	l/r_i =20	l/r_i =1000	
0.1	0.082	1.4449	1.583	1.5939
0.19	0.0611	1.0514	1.1514	1.1486
0.28	0.0636	0.9868	1.0788	1.0772
0.37	0.0741	0.9776	1.0656	1.0646
0.46	0.0911	0.984	1.0685	1.0677
0.55	0.1153	0.9961	1.0765	1.0759
0.64	0.1483	1.0104	1.0862	1.0857
0.73	0.1915	1.0256	1.0963	1.0959
0.82	0.2459	1.0408	1.1063	1.1059
0.91	0.31	1.0555	1.1159	1.1156
1	0.3767	1.0692	1.1251	1.1248

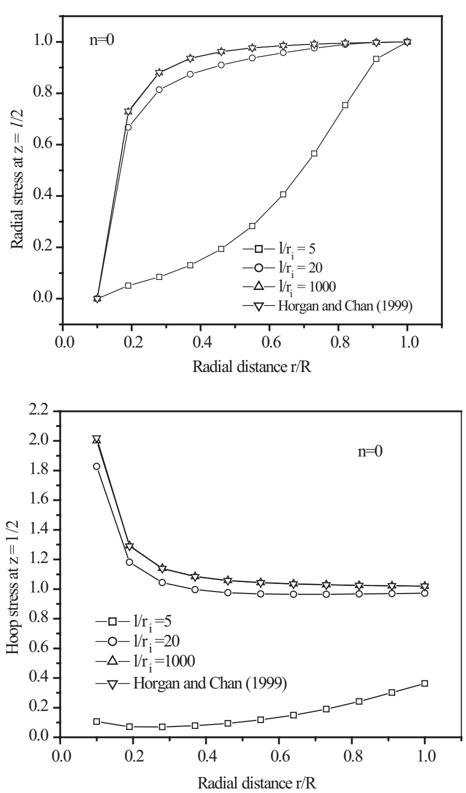

Fig. 2 Distribution of radial and hoop stress through thickness for n=0 and r_i/r_o =1/10 under pressure loading.

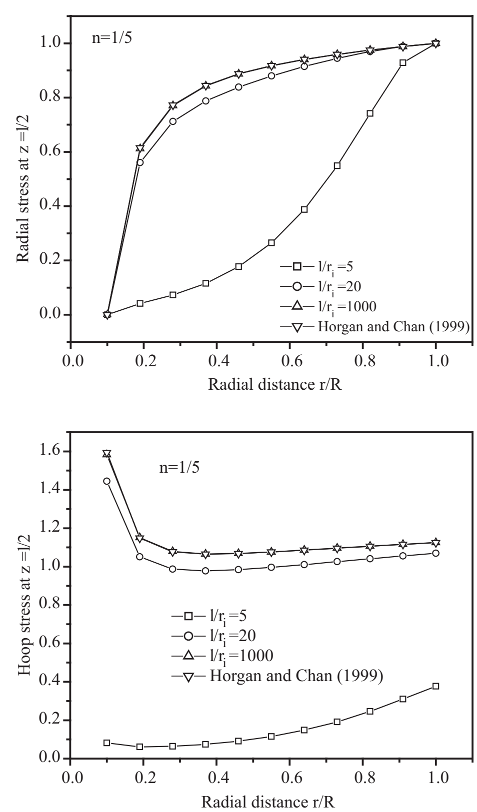

Fig. 3 Distribution of radial and hoop stress through thickness for n=1/5 and r_i/r_o =1/10 under pressure loading

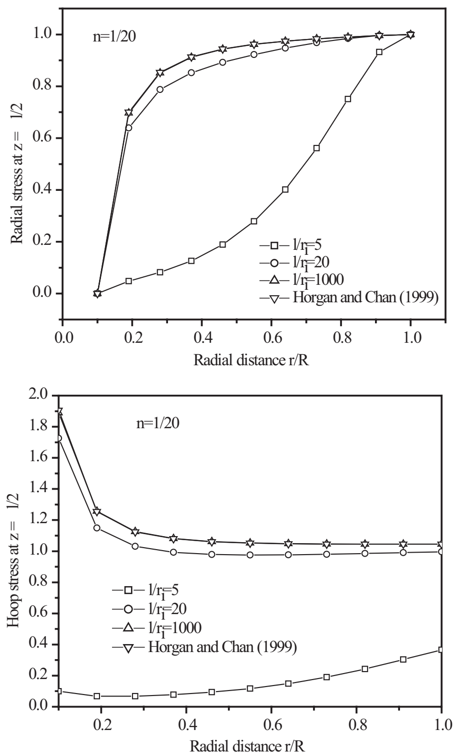

Fig. 4 Distribution of radial and hoop stress through thickness for n=1/20 and r_i/r_o =1/10 under pressure loading.

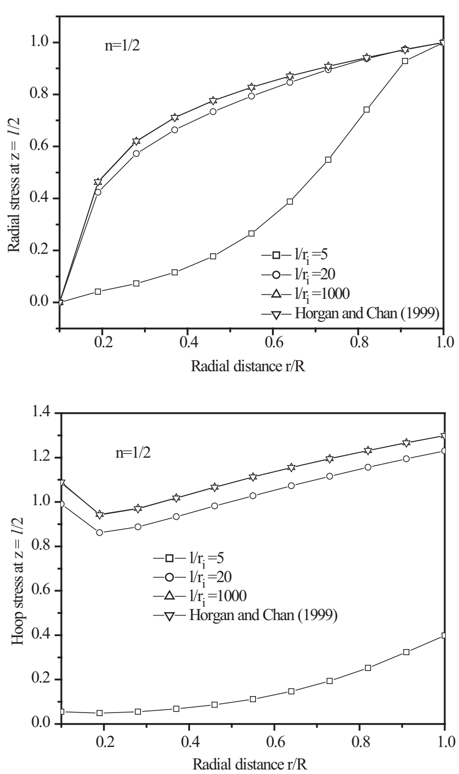

Fig. 5 Distribution of radial and hoop stress through thickness for n=1/2 and r_i/r_o =1/10 under pressure loading.

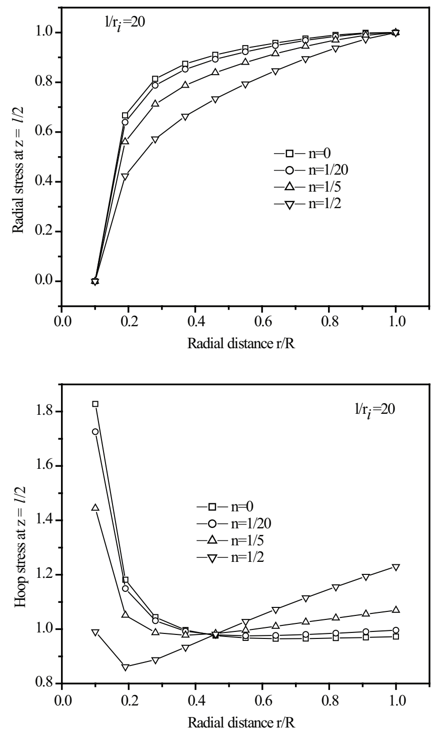

Fig. 6 Comparison of radial and hoop stress through thickness with different non-homogeneity parameters for r_i/r_o = 1/10 and l/r_i =20 under pressure loading.

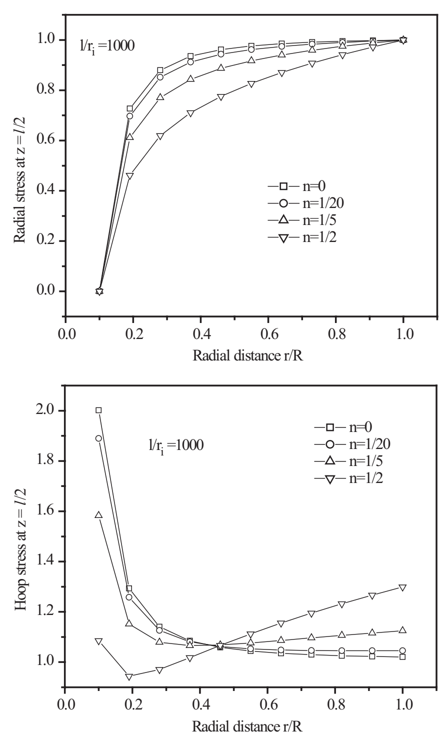

Fig. 7 Comparison of radial and hoop stress through thickness with different non-homogeneity parameters for r_i/r_o =1/10 and for l/r_i = 1000 under pressure loading.

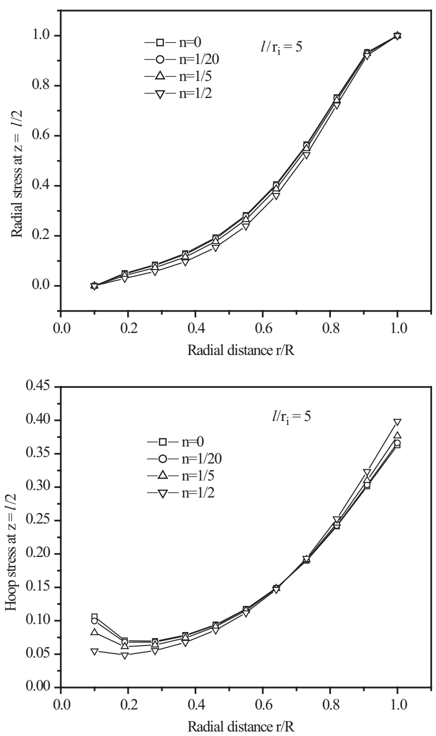

Fig. 8 Comparison of radial and hoop stress through thickness with different non-homogeneity parameters for r_i/r_o =1/10 for l/r_i =5 under pressure loading.

$$\sigma_r = -\frac{p_0 b^{(2+k-n)/2}}{b^k - a^k}\left[r^{(-2+k+n)/2} - a^k r^{(-2-k+n)/2}\right]$$

$$\sigma_\theta = -\frac{p_0 b^{(2+k-n)/2}}{b^k - a^k}\left[\frac{2+kv-nv}{k-n+2v} r^{(-2+k+n)/2} + \frac{2-kv-nv}{k+n-2v} a^k r^{(-2-k+n)/2}\right] \quad (13)$$

where $k=(n^2+4-4nv)^{1/2}$,

a=inner radius, b=outer radius

Example 2: FGM cylinder under thermal loading

A hollow cylinder is analyzed by taking two r_o/r_i ratios, 1.05 and 1.5, which cover both thick and thin cases. Material properties for isotropic material are taken as follows.

v=0.3, E_0=2×10^8 KN/m^2, α_0=2.306 × 10^{-6} l/°C (14)

Non-dimesnional parameters are chosen as follows under thermal loading

$$\bar{r} = \frac{r}{r_i}, (\bar{u}, \bar{w}) = \frac{1}{\alpha^o T r_i}(u, w), (\overline{\sigma_r}, \overline{\sigma_\theta}, \overline{\sigma_z}, \overline{\tau_{rz}}) = \frac{1}{E^o \alpha^o T}(\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}) \quad (15)$$

For thermal loading, numerical analysis is carried out taking various inhomogeniety parameter *n* as 2, 1, -1 and -2 for both l/ri ratios 2 and 50. It is seen from Figs. 9-20 that inhomogeniety parameter has greater effect on distribution of thermoelastic stresses. Negative n gives compressive nature of radial stress. Axial displacement is constant over the thickness for both r_o/r_i ratios. Radial displacement is linear through thickness, and assumes higher value for positive n as compared to negative n. Parabolic variation of shear stress and radial stress is seen in both thin and thick cases with zero values at inner and outer surfaces of the cylinder. Hoop and axial stresses are linear through the thickness in the case of a thin cylinder. This change to nonlinear parabolic when the cylinder turns thick.

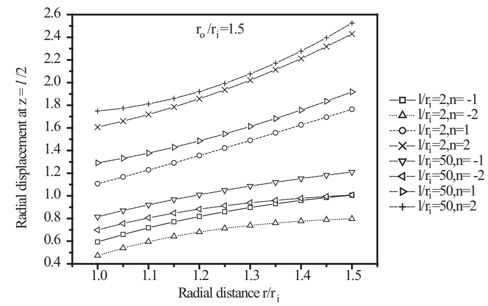

Fig. 9 Distribution of radial displacement $\bar{u}$ through thickness for functionally graded thick cylinder under thermal loading.

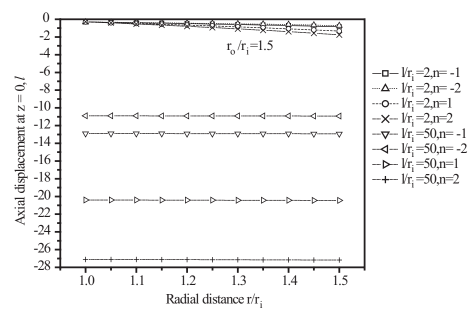

Fig. 10 Distribution of axial displacement $\bar{w}$ through thickness for functionally graded thick cylinder under thermal loading.

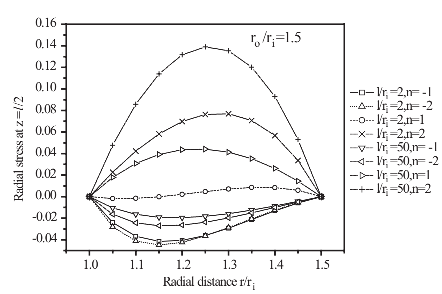

Fig. 11 Distribution of radial stress $\overline{\sigma_r}$ through thickness for functionally graded thick cylinder under thermal loading.

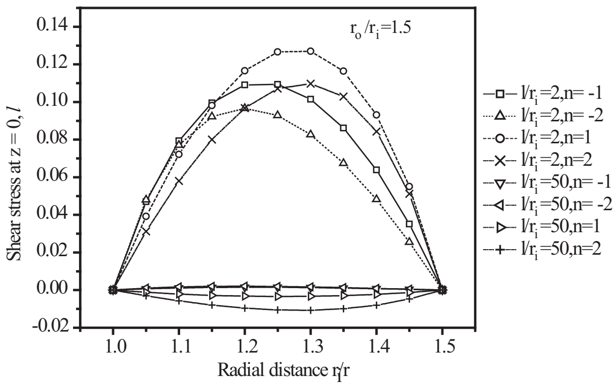

Fig. 12 Distribution of shear stress $\overline{\tau_{rz}}$ through thickness for functionally graded thick cylinder under thermal loading

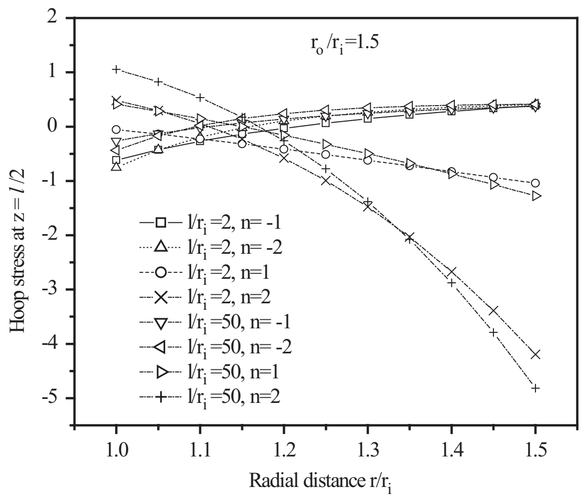

Fig. 13 Distribution of hoop stress $\overline{\sigma_\theta}$ through thickness for functionally graded thick cylinder under thermal loading.

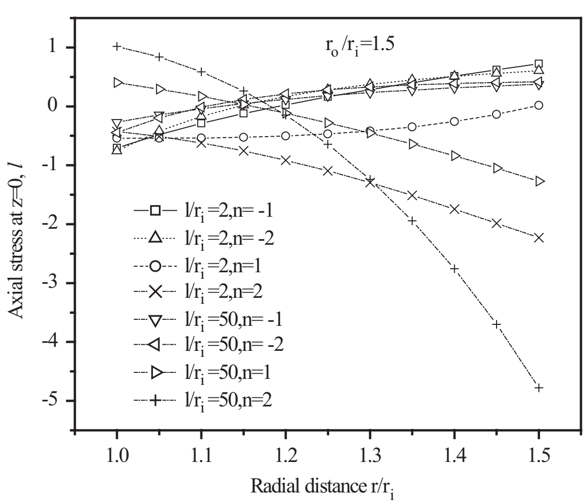

Fig. 14 Distribution of axial stress $\overline{\sigma_z}$ through thickness for functionally graded thick cylinder under thermal loading.

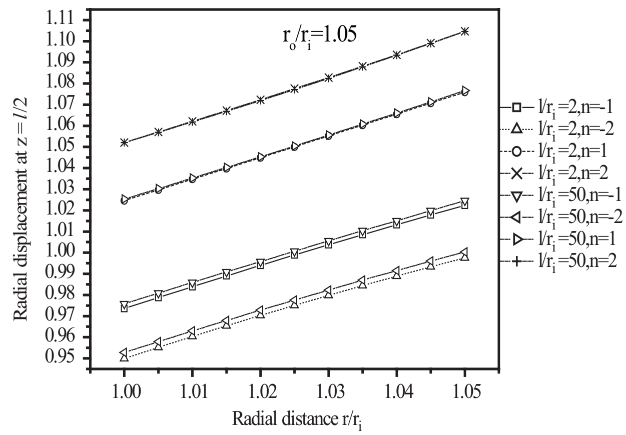

Fig. 15 Distribution of radial displacement $\overline{u}$ through thickness for functionally graded thin cylinder under thermal loading.

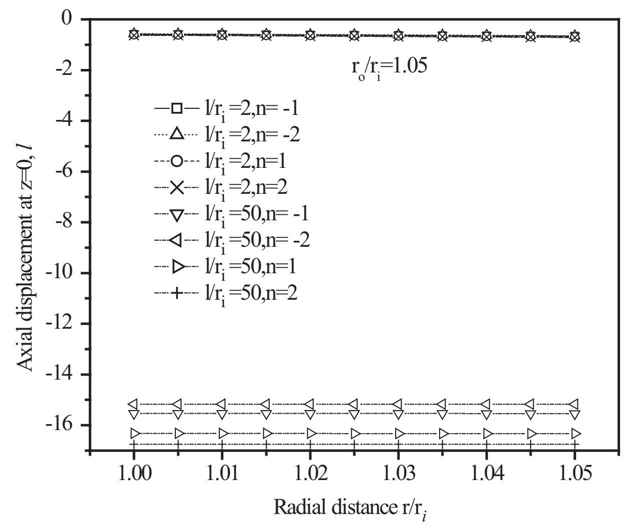

Fig. 16 Distribution of axial displacement $\overline{w}$ through thickness for functionally graded thin cylinder under thermal loading.

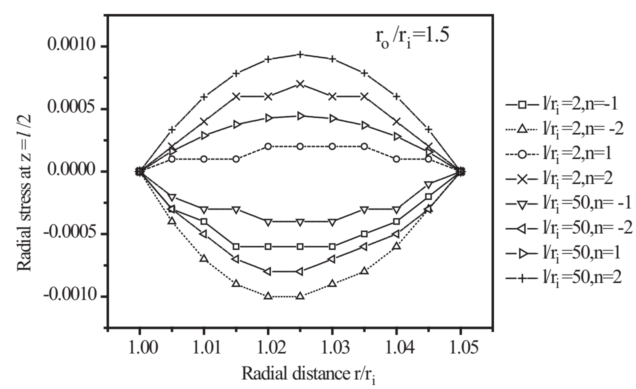

Fig. 17 Distribution of radial stress $\overline{\sigma_r}$ through thickness for functionally graded thin cylinder under thermal loading.

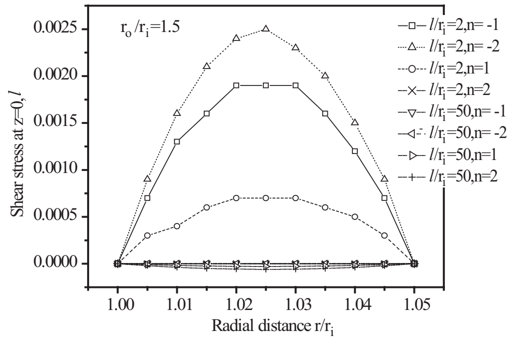

Fig. 18 Distribution of shear stress $\overline{\tau_{rz}}$ through thickness for functionally graded thin cylinder under thermal loading.

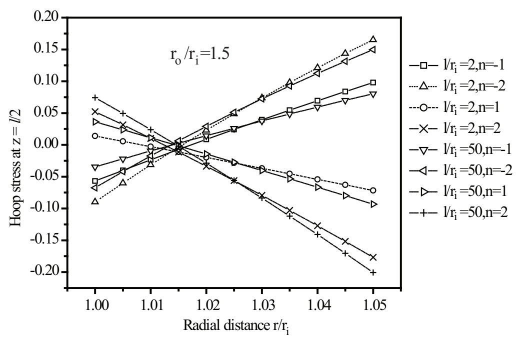

Fig. 19 Distribution of hoop stress $\overline{\sigma_{\theta}}$ through thickness for functionally graded thin cylinder under thermal loading

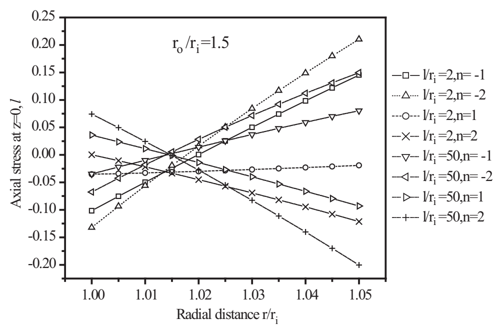

Fig. 20 Distribution of axial stress $\overline{\sigma_{z}}$ through thickness for functionally graded thin cylinder under thermal loading.

NOMENCLATURE

r, θ, z	Cylindrical coordinates
u,v,w	Displacement components
$\sigma_x, \sigma_\theta, \sigma_z$	Normal stress components parallel to r, θ, and z axis
τ_{Zr}	Shearing stress in cylindrical coordinates
$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$	Unit elongations (normal strain) components in cylindrical coordinates
γ_{zr}	Shearing strain component in cylindrical coordinates
$E(r)$	Power law variation of Young's modulus
$\alpha(r)$	Power law variation of Coefficient of thermal expansion
E^0	Young's modulus at inner radius
α^0	Coefficient of thermal expansion at inner radius
T	Temperature rise at any point in a cylinder
v	Poisson's ratio
r_i	Inner radius of the cylinder
r_0	Outer radius of the cylinder
l	Length of the cylinder
T_m	Initial reference temperature
$\bar{u}, \bar{w}$	Nondimensionalized displacement components
$\overline{\sigma_r}, \overline{\sigma_\theta}, \overline{\sigma_z}$,	Nondimensionalized normal stress components parallel to r, θ, and z axis
$\overline{\tau_{rz}}$	Nondimensionalized shearing stress in cylindrical coordinates
$\bar{r}$	Nondimensionalized radius
R	Mean radius $\frac{(r_0 + r_i)}{2}$

CONCLUSION

An attempt is made here to analyze the FG cylinders which are subjected to elastostatic and temperature fields through exact semi analytical cum numerical approach which differs from conventional approximate finite element approach and is also free from any assumptions in the theory. Results are very useful when one is designing pressurized cylinders made up of FG materials. This approach can be applied to very thick cylinders. Technique is very convenient to obtain the stresses with an ease, since no separate integration is required to account the non homogeneity effect occurred due to gradation. This is an important feature of the proposed model. Also, it involves mixed variables in the derivations, both stresses and displacements are obtained accurately simultaneously. Systematic development of mathematical model has significantly contributed in understanding the behavioural phenomenon of Graded cylinders under extreme loading environment of thermo-mechanical loadings. Mathematical model developed here is simple in nature and easily applicable for the large class of shell problems. Choice of fundamental variables is an important task for developing the model.

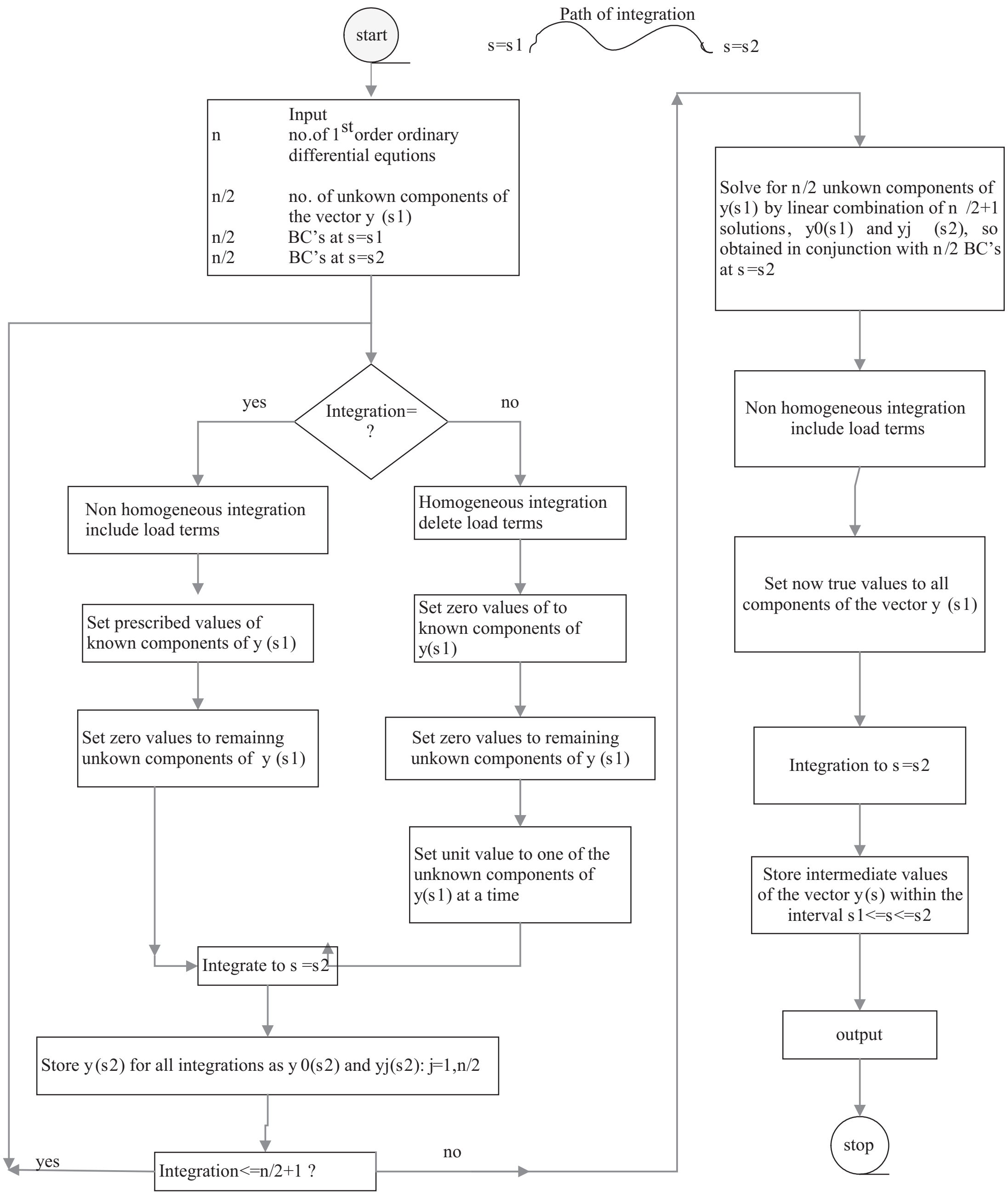

Fig. 21 Flowchart of numerical integration

Numerical results presented for different r_o/r_i and l/r_i ratios will be useful for future reference and can be used as benchmark results.

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APPENDIX

Mathematical model for infinitely long cylinder

- Equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1A}$$

- Strain-displacement relations

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \tag{2A}$$

- Stresses in terms of strains are

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_r + v\varepsilon_\theta\right] - \frac{\alpha TE}{1-2v}$$
$$\sigma_\theta = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_\theta + v\varepsilon_r\right] - \frac{\alpha TE}{1-2v} \quad (3A)$$

- First order two simultaneous ordinary differential equations

$$\frac{du}{dr} = \frac{\sigma_r}{\lambda(1-v)} - \frac{v}{1-v}\frac{u}{r} + \frac{\alpha TE}{\lambda(1-v)(1-2v)}$$
$$\frac{d\sigma_r}{dr} = \frac{\sigma_r}{r}\left(\frac{v}{1-v} - 1\right) + \frac{u}{r^2}\left(\frac{\lambda(1-2v)}{(1-v)}\right) + \frac{\alpha TE}{r(1-2v)}\left(\frac{v}{1-v} - 1\right) \quad (4A)$$

Where

$$E = E(r) = E^o\left(\frac{r}{r_i}\right)^n, \alpha = \alpha(r) = \alpha^o\left(\frac{r}{r_i}\right)^n$$

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